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BOUSFIELD–SEGAL SPACES

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(*communicated by Emily Riehl*)

Abstract

This paper is a study of Bousfield–Segal spaces, a notion introduced by Julie Bergner drawing on ideas about Eilenberg–Mac Lane objects due to Bousfield. In analogy to Rezk's Segal spaces, they are defined in such a way that Bousfield–Segal spaces naturally come equipped with a homotopy-coherent fraction operation in place of a composition.

In this paper we show that Bergner's model structure for Bousfield–Segal spaces in fact can be obtained from the model structure for Segal spaces both as a localization and a colocalization. We thereby prove that Bousfield–Segal spaces really are Segal spaces, and that they characterize exactly those with invertible arrows. We note that the complete Bousfield–Segal spaces are precisely the homotopically constant Segal spaces, and deduce that the associated model structure yields a model for both ∞-groupoids and Homotopy Type Theory.

1. Introduction

In [**1**, Section 6], Julie Bergner introduced two model structures on the category $s\mathbf{S}$ of bisimplicial sets. Characterized by their fibrant objects, these are the model structure $(s\mathbf{S}, \mathrm{B})$ for Bousfield–Segal spaces and the model structure $(s\mathbf{S}, \mathrm{CB})$ for complete Bousfield–Segal spaces. A reduced and pointed version of Bousfield–Segal spaces itself originated in Bousfield's work [**3**] under the name "very special bisimplicial sets of type 1" (representing what he calls Eilenberg–Mac Lane objects of type 1 in the homotopy category of spaces). A pointed version of complete Bousfield–Segal spaces appeared in the same unpublished note under the name "very special bisimplicial sets of type 0".

Both notions were defined in the last section of Bergner's paper, proposing model structures whose fibrant objects are to be thought of as ∞-groupoidal "Segal-like" spaces. The primary topic of this paper is to study these two model structures and relate them to Rezk's model structures for Segal spaces and complete Segal spaces as introduced in [**18**].

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In essence, the relation between Rezk's notions and Bergner's notions is the relation between formal multiplication and formal division. In 1-dimensional algebra, taking fractions in a group yields an operation $(g,h) \mapsto g/h$ on the underlying set of the group which is defined by the multiplication gh^{-1}. This operation satisfies various properties which may be abstractly axiomatized so as to define a formal fraction operation on any set without the assumption of a multiplication in the first place (see [**3**, 1.4] or [**14**, 1.3]).

To this effect, Bousfield–Segal spaces as defined in the work of Bergner are Reedy fibrant bisimplicial sets X which come equipped with a homotopy-coherent and many-sorted fraction operation. If we denote the space of edges between two vertices $x, y \in X_{00}$ in X by $X_1(x,y)$, that is a family of arrows

$$\_\,/\,\_ : X_1(x,z) \times X_1(x,y) \to X_1(y,z)$$

induced by associated *Bousfield maps*, in a very similar way as Segal spaces are Reedy fibrant bisimplicial sets Y equipped with a composition operation

$$\circ \colon Y_1(y,z) \times Y_1(x,y) \to Y_1(x,z)$$

induced by associated Segal maps. It is a classical result of ordinary group theory that fraction operations (subject to suitable axioms) and group structures yield equivalent data on any given set. Accordingly, we will exhibit Bousfield–Segal spaces as exactly those Segal spaces in which every edge is an equivalence.

To explain how the latter arise as the fibrant objects in a left Bousfield localization of Rezk's model structure for Segal spaces in a very natural way, let us recall that the category **Gpd** of (small) groupoids arises as a localization of the category **Cat** of (small) categories. If by $I[1]$ we denote the free groupoid generated by the walking arrow $[1]$, then **Gpd** is the localization of **Cat** at the inclusion $[1] \to I[1]$. The model structure for Kan complexes can be obtained similarly as the left Bousfield localisation of the model structure for quasi-categories at the inclusion $\Delta^1 \to N(I[1])$, and, indeed, Kan complexes are exactly the quasi-categories with invertible edges. Analogously, we will see that the model structure for Bousfield–Segal spaces is the left Bousfield localization of the model structure for Segal spaces at a canonical map induced by the inclusion $[1] \to I[1]$.

Joyal and Tierney have shown in [**13**, Theorem 4.11] that the model structure $(s\mathbf{S}, \mathrm{CS})$ for complete Segal spaces is a model for $(\infty,1)$-category theory equivalent to the model category for quasi-categories. It hence follows that the model structure $(s\mathbf{S}, \mathrm{CB})$ for complete Bousfield–Segal spaces is a model for ∞-groupoids equivalent to the one associated to Kan complexes, as stated in [**1**, Theorem 6.12]. We will furthermore see that $(s\mathbf{S}, \mathrm{CB})$ is right proper and supports a model of Homotopy Type Theory with univalent universes in the sense of [**22**], using that Bousfield–Segal spaces are complete if and only if they are homotopically constant.

