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(CO)HOMOLOGY SELF-CLOSENESS NUMBERS OF SIMPLY-CONNECTED SPACES

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Abstract

The (co)homology self-closeness number of a simply-connected based CW-complex X is the minimal number k such that any self-map f of X inducing an automorphism of the (co)homology groups for dimensions $\leqslant k$ is a self-homotopy equivalence. These two numbers are homotopy invariants and have a close relation with the group of self-homotopy equivalences. In this paper, we compare the (co)homology self-closeness numbers of spaces in certain cofibrations, define the mod p (co)homology self-closeness number of simply-connected p-local spaces with finitely generated homologies and study some properties of the (mod p) (co)homology self-closeness numbers.

1. Introduction

The group of self-homotopy equivalences of a space, and its subgroups, have been extensively studied by many mathematicians in history, such as Arkowitz [2, 4], Rutter [16, 14], Maruyama [9]. The groups of self-homotopy equivalences are usually difficult to compute. In 2015 Choi and Lee [6] introduced the *self-closeness number* $N\mathcal{E}(X)$ of a space X to investigate the group $\mathcal{E}(X)$ of self-homotopy equivalences of X. The *self-closeness number* $N\mathcal{E}(X)$, which is denoted by $N_\sharp\mathcal{E}(X)$ in this paper, is defined by

$$N_\sharp\mathcal{E}(X) := \min\{k \mid \mathcal{A}^k_\sharp(X) = \mathcal{E}(X)\},$$

where $\mathcal{A}^k_\sharp(X) := \{f \in [X,X] \mid f_\sharp\colon \pi_i(X) \xrightarrow{\cong} \pi_i(X) \text{ for } i \leqslant k\}$. Oda and Yamaguchi [11] continued the study of the self-closeness number and proved inequalities among the self-closeness numbers of spaces of a cofibration of the type:

$$S^{m+1} \xrightarrow{\gamma} B \xrightarrow{i} X \xrightarrow{p} S^{m+2}$$

and gave dual results of the comparison of self-closeness numbers of spaces in a fibration of the type, [12]:

$$K(G, m+1) \xrightarrow{q} X \xrightarrow{i} Y \xrightarrow{\gamma} K(G, m+2).$$

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Recently they published a paper involving the homology and cohomology self-closeness numbers of a space, see Section 6 of [13]. I avoid repeating the overlaps of some results and directly quote their results in this paper. The following notation is needed to make sense of the introduction.

We agree once and for all that all spaces are simply-connected based CW-complexes and maps are thought of as the homotopy classes with the given representative. In notation, let $\mathcal{CW}_{sc}$ be the homotopy category of simply-connected based CW-complexes. Let $X, Y \in \mathcal{CW}_{sc}$, $[X, Y]$ denote the set of homotopy classes of based maps from X to Y; identify a map f with its homotopy classes ($f = [f]$) and understand $f = g$ as meaning $f \simeq g$. Let $H_i(X; G)$ be the i-th reduced homology group of X with coefficient group G and let $H_i(X) = H_i(X; \mathbb{Z})$. For a map $f\colon X \to Y$, denote by f_* or $H_i(f; G)\colon H_i(X; G) \to H_i(Y; G)$ the corresponding induced homomorphism. Similar notation is used for cohomology.

For simply-connected spaces, the Whitehead theorem and the universal coefficient theorem for cohomology indicate that a map $f\colon X \to Y$ is a homotopy equivalence if and only if 1. f is a homology equivalence: $f_*\colon H_i(X) \xrightarrow{\cong} H_i(X)$ for all i; or 2. f is a cohomology equivalence: $f^*\colon H^i(X) \xrightarrow{\cong} H^i(X)$ for all i. This motivates us to define the *homology and cohomology self-closeness numbers*.

Let X be a based CW-complex and consider the following subsets of $[X, X]$:

$$\mathcal{A}_*^k(X) := \{f \in [X, X] \mid f_*\colon H_i(X) \xrightarrow{\cong} H_i(X) \text{ for } i \leqslant k\},\ \mathcal{A}_*^\infty(X) := \lim_{k\to+\infty} \mathcal{A}_*^k(X);$$
$$\mathcal{A}^*_k(X) := \{f \in [X, X] \mid f^*\colon H^i(X) \xrightarrow{\cong} H^i(X) \text{ for } i \leqslant k\},\ \mathcal{A}^*_\infty(X) := \lim_{k\to+\infty} \mathcal{A}^*_k(X).$$

If $n \leqslant k$, by the Whitehead theorem there is a chain of monoids by inclusion:

$$\mathcal{E}(X) \subseteq \mathcal{A}_*^\infty(X) \subseteq \mathcal{A}^*_k(X) \subseteq \mathcal{A}^*_n(X) \subseteq [X, X].$$

There is a similar chain in the cohomology case. The *homology self-closeness number* $N_*\mathcal{E}(X)$ and the *cohomology self-closeness number* $N^*\mathcal{E}(X)$ of X are defined by:

$$N_*\mathcal{E}(X) := \min\{k \mid \mathcal{A}_*^k(X) = \mathcal{E}(X)\} \text{ and } N^*\mathcal{E}(X) := \min\{k \mid \mathcal{A}^*_k(X; \mathbb{Z}) = \mathcal{E}(X)\}.$$

They are both well-defined homotopy invariants (Proposition 37 of [13]).

Remark 1.1.
1. If $X \in \mathcal{CW}_{sc}$, $N_*\mathcal{E}(X), N^*\mathcal{E}(X)$ take values in the range $\mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$. $\mathcal{E}_*(X) = 0$ if and only if X is contractible, which is denoted by $X = \{*\}$. $N^*\mathcal{E}(\bigvee_{n\geqslant 2} S^n) = N_*\mathcal{E}(\bigvee_{n\geqslant 2} S^n) = N_\sharp\mathcal{E}(\bigvee_{n\geqslant 2} S^n) = +\infty$ (Example 39 of [13]).
2. If X is not simple or simply-connected, it may happen that a self-map f is a homology equivalence but not a homotopy equivalence, see Example 4.35 of [8]; in this case $\mathcal{A}_*^k(X) \neq \mathcal{E}(X)$ for any integer $k \geqslant 0$ and we denote by $N_*\mathcal{E}(X) = -\infty$.

The connectivity degree of X is denoted by $\operatorname{conn}(X)$, which means that $\pi_i(X) = 0$ if $i \leqslant \operatorname{conn}(X)$. Let

$$H^*\text{-}\dim(X) := \max\{i \geqslant 0 \mid H^i(X) \neq 0\}, \quad H_*\text{-}\dim(X) := \max\{i \geqslant 0 \mid H_i(X) \neq 0\}$$

be the homology dimension and cohomology dimension of X, respectively. It is easy to prove that if $\{*\} \neq X \in \mathcal{CW}_{sc}$,

$$\operatorname{conn}(X) + 1 \leqslant N_*\mathcal{E}(X) \leqslant H_*\text{-}\dim(X), \quad \operatorname{conn}(X) + 1 \leqslant N^*\mathcal{E}(X) \leqslant H^*\text{-}\dim(X).$$

We can compare these three types of self-closeness numbers of a simply-connected space and prove some inequalities among them, refer to Section 6 of [**13**]. In this paper I choose the cohomology self-closeness number of simply-connected spaces to be the protagonist, since there are richer structures in cohomology theory, such as the cohomology ring and the Steenrod operations. The paper is arranged as follows.

In Section 2, motivated by Oda and Yamaguchi's paper [**12**], I quote some of Rutter's results about extension of ladders of cofibrations [**15**] and give a dual discussion on the cohomology self-closeness numbers of spaces in a generic cofibration $A \xrightarrow{\gamma} B \xrightarrow{i} X$. By Theorem 2.7, the inequality $N^*\mathcal{E}(B) \leqslant N^*\mathcal{E}(X)$ holds if the following conditions hold:

1. $n-1 \leqslant \mathrm{conn}(A)$, $H^*\text{-}\dim(B) \leqslant n$;
2. $\gamma\colon A \to B$ induces a surjection: $\gamma_*\colon [A,A] \to [A,B]$.

