Abstract

We modify Grayson’s model of $K_1$ of an exact category to give a presentation whose generators are binary acyclic complexes of length at most $k$ for any given $k \geq 2$. As a corollary, we obtain another, very short proof of the identification of Nenashev’s and Grayson’s presentations.

1. Introduction

Let $\mathcal{N}$ be an exact category. Algebraic descriptions of Quillen’s $K_1$-group of $\mathcal{N}$ in terms of explicit generators and relations have been given by Nenashev [Nen98] and Grayson [Gra12]. The generators in both descriptions are so-called binary acyclic complexes in $\mathcal{N}$. While Nenashev uses complexes of length at most 2, Grayson’s generators are of arbitrary (finite) length. An algebraic proof of the fact that these two descriptions agree has been given in [KW]. In this paper, we will give another presentation of $K_1(\mathcal{N})$; this time the generators are binary acyclic complexes of length at most $k$ for any $k \geq 2$, see Definition 2.2 and Theorem 2.4. A motivating question behind this new description is to determine the precise relations that in addition to Grayson’s relations need to be divided out when restricting the generators in Grayson’s description to complexes of length at most $k$. If $k = 2$, our relations are special cases of Nenashev’s relations. In this sense, our presentation simplifies Nenashev’s presentation. All this leads to a new, natural and sleek algebraic proof of the fact that Nenashev’s and Grayson’s descriptions agree, see Section 4.

The proof of our main result, see Section 3, basically proceeds by induction on $k$. The crucial ingredient in the inductive step is a shortening procedure for binary acyclic complexes, see Definition 3.5, as discovered by Grayson and also used in [KW]. The main new idea in this paper is the comparatively short and simple way of showing how this shortening procedure yields an inverse to passing from complexes of length $k$ to complexes of length $k + 1$, see Proposition 3.9.
Acknowledgments

We thank the referee for their quick and careful report and for a question, which we answer in Remark 4.3.

2. Background and statement of main theorem

We recall a binary acyclic complex $P = (P_*, d, d')$ in $\mathcal{N}$ is a graded object $P_*$ in $\mathcal{N}$ supported on a finite subset of $[0, \infty)$ together with two maps $d, d': P_* \to P_*$ of degree $-1$ such that both $(P, d)$ and $(P_*, d')$ are acyclic chain complexes in $\mathcal{N}$. Here, acyclic means that each differential $d_n: P_n \to P_{n-1}$ admits a factorisation into an admissible epimorphism followed by an admissible monomorphism $P_n \twoheadrightarrow J_n \rightarrowtail P_{n-1}$ such that $J_n \rightarrowtail P_n \twoheadrightarrow J_n$ is a short exact sequence in $\mathcal{N}$ for every $n$. The differentials $d$ and $d'$ are called the top and bottom differential. We also write $P^\top$ and $P^\bot$ for the complexes $(P_*, d)$ and $(P_*, d')$. If $d = d'$, we call $P$ a diagonal binary acyclic complex.

A morphism between binary acyclic complexes $P$ and $Q$ is a degree 0 map between the underlying graded objects which is a chain map with respect to both differentials. According to [Gra12, Section 3], the obvious definition of short exact sequences turns the category of binary acyclic complexes into an exact category. We denote its Grothendieck group by $B_1(\mathcal{N})$.

Definition 2.1.

(a) A binary ladder is a quadruple $(P, Q, \sigma, \tau)$ consisting of two binary complexes $P$ and $Q$ together with isomorphisms $\sigma: P^\top \cong Q^\top$ and $\tau: P^\bot \cong Q^\bot$.

(b) Any two isomorphisms $\alpha, \beta: P \cong Q$ in $\mathcal{N}$ define a binary acyclic complex $P \xrightarrow[\alpha]{\beta} Q$ in $\mathcal{N}$ supported on $[0, 1]$. The corresponding element in any of the groups $B_1(\mathcal{N})$, $K_1(\mathcal{N})$ or $B_k^\top(\mathcal{N})$, $K_k^\top(\mathcal{N})$ for $k \geq 1$ is denoted by $\langle \alpha, \beta \rangle$.

