Homology, Homotopy and Applications, vol. 23(2), 2021, pp.375–378

ERRATUM TO "SMALL PRESENTATIONS OF MODEL CATEGORIES AND VOPĚNKA'S PRINCIPLE"

G. RAPTIS AND J. ROSICKÝ

(*communicated by J. Daniel Christensen*)

Abstract

The proof of [**6**, Theorem 3.1(3)] turns out to be invalid due to a recent correction [**3**] of a result from [**1**] which was used in our proof. We revise the definition of a strictly cofibrantly generated weak factorization system $(\mathcal{L}, \mathcal{R})$ in a cocomplete category $\mathcal{M}$ by requiring that the codomains of the generator $\mathcal{X} \subseteq \mathcal{L}$ are also strictly small relative to $\mathcal{X}$. Using this slightly stronger definition, we give a new proof of [**6**, Theorem 3.1(3)] and explain that the rest of the results of [**6**] hold with essentially the same proofs.

1. Recollections

We recall that our article [**6**] had two main and related goals.

The first main goal was concerned with the problem of finding small presentations of model categories. Our strategy for addressing this problem was based on the work of Dugger [**4**] who proved that combinatorial model categories admit small presentations. Dugger's method was revisited and formulated more abstractly in [**6**, Subsection 4.1].

Our main theorem [**6**, Theorem 4.5] formulated two such existence results for small presentations of model categories. Assuming that Vopěnka's principle holds, the first result [**6**, Theorem 4.5(1)] claimed that a model category $(\mathcal{M}, \mathcal{C}of, \mathcal{W}, \mathcal{F}ib)$ admits a small presentation if the weak factorization system $(\mathcal{C}of, \mathcal{W} \cap \mathcal{F}ib)$ is *strictly cofibrantly generated* (in the sense of [**6**, Definition 2.9]). We recall that a strictly cofibrantly generated weak factorization system is also cofibrantly generated in the standard sense – the more refined notion was inspired by the fat small-object argument (see [**6**, Section 2] for more details).

The second existence result for small presentations [**6**, Theorem 4.5(2)] required stronger assumptions on $\mathcal{M}$, but it is independent of Vopěnka's principle. The present note is not concerned with this second existence result.

A model category which admits a small presentation is clearly Quillen equivalent to a combinatorial model category – the converse is also true. The problem of whether a general cofibrantly generated model category is Quillen equivalent to a combinatorial model category was addressed in previous articles by the authors [**5, 7**]. Specifically, the first author had claimed in [**5**] that, assuming Vopěnka's principle,

Received October 10, 2020; published on September 29, 2021.
2020 Mathematics Subject Classification: 18N40, 18C35, 55U35.
Key words and phrases: cofibrantly generated model category, colimit–dense subcategory, Vopěnka's principle.
Article available at http://dx.doi.org/10.4310/HHA.2021.v23.n2.a20

every cofibrantly generated model category is Quillen equivalent to a combinatorial model category. The proof in [**5**] contained a mistake and is only valid under further additional assumptions.

The second main goal of our article [**6**] was to point out the mistake in [**5**] and revise the statement in [**5**] accordingly – this was done in [**6**, Section 5]. The original claim in [**5**] remains open; the closely related aforementioned result [**6**, Theorem 4.5(1)], which is obtained using different methods, appears to be slightly weaker in general (cf. [**6**, Corollary 4.6, Remark 4.7]). The present note is not concerned with the results of [**6**, Section 5].

2. Colimit–dense subcategories

The proof of [**6**, Theorem 4.5(1)] is heavily based on the technical results of [**6**, Section 3] in order to ensure that Dugger's general method can be applied. Specifically, it is based on [**6**, Theorem 3.1(3)], either directly, or via one of its important consequences (for example, [**6**, Corollary 3.14]).