Therefore, Section 2 recalls the Reedy model structure $(s\mathbf{S}, R_v)$ on bisimplicial sets and some of its associated Joyal–Tierney calculus from [**13**, Section 2]. Section 3 introduces Bousfield–Segal spaces in the sense of [**1**]. Here, we explain how every Bousfield–Segal space X comes equipped with a contractible choice of fraction operations which induce an associated homotopy groupoid $\mathrm{Ho}_B(X)$.

In Section 4 we will show that every Bousfield–Segal space is not just "Segal-like" but in fact a Segal space and that the model structure $(s\mathbf{S}, \mathrm{B})$ is a left Bousfield localization of $(s\mathbf{S}, \mathrm{S})$. We will also see that the homotopy category $\mathrm{Ho}(X)$ of a Bousfield–Segal space X associated to it *as a Segal space* (following [**18**, 5.5]) is a groupoid and coincides with the construction $\mathrm{Ho}_B(X)$. Hence, many of Rezk's results in [**18**] and Joyal and Tierney's results in [**13**] carry over to the model structure for Bousfield–Segal spaces.

In Section 5 we use this to describe Bousfield–Segal spaces as the Segal spaces with invertible edges in a precise way. We furthermore define the core of a Segal space X as the largest Bousfield–Segal space contained in X and show that this construction exhibits $(s\mathbf{S}, \mathrm{B})$ as a homotopy colocalization of $(s\mathbf{S}, \mathrm{S})$ as well.

In Section 6 we study complete Bousfield–Segal spaces and show that they are exactly the Reedy fibrant homotopically constant bisimplicial sets. The model structure for homotopically constant bisimplicial sets is contained in different classes of well understood model structures treated in the literature of [**20**], [**7**] and [**6**] respectively. It follows that the vertical projection, the horizontal projection and the diagonal functor all are part of Quillen equivalences between the model category $(s\mathbf{S}, \mathrm{CB})$ for complete Bousfield–Segal spaces and the model category $(\mathbf{S}, \mathrm{Kan})$ for Kan complexes. In Section 7 we conclude that $(s\mathbf{S}, \mathrm{CB})$ is a type theoretic model category with as many univalent fibrant universes as there are inaccessible cardinals. We give a direct proof that $(s\mathbf{S}, \mathrm{CB})$ is right proper using a symmetry argument.

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2. Preliminaries on bisimplicial sets

A bisimplicial set $X \in s\mathbf{S}$ can be understood as a functor $X\colon \Delta^{op} \times \Delta^{op} \to \mathrm{Set}$, and whenever done so, will be denoted by $X_{\bullet\bullet}$ to highlight its two components. Taking its exponential transpose associated to the product with Δ^{op} on the left or on the right yields simplicial objects in the category $\mathbf{S}$ of simplicial sets, whose evaluation at an object $[n] \in \Delta^{op}$ is the n-th column $X_n := X_{n\bullet}$ and the n-th row $X_{\bullet n}$ respectively.

2.1. The box product and its adjoints

To recall some constructions which are very convenient in describing the generating sets for the model structures on bisimplicial sets we are interested in, we briefly summarize some constructions from [**13**, Section 2].

By left Kan extension of the Yoneda embedding $y\colon \Delta\times\Delta\to s\mathbf{S}$ along the product of Yoneda embeddings $y\times y\colon \Delta\times\Delta\to\mathbf{S}\times\mathbf{S}$ one obtains a bicontinuous functor $\_ \,\square\, \_ \colon \mathbf{S}\times\mathbf{S}\to s\mathbf{S}$, often called the *box product*. The box product is divisible on both sides, i.e. gives rise to adjoint pairs

$$A\,\square\,\_ \colon \mathbf{S} \rightleftarrows s\mathbf{S}\colon A\setminus \_$$

and

$$\_ \,\square\, B\colon \mathbf{S} \rightleftarrows s\mathbf{S}\colon \_/B$$

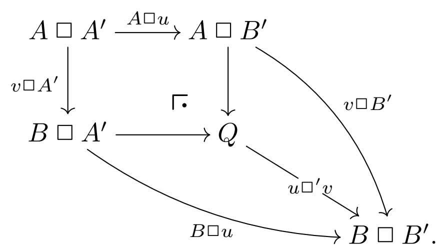
for all simplicial sets A and B. In particular, for any bisimplicial set X, the simplicial set $\Delta^n\setminus X\cong X_n$ is the n-th column and $X/\Delta^n\cong X_{\bullet n}$ is the n-th row of X. Vice versa, for a given $X\in s\mathbf{S}$, the induced functors

$$\_ \setminus X\colon \mathbf{S}^{op} \rightleftarrows \mathbf{S}\colon X/\_$$

form an adjoint pair with left adjoint $\_ \setminus X$. The box product induces a functor

$$\_ \,\square'\, \_ \colon \mathbf{S}^{[1]}\times\mathbf{S}^{[1]}\to (s\mathbf{S})^{[1]}$$

on arrow-categories via the pushout-product construction. That means, it takes a pair of arrows $u\colon A\to B$, $v\colon A'\to B'$ in $\mathbf{S}$ to the natural map $u\,\square'\,v$ in the diagram



The functor $\_ \,\square'\, \_$ is divisible on both sides, too; the right adjoints to the functors $f\,\square'\,\_$ and $\_ \,\square'\, f$ for a given map $f\in s\mathbf{S}$ are denoted by

$$\langle f\setminus \_\rangle, \langle \_/f\rangle\colon (s\mathbf{S})^{[1]}\to\mathbf{S}^{[1]}$$

respectively. In the following, given arrows u,v in $\mathbf{S}$ we write $u\pitchfork v$ to denote that u has the left lifting property against v.

Proposition 2.1 ([**13**, Proposition 2.1])**.** *For any two maps $u,v\in\mathbf{S}$ and any map $f\in s\mathbf{S}$, we have*

$$(u\,\square'\,v)\pitchfork f \iff u\pitchfork\langle f/v\rangle \iff v\pitchfork\langle u\setminus f\rangle.$$

□

2.2. The vertical and horizontal Reedy model structures

It is well known that the Reedy and injective model structures on $s\mathbf{S}$ coincide. This coincidence boils down to the fact that the simplex-category Δ is an elegant

Reedy category as treated by Bergner and Rezk in [**2**, 3] (in fact it is the archetype of such a Reedy category). We loosely follow the language and structure of [**13**] and call this model structure the *vertical Reedy model structure*, denoted by R_v. Its cofibrations are the (pointwise) monomorphisms, its weak equivalences the pointwise weak equivalences and its fibrations the maps with the right lifting property with respect to those cofibrations which are also weak equivalences. The classes of vertical Reedy cofibrations, weak equivalences and fibrations will be denoted by $\mathcal{C}_v$, $\mathcal{W}_v$ and $\mathcal{F}_v$, respectively. We call the fibrations in the vertical Reedy model structure "v-fibrations" for short.

For $n \geqslant 0$ we denote by $\delta_n\colon \partial\Delta^n \hookrightarrow \Delta^n$ the boundary inclusion of the n-simplex $\Delta^n \in \mathbf{S}$ and, for $0 \leqslant i \leqslant n$, by $h^n_i\colon \Lambda^n_i \hookrightarrow \Delta^n$ the corresponding i-th horn inclusion. Recall that the set $\{\delta_n \mid n \geqslant 0\}$ of boundary inclusions generates the class of cofibrations and the set $\{h^n_i \mid 0 \leqslant i \leqslant n, 1 \leqslant n\}$ of horn inclusions generates the class of acyclic cofibrations in the Quillen model structure $(\mathbf{S}, \mathrm{Kan})$. These acyclic cofibrations are often referred to as the *anodyne* maps, and their corresponding fibrations are the *Kan fibrations*.

In terms of the general calculus of Reedy structures as presented for example in [**11**, Section 5.2], the object $\partial\Delta^n \setminus X$ is the n-th matching object of X. Hence, by [**11**, Theorem 5.2.5], a map $f\colon X \to Y$ in $(s\mathbf{S}, R_v)$ is an (acyclic) v-fibration if and only if the associated maps

$$\langle \delta_n \setminus f\rangle\colon X_n \to Y_n \times_{(\partial\Delta^n \setminus Y)} (\partial\Delta^n \setminus X)$$

are (acyclic) Kan fibrations in $\mathbf{S}$. One can show that the class of cofibrations $\mathcal{C}_v$ of $(s\mathbf{S}, R_v)$ is generated by the set

$$\mathcal{I}_v := \{\delta_n \,\Box'\, \delta_m\colon (\Delta^n \,\Box\, \partial\Delta^m) \cup_{\partial\Delta^n \Box \partial\Delta^m} (\partial\Delta^n \,\Box\, \Delta^m) \to (\Delta^n \,\Box\, \Delta^m) \mid 0 \leqslant m, n\},$$

and the class $\mathcal{W}_v \cap \mathcal{C}_v$ of acyclic cofibrations is generated by the set

$$\mathcal{J}_v := \{\delta_n \,\Box'\, h^m_i\colon (\Delta^n \,\Box\, \Lambda^m_i) \cup_{\partial\Delta^n \Box \Lambda^m_i} (\partial\Delta^n \,\Box\, \Delta^m) \to (\Delta^n \,\Box\, \Delta^m) \mid 0 \leqslant n, 1 \leqslant m, 0 \leqslant i \leqslant m\}.$$

It is easy to show that properness of $(\mathbf{S}, \mathrm{Kan})$ implies properness of $(s\mathbf{S}, R_v)$, since every v-(co)fibration is also a pointwise (co)fibration, and the Reedy weak equivalences are exactly the pointwise weak equivalences.

Lastly, we recall that the vertical Reedy model structure is simplicially enriched as follows. The projection $p_2\colon \Delta \times \Delta \to \Delta$ onto the second component and the corresponding inclusion $\iota_2 = ([0], \mathrm{id})\colon \Delta \to \Delta \times \Delta$ constitute an adjoint pair $p_2 \dashv \iota_2$, and hence give rise to an adjoint pair

$$p_2^*\colon \mathbf{S} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} s\mathbf{S}\colon \iota_2^*,$$

with $(p_2^* A)_n = A$ for all $n \geqslant 0$, and $\iota_2^* X = X_0$ the 0-th column of X. The category $s\mathbf{S}$ is a presheaf category and as such cartesian closed. Thus, for bisimplicial sets X and Y we obtain an exponential $Y^X \in s\mathbf{S}$, and thereby a simplicial enrichment of $s\mathbf{S}$ via $\mathrm{Hom}_2(X, Y) := \iota_2^*(Y^X)$.

Proposition 2.2 ([**13**, Propositions 2.4 and 2.6])**.** *The simplicial enrichment* Hom_2 *on* $s\mathbf{S}$ *turns* $(s\mathbf{S}, R_v)$ *into a simplicial model category.* □

It follows that the Reedy model structure $(s\mathbf{S}, R_v)$ is a simplicial and left proper cellular model category in the sense of [10, Chapter 12]. Hence, by [10, Theorem 4.1.1], given a set of maps $A \subset s\mathbf{S}$, the left Bousfield localization of $(s\mathbf{S}, R_v)$ at A exists, and is simplicial, left proper and cellular again. We will denote this localization by $\mathcal{L}_A(s\mathbf{S}, R_v)$. Its fibrant objects are exactly the A-local v-fibrant objects.

We hereby conclude the discussion of the vertical Reedy model structure.

The permutation $\sigma := (p_2, p_1)\colon \Delta \times \Delta \to \Delta \times \Delta$ induces an isomorphism $\sigma^*\colon s\mathbf{S} \to s\mathbf{S}$ which transports the vertical Reedy model structure into the *horizontal Reedy model structure* R_h. Its class of cofibrations is given by $\mathcal{C}_h = \{\text{monomorphisms in } s\mathbf{S}\}$, its weak equivalences are the rowwise weak homotopy equivalences,

$$\mathcal{W}_h = \{f\colon X \to Y \mid f_{\bullet n}\colon X_{\bullet n} \to Y_{\bullet n} \text{ is a weak homotopy equivalence for all } n \geqslant 0\}.$$

Its cofibrations and acyclic cofibrations are generated by the sets $\mathcal{I}_h = \mathcal{I}_v$ and

$$\mathcal{J}_h = \{h_i^n \,\square'\, \delta_m\colon (\Delta^n \,\square\, \partial\Delta^m) \cup_{\Lambda_i^n \square \partial\Delta^m} (\Lambda_i^n \,\square\, \Delta^m) \to (\Delta^n \,\square\, \Delta^m) \mid$$
$$0 \leqslant n, 1 \leqslant m, 0 \leqslant i \leqslant m\}$$

respectively. We denote its class of fibrations by $\mathcal{F}_h$. In analogy to the pair $p_2^* \dashv \iota_2^*$, we have an adjunction

$$p_1^*\colon \mathbf{S} \underset{\perp}{\rightleftarrows} s\mathbf{S}\colon \iota_1^*,$$

with $(p_1^* A)_{\bullet n} = A$ for all $n \geqslant 0$, and $\iota_1^* X = X_{\bullet 0}$ the 0-th row of X.

Joyal and Tierney show in [13] that $(s\mathbf{S}, R_v)$ naturally comes equipped with two orthogonal projections, a Quillen right adjoint $\iota_1^*\colon (s\mathbf{S}, R_v) \to (\mathbf{S}, \mathrm{Kan})$ on the one hand, and a mere right adjoint $\iota_2^*\colon s\mathbf{S} \to \mathbf{S}$ on the other. In order to construct a homotopy theory of $(\infty, 1)$-categories in $s\mathbf{S}$, they localize $(s\mathbf{S}, R_v)$ at a suitable set of maps such that the horizontal projection $i_2^*\colon s\mathbf{S} \to \mathbf{S}$ becomes a Quillen right adjoint (and in fact part of a Quillen equivalence) to the Joyal model structure $(\mathbf{S}, \mathrm{QCat})$. In the process, Segal spaces arise naturally as an intermediate step; their individual horizontal projections already yield quasi-categories objectwise.

In order to construct a homotopy theory of ∞-*groupoids*, one can localize $(s\mathbf{S}, R_v)$ at a larger class of maps such that the horizontal projection $\iota_2^*\colon s\mathbf{S} \to \mathbf{S}$ becomes a Quillen right adjoint (and in fact part of a Quillen equivalence) to the model structure for *Kan complexes* $(\mathbf{S}, \mathrm{Kan})$ as we will see in Section 6. The corresponding intermediate objects, the Segal spaces with invertible arrows, are the subject of the following three sections.

3. Bousfield–Segal spaces

In this section, we give the definition of Bousfield–Segal spaces as introduced in [1, Section 6] and describe an associated fraction operation they naturally come equipped with. The definition is phrased in terms of *Bousfield maps* associated to a bisimplicial set, and in order to motivate these let us first recall the definition of Segal spaces and their associated Segal maps.

Let $\mathrm{sp}_n\colon \mathrm{Sp}_n \hookrightarrow \Delta^n$ be the n-th spine-inclusion, i.e.

$$\mathrm{Sp}_n = \bigcup_{i<n} \zeta_i[\Delta^1]$$

for $\zeta_i\colon [1] \to [n]$, $0 \mapsto i$, $1 \mapsto i+1$. Localizing $(s\mathbf{S}, R_v)$ at the set of horizontally constant diagrams

$$\mathrm{S} := \{p_1^*(\mathrm{sp}_n)\colon p_1^*(\mathrm{Sp}_n) \hookrightarrow p_1^*(\Delta^n) \mid 2 \leqslant n\}$$

yields the left proper cellular simplicial model structure $(s\mathbf{S}, S) := \mathcal{L}_S(s\mathbf{S}, R_v)$ whose fibrant objects are the *Segal spaces* as defined in [**18**, Section 4.1] and [**13**, Definition 3.1]. By construction, these are v-fibrant bisimplicial sets X such that the maps

$$(p_1^*(\mathrm{sp}_n))^*\colon \mathrm{Hom}_2(p_1^*(\Delta^n), X) \to \mathrm{Hom}_2(p_1^*(\mathrm{Sp}_n), X) \tag{1}$$

are weak homotopy equivalences for all $n \geqslant 2$.

For each $n \geqslant 2$, the map (1) is isomorphic to $\mathrm{sp}_n \setminus X\colon \Delta^n \setminus X \to \mathrm{Sp}_n \setminus X$. Thus, its codomain is the pullback $X_1 \times_{X_0} \cdots \times_{X_0} X_1$ taken along the boundaries $d_0 \setminus X$ and $d_1 \setminus X$ successively. In the following, we denote this pullback by $X_1 \times^S_{X_0} \cdots \times^S_{X_0} X_1$. Then we define the *Segal maps*

$$\xi_n\colon X_n \to X_1 \times^S_{X_0} \cdots \times^S_{X_0} X_1 \tag{2}$$

via $\xi_n := \mathrm{sp}_n \setminus X$ for $n \geqslant 2$, such that Segal spaces are the v-fibrant bisimplicial sets whose associated Segal maps are acyclic fibrations. We can think of a Segal space X as a simplicial collection of Kan complexes, where X_0 is its space of objects and X_1 is its space of edges. It comes equipped with a horizontal weak composition in the form of the diagram $X_1 \times^S_{X_0} X_1 \xleftarrow[\sim]{\xi_2} X_2 \xrightarrow{d_1} X_1$ whose higher compositional laws are encoded in the Segal maps ξ_n for $n > 2$.

In a similar fashion, v-fibrant bisimplicial sets can carry a weak horizontal fractional structure as referred to in the Introduction. It is induced by acyclicity of their associated Bousfield maps defined as follows. Consider the function $\gamma_i\colon [1] \to [n]$ mapping $0 \mapsto 0$, $1 \mapsto i$ and let

$$C_n := \bigcup_{0<i} \gamma_i[\Delta^1]$$

be the 1-skeletal cone whose pinnacle is the initial vertex $0 \in \Delta^n$. We will refer to its edges as the *initial edges* of Δ^n and let $c_n\colon C_n \hookrightarrow \Delta^n$ denote the canonical inclusion. Localizing $(s\mathbf{S}, R_v)$ at the set of horizontally constant diagrams

$$B := \{p_1^*(c_n)\colon p_1^*(C_n) \to p_1^*(\Delta^n) \mid n \geqslant 2\}$$

yields a model structure $(s\mathbf{S}, B) := \mathcal{L}_B(s\mathbf{S}, R_v)$. This model structure was introduced in [**1**, Section 6], Bergner calls its fibrant objects *Bousfield–Segal spaces.* A v-fibrant bisimplicial set X is B-local if and only if the fibrations $c_n \setminus X\colon \Delta^n \setminus X \twoheadrightarrow C_n \setminus X$ are weak homotopy equivalences. Here, $C_n \setminus X \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is the n-fold fibre product of X_1 over X_0 along d_1 everywhere. We distinguish this pullback notationally by $X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1$. We define the *Bousfield maps*

$$\beta_n\colon X_n \to X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1$$

of X via $\beta_n := c_n \setminus X$.

Definition 3.1. Let X be a v-fibrant bisimplicial set. We say that X is a *Bousfield–Segal space* if the Bousfield maps

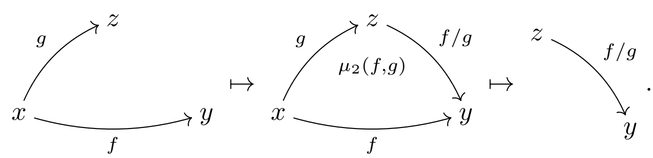
$$\beta_n \colon X_n \to X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1 \tag{3}$$

are weak homotopy equivalences for all $n \geqslant 2$.

Given a Bousfield–Segal space X, the fibration $\beta_2 \colon X_2 \twoheadrightarrow X_1 \times^B_{X_0} X_1$ admits a section μ_2 and thus yields a composite map

$$\_\,/\,\_ \colon X_1 \times^B_{X_0} X_1 \xrightarrow{\mu_2} X_2 \xrightarrow{d_0} X_1. \tag{4}$$

From now on we refer to this map as the *fraction* operation associated to X. On the horizontal simplicial sets $X_{\bullet m}$ it may be illustrated as follows:



Definition 3.2. Given a bisimplicial set X, for vertices $x \in X_{00}$ we write $1_x := s_0 x$ and for $v, w \in X_{n0}$ we write $v \sim w$ if $[v] = [w] \in \pi_0 X_n$. If X is v-fibrant and $x, y \in X_{00}$ are vertices, the *hom-space* $X(x, y)$ denotes the pullback of $(d_1, d_0) \colon X_1 \to X_0 \times X_0$ along $(x, y) \in X_{00} \times X_{00}$.

The following lemma shows that the operation (4) satisfies the axioms of an abstract fraction operation emerging in ordinary group theory in a many-sorted and homotopy-coherent manner.

Lemma 3.3. *For any Bousfield–Segal space X and $x, y, z \in X_{00}$, the fraction operation restricts to a map*

$$\_\,/\,\_ \colon X_1(x, y) \times X_1(x, z) \to X_1(z, y).$$

Then

(1) $f/f \sim 1_y$ for all vertices $f \colon x \to y$ in X_1,
(2) $f/1_x \sim f$ for all vertices $f \colon x \to y$ in X_1,
(3) $f/g \sim (f/h)/(g/h)$ for all triples $(f, g, h) \in X_1 \times^B_{X_0} X_1 \times^B_{X_0} X_1$.

Proof. Straightforward calculation. □

The maps μ_2 and d_0 are natural transformations of simplicial sets, hence the operation $\_\,/\,\_$ descends to homotopy classes. Therefore, for the family of sets

$$\mathrm{Ho}_B(X) := (\pi_0 X_1(x, y) \mid (x, y) \in X_{00} \times X_{00})$$

we obtain the following corollary.

Corollary 3.4. *The family $\mathrm{Ho}_B(X)$ together with the operation $\_ \circ \_$, defined as the composite*

$$\mathrm{Ho}_B(X)(y, z) \times \mathrm{Ho}_B(X)(x, y) \to \mathrm{Ho}_B(X)(y, z) \times \mathrm{Ho}_B(X)(y, x) \to \mathrm{Ho}_B(X)(x, z)$$

$$([g], [f]) \mapsto [g]/([1_x]/[f]) = [g/(1_x/f)],$$

is a groupoid.

Proof. Straightforward calculation. □

This fraction operation on Bousfield–Segal spaces is in its essence already present in [**3**]. Corollary 3.4 says that the fraction operation on a Bousfield–Segal space X induces an invertible composition on the quotient of X under some of its homotopical data. In the course of the following two sections we lift this statement to the level of the homotopy-coherent data itself.

4. Bousfield–Segal spaces are B-local Segal spaces

Despite the suggestive name it is not clear a priori that Bousfield–Segal spaces as defined in the previous section are in fact Segal spaces. In this section we show that Bousfield–Segal spaces are exactly the B-local Segal spaces.

Let X be a Bousfield–Segal space and recall the notation from (2) and (3) for its associated Segal and Bousfield maps respectively. Then by Definition 3.1, its Bousfield maps $\beta_n\colon X_n \twoheadrightarrow X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1$ are acyclic fibrations. In order to show that X is a Segal space, we have to derive that its Segal maps $\xi_n\colon X_n \twoheadrightarrow X_1 \times^S_{X_0} \cdots \times^S_{X_0} X_1$ are acyclic as well.

This means we want to show that the functor $\_ \setminus X\colon \mathbf{S}^{op} \to \mathbf{S}$ takes the spine inclusions sp_n to acyclic fibrations whenever it takes the left cone inclusions c_n to such. The proof can thus be boiled down to verifying that every class of morphisms in $\mathbf{S}$ which is subject to suitable closure properties contains the spine inclusions whenever it contains the left cone inclusions. Therefore, the discussion in this section will take place almost entirely in the category of simplicial sets.

We proceed in the following steps. Let A be a class of arrows in $\mathbf{S}$ which contains the left cone inclusions and furthermore is saturated and satisfies 3-for-2 for monomorphisms (these are the suitable closure properties referred to above). Then, first, via a reduction to the left horn inclusions, we show that for every $n \geqslant 2$ the canonical embedding of $\Delta^n = N([n])$ into the free groupoid $I\Delta^n = N(I[n])$ generated by it is contained in A as well. In fact, we will need only the case $n = 2$ here, but the full result will be applied later in Section 5. The case $n = 2$ will be used to show that $c_2 \in A \Leftrightarrow I[c_2] \in A$ and that $\mathrm{sp}_2 \in A \Leftrightarrow I[\mathrm{sp}_2] \in A$, where the maps on the right hand side are canonically induced inclusions into the free groupoid $I\Delta^2$. Now, while the automorphism group of the category $[2]$ is trivial, the groupoid $I[2]$ comes equipped with a non-trivial automorphism which interchanges its associated spine and cone inclusions $I[\mathrm{sp}_2]$ and $I[c_2]$. That means $I[c_2] \in A \Leftrightarrow I[\mathrm{sp}_2] \in A$, which together with the two equivalences above proves that $\mathrm{sp}_2 \in A$. The case $n > 2$ will then follow from $n = 2$ by a factorization argument.

With this road map in mind, we first relate the left cone inclusions and the left horn inclusions. The following lemma is a variation of [**13**, Lemma 3.5] which is a similar statement for essential edges.

Lemma 4.1. *Let A be a saturated class of morphisms in $\mathbf{S}$. Suppose further that A has the right cancellation property for monomorphisms, i.e. $vu \in A$ and $u \in A$ imply $v \in A$ for all monomorphisms $u, v \in \mathbf{S}$. Then $\{h^n_0 \mid n \geqslant 2\} \subseteq A$ if and only if $\{c_n \mid n \geqslant 2\} \subseteq A$.*

Proof. Let $k_n\colon C_n \to \Lambda^n_0$ be the canonical inclusion of simplicial sets, such that the map $c_n\colon C_n \hookrightarrow \Delta^n$ factors through the inclusions

$$C_n \xrightarrow{k_n} \Lambda^n_0 \xrightarrow{h^n_0} \Delta^n.$$

By assumption, it suffices to show that $k_n \in A$ for all $n \geqslant 2$ for both directions. Therefore, we show that the inclusions k_n can be constructed from the lower dimensional left cone inclusions c_m by pasting them together recursively in such a way that whenever $c_m \in A$ for all $m < n$ we stay inside A at each step along the way. The two implications will then be shown to follow from this construction in the last paragraph.

For $n \geqslant 2$ and $0 < i \leqslant n$, consider the inclusion

$$C_n \hookrightarrow C_n \cup \bigcup_{0<j\leqslant i} d^j[\Delta^{n-1}]. \tag{5}$$

Note that for $i = n$ this inclusion is exactly k_n. We show by induction that the inclusion (5) is contained in A for all $0 < i \leqslant n$ whenever the set $\{c_m \mid 2 \leqslant m < n\}$ is contained in A.

For $n = 2$ and $0 < i \leqslant 2$ the inclusion (5) is exactly the identity $k_2\colon C_2 \to \Lambda^2_0$ and is hence contained in A. Given $n \geqslant 2$, assume the inclusion (5) is contained in A for every $0 < i \leqslant n$. We now show that the inclusion

$$C_{n+1} \hookrightarrow C_{n+1} \cup \bigcup_{0<j\leqslant i} d^j[\Delta^n]$$

is contained in A for every $0 < i \leqslant n+1$. There is a pushout square

$$\begin{array}{ccccc}
C_n & \xrightarrow[d^1]{\cong} & C_{n+1} \cap d^1[\Delta^n] & \hookrightarrow & C_{n+1} \\
{\scriptstyle c_n}\big\downarrow & & \big\downarrow & & \big\downarrow \\
\Delta^n & \xrightarrow[d^1]{\cong} & d^1[\Delta^n] & \hookrightarrow & C_{n+1} \cup d^1[\Delta^n],
\end{array}$$

where the boundaries d^1 in the left square are isomorphisms, because the coboundary $d^1\colon [n] \to [n+1]$ is a monomorphism. This implies that the canonical inclusion

$$c_{(1,n+1)}\colon C_{n+1} \hookrightarrow C_{n+1} \cup d^1[\Delta^n]$$

is contained in A. Similarly, for $0 < i \leqslant n$ we have a pushout square

$$\begin{array}{ccc}
d^{i+1}[\Delta^n] \cap (C_{n+1} \cup \bigcup_{0<j\leqslant i} d^j[\Delta^n]) & \hookrightarrow & C_{n+1} \cup \bigcup_{0<j\leqslant i} d^j[\Delta^n] \\
\big\downarrow & & \big\downarrow \\
d^{i+1}[\Delta^n] & \hookrightarrow & C_{n+1} \cup \bigcup_{0<j\leqslant i+1} d^j[\Delta^n]
\end{array} \tag{6}$$

and isomorphisms

$$\begin{array}{ccc}
C_n \cup \bigcup_{0<j\leqslant i} d^j[\Delta^{n-1}] & \xrightarrow[d^{i+1}]{\cong} & d^{i+1}[\Delta^n] \cap (C_{n+1} \cup \bigcup_{0<j\leqslant i} d^j[\Delta^n]) \\
\big\downarrow & & \big\downarrow \\
\Delta^n & \xrightarrow[d^{i+1}]{\cong} & d^{i+1}[\Delta^n].
\end{array} \tag{7}$$

Here, the upper boundary d^{i+1} is an isomorphism, because

$$\begin{aligned} d^{i+1}[\Delta^n] \cap (C_{n+1} \cup \bigcup_{0<j\leqslant i} d^j[\Delta^n]) &= (d^{i+1}[\Delta^n] \cap C_{n+1}) \cup \bigcup_{0<j\leqslant i} (d^{i+1}[\Delta^n] \cap d^j[\Delta^n]) \\ &= (d^{i+1}[\Delta^n] \cap C_{n+1}) \cup \bigcup_{0<j\leqslant i} d^{i+1}d^j[\Delta^{n-1}] \\ &= d^{i+1}[C_n \cup \bigcup_{0<j\leqslant i} d^j[\Delta^{n-1}]]. \end{aligned}$$

By the inductive hypothesis, the inclusion $C_n \hookrightarrow C_n \cup \bigcup_{0<j\leqslant i} d^j[\Delta^n]$ is contained in A. But then, by the right cancellation property of A, the inclusion

$$C_n \cup \bigcup_{0<j\leqslant i} d^j[\Delta^n] \hookrightarrow \Delta^n$$

is contained in A, too. Therefore, via (6) and (7), the canonical inclusion

$$c_{(i+1,n+1)} \colon C_{n+1} \cup \bigcup_{0<j\leqslant i} d^j[\Delta^n] \hookrightarrow C_{n+1} \cup \bigcup_{0<j\leqslant i+1} d^j[\Delta^n]$$

is contained in A for every $0 < i \leqslant n+1$. It follows that the composition

$$c_{(i+1,n+1)} \circ \cdots \circ c_{(2,n+1)} \circ c_{(1,n+1)} \colon C_{n+1} \hookrightarrow \bigcup_{0<j\leqslant i+1} d^j[\Delta^n]$$

is contained in A. In particular, since k_{n+1} is the composition of all $c_{(i,n+1)}$ for $0 < i \leqslant n+1$, the map k_{n+1} is contained in A. This finishes the induction.

Thus, on the one hand, $\{c_n \mid n \geqslant 2\} \subseteq A$ implies $\{k_n \mid n \geqslant 3\} \subseteq A$, and $k_2 = \mathrm{id}_{C_2}$ is contained in A trivially. On the other hand, in order to prove that $\{h_0^n \mid n \geqslant 2\} \subseteq A$ implies $\{k_n \mid n \geqslant 2\} \subseteq A$ and hence $\{c_n \mid n \geqslant 2\} \subseteq A$, assume that A contains all left horn inclusions. Since $C_2 = \Lambda_0^2$ and $h_0^2 = c_2$, the inclusion c_2 is contained in A. Suppose $n \geqslant 2$ and $c_m \in A$ for all $2 \leqslant m \leqslant n$. As we have seen above, this implies $k_{n+1} \in A$. This in turn implies $c_{n+1} \in A$, because $c_{n+1} = h_0^{n+1} \circ k_{n+1}$. □

Corollary 4.2. *Let $X \in s\mathbf{S}$ be v-fibrant. The following two conditions are equivalent:*

(1) $c_n \setminus X$ is an acyclic fibration in $\mathbf{S}$ for all $n \geqslant 2$.

(2) $h_0^n \setminus X$ is an acyclic fibration in $\mathbf{S}$ for all $n \geqslant 2$.

Proof. Let X be v-fibrant. The class

$$A := \{f \in \mathbf{S} \mid f \text{ is a monomorphism and } f \setminus X \text{ is an acyclic fibration}\}$$

has the right cancellation property for monomorphisms. It is saturated by Proposition 2.1 and the fact that the class of monomorphisms in $\mathbf{S}$ is saturated. Therefore, the two statements are equivalent by Lemma 4.1. □

Next, we show that the canonical embeddings $\Delta^n \to I\Delta^n$ into the nerve $I\Delta^n := N(I[n])$ of the free groupoid generated by $[n]$ are cellular cofibrations with respect to the set $\{h_0^n \mid n \geqslant 2\}$. By that we mean that each map $\Delta^n \to I\Delta^n$ is the transfinite composition of pushouts of coproducts of