(c) Define $L_k^\top(\mathcal{N})$ to be the quotient of $K_k^\top(\mathcal{N})$ obtained by additionally imposing
the relation

\[ Q - P = \sum_{i=0}^{k} (-1)^i \langle \sigma_i, \tau_i \rangle \]

for every binary ladder \((P, Q, \sigma, \tau)\) in \(N\) such that \(P\) and \(Q\) are supported on \([0, k]\), \(P_i = Q_i\) and all \(\sigma_i\) and \(\tau_i\) are involutions, i.e. \(\sigma_i^2 = \text{id} = \tau_i^2\).

**Remark 2.3.** The object \(L^k_1(N)\) has of course nothing to do with \(L\)-Theory; the \(L\) here is rather meant to refer to ‘ladder’. It will follow from Proposition 3.9, that imposing the ladder relation in Definition 2.2(c) not just for \(\sigma_i, \tau_i\) involutions but for all automorphisms yields an isomorphic group.

The following theorem is the precise formulation of our main result. Note that Lemma 3.3 below implies that we have a natural map \(L^k_1(N) \to K_1(N)\).

**Theorem 2.4.** The canonical map

\[ L^k_1(N) \to K_1(N) \]

is an isomorphism for every \(k \geq 2\).

Section 3 contains the proof of Theorem 2.4.

### 3. Proof of the main theorem

**Definition 3.1.** For any object \(P\) in \(N\) let

\[ \tau_P := \left( \begin{array}{cc} 0 & \text{id} \\ \text{id} & 0 \end{array} \right) : P \oplus P \xrightarrow{\sim} P \oplus P \]

denote the automorphism of \(P \oplus P\) which switches the two summands.

Note that assigning \(\langle \text{id}_{P \oplus P}, \tau_P \rangle\) with any object \(P \in N\) defines a homomorphism \(K^0_0(N) \to B^1_1(N)\). In particular, \(\langle \text{id}_{P \oplus P}, \tau_P \rangle = \langle \text{id}_{Q \oplus Q}, \tau_Q \rangle\) if \(P\) and \(Q\) represent the same element in \(K^0_0\). As an aside, we remark that \(\langle \text{id}_{P \oplus P}, \tau_P \rangle\) is equal to \(\langle \text{id}_{P \oplus P}, -\text{id}_{P} \rangle\) in \(B^1_1(N)\) and of order at most two in \(L^1_1(N)\); both of these two facts are easy to prove but won’t be used in this paper.

**Lemma 3.2.** Let \(P\) be a binary acyclic complex in \(N\) supported on \([0, k]\) and let \(\text{sw}(P)\) denote the binary complex obtained from \(P\) by switching top and bottom differential. Then we have

\[ \text{sw}(P) = -P \quad \text{in} \quad L^k_1(N). \]

**Proof.** We have \(P + \text{sw}(P) = P \oplus \text{sw}(P)\) in \(B^k_1(N)\). The latter complex represents \(0\) in \(L^k_1(N)\). To see this, consider the binary ladder \((P \oplus \text{sw}(P), D, \sigma, \tau)\) where \(D\) is the diagonal complex with \(D^\top = D^\perp = (P \oplus \text{sw}(P))^\top, \sigma = \text{id}\) and \(\tau\) switches the two summands \(P\) and \(\text{sw}(P)\), and note that \(\sum_{i=0}^{k} (-1)^i P_i = 0\) in \(K^0_0(N)\).

Regarding binary acyclic complexes supported on \([0, k]\) as complexes supported on \([0, k + 1]\] defines a homomorphism

\[ i_k : L^k_1(N) \to L^{k+1}_1(N). \]

The following lemma is basically a special case of the generalised Nenashev relation, see Definition 4.1 below and [Har15, Proposition 2.12]. We include a short proof to
convince the reader that complexes supported on \([0, k+1]\) suffice to prove the desired relation. As usual, we write \(\mathbb{P}[1]\) for the complex shifted by 1 (without changing the sign of the differentials \(d\) and \(d'\)) so that \(\mathbb{P}[1]_0 = 0\).

**Lemma 3.3.** The homomorphism \(i_k\) naturally factorises as

\[ i_k : L_1^k(\mathcal{N}) \to K_1^{k+1}(\mathcal{N}) \to L_1^{k+1}(\mathcal{N}), \]

where the second map is the canonical epimorphism.