By Theorem 2.8, $N^*\mathcal{E}(X) \leqslant N^*\mathcal{E}(B)$ holds under the following assumptions

1′. $m-1 \leqslant \mathrm{conn}(B) < H^*\text{-}\dim(B) \leqslant n-1 \leqslant \mathrm{conn}(A) < \dim(A) \leqslant n+m-2$.

2′. If there exist maps $h \in [A,A]$ and $g \in \mathcal{E}(B)$ such that $g\gamma = \gamma h$, then $h \in \mathcal{E}(A)$.

Moreover, if 2′ is substituted by the assumption that the induced map $\gamma_*\colon [A,A] \to [A,B]$ is bijective, then $N^*\mathcal{E}(X) = N^*\mathcal{E}(B)$, see Theorem 2.10. It follows that if B is atomic ($N^*\mathcal{E}(B) = \mathrm{conn}(B)+1$), then so is X (Corollary 2.13). Consider the case $m=2,\ A=S^n$ and $m=3, A=P^{n+1}(q)=S^n\cup_q e^{n+1}$ respectively, we get Corollary 2.14 2.15; particularly, Corollary 2.14 is a cohomology version of Theorem 6 of [**13**]. The special case where $[A,A]$ is a cyclic group $\mathbb{Z}$ or $\mathbb{Z}/q$ ($q \geqslant 2$) (Theorem 2.17) can be viewed as a generalization Theorem 5 ($A=S^n$) of [**13**].

In Section 3, we define the *mod* p *homology self-closeness number* $N_*\mathcal{E}(X;p)$ and the *mod* p *cohomology self-closeness number* $N^*\mathcal{E}(X;p)$ of a simply-connected p-local space X with finitely generated homology. They are also well-defined homotopy invariants. For such a space X, we have $N_*\mathcal{E}(X;p) = N^*\mathcal{E}(X;p)$ (Proposition 3.4) and $N_*\mathcal{E}(X) = N_*\mathcal{E}(X;p)$ (Proposition 3.7).

In Section 4 we prove some properties of (mod p) homology and cohomology self-closeness numbers. Let p be a prime or $p=0$, let X be a simply-connected space with finitely generated homology if $p=0$ and further let X be p-local if p is a prime. Denote by $N^*\mathcal{E}(X;0) = N^*\mathcal{E}(X)$. By Propositions 4.1, 4.3, we have the following inequalities:

$$N^*\mathcal{E}(\Sigma X;p) \geqslant N^*\mathcal{E}(X;p)+1;$$
$$N^*\mathcal{E}(X\times Y;p),\ N^*\mathcal{E}(X\wedge Y;p),\ N^*\mathcal{E}(X\vee Y;p) \geqslant \max\{N^*\mathcal{E}(X;p), N^*\mathcal{E}(Y;p)\};$$
$$N^*\mathcal{E}(\Sigma(X\times Y);p) \geqslant N^*\mathcal{E}(\Sigma(X\wedge Y);p).$$

The above inequalities are also true for (mod p) homology self-closeness numbers. If the cohomology ring $H^*(X;\mathbb{Z}/p)$ ($\mathbb{Z}/0=\mathbb{Z}$) is generated by classes $x_i \in H^{|x_i|}(X;\mathbb{Z}/p)$, $1,\dots,m$, then by Proposition 4.6 we have

$$N^*\mathcal{E}(X;p) \leqslant \max\{|x_1|,\dots,|x_m|\}.$$

Finally, I exhibit a result of Haibao Duan, Theorem 4.9, which states that for a simply-connected compact Kähler manifold M with torsion-free cohomology and $H^2(M)\cong\mathbb{Z}$, we have $N^*\mathcal{E}(M)=2$.

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2. Cohomology self-closeness number and cofibrations

In this section we consider a generic cofibration $A \xrightarrow{\gamma} B \xrightarrow{i} X \xrightarrow{p} \Sigma A$ and discuss conditions for $N^*\mathcal{E}(B) \leqslant N^*\mathcal{E}(X)$ and $N^*\mathcal{E}(X) \leqslant N^*\mathcal{E}(B)$.

2.1. Some lemmas

Lemma 2.1. *Let $f\colon X \to Y$ be a map between simply-connected spaces, let F_f be the homotopy fiber of f, and C_f the homotopy cofiber of f. Then the following are equivalent:*

1. *f is n-connected.*
2. *F_f is $(n-1)$-connected.*
3. *C_f is n-connected.*

Proof. By Lemma 6.4.11 and Proposition 6.4.14 of [3]. □

By the long exact sequence of (co)homology groups, the five-lemma and the Whitehead theorem, it is clear that

Lemma 2.2. *In the following homotopy commutative diagram with fibering rows of simply-connected spaces:*

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "h"'] & B \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "f"] \\ A \arrow[r, "\gamma"] & B \arrow[r, "i"] & X \end{tikzcd}$$

if any two of the vertical maps f, g, h are self-homotopy equivalences, so is the third one.

Lemma 2.3. *Let $r \geqslant 1, n \geqslant 2$, $A \xrightarrow{\gamma} B \xrightarrow{i} X \xrightarrow{p} \Sigma A$ be a cofibration with $\mathrm{conn}(A) \geqslant n-1$, $\mathrm{conn}(X) \geqslant r$ and $\dim(A) \leqslant r+n-1$. Given self-maps $g\colon B \to B$, $f\colon X \to X$ such that $fi = ig$, there exists a map $h \in$ such that $\gamma h = g\gamma$.*

Proof. A direct result of Proposition 4.4 of [15]. □

Corollary 2.4. *Let $r \geqslant n \geqslant 2$. If $n \leqslant \mathrm{conn}(A) + 1 \leqslant \dim(A) \leqslant r \leqslant \mathrm{conn}(X)$, then given a self-map $g\colon B \to B$, there exist maps $f\colon X \to X$, $h\colon A \to A$ such that the following diagram is homotopy commutative:*

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, dashed, "h"'] & B \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, dashed, "f"] \\ A \arrow[r, "\gamma"] & B \arrow[r, "i"] & X \end{tikzcd}$$

Proof. The condition $\dim(A) \leqslant \mathrm{conn}(X)$ implies that the map

$$i^*\colon [X,X] \to [B,X]$$

is surjective, there exists a map $f\colon X \to X$ such that $fi = i^*(f) = ig$. Then apply Lemma 2.3. ☐

Lemma 2.5. *Let $m,n \geqslant 2$ and let $A \xrightarrow{\gamma} B \xrightarrow{i} X \xrightarrow{p} \Sigma A$ be a cofibration with $\mathrm{conn}(A) \geqslant n-1$, $\mathrm{conn}(B) \geqslant m-1$ and $\dim(A) \leqslant m+n-2$. Suppose there is a commutative diagram:*

$$\begin{tikzcd}
A \arrow[r, "\gamma"] & B \arrow[r, "i"] \arrow[d, "g"'] & X \arrow[r, "p"] \arrow[d, "f"] & \Sigma A \arrow[d, "\Sigma h"] \\
A \arrow[r, "\gamma"] & B \arrow[r, "i"] & X \arrow[r, "p"] & \Sigma A
\end{tikzcd}$$

Then $g\gamma = \gamma h$.