Among other things, this theorem [**6**, Theorem 3.1(3)] (or [**6**, Corollary 3.4(3)]) claimed that the full subcategory of cofibrant objects $\mathcal{M}_c \subseteq \mathcal{M}$ is preaccessible – or, equivalently, every cofibrant object is presentable in $\mathcal{M}_c$. The proof of this claim in [**6**, Theorem 3.1(3)] used [**1**, 6.35] which asserts that, assuming Vopěnka's principle, every category with a small colimit–dense subcategory has a small dense subcategory. Unfortunately, it has recently turned out that this result [**1**, 6.35] is false in this generality. This was discussed in [**3**] and a corrected weaker statement was obtained [**3**, Theorem 2.2].

Thus, due to our application of [**1**, 6.35] in the proof of [**6**, Theorem 3.1(3)], the proof of the first claim in [**6**, Theorem 3.1(3)] is rendered invalid. The present note is concerned with a correction of [**6**, Theorem 3.1(3)], and consequently of those statements in [**6**] which made use of this result (e.g., [**6**, Theorem 4.5(1)]).

3. Corrections

For the conclusion of [**6**, Theorem 3.1(3)] to hold, we consider the following strengthening of [**6**, Definition 2.9(d)]:

Definition 3.1. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a cocomplete category $\mathcal{M}$ and let λ be a regular cardinal. We say that $(\mathcal{L}, \mathcal{R})$ is *strictly* (λ-)*cofibrantly generated* if there is a generator $\mathcal{X} \subseteq \mathcal{L}$ for $\mathcal{R}$ such that the domains *and the codomains* of the morphisms in $\mathcal{X}$ are strictly (λ-)small relative to $\mathcal{X}$.

With this definition, the statement of [**6**, Proposition 2.11] does not apply for obvious reasons, but it can be revised as follows (with the same proof):

Proposition 3.2. *Let $(\mathcal{L}, \mathcal{R})$ be a cofibrantly generated weak factorization system in $\mathcal{M}$ generated by a set of morphisms $\mathcal{X}$ whose domains and codomains are $\aleph_0$-small relative to $\mathcal{X}$. Then $(\mathcal{L}, \mathcal{R})$ is strictly $\aleph_0$-cofibrantly generated.*

Moreover, with the above strengthening of [**6**, Definition 2.9(d)], we also recover the first claim in [**6**, Theorem 3.1(3)] as follows (we refer to [**6**, p. 311] for the details about terminology and notation):

Theorem 3.3. *Let $\mathcal{M}$ be a cocomplete category and $(\mathcal{L}, \mathcal{R})$ a strictly κ-cofibrantly generated weak factorization system in $\mathcal{M}$ with generating set $\mathcal{X}$, where κ is regular cardinal. Then, assuming that Vopěnka's principle holds, $\mathcal{M}_c$ is preaccessible.*

Proof. Since the domains and the codomains of the generator $\mathcal{X}$ are strictly κ-small relative to $\mathcal{X}$, every object in $\widetilde{\mathcal{A}}(\mathcal{X}, \kappa)$ (= the closure of the collection of domains and codomains of the morphisms in $\mathcal{X}$ under κ-small colimits [**6**, Subsection 3.1]) is again strictly κ-small. To see this, consider a κ-small colimit $M = \operatorname{colim}_i M_i$ in $\mathcal{M}$, where each M_i is strictly κ-small, and let $N = \operatorname{colim}_j N_j$ be the colimit of a κ-directed good diagram in $\mathcal{M}$ whose links are $\mathcal{X}$-cellular morphisms. Then note the canonical isomorphisms:

$$\begin{aligned}
\mathcal{M}(M, N) &\cong \mathcal{M}(\operatorname{colim}_i M_i, \operatorname{colim}_j N_j) \cong \lim_i \mathcal{M}(M_i, \operatorname{colim}_j N_j) \\
&\cong \lim_i \operatorname{colim}_j \mathcal{M}(M_i, N_j) \quad \cong \operatorname{colim}_j \lim_i \mathcal{M}(M_i, N_j) \\
&\cong \operatorname{colim}_j \mathcal{M}(\operatorname{colim}_i M_i, N_j) \cong \operatorname{colim}_j \mathcal{M}(M, N_j).
\end{aligned}$$

Note that we used here that κ-small limits commute with κ-directed colimits in **Set**. In particular, every object in the full subcategory $\mathcal{A}(\mathcal{X}, \kappa) \subseteq \widetilde{\mathcal{A}}(\mathcal{X}, \kappa)$, which consists of those objects which are cofibrant, is strictly κ-small.