**Proof.** Let \((\mathbb{P}, \mathbb{Q}, \sigma, \tau)\) be a binary ladder with \(\mathbb{P}, \mathbb{Q}\) supported on \([0, k]\). Then all rows and columns of the diagram

\[ \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \]
\[ \sigma_2 \downarrow \tau_2 \downarrow \sigma_1 \downarrow \tau_1 \downarrow \sigma_0 \downarrow \tau_0 \]
\[ \cdots \rightarrow Q_2 \rightarrow Q_1 \rightarrow Q_0 \]

are binary acyclic complexes, top differentials commute with top differentials and bottom differentials commute with bottom differentials. Filtering the associated total complex \(T\) (which is a binary acyclic complex supported on \([0, k+1]\)) “horizontally and vertically” then yields the relation

\[ \mathbb{Q} + \mathbb{P}[1] = T = \sum_{i=0}^{k} \langle \sigma_i, \tau_i \rangle[i] \]

in \(B_{1}^{k+1}(\mathcal{N})\). If \(\mathbb{P} = \mathbb{Q}\) and \(\sigma = \tau = \text{id}\), this shows that

\[ \mathbb{P}[1] = -\mathbb{P} \]

(3.4)

in \(K_{1}^{k+1}(\mathcal{N})\). The two equalities above finally show that

\[ \mathbb{Q} - \mathbb{P} = \sum_{i=0}^{k} (-1)^i \langle \sigma_i, \tau_i \rangle \]

in \(K_{1}^{k+1}(\mathcal{N})\), as desired. \(\square\)

**Definition 3.5.** Let \(k \geq 2\) and let \(\mathbb{P} = (P_*, d, d')\) be a binary acyclic complex supported on \([0, k+1]\), and choose factorisations

\[ d_2 : P_2 \rightarrow J \rightarrow P_1 \quad \text{and} \quad d'_2 : P_2 \rightarrow K \rightarrow P_1 \]

witnessing that \((P_*, d)\) and \((P_*, d')\) are acyclic. In the following, we denote the maps \(P_2 \rightarrow J\) and \(P_2 \rightarrow K\) again by \(d_2\) and \(d'_2\).

The **Grayson shortening of** \(\mathbb{P}\) is the binary acyclic complex \(\text{sh}(\mathbb{P})\) supported on \([0, k]\) whose top component is given by

\[ \cdots \rightarrow P_3 \rightarrow P_2 \rightarrow J \rightarrow P_1 \rightarrow P_0 \]
\[ \oplus \rightarrow K \rightarrow J \rightarrow P_0 \]
\[ \oplus \rightarrow P_0 \rightarrow K \rightarrow P_0 \]
and whose bottom component is

\[
\cdots \rightarrow P_3 \xrightarrow{d_3'} P_2 \xrightarrow{d_2'} K \\
\oplus J \xrightarrow{id} J \\
K \xrightarrow{id} K \\
J \xrightarrow{id} P_1 \xrightarrow{d_1} P_0
\]

Note that we have permuted the summands in the bottom component for better legibility, but that we consider the summation order in the top component to be the definitive one.

**Remark 3.6.** Note that if \( P \) is supported on \([0, 1]\) then \( \text{sh}(P) = \text{sw}(P) \) and thus \( \text{sh}(P) = -P \in L^1_1(N) \) in this case by Lemma 3.2.

**Remark 3.7.** The complex \( \text{sh}(P) \) appears in handwritten notes by Grayson and has also been used in [KW, Section 5]. Note that our definition of \( \text{sh}(P) \) includes a shift by \(-1\) so \( \text{sh}(P) \) is supported on \([0, k]\) rather than on \([1, k + 1]\). This avoids bulky notations later.

For \( P, J \) and \( K \) as in Definition 3.5, we have \( \langle \text{id}, \tau_J \rangle = \langle \text{id}, \tau_K \rangle \) because \( J = K \) in \( K_0(N) \). We denote the latter element by \( \tau_P \). If, in fact, \( J \cong K \), we replace the morphisms \( P_2 \rightarrow K \) and \( K \rightarrow P_1 \) with \( P_2 \rightarrow J \) and \( J \rightarrow P_1 \) by composing them with a fixed isomorphism between \( J \) and \( K \). Then the ordinary (non-naive) truncations

\[
t_{\geq 1}(P) := ( \cdots \rightarrow P_3 \rightarrow J )
\]

and

\[
t_{\leq 2}(P) := ( \rightarrow P_1 \rightarrow P_0 )
\]

are binary acyclic complexes again. In this case, the following crucial lemma computes \( \text{sh}(P) \in L^k_1(N) \) in terms of these truncations and \( \tau_P \).

**Lemma 3.8.** Let \( P \) be a binary acyclic complex supported on \([0, k + 1]\) and suppose that \( J \cong K \). Then we have

\[
\text{sh}(P) = t_{\geq 1}(P)[-1] - t_{\leq 2}(P) - \tau_P \quad \text{in} \quad L^k_1(N).
\]

In particular:

(a) If \( P \) is a diagonal complex, then \( \text{sh}(P) = -\tau_P \) in \( L^k_1(N) \).

(b) If \( P_0 = 0 \), then \( \text{sh}(P) = P[-1] - \tau_P \) in \( L^k_1(N) \).

**Proof.** Without permuting any summands in the bottom component, \( \text{sh}(P) \) has top component

\[
\cdots \rightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} J \\
\oplus J \xrightarrow{id} J \\
J \xrightarrow{id} P_1 \xrightarrow{d_1} P_0
\]

and bottom component

\[ \cdots \rightarrow P_3 \xrightarrow{d_3'} P_2 \xrightarrow{d_2'} J \oplus P_1 \xrightarrow{d_1} P_0 \]

Now consider the binary ladder

\[(\text{sh}(\mathcal{P}), t_{\geq 1}(\mathcal{P})[-1] \oplus J \oplus [1] \oplus \text{sw}(t_{\leq 2}(\mathcal{P})), \sigma, \tau),\]

where \(J\) is the diagonal binary complex supported on \([0, 1]\) given by \(J^\top = (J \xrightarrow{\text{id}} J) = J^\bot\), \(\sigma = \text{id}\) and \(\tau\) is the automorphism switching the two copies of \(J\) in degrees 0, 1 and 2 and the identity in all higher degrees. From this binary ladder we obtain the following equality in \(L^k_1(\mathcal{N})\):

\[ t_{\geq 1}(\mathcal{P})[-1] + \text{sw}(t_{\leq 2}(\mathcal{P})) = \text{sh}(\mathcal{P}) + \tau_\mathcal{P}. \]

Using Lemma 3.2, we finally obtain the desired equality in \(L^k_1(\mathcal{N})\):

\[ \text{sh}(\mathcal{P}) = t_{\geq 1}(\mathcal{P})[-1] - t_{\leq 2}(\mathcal{P}) - \tau_\mathcal{P}. \]

If \(\mathcal{P}\) is a diagonal complex, both truncations are diagonal again and part (a) follows. If \(P_0 = 0\), then \(t_{\geq 1}(\mathcal{P}) = \mathcal{P}\), \(J = P_1\) and \(t_{\leq 2}(\mathcal{P}) = \langle \text{id}_{P_1}, \text{id}_{P_1} \rangle[1] = 0\) in \(L^2_1(\mathcal{N})\); this shows part (b).

**Proposition 3.9.** The homomorphism

\[ i_k : L^k_1(\mathcal{N}) \rightarrow L^{k+1}_1(\mathcal{N}) \]

is an isomorphism for all \(k \geq 2\). Its inverse is induced by the assignment

\[ \mathcal{P} \mapsto -\text{sh}(\mathcal{P}) - \tau_\mathcal{P}. \]

**Proof.** We begin by showing that the assignment \(\mathcal{P} \mapsto -\text{sh}(\mathcal{P}) - \tau_\mathcal{P}\) induces a well-defined homomorphism \(p_k : L^{k+1}_1(\mathcal{N}) \rightarrow L^k_1(\mathcal{N})\).

If \(\mathcal{P}' \rightarrow \mathcal{P} \rightarrow \mathcal{P}''\) is a short exact sequence of binary acyclic complexes supported on \([0, k+1]\), then we have induced short exact sequences \(J' \rightarrow J \rightarrow J''\) and \(K' \rightarrow K \rightarrow K''\) by [Büh10, Corollary 3.6]. In particular, we obtain a short exact sequence \(\text{sh}(\mathcal{P}') \rightarrow \text{sh}(\mathcal{P}) \rightarrow \text{sh}(\mathcal{P}'')\). Since we also have \(\tau_\mathcal{P} = \tau_{\mathcal{P}'} + \tau_{\mathcal{P}''}\), we obtain an induced homomorphism

\[ p'_k : B^{k+1}_1(\mathcal{N}) \rightarrow L^k_1(\mathcal{N}). \]

Let \(\mathcal{P}\) be a diagonal binary acyclic complex supported on \([0, k+1]\), then we have \(p'_k(\mathcal{P}) = 0\) in \(L^k_1(\mathcal{N})\) by Lemma 3.8(a). In particular, \(p'_k\) induces a homomorphism

\[ p'_k : K^{k+1}_1(\mathcal{N}) \rightarrow L^k_1(\mathcal{N}). \]

If \((\mathcal{P}, Q, \sigma, \tau)\) is a binary ladder in \(\mathcal{N}\) with \(\mathcal{P}\) and \(Q\) supported on \([0, k+1]\), then \(\sigma_1\) and \(\tau_1\) induce isomorphisms \(\sigma_J : J_\mathcal{P} \xrightarrow{\sim} J_\mathcal{Q}\) and \(\tau_K : K_\mathcal{P} \xrightarrow{\sim} K_\mathcal{Q}\), respectively. If \(\sigma_1\) and
\[ \tau_2 \text{ are involutions, so are } \sigma^J \text{ and } \tau^K. \] These define a binary ladder
\[ (\sh(P), \sh(Q), \sh(\sigma), \sh(\tau)) \]
with
\[ \sh(\sigma)_0 = \sigma^J \oplus \tau^K \oplus \tau_0, \quad \sh(\sigma)_1 = \sigma_2 \oplus \tau^K \oplus \sigma^J \oplus \tau_1, \quad \sh(\sigma)_2 = \sigma_3 \oplus \sigma^J \oplus \tau^K, \]
\[ \sh(\tau)_0 = \sigma^J \oplus \tau^K \oplus \sigma_0, \quad \sh(\tau)_1 = \tau_2 \oplus \tau^K \oplus \sigma_1, \quad \sh(\tau)_2 = \tau_3 \oplus \sigma^J \oplus \tau^K, \]
and \( \sh(\sigma)_i = \sigma_{i+1} \) and \( \sh(\tau)_i = \tau_{i+1} \) for \( i \geq 3 \). Hence we have
\[
\sum_{i=0}^{k}(1)^{i}(\sh(\sigma)_i, \sh(\tau)_i) = (\tau_0, \sigma_0) - (\sigma_2, \tau_2) - (\tau_1, \sigma_1) + (\sigma_3, \tau_3) + \sum_{i=3}^{\infty}(1)^{i}(\sigma_{i+1}, \tau_{i+1})
\]
\[
= -\sum_{i=0}^{k+1}(1)^{i}(\sigma_i, \tau_i)
\]
in \( L^1(N) \) by Lemma 3.2. If the given ladder is as in Definition 2.2(c), it follows that
\[
p_k'(Q - P) = -\sh(Q) - \tau_Q + \sh(P) + \tau_Q = -\sum_{i \geq 0}(1)^{i}(\sh(\sigma)_i, \sh(\tau)_i)
\]
\[
= p_k' \left( \sum_{i \geq 0}(1)^{i}(\sigma_i, \tau_i) \right)
\]
in \( L^k(N) \) since \( J_P \cong J_Q \), hence \( \tau_Q = \tau_Q \), and since, by Remark 3.6, \( p_k' \) maps \( (\sigma_i, \tau_i) \in K^{k+1}(N) \) to \( (\sigma_i, \tau_i) \in L^k(N) \) for each \( i \). Consequently, \( p_k' \) induces the desired homomorphism
\[
p_k: L^{k+1}(N) \to L^k(N).
\]
We now show that \( p_k \circ i_k = id \) for all \( k \geq 2 \). Let \( P \) be a binary acyclic complex in \( N \) supported on \([0, k]\). By Equation (3.4) and Lemma 3.8(b), we have
\[
p_k(i_k(P)) = -p_k(P[1]) = \sh(P[1]) + \tau_P[1] = (P - \tau_P[1]) + \tau_P[1] = P
\]
in \( L^k(N) \), as was to be shown.

We are left with showing that \( p_k \) is also a right-inverse to \( i_k \). Let \( P \) be a binary acyclic complex in \( N \) supported on \([0, k+1]\). Since the definition of \( \sh(P) \) is independent of whether we regard \( P \) as a complex supported on \([0, k+1]\) or \([0, k+2]\), we have
\[
i_k(p_k(P)) = p_{k+1}(i_{k+1}(P)) = P
\]
in \( L^{k+1}(N) \), as desired. \( \square \)

Proof of Theorem 2.4. From Lemma 3.3 we obtain the directed system
\[
K^2_1(N) \to L^2_1(N) \to K^3_1(N) \to L^3_1(N) \to \cdots.
\]
Since the colimit of the cofinal sub-system \( K^2_1(N) \to K^3_1(N) \to \cdots \) is \( K_1(N) \), the colimit of the displayed system is \( K_1(N) \) as well. Hence, the colimit of the cofinal sub-system \( L^2_1(N) \to L^3_1(N) \to \cdots \) is also \( K_1(N) \). Furthermore, all the connecting maps in this sub-system are isomorphisms by Proposition 3.9. The claim follows. \( \square \)
Remark 3.10. Grayson shows in the handwritten notes mentioned in Remark 3.7 that \( \text{sh}(P) \in K_1(N) \) differs from \( P \) by classes of binary acyclic complexes of length at most 2. By induction, this proves that the canonical map \( K_1^2(N) \rightarrow K_1(N) \) is surjective. While Grayson uses slightly involved double complex arguments, we use simpler and at the same time more potent arguments and also prove the simple relation \( P + \text{sh}(P) = -\tau P \) in \( K_1(N) \).

Corollary 3.11 ([KW, Theorem 1.4]). The canonical map \( K_1^2(N) \rightarrow K_1(N) \) is onto and admits a canonical section.

Proof. The right inverse is given by the inverse of the bijection \( L_1^2(N) \rightarrow K_1(N) \) from Theorem 2.4 composed with the map \( L_1^2(N) \rightarrow K_1^2(N) \) from Lemma 3.3. \( \Box \)

Remark 3.12. The inverse of the isomorphism \( L_1^2(N) \rightarrow K_1(N) \) from Theorem 2.4 admits an explicit description. This agrees with the map \( \Psi \) appearing in the proof of [KW, Theorem 1.1].

4. The relation to Nenashev’s description

In this section, we compare Nenashev’s and Grayson’s descriptions of \( K_1 \).

Definition 4.1. Nenashev’s \( K_1 \)-group \( K_1^N(N) \) of \( N \) is defined as the abelian group generated by binary acyclic complexes \( P \) of length 2 subject to the following relations:

1. If \( P \) is a diagonal complex, then \( P = 0 \).
2. If

\[
\begin{array}{c}
P_2 \\
\downarrow \\
P_2''
\end{array} \longrightarrow
\begin{array}{c}
\longrightarrow \quad P_1' \\
\downarrow \\
\downarrow \\
\downarrow
\end{array} \longrightarrow
\begin{array}{c}
P_0' \\
\downarrow
\end{array}
\]

is a diagram in \( N \) such that all rows and columns are binary acyclic complexes, top differentials commute with top differentials and bottom differentials commute with bottom differentials, then

\( P_0 - P_1 + P_2 = P' - P + P'' \).

Nenashev proves in [Nen98] that \( K_1^N(N) \) is canonically isomorphic to Quillen’s \( K_1 \)-group of \( N \). The following corollary purely algebraically proves that \( K_1^N(N) \) is isomorphic to \( K_1(N) \), i.e., to Grayson’s \( K_1 \)-group of \( N \). By [Gra12, Remark 8.1], regarding a binary acyclic complex of length 2 as a class in \( K_1(N) \) defines a map \( K_1^N(N) \rightarrow K_1(N) \).

Corollary 4.2 ([KW, Theorem 1.1]). The canonical map

\( K_1^N(N) \rightarrow K_1(N) \)

is an isomorphism.
**Proof.** Since the relations used to define $L_2^2(\mathcal{N})$ are special cases of Nenashev’s relation, the canonical surjection $K_2^2(\mathcal{N}) \to K_1^N(\mathcal{N})$ factors via $L_2^1(\mathcal{N})$, yielding a surjection $L_2^1(\mathcal{N}) \to K_1^N(\mathcal{N})$. Since we have a commutative diagram

$$
\begin{array}{ccc}
L_2^1(\mathcal{N}) & \longrightarrow & K_1(\mathcal{N}) \\
\downarrow & & \downarrow \\
K_1^N(\mathcal{N}) & \longrightarrow & K_1(\mathcal{N})
\end{array}
$$

it follows from Theorem 2.4 that $L_2^2(\mathcal{N}) \to K_1^N(\mathcal{N})$ and $K_1^N(\mathcal{N}) \to K_1(\mathcal{N})$ are isomorphisms.

**Remark 4.3.** The bijectivity of the map $L_2^2(\mathcal{N}) \to K_1^N(\mathcal{N})$ in the proof of the previous corollary means that Nenashev’s relation (Definition 4.1(2)) can be expressed in $K_2^2(\mathcal{N})$ as a linear combination of relations arising from binary ladders that we used to define $L_2^1(\mathcal{N})$ (Definition 2.2(c)). The object of this remark is to explicitly write down such a linear combination.

Let

$$
\begin{array}{ccc}
M_2 & \longrightarrow & M_1 & \longrightarrow & M_0 \\
\downarrow & & \downarrow & & \downarrow \\
N_2 & \longrightarrow & N_1 & \longrightarrow & N_0 \\
\downarrow & & \downarrow & & \downarrow \\
P_2 & \longrightarrow & P_1 & \longrightarrow & P_0
\end{array}
$$

be a diagram in $\mathcal{N}$ as in Definition 4.1(2). Let $T$ denote the associated binary total complex (of length 4). Choose factorisations

$$
M_1 \oplus N_2 \rightarrow J_3 \rightarrow M_0 \oplus N_1 \oplus P_2 \rightarrow J_2 \rightarrow N_0 \oplus P_1
$$

of the second and third top differential of $T$, and define $K_3$ and $K_2$ analogously in terms of the bottom differentials of $T$. The next step, according to our earlier constructions, would be to apply the Grayson shortening twice to $T$, but this results in a complicated complex with superfluous terms. We rather apply twice just the idea behind the Grayson shortening in order to obtain the following complex $T'$ of length 2 with obvious differentials (top differentials on the left hand side, bottom differentials on the right hand side):

$$
\begin{array}{ccc}
M_2 & \rightarrow & M_1 \oplus N_2 & \rightarrow & J_3 \\
\oplus & \rightarrow & K_3 & \rightarrow & K_3 \\
J_3 & \rightarrow & J_3 & \rightarrow & J_3 \\
\oplus & \rightarrow & \oplus & \rightarrow & \oplus \\
K_3 & \rightarrow & M_0 \oplus N_1 \oplus P_2 & \rightarrow & K_2 \\
\oplus & \rightarrow & \oplus & \rightarrow & \oplus \\
J_2 & \rightarrow & J_2 & \rightarrow & J_2 \\
\oplus & \rightarrow & \oplus & \rightarrow & \oplus \\
K_2 & \rightarrow & K_2 & \rightarrow & K_2 \\
\oplus & \rightarrow & \oplus & \rightarrow & \oplus \\
J_2 & \rightarrow & N_0 \oplus P_1 & \rightarrow & P_0 \\
\oplus & \rightarrow & \oplus & \rightarrow & \oplus \\
M_2 & \rightarrow & M_1 \oplus N_2 & \rightarrow & J_3 \\
\oplus & \rightarrow & K_3 & \rightarrow & K_3 \\
J_3 & \rightarrow & J_3 & \rightarrow & J_3 \\
\oplus & \rightarrow & \oplus & \rightarrow & \oplus \\
K_3 & \rightarrow & M_0 \oplus N_1 \oplus P_2 & \rightarrow & K_2 \\
\oplus & \rightarrow & \oplus & \rightarrow & \oplus \\
J_2 & \rightarrow & J_2 & \rightarrow & J_2 \\
\oplus & \rightarrow & \oplus & \rightarrow & \oplus \\
K_2 & \rightarrow & K_2 & \rightarrow & K_2 \\
\oplus & \rightarrow & \oplus & \rightarrow & \oplus \\
J_2 & \rightarrow & N_0 \oplus P_1 & \rightarrow & P_0
\end{array}
$$
The obvious admissible monomorphisms $P_2 \to J_2$ and $P_2 \to K_2$ (the cokernel of both is $N_0$) define an admissible monomorphism from the binary acyclic complex $T_b(P)$ to (the bottom half of) $T'$. Similarly, we have an admissible epimorphism from (the top half of) $T'$ to the binary acyclic complex $T_f(M)$.

\[
\begin{array}{ccc}
P_2 & \to & P_2 \\
\oplus & \to & \oplus \\
P_2 & \to & P_2 \\
\oplus & \to & \oplus \\
P_2 & \to & P_2 \\
\oplus & \to & \oplus \\
P_2 & \to & P_1 \\
\oplus & \to & \oplus \\
P_2 & \to & P_0 \\
\oplus & \to & \oplus \\
\end{array}
\]

\[
\begin{array}{ccc}
P_2 & \to & P_2 \\
\oplus & \to & \oplus \\
P_2 & \to & P_2 \\
\oplus & \to & \oplus \\
P_2 & \to & P_1 \\
\oplus & \to & \oplus \\
P_2 & \to & P_0 \\
\oplus & \to & \oplus \\
\end{array}
\]

which obviously factors modulo $T_b(P)$. The resulting epimorphism $T'/T_b(P) \to T_f(M)$ has kernel $T_{b,f}(\text{sw}(N))$ which is obtained from $N$ by first switching top and bottom differential and then, similarly to $T_b(P)$ and $T_f(M)$, by adding copies of $N_2$ above $\text{sw}(N)$ and copies of $N_0$ below $\text{sw}(N)$. Hence we have

\[
T' = T_b(P) + T_{b,f}(\text{sw}(N)) + T_f(M) \quad \text{in} \quad B_1(N).
\]

Applying the switching automorphism $\tau_{P_2}$ at the appropriate place in all three degrees of $T_b(P)$ produces the direct sum of $P$ and a diagonal complex, so we obtain the relation

\[
T_b(P) = P + \tau_{P_2} \quad \text{in} \quad L_2^2(N).
\]  (4.4)

Similarly, we obtain $T_f(M) = M + \tau_{M_0}$ and, also using Lemma 3.2, $T_{b,f}(\text{sw}(N)) = -N + \tau_{N_0} + \tau_{N_2}$. Hence we have

\[
T' = P - N + M + \tau_{P_2} + \tau_{N_0} + \tau_{N_2} + \tau_{M_0} \quad \text{in} \quad L_1^2(N).
\]

Let $C_i$ denote the binary acyclic complex $M_i \Rightarrow N_i \Rightarrow P_i$. Filtering the total complex by columns, we similarly obtain

\[
T' = C_0 - C_1 + C_2 + \tau_{M_0} + \tau_{P_1} + \tau_{M_1} + \tau_{P_2} \quad \text{in} \quad L_1^2(N).
\]

The exact sequences $N$ and $C_1$ furthermore imply the relations

\[
\tau_{N_0} + \tau_{N_2} = \tau_{N_1} = \tau_{P_1} + \tau_{M_1} \quad \text{in} \quad B_1(N).
\]

Hence we finally obtain the Nenashev relation

\[
P - N + M = C_0 - C_1 + C_2 \quad \text{in} \quad L_1^2(N).
\]  (4.5)

Put slightly differently, in $B_1(N)$, the difference of the two sides in (4.5) is equal to the sum of (the negative of) the relation given by (4.4) and the analogous relations for $T_f(M)$, $T_{b,f}(\text{sw}(N))$, $T_b(C_0)$, $T_f(C_2)$ and $T_{b,f}(\text{sw}(C_1))$. Each of these relations in
turn is made up of a ladder and a diagonal relation apart from those for $T_{b,f}(sw(\mathbb{N}))$ and $T_{b,f}(sw(G_1))$ which in addition involve the ladder and diagonal relation occurring in the proof of Lemma 3.2.

References


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