Proof. A direct result of Theorem 4.6 of [15]. ☐

Corollary 2.6. *Let $n \geqslant m \geqslant 2$. If $m \leqslant \mathrm{conn}(B)+1 \leqslant \dim(B) \leqslant n \leqslant \mathrm{conn}(A)+1 \leqslant \dim(A) \leqslant n+m-2$, then given a map $f\colon X \to X$, there exists maps $h\colon A \to A$, $g\colon B \to B$ such that the following diagram is homotopy commutative, in which rows are cofibrations:*

$$\begin{tikzcd}
A \arrow[r, "\gamma"] \arrow[d, dotted, "h"'] & B \arrow[r, "i"] \arrow[d, dashed, "g"] & X \arrow[r, "p"] \arrow[d, "f"] & \Sigma A \arrow[d, dashed, "\Sigma h"] \\
A \arrow[r, "\gamma"] & B \arrow[r, "i"] & X \arrow[r, "p"] & \Sigma A
\end{tikzcd}$$

Proof. The condition $\dim(A) \leqslant n+m-2 \leqslant 2n-2 \leqslant 2\cdot \mathrm{conn}(A)$ implies that the suspension map

$$\Sigma\colon [A,A] \to [\Sigma A, \Sigma A]$$

is bijective, by Theorem 1.21 of [7]. Then the result follows from Corollary 2.4 and Lemma 2.5. ☐

Let $A \xrightarrow{\gamma} B \xrightarrow{i} X$ be a cofibration of simply-connected spaces. In the remainder of this section we shall investigate conditions for the comparison of $N^*\mathcal{E}(B)$ and $N^*\mathcal{E}(X)$.

2.2. Conditions for $N^*\mathcal{E}(B) \leqslant N^*\mathcal{E}(X)$

Theorem 2.7. *Let $n \geqslant 2$, $A \xrightarrow{\gamma} B \xrightarrow{i} X$ be a cofibration in $\mathcal{CW}_{sc}$. If the following conditions hold:*

1. *$n-1 \leqslant \mathrm{conn}(A)$, H^*-$\dim(B) \leqslant n-1$.*
2. *$\gamma\colon A \to B$ induces a surjection: $\gamma_*\colon [A,A] \to [A,B]$.*

Then $N^*\mathcal{E}(B) \leqslant N^*\mathcal{E}(X)$.

Proof. Since $N^*\mathcal{E}(B) \leqslant H^*$-$\dim(B) \leqslant n-1$, we may suppose that $N^*\mathcal{E}(X) = k \leqslant n-1$ and $g \in \mathcal{A}_k^*(B)$.

By the long exact sequence of cohomology groups and $\mathrm{conn}(A) \geqslant n-1$, the induced homomorphism $i^*\colon H^d(X) \to H^d(B)$ is an isomorphism for $d \leqslant n-1$.

The surjectivity of $\gamma_*\colon [A, A] \to [A, B]$ implies that there exists a map $h \in [A, A]$ such that $g\gamma = \gamma h$ and hence there is a map $f \in [X, X]$ such that $fi = ig$.

Consider the following commutative diagram for $d \leqslant k \leqslant n-1$:

$$\begin{tikzcd} H^d(X) \arrow[r, "i^*", "\cong"'] \arrow[d, "f^*"'] & H^d(B) \arrow[d, "g_*"', "\cong"] \\ H^d(X) \arrow[r, "i^*", "\cong"'] & H^d(B) \end{tikzcd}$$

Then $g \in \mathcal{A}^*_k(B)$ implies that $f \in \mathcal{A}^*_k(X) = \mathcal{E}(X)$.

Since $H^*\text{-}\dim(B) \leqslant n-1$, the induced homomorphism

$$\partial\colon H^{d-1}(A) \to H^d(X) \text{ is an isomorphism for } d \geqslant n+1.$$

Then the commutative diagram for $d \geqslant n+1$:

$$\begin{tikzcd} H^{d-1}(A) \arrow[r, "\partial", "\cong"'] \arrow[d, "h^*"'] & H^d(X) \arrow[d, "\cong"', "f^*"] \\ H^{d-1}(A) \arrow[r, "\partial", "\cong"'] & H^d(X) \end{tikzcd}$$

implies that $h^*\colon H^d(A) \to H^d(A)$ is an isomorphism for $d \geqslant n$ and hence for all $d \geqslant 0$, since $\mathrm{conn}(A) \geqslant n-1$. By the Whitehead theorem, $h \in \mathcal{E}(A)$. Hence $g \in \mathcal{E}(B)$ by Lemma 2.2 and therefore $N^*\mathcal{E}(B) \leqslant k = N^*\mathcal{E}(X)$. $\square$

2.3. Conditions for $N^*\mathcal{E}(X) \leqslant N^*\mathcal{E}(B)$

Note that for a simply-connected CW-complex B and $n \geqslant 2$, the condition $H^*\text{-}\dim(B) \leqslant n-1$ implies $H_*\text{-}\dim(B) \leqslant n-1$, by the universal coefficient theorem for cohomology. Then, by Proposition 4C.1 of [8], B admits a cell structure of dimension at most n.

Theorem 2.8. *Let $n, m \geqslant 2$ and let $A \xrightarrow{\gamma} B \xrightarrow{i} X$ be a cofibration. Consider the following assumptions:*

1. $m-1 \leqslant \mathrm{conn}(B) < H^*\text{-}\dim(B) \leqslant n-1 \leqslant \mathrm{conn}(A) < \dim(A) \leqslant n+m-2.$
2. *If there exist maps $h \in [A, A]$ and $g \in \mathcal{E}(B)$ such that $g\gamma = \gamma h$, then $h \in \mathcal{E}(A)$.*

If assumptions 1 and 2 hold, then $N^*\mathcal{E}(X) \leqslant N^*\mathcal{E}(B)$.

Proof. Since $\mathrm{conn}(A) \geqslant n-1$, $i\colon B \to X$ is n-connected. The induced homomorphism

$$i^*\colon H^d(X) \to H^d(B)$$

is an isomorphism for $d \leqslant n-1$ and an injection for $d = n$.

Suppose that $N^*\mathcal{E}(B) = k$ and $f \in \mathcal{A}^*_k(X)$. Then

$$m \leqslant k \leqslant H^*\text{-}\dim(B) \leqslant n-1.$$

By Corollary 2.6, there exist self-maps $h\colon A \to A$, $g\colon B \to B$ filling in the homotopy commutative diagram:

$$\begin{tikzcd} A \arrow[r, "a\cdot\gamma"] \arrow[d, dotted, "h"'] & B \arrow[r, "i"] \arrow[d, dashed, "g"] & X \arrow[r, "p"] \arrow[d, "f"] & \Sigma A \arrow[d, dashed, "\Sigma h"] \\ A \arrow[r, "a\cdot\gamma"] & B \arrow[r, "i"] & X \arrow[r, "p"] & \Sigma A \end{tikzcd} \tag{1}$$

Consider the induced commutative diagram for $d \leqslant k \leqslant n-1$:

$$\begin{tikzcd} H^d(X) \arrow[r, "i^*", "\cong"'] \arrow[d, "f^*"', "\cong"] & H^d(B) \arrow[d, "g^*"'] \\ H^d(X) \arrow[r, "i^*", "\cong"'] & H^d(B) \end{tikzcd} \tag{2}$$

Then $f \in \mathcal{A}^*_k(X)$ implies that $g \in \mathcal{A}^*_k(B) = \mathcal{E}(B)$. By assumption 2 we then have $h \in \mathcal{E}(A)$ and hence $f \in \mathcal{E}(X)$, by Lemma 2.2. Therefore, $N^*\mathcal{E}(X) \leqslant k = N^*\mathcal{E}(B)$. $\square$

Lemma 2.9. *If $\gamma\colon A \to B$ induces a bijection: $\gamma_*\colon [A, A] \to [A, B]$, then given a map $h\colon A \to A$ and $g \in \mathcal{E}(B)$ such that $g\gamma = \gamma h$, we have $h \in \mathcal{E}(A)$.*

Proof. Let $\bar{g} \in \mathcal{E}(B)$ be the homotopy inverse of g; that is, $\bar{g}g = 1_B = g\bar{g}$. By the surjectivity of $\gamma_*\colon [A, A] \to [A, B]$, there exists a map $\bar{h} \in [A, A]$ satisfying $\bar{g}\gamma = \gamma\bar{h}$. We then have

$$\gamma = \bar{g}g\gamma = \bar{g}\gamma h = \gamma\bar{h}h \qquad \text{and} \qquad \gamma = g\bar{g}\gamma = g\gamma\bar{h} = \gamma h\bar{h}.$$

Then, by the injectivity of $\gamma_*\colon [A, A] \to [A, B]$, we get $\bar{h}h = 1_A = h\bar{h}$ and hence $h \in \mathcal{E}(B)$. $\square$

Theorem 2.10. *Let $n, m \geqslant 2$ and let $A \xrightarrow{\gamma} B \xrightarrow{i} X$ be a cofibration satisfying the following conditions:*

1. *$m-1 \leqslant \mathrm{conn}(B) < H^*\text{-}\dim(B) \leqslant n-1 \leqslant \mathrm{conn}(A) < \dim(A) \leqslant n+m-2$.*
2. *$\gamma\colon A \to B$ induces a bijection: $\gamma_*\colon [A, A] \to [A, B]$.*

Then $N^*\mathcal{E}(X) = N^*\mathcal{E}(B)$.

Proof. $N^*\mathcal{E}(X) \geqslant N^*\mathcal{E}(B)$ by Theorem 2.7; $N^*\mathcal{E}(X) \leqslant N^*\mathcal{E}(B)$ by Theorem 2.8 and Lemma 2.9. $\square$

Definition 2.11. A CW-complex X is called *atomic* if $N_\sharp\mathcal{E}(X) = \mathrm{conn}(X) + 1$.

It is immediate that

Lemma 2.12. *If $X \in \mathcal{CW}_{sc}$, the following are equivalent:*

1. *X is atomic.*
2. *$N^*\mathcal{E}(X) = \mathrm{conn}(X) + 1$.*
3. *$N_*\mathcal{E}(X) = \mathrm{conn}(X) + 1$.*

Corollary 2.13. *Let $n, m \geqslant 2$ and let $A \xrightarrow{\gamma} B \xrightarrow{i} X$ be a cofibration satisfying the following conditions:*

1. *$m-1 \leqslant \mathrm{conn}(B) < H^*\text{-}\dim(B) \leqslant n-1 \leqslant \mathrm{conn}(A) < \dim(A) \leqslant n+m-2$.*

2. $\gamma\colon A \to B$ induces a bijection: $\gamma_*\colon [A,A] \to [A,B]$.

If B is atomic, then so is X.

Proof. Since ΣA is n-connected, $i_*\colon B \to X$ is n-connected, $i_*\colon \pi_i(B) \to \pi_i(X)$ is an isomorphism for $i \leqslant n-1$. Since $m-1 \leqslant \mathrm{conn}(B) \leqslant n-1$, we have $\mathrm{conn}(X) = \mathrm{conn}(B)$ and hence

$$N^*\mathcal{E}(X) = N^*\mathcal{E}(B) = \mathrm{conn}(B) + 1 = \mathrm{conn}(X) + 1. \qquad \square$$

Let $m = 2$ and $A = S^n$. Then we have

Corollary 2.14 (A cohomological version of Theorem 6 of [13])**.** *Let $n \geqslant 2$, $a \neq 0$, let B be 1-connected with $H^*\text{-}\dim(B) \leqslant n-1$ and let $S^n \xrightarrow{a\cdot\gamma} B \xrightarrow{i} X$ be a cofibration. If $\pi_n(B) \cong \mathbb{Z}\langle\gamma\rangle$, then $N^*\mathcal{E}(X) = N^*\mathcal{E}(B)$.*

Let $m = 3$ and $A = P^{n+1}(\mathbb{Z}/q) = S^n \cup_q e^{n+1}$. Then we have

Corollary 2.15. *Let $n \geqslant 3$, $q \geqslant 2$, $(a,q) = 1$, let B be 2-connected with $H^*\text{-}\dim(B) \leqslant n-1$ and let $P^{n+1}(\mathbb{Z}/q) \xrightarrow{a\cdot\gamma} B \xrightarrow{i} X$ be a cofibration. If $[P^{n+1}(\mathbb{Z}/q), B] \cong \mathbb{Z}/q\langle\gamma\rangle$, then $N^*\mathcal{E}(X) = N^*\mathcal{E}(B)$.*

Remark 2.16. The above results are also true for homotopy and homology self-closeness after every $H^*\text{-}\dim(B)$ is substituted by $H_*\text{-}\dim(B)$, and every N^* by $N_\sharp$ and N_*, respectively.

2.4. A special case

Let $q \in \mathbb{Z}$. If $q > 1$, denote the set of prime factors of q by $Pr(q)$:

$$Pr(q) := \{p_1, \dots, p_l \mid q = p_1^{r_1} \cdot \dots \cdot p_l^{r_l}, p_i \text{ are primes}, r_i \geqslant 1\}.$$

Theorem 2.17. *Let $n, m \geqslant 2$, let $A = \Sigma^2 A'$ and let $A \xrightarrow{a\cdot\gamma} B \xrightarrow{i} X \xrightarrow{p} \Sigma A$ be a cofibration with $a \cdot \gamma \in [A,B]$ nontrivial. If the following assumptions hold:*

1. *$m - 1 \leqslant \mathrm{conn}(B) < H^*\text{-}\dim(B) \leqslant n-1 \leqslant \mathrm{conn}(A) < \dim(A) \leqslant n+m-2$.*
2. *$[A,A] \cong \mathbb{Z}/q\langle 1_A\rangle$ and $\gamma \in [A,B]$ is a generator of a direct summand $\mathbb{Z}/q'$, where q, q' satisfy the conditions:*

$$\begin{cases} q' = q, & q = 0; \\ q' | q, Pr(q') = Pr(q) & q \neq 0, \end{cases} \tag{3}$$

then $N^*\mathcal{E}(X) \leqslant N^*\mathcal{E}(B)$. Equality holds if, additionally, $[A,B] \cong \mathbb{Z}/q'\langle\gamma\rangle$.

Proof. For the inequality $N^*\mathcal{E}(X) \leqslant N^*\mathcal{E}(B)$, it suffices to show the new assumption 2 above implies the "old" 2 in Theorem 2.8.

Let $\bar{g}$ be the inverse of g. By assumption 2 we may put

$$h = s \cdot 1_A, \bar{g}\gamma = t \cdot \gamma + u$$

for some $s \in \mathbb{Z}/q, t \in \mathbb{Z}/q'$ and $u \in [A,B]/\mathbb{Z}/q'$. By (1) we have

$$a \cdot \gamma = \bar{g}g(a \cdot \gamma) = ast \cdot \gamma + as \cdot u.$$

It follows that $s = 1$ if $q = 0$ and $s \equiv 1 \pmod{q}$ if $q \neq 0$ by condition (3). Thus $h \in \mathcal{E}(A)$ and therefore $f \in \mathcal{E}(X)$ by Lemma 2.2.

If, in addition, $[A,B]\cong \mathbb{Z}/q'\langle\gamma\rangle$, we show that $N^*\mathcal{E}(B)\leqslant N^*\mathcal{E}(X)$.

Suppose that $N^*\mathcal{E}(X)=l$ and $g\in\mathcal{A}^*_l(B)$. Then

$$l=N^*\mathcal{E}(X)\leqslant N^*\mathcal{E}(B)\leqslant H^*\text{-}\dim(B)\leqslant n-1$$

Let $h=s'\cdot 1_A\colon A\to A$. Since $[A,B]\cong\mathbb{Z}/q'\langle\gamma\rangle$, we have $g\gamma=t'\cdot\gamma$ for some $t'\in\mathbb{Z}/q'$. Then there exists a map $f\colon X\to X$ such that

$$fi=ig,(\Sigma h)p=pf.$$

From the commutative diagram (2) in the proof of Theorem 2.8 for $d\leqslant l$, we see that $g\in\mathcal{A}^*_l(B)$ implies $f\in\mathcal{A}^*_l(X)=\mathcal{E}(X)$, which in turn implies $g\in\mathcal{E}(B)$ by the commutative diagram (2) for $d\leqslant n-1$. Therefore $N^*\mathcal{E}(B)\leqslant N^*\mathcal{E}(X)$. □

Let $m=2$ and $A=S^n$. Then we get a cohomological version of Theorem 5 of [**13**].

Corollary 2.18. *Let $n\geqslant 2$ and let $S^n\xrightarrow{a\cdot\gamma}B\xrightarrow{i}X$ be a cofibration, in which B is 1-connected and H^*-$\dim(B)\leqslant n-1$. If $0\neq a\in\mathbb{Z}$, and γ is a generator of a direct summand $\mathbb{Z}\subseteq\pi_{n+1}(B)$, then $N^*\mathcal{E}(X)\leqslant N^*\mathcal{E}(B)$.*

Let 1_P be the identity of $P^n(q)$. It is well known that if $q\equiv 1\pmod 2$, $n\geqslant 4$,

$$[P^n(q),P^n(q)]\cong\mathbb{Z}/q\langle 1_P\rangle.$$

Let $m=3$ and $A=P^{n+1}(q)$. Then we have:

Corollary 2.19. *Let $n\geqslant 2$ and let $q,q'>1$ be odd integers such that $Pr(q)=Pr(q')$ and let $P^{n+1}(q)\xrightarrow{a\cdot\gamma}B\xrightarrow{i}X$ be a cofibration, in which B is 2-connected and H^*-$\dim(B)\leqslant n-1$. If $\langle\gamma\rangle\subseteq\pi_{n+1}(B;\mathbb{Z}/q)\cong\mathbb{Z}/q'$ is a direct summand and $a\cdot\gamma\neq 0$, then $N^*\mathcal{E}(X)\leqslant N^*(B)$, and equality holds if $\pi_{n+1}(B;\mathbb{Z}/q)\cong\mathbb{Z}/q'\langle\gamma\rangle$.*

2.5. Another condition for $N^*\mathcal{E}(B)\leqslant N^*\mathcal{E}(X)$

Theorem 2.20 (a cohomological version of Theorem 9 of [**11**]). *Let $r\geqslant n\geqslant 2$ and $A\xrightarrow{\gamma}B\xrightarrow{i}X$ be a cofibration. If one of the following conditions holds*

1. *$n\leqslant \mathrm{conn}(A)+1\leqslant\dim(A)\leqslant r\leqslant\mathrm{conn}(X)$ and $H^r(A)\cong H^r(B)$,*
2. *$n\leqslant \mathrm{conn}(A)+1\leqslant\dim(A)<r\leqslant\mathrm{conn}(X)$,*

then $N^*\mathcal{E}(B)\leqslant N^*\mathcal{E}(X)$.

Proof. Suppose that condition 2 holds. Since X is r-connected and $H^r(A)=0$, by the long exact sequence of cohomology groups, the induced homomorphism

$$\gamma^*\colon H^d(B)\to H^d(A)\text{ is an isomorphism for } d\leqslant r. \tag{4}$$

Since $\dim(A)<r$, the induced homomorphism

$$i^*\colon H^d(X)\to H^d(B)\text{ is an isomorphism for } d\geqslant r+1. \tag{5}$$

If 1 holds, we can also get the above (4), (5).

Suppose that $N^*\mathcal{E}(X) = k \geqslant \mathrm{conn}(X)+1 \geqslant r+1$ and $g \in \mathcal{A}^*_k(B)$. By Corollary 2.4, there exist self-maps $f\colon X \to X$ and $h\colon A \to A$ such that

$$fi = ig, g\gamma = \gamma h.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
H^d(X) \arrow[r, "i^*"] \arrow[d, "f^*"'] & H^d(B) \arrow[r, "\gamma^*"] \arrow[d, "g^*"', "\cong"] & H^d(A) \arrow[d, "h^*"] \\
H^d(X) \arrow[r, "i^*"] & H^d(B) \arrow[r, "\gamma^*"] & H^d(A)
\end{tikzcd}$$

Then for $d \leqslant r < k$, by the second square above and (4), $g \in \mathcal{A}^*_k(B) \subseteq \mathcal{A}^*_r(B)$ implies that $h \in \mathcal{A}^*_r(A) = \mathcal{E}(A)$. For $r+1 \leqslant d \leqslant k$, by the first square above and (5), $g \in \mathcal{A}^*_k(B)$ implies that $f^*\colon H^d(X) \to H^d(X)$ is an isomorphism. Since $\mathrm{conn}(X) \geqslant r$, we get $f \in \mathcal{A}^*_k(X) = \mathcal{E}(X)$. Hence $g \in \mathcal{E}(B)$ and therefore $N^*\mathcal{E}(B) \leqslant k = N^*\mathcal{E}(X)$. □

3. mod p (co)homology self-closeness numbers

Let p be a prime, let $\mathbb{Z}/p$ be the set of integers modulo p and let $\mathbb{Z}_p$ be the set of integers localized at p. Let $\mathcal{CW}_{scpft}$ be the category of simply connected p-local CW-complexes with finitely generated homology groups over $\mathbb{Z}_p$ in each dimension.

We shall use the following universal coefficient theorem for cohomology:

Lemma 3.1. *For each $i \geqslant 1$ and a CW-complex X, there is an isomorphism:*

$$H^i(X;\mathbb{Z}/p) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}/p}(H_i(X;\mathbb{Z}/p),\mathbb{Z}/p).$$

There is an easier criterion to determine a homotopy equivalence in $\mathcal{CW}_{scpft}$:

Lemma 3.2. *Let p be a prime and $f\colon X \to Y$ a map (morphism) in the category $\mathcal{CW}_{scpft}$. Then the following are equivalent:*

1. *f is a homotopy equivalence.*
2. *$f_*\colon H_i(X;\mathbb{Z}/p) \to H_i(Y;\mathbb{Z}/p)$ is an isomorphism for all $i \geqslant 0$.*
3. *$f^*\colon H^i(Y;\mathbb{Z}/p) \to H^i(X;\mathbb{Z}/p)$ is an isomorphism for all $i \geqslant 0$.*

Proof. 1⇔2 is a restatement of Lemma 1.3 of [**19**]; 2⇔3 by Lemma 3.1. □

Hence for $X \in \mathcal{CW}_{scpft}$, we can detect self-homotopy equivalences of X by the induced automorphisms of $H_i(X;\mathbb{Z}/p)$ or $H^i(X;\mathbb{Z}/p)$.

Definition 3.3. Let $X \in \mathcal{CW}_{scpft}$.

$$\mathcal{A}^*_k(X;p) := \{f \in [X,X] \mid f^*\colon H^i(X;\mathbb{Z}/p) \xrightarrow{\cong} H^i(X;\mathbb{Z}/p) \text{ for } i \leqslant k\}.$$

The *mod-p cohomology self-closeness number* $N^*\mathcal{E}(X;p)$ is defined by:

$$N^*\mathcal{E}(X;p) := \min\{k \mid \mathcal{A}^*_k(X;p) = \mathcal{E}(X)\}.$$

The monoids $\mathcal{A}^k_*(X;p)$ and the *mod-p homology self-closeness number* $N_*\mathcal{E}(X;p)$ are defined after replacing cohomology by homology.

It is easy to see that $N^*\mathcal{E}(X;p), N^*\mathcal{E}(X;p)$ are homotopy invariants, by a parallel proof of Proposition 37 of [13].

Proposition 3.4. *Let p be a prime, $X \in \mathcal{CW}_{scpft}$. Then $N^*\mathcal{E}(X;p) = N_*\mathcal{E}(X;p)$.*

Proof. By Lemma 3.1 we have $\mathcal{A}_k^*(X;p) = \mathcal{A}_*^k(X;p)$ for each $k \geqslant 0$. Then the equality in the proposition follows. □

Proposition 3.5. *Let $X \in \mathcal{CW}_{sc}$ such that $H_i(X)$ is finitely generated for each i. Then $N_*\mathcal{E}(X) \leqslant N^*\mathcal{E}(X) \leqslant N_*\mathcal{E}(X) + 1$; $N_*\mathcal{E}(X) = N^*\mathcal{E}(X)$ if $H^{k+1}(X)$ is free for $k = N_*\mathcal{E}(X)$.*

Proof. By Propositions 41, 43, 45 of [13]. □

Example 3.6. Let $q \neq 0, n \geqslant 2$, then $N_*\mathcal{E}(P^n(q)) = n-1 < N^*\mathcal{E}(P^n(q)) = n$.

Proposition 3.7. *Let p be a prime and $X \in \mathcal{CW}_{scpft}$. Then $N_*\mathcal{E}(X) = N_*\mathcal{E}(X;p)$.*

Proof. Suppose that $f\colon X \to X$. By the naturality of the universal coefficient theorem for homology, there is a commutative diagram for each k:

$$\begin{tikzcd}
H_k(X)\otimes \mathbb{Z}/p \arrow[r, tail] \arrow[d, "f_*\otimes\mathbb{Z}/p"'] & H_k(X;\mathbb{Z}/p) \arrow[r, two heads] \arrow[d, "f_*"] & \mathrm{Tor}_1^{\mathbb{Z}}(H_{k-1}(X),\mathbb{Z}/p) \arrow[d, "\mathrm{Tor}(f_*,\mathbb{Z}/p)"] \\
H_k(X)\otimes \mathbb{Z}/p \arrow[r, tail] & H_k(X;\mathbb{Z}/p) \arrow[r, two heads] & \mathrm{Tor}_1^{\mathbb{Z}}(H_{k-1}(X),\mathbb{Z}/p)
\end{tikzcd}$$

It follows that $\mathcal{A}_*^k(X) \subseteq \mathcal{A}_*^k(X;p)$ and hence $N_*\mathcal{E}(X) \leqslant N_*\mathcal{E}(X;p)$.

Suppose that $N_*\mathcal{E}(X) = l$ and $f \in \mathcal{A}_*^l(X;p)$. Then by the long exact sequence of homology groups we have $H_i(C_f;\mathbb{Z}/p) = 0$ for $i \leqslant l$ and hence $H_i(C_f)\otimes\mathbb{Z}/p = 0$ for $i \leqslant l$, by the universal coefficient theorem for homology. Since X is p-local, so is C_f. It follows that $H_i(C_f) = 0$ for $i \leqslant l$ and hence the homomorphism $f_*\colon H_i(X) \to H_i(X)$ is an isomorphism for $i \leqslant l-1$ and an epimorphism for $i = l$. Since $H_k(X)$ is finitely generated, $f \in \mathcal{A}_*^l(X) = \mathcal{E}(X)$. Therefore $N_*\mathcal{E}(X;p) \leqslant l = N_*\mathcal{E}(X)$. □

Example 3.8. Let $n \geqslant 3, t, r \geqslant 1$, let $C_r^{n+2,t} = P^{n+1}(2^r) \cup_{i\eta q} \mathbf{C}P^{n+1}(2^t)$ be the four-cell Chang complex, where $\eta \in \pi_1^s$ is the suspension of the Hopf map and $i\colon S^n \to P^{n+1}(2^t)$ and $q\colon P^{n+1}(2^r) \to S^{n+1}$ are the canonical inclusion and quotient maps. We have

$$N_*\mathcal{E}(C_r^{n+2,t}) = N_*\mathcal{E}(C_r^{n+2,t};2) = n.$$

Proof. The proof is parallel to that of Lemma 3.1 of [20]. □

4. More properties of self-closeness numbers

Let $\mathcal{CW}_{sc0ft}$ be the category of simply connected CW-complexes with finitely generated homology group in each dimension. $\mathbb{Z}/0 = \mathbb{Z}$. We temporarily adopt the following notation:

$$\mathcal{A}_k^*(X;0) := \mathcal{A}_k^*(X), N^*\mathcal{E}(X;0) := N^*\mathcal{E}(X);$$
$$\mathcal{A}_*^k(X;0) := \mathcal{A}_*^k(X), N_*\mathcal{E}(X;0) := N_*\mathcal{E}(X).$$

Proposition 4.1. *Let p be a prime or $p = 0$ and $\{*\} \neq X \in \mathcal{CW}_{scpft}$. Then*

1. $N^*\mathcal{E}(\Sigma X;p) \geqslant N^*\mathcal{E}(X;p)+1$; *equality holds if* $\dim(X) \leqslant 2\cdot \mathrm{conn}(X)+1$.
2. $N_*\mathcal{E}(\Sigma X;p) \geqslant N_*\mathcal{E}(X;p)+1$; *equality holds if* $\dim(X) \leqslant 2\cdot \mathrm{conn}(X)+1$.

Proof. 1. Suppose that $N_*\mathcal{E}(\Sigma X;p)=k+1$ for some $k\geqslant 0$ and $f\in\mathcal{A}^k_*(X;p)$. By the natural isomorphism $H^i(X;\mathbb{Z}/p)\to H^{i+1}(\Sigma X;\mathbb{Z}/p)$, $\Sigma f\in\mathcal{A}^{k+1}_*(\Sigma X;p)=\mathcal{E}(\Sigma X)$. By naturality again, we get

$$f^*\colon H^i(X;\mathbb{Z}/p)\xrightarrow{\cong} H^i(X;\mathbb{Z}/p),\ \forall i\geqslant 0.$$

Thus $f\in\mathcal{E}(X)$ by Lemma 3.2 and therefore $N^*\mathcal{E}(X;p)\leqslant k=N^*\mathcal{E}(\Sigma X;p)-1$.

If $\dim(X)\leqslant 2\cdot\mathrm{conn}(X)+1$, by Theorem 1.21 of [7], the suspension map

$$\Sigma\colon [X,X]\longrightarrow[\Sigma X,\Sigma X]$$

is a surjection. Suppose that $N^*\mathcal{E}(X;p)=l$ and $F\in\mathcal{A}^*_{l+1}(\Sigma X;p)$ such that $F=\Sigma f$ for some $f\in[X,X]$. Then we have $f\in\mathcal{A}^*_l(X;p)=\mathcal{E}(X)$ and hence $F=\Sigma f\in\mathcal{E}(\Sigma X)$. Thus $N^*\mathcal{E}(\Sigma X;p)\leqslant l+1=N^*\mathcal{E}(X;p)+1$.

2. The proof of 2 is completed after replacing "cohomology" with "homology" in 1 above. □

It is easy to get $N_\sharp\mathcal{E}(\mathbb{C}P^n)=N_*\mathcal{E}(\mathbb{C}P^n)=N^*\mathcal{E}(\mathbb{C}P^n)=2$.

Example 4.2. $N_\sharp\mathcal{E}(\Sigma\mathbb{C}P^2)=N_*\mathcal{E}(\Sigma\mathbb{C}P^2)=N^*\mathcal{E}(\Sigma\mathbb{C}P^2)=5$.

Proof. Write $C^5_\eta=\Sigma\mathbb{C}P^2=S^3\cup_\eta e^5$. By Theorems 41, 45 of [13], we have

$$N_\sharp\mathcal{E}(C^5_\eta)=N^*\mathcal{E}(C^5_\eta)=N_*\mathcal{E}(C^5_\eta).$$

By Section 8 of [1], $[C^5_\eta,C^5_\eta]\cong\mathbb{Z}\langle 1_\eta\rangle\oplus\mathbb{Z}\langle i_3\bar{\zeta}\rangle$, where 1_η is the identity of C^5_η, $i_3\colon S^3\to C^5_\eta$ is the canonical inclusion map and $\bar{\zeta}\in[C^5_\eta,S^3]$ and $\tilde{\zeta}\in[S^5,C^5_\eta]$ satisfy the relations (relations (8.3) and (8.4) of [1]):

$$\bar{\zeta}i_3=2\cdot 1_3,\qquad q_5\tilde{\zeta}=2\cdot 1_5,\qquad i_3\bar{\zeta}+\tilde{\zeta}q_5=2\cdot 1_\eta, \tag{6}$$

where 1_n is the identity of S^n and $q_5\colon C^5_\eta\to S^5$ is the canonical quotient map.

Let $\sigma_n 1$ be the image of $1\in H_0(S^0)$ under the suspension: $H_0(S^0)\underset{\cong}{\xrightarrow{\Sigma^n}}H_n(S^n)$. We have

$$H_k(C^5_\eta)\cong\begin{cases}\mathbb{Z}\ \langle a_\eta\rangle, & k=3;\\ \mathbb{Z}\ \langle b_\eta\rangle, & k=5;\\ \quad 0, & otherwise,\end{cases}$$

where $a_\eta=(i_3)_*(\sigma_3 1), b_\eta=(q_5)^{-1}_*(\sigma_5 1)$. It follows that $N_*\mathcal{E}(C^5_\eta)=3$ or 5.

By the relations (6), it is easy to get that

$$(\overline{\zeta})_*(a_\eta)=2\cdot\sigma_3 1,\qquad (i_3\overline{\zeta})_*(b_\eta)=0,\qquad (\widetilde{\zeta})_*(\sigma_5 1)=2\cdot b_\eta.$$

We compute that $f=x\cdot 1_\eta+y\cdot i_n\overline{\zeta}\in\mathcal{A}^3_*(C^5_\eta)$ with $x,y\in\mathbb{Z}$ if and only if $x+2y=\pm1$. Note that $f=3\cdot 1_\eta-i_3\overline{\zeta}\notin\mathcal{E}(C^5_\eta)$: $f_*(b_\eta)=3\cdot b_\eta$, so we get

$$\mathcal{E}(C^5_\eta)=\mathcal{A}^5_*(C^5_\eta)\subsetneqq\mathcal{A}^3_*(C^5_\eta),\qquad N_*\mathcal{E}(C^5_\eta)=5.$$ □

Proposition 4.3. *Let p be a prime or $p=0$ and $X,Y\in\mathcal{CW}_{scpft}$. Then*

1. $N^*\mathcal{E}(X\vee Y;p)\geqslant\max\{N^*\mathcal{E}(X;p),N^*\mathcal{E}(Y;p)\}$.

2. $N^*\mathcal{E}(X \wedge Y; p), N^*\mathcal{E}(X \times Y; p) \geqslant \max\{N^*\mathcal{E}(X; p), N^*\mathcal{E}(Y; p)\}$.
3. $N^*\mathcal{E}(\Sigma(X \times Y); p) \geqslant N^*\mathcal{E}(\Sigma(X \wedge Y); p)$.

Similar results hold for mod p homology self-closeness numbers.

Proof. 1. Assume that $N^*\mathcal{E}(X \vee Y) = k < \max\{N^*\mathcal{E}(X), N^*\mathcal{E}(Y)\} = N^*\mathcal{E}(X)$. Suppose $f \in \mathcal{A}^*_k(X; p), g \in \mathcal{A}^*_k(Y; p)$. By the natural isomorphism: $H^d(X \vee Y; \mathbb{Z}/p) \cong H^d(X; \mathbb{Z}/p) \oplus H^d(Y; \mathbb{Z}/p)$, we have $f \vee g \in \mathcal{A}^*_k(X \vee Y; p) = \mathcal{E}(X \vee Y)$. It follows that $f \in \mathcal{E}(X), g \in \mathcal{E}(Y)$ and hence $\mathcal{A}^*_k(X) = \mathcal{E}(X)$. Therefore $N^*\mathcal{E}(X) \leqslant k = N^*\mathcal{E}(X \vee Y)$, which contradicts the assumption.

2.The proof is similar to that of Proposition 46 (2) of [**13**], using the general Künneth formula for cohomology with coefficients $\mathbb{Z}/p$.

3. By Proposition 4I.1 of [**8**], there is a homotopy equivalence:

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$

The inequality then follows from 1 and 2.

Replacing N^* by N_* and "cohomology" by the dual "homology", we get the proof of the corresponding results for (mod p) homology self-closeness numbers. $\square$

Example 4.4. Let $n \geqslant m \geqslant 2$. We have

$$N^*\mathcal{E}(\Sigma(S^m \times S^n)) = m + n + 1 > N^*\mathcal{E}(S^m \times S^n) = N^*\mathcal{E}(S^m \vee S^n) = n.$$

Proof. By Proposition 4.3 we have

$$m + n + 1 \geqslant N^*\mathcal{E}(\Sigma(S^m \times S^n)) \geqslant N^*\mathcal{E}(\Sigma(S^m \wedge S^n)) = m + n + 1.$$

$N^*\mathcal{E}(S^m \times S^n) = N^*\mathcal{E}(S^m \vee S^n) = n$ follows from Theorem 41, Theorem 45 of [**13**] and Proposition 5 of [**11**]. One can also prove this by applying Theorem 2.10 to the cofibration:

$$S^{m+n-1} \xrightarrow{[i_1,i_2]} S^m \vee S^n \xrightarrow{i} S^m \times S^n,$$

where $i_1\colon S^m \hookrightarrow S^m \vee S^n$, $i_2\colon S^n \hookrightarrow S^m \times S^n$ are the canonical inclusion maps and $[i_1, i_2] \in \pi_{m+n-1}(S^m \vee S^n)$ is their Whitehead product, a generator of a direct summand $\mathbb{Z}$. $\square$

Proposition 4.5. *Let p be a prime or $p = 0$, $X \in \mathcal{CW}_{sc}$ and let $l_p\colon X \to X_p$ be the localization at p. Then*

1. $N_*\mathcal{E}(X) \leqslant \max\big\{N_*\mathcal{E}(X_p) \mid p \in \{\text{primes}, 0\}\big\} \leqslant H_*\text{-}\dim(X)$.
2. $N_\sharp\mathcal{E}(X) \leqslant \max\big\{N_\sharp\mathcal{E}(X_p) \mid p \in \{\text{primes}, 0\}\big\} \leqslant H_*\text{-}\dim(X) + 1$.
   If, in addition, $H_n(X)$ is finitely generated for $n = H_*\text{-}\dim(X)$, then
   $$\max\big\{N_\sharp\mathcal{E}(X_p) \mid p \in \{\text{primes}, 0\}\big\} \leqslant H_*\text{-}\dim(X).$$
3. *If X is a torsion space ($X_0 = \{*\}$ or $\pi_i(X) \otimes \mathbb{Q} = 0$), then*
   $$N_\square\mathcal{E}(X) = \max\big\{N_\square\mathcal{E}(X_p) \mid p \in \{\text{primes}\}\big\}, \square = *, \sharp.$$

Proof. 1. Suppose that $\max\big\{N_*\mathcal{E}(X_p) \mid p \in \{\text{primes}, 0\}\big\} = k$ and $f \in \mathcal{A}^k_*(X)$. For each $p \in \{\text{primes}, 0\}$, by the universal property of localization, there is a unique (up to homotopy) map $f_p\colon X_p \to X_p$ such that $l_p f = f_p l_p$. Consider the following commutative diagram:

$$\begin{tikzcd} H_i(X) \arrow[r, "l_{p*}"] \arrow[d, "f_*"'] \arrow[rr, bend left, "-\otimes\mathbb{Z}_p"] & H_i(X_p) \arrow[r, "\cong"] \arrow[d, "f_{p*}"] & H_i(X)\otimes\mathbb{Z}_p \arrow[d, "f_*\otimes\mathbb{Z}_p"] \\ H_i(X) \arrow[r, "l_{p*}"] \arrow[rr, bend right, "-\otimes\mathbb{Z}_p"'] & H_i(X_p) \arrow[r, "\cong"] & H_i(X)\otimes\mathbb{Z}_p \end{tikzcd}$$

Since $-\otimes\mathbb{Z}_p$ is an exact functor, $f \in \mathcal{A}_*^k(X)$ implies that $f_*\otimes\mathbb{Z}_p$ is an isomorphism for $i \leqslant k$ and hence $f_p \in \mathcal{A}_*^k(X_p) = \mathcal{E}(X_p)$ for all p. Thus $f \in \mathcal{A}_*^\infty(X) = \mathcal{E}(X)$ and $N_*\mathcal{E}(X) \leqslant k = \max\left\{N_*\mathcal{E}(X_p) \mid p \in \{\text{primes}, 0\}\right\}$.

For the second "$\leqslant$", since $H_i(X_p) \cong H_i(X)\otimes\mathbb{Z}_p$ for each prime p or $p=0$, we have

$$N_*\mathcal{E}(X_p) \leqslant H_*\text{-}\dim(X_p) \leqslant H_*\text{-}\dim(X).$$

2. The proof of the first "$\leqslant$" is similar and the second "$\leqslant$" follows from Theorem 3 of [**13**].

3. If X is a torsion space, by (5) of [**18**, page 41], there is a product decomposition:

$$X = X_p \times \prod_{q\neq p} X_q.$$

Thus $N_\Box\mathcal{E}(X) \geqslant \max\left\{N_\Box\mathcal{E}(X_p) \mid p \in \{\text{primes}\}\right\}$, by Proposition 4.3 if $\Box = *$ and by Theorem 3 of [**6**] if $\Box = \sharp$. $\square$

Proposition 4.6. *Let p be a prime or $p=0$ and let $X \in \mathcal{CW}_{scpft}$. If the cohomology ring $H^*(X;\mathbb{Z}/p)$ is generated by cohomology classes $x_i \in H^{k_i}(X;\mathbb{Z}/p)$ $(i=1,\dots,m)$ with $k_1 \leqslant \cdots \leqslant k_m$, then $N^*\mathcal{E}(X;p) \leqslant k_m$.*

Proof. Suppose that $f \in \mathcal{A}^*_{k_m}(X;p)$. Then the induced ring homomorphism

$$f^*\colon H^*(X;\mathbb{Z}/p) \longrightarrow H^*(X;\mathbb{Z}/p)$$

is surjective, since all generators x_i are in the image. Then in each degree $H^i(X;\mathbb{Z}/p)$ is finitely generated, which implies that the induced epimorphism $f^*\colon H^i(X;\mathbb{Z}/p) \to H^i(X;\mathbb{Z}/p)$ is an isomorphism for all i. Thus $f \in \mathcal{A}^*_\infty(X;p) = \mathcal{E}(X)$ by Lemma 3.2. $\square$

Lemma 4.7. *Let M be a closed simply-connected manifold of dimension $2n$. If $f\colon M \to M$ is a map of degree ±1, then $f \in \mathcal{E}(M)$ if and only if $f \in \mathcal{A}^*_n(M)$.*

Proof. Suppose that $f \in \mathcal{A}^*_n(M)$. By 12 Theorem (p. 248) of [**17**], there is a natural short exact sequence:

$$0 \longrightarrow \operatorname{Ext}(H^{i+1}(M),\mathbb{Z}) \longrightarrow H_i(M) \longrightarrow \operatorname{Hom}(H^i(M),\mathbb{Z}) \longrightarrow 0.$$

Hence $f \in \mathcal{A}^*_n(M)$ implies that $f \in \mathcal{A}_*^{n-1}(M)$. Then by the natural Poincáre duality $H^{n+i}(M) \cong H_{n-i}(M)$ for $i = 1,\dots,n$, we get $f \in \mathcal{A}^*_{2n-1}(M)$. Since $\deg(f) = \pm1$, $f^*\colon H^{2n}(M) \to H^{2n}(M)$ is an isomorphism and hence $f \in \mathcal{A}^*_{2n}(M) = \mathcal{E}(M)$. $\square$

We end the paper with a theorem given by Professor Haibao Duan.

Lemma 4.8 (The Hard Lefschetz Theorem). *Let M be a simply-connected compact Kähler manifold of real dimension $2n$ with the Kähler class $d \in H^2(M, \mathbb{Q})$. Then the multiplication*

$$d^{n-r} \cup -\colon H^r(M;\mathbb{Q}) \longrightarrow H^{2n-r}(M;\mathbb{Q})$$

is an isomorphism for $0 \leqslant r \leqslant n$.

Theorem 4.9 (Duan). *Let M be a simply-connected compact Kähler manifold with torsion-free cohomology and $H^2(M)$ a cyclic group. Then $N^*\mathcal{E}(M) = 2$.*

Proof. Let $\dim(M) = 2n$. We may choose a Kähler class d of M such that (M, d) is a Kähler manifold with $H^2(M;\mathbb{Z}) \cong \mathbb{Z}\langle d\rangle$. By Lemma 4.7, it suffices to show that a self-map f of M satisfying $f^*(d) = \varepsilon \cdot d(\varepsilon = \pm 1)$ belongs to $\mathcal{A}^*_n(M)$.

For each $2 \leqslant r \leqslant n$, since $H^r(M)$ is torsion free, there exist cohomology classes $x_1, \ldots, x_{m_r}$ such that $H^r(M) \cong \bigoplus_{i=1}^{m_r} \mathbb{Z}\langle x_i\rangle$. Then $\{x_i\}_{i=1}^{m_r}$ is also a $\mathbb{Q}$-basis of $H^r(M;\mathbb{Q})$. By Lemma 4.8, $\{d^{n-r}x_i\}_{i=1}^{m_r}$ is a basis of $H^{2n-r}(M;\mathbb{Q})$. There are relations:

$$d^{n-r}x_i x_j = a_{ij}d^n, a_{ij} \in \mathbb{Q}, 1 \leqslant i, j \leqslant m_r.$$

Then $A = (a_{ij})_{m_r \times m_r}$ is a non-singular matrix by the Poincáre duality.

Let $f^*(x_i) = \sum_{k=1}^{m_r} b_{ik}x_k$ and put $B_r = (b_{ij}) \in M_{m_r}(\mathbb{Z})$. Applying the ring homomorphism f^* to the above relations, we have

$$\varepsilon^{n-r}d^{n-r}(\sum_{k=1}^{m_r} b_{ik}x_k)(\sum_{k=1}^{m_r} b_{jk}x_k) = a_{ij}\varepsilon^n d^n.$$

Let B_r^T denote the transpose of B_r. Then we get an equality of matrices:

$$\varepsilon^{n-r}B_r A B_r^T = \varepsilon^n A.$$

The non-singularity of A then implies that $det(B_r)^2 = \varepsilon^{rm_r} = 1$. Thus B_r is non-singular and therefore $f \in \mathcal{A}^*_n(M)$. $\square$

Example 4.10. Let $n \leqslant m < \infty$, $G_n(\mathbb{C}^m)$ be the Grassmannian of n-dimension vector subspaces of $\mathbb{C}^m$. By 4.10 Example of [**5**], $G_n(\mathbb{C}^m)$ is a Kähler manifold and by Chapters 6, 14 of [**10**], $G_n(\mathbb{C}^m)$ satisfies the other conditions in Theorem 4.9. Thus $N^*(G_n(\mathbb{C}^m)) = 2$.

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