Let $\mathcal{M}_{ce}$ denote the full subcategory of $\mathcal{X}$-cellular objects in $\mathcal{M}$. The proof of [**6**, Theorem 3.1(1)] showed that every $\mathcal{X}$-cellular object in $\mathcal{M}$ is a κ-directed good colimit of a diagram in $\mathcal{M}$ consisting of objects in $\mathcal{A}(\mathcal{X}, \kappa)$ and $\mathcal{X}$-cellular morphisms. Thus, $\mathcal{A}(\mathcal{X}, \kappa)$ is consistently colimit–dense in $\mathcal{M}_{ce}$ in the sense of [**3**, Definition 2.1]. By [**3**, Theorem 2.2], it follows that $\mathcal{M}_{ce}$ has a small dense subcategory. Since the category of cofibrant objects $\mathcal{M}_c$ is the closure of $\mathcal{M}_{ce}$ under split idempotents, it follows that $\mathcal{M}_c$ also has a small dense subcategory (see the proof at the end of [**3**, Remark 3.10(b)]). Therefore, assuming Vopěnka's principle, it follows also that $\mathcal{M}_c$ is preaccessible by [**2**, Theorem 1]. □

Remark 3.4. We mention that [**6**, Theorem 3.1(1)] remains valid – without the stronger definition above in place of [**6**, Definition 2.9(d)]. We do not know if [**6**, Theorem 3.1(3)] is false as originally stated, that is, using the original [**6**, Definition 2.9(d)].

With the definition given above in place of [**6**, Definition 2.9(d)], the rest of the statements in [**6**, Section 3] remain valid with the same proofs. More specifically, this applies to the following statements in [**6**]: Corollary 3.3(1), Corollary 3.4(3+4), Lemma 3.9, Proposition 3.12 and Corollary 3.14.

Lastly, with the definition given above in place of [**6**, Definition 2.9(d)], our main result [**6**, Theorem 4.5(1)] is valid with the same proof. The statement of [**6**, Corollary 4.6(2)] must be modified as follows (cf. Proposition 3.2 above):

Corollary 3.5. *Let $(\mathcal{M}, \mathcal{C}of, \mathcal{W}, \mathcal{F}ib)$ be a cofibrantly generated model category. Suppose that $(\mathcal{C}of, \mathcal{W} \cap \mathcal{F}ib)$ admits a generating set $\mathcal{X}$ whose domains and codomains are $\aleph_0$-small relative to $\mathcal{X}$. Then, assuming Vopěnka's principle, $\mathcal{M}$ admits a small presentation.*

References

[**1**] J. Adámek and J. Rosický, *Locally presentable and accessible categories.* London Math. Soc. Lecture Note Ser., 189. Cambridge University Press, Cambridge, 1994.

[**2**] J. Adámek and J. Rosický, *On preaccessible categories*, J. Pure Appl. Algebra 105 (1995), no. 3, 225–232.

[**3**] J. Adámek, A. Brooke-Taylor, T. Campion, L. Positselski and J. Rosický, *Colimit-dense subcategories*, Comment. Math. Univ. Carolin. 60 (2019), no. 4, 447–462.

[**4**] D. Dugger, *Combinatorial model categories have presentations*, Adv. Math. 164 (2001), no. 1, 177–201.

[**5**] G. Raptis, *On the cofibrant generation of model categories*, J. Homotopy Relat. Struct. 4 (2009), no. 1, 245–253.

[**6**] G. Raptis and J. Rosický, *Small presentations of model categories and Vopěnka's principle*, Homology Homotopy Appl. 20 (2018), no. 1, 303–328.

[**7**] J. Rosický, *Are all cofibrantly generated model categories combinatorial?*, Cah. Topol. Géom. Différ. Catég. 50 (2009), no. 3, 233–238.

G. Raptis `georgios.raptis@ur.de`

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

J. Rosický `rosicky@math.muni.cz`

Department of Mathematics and Statistics, Faculty of Sciences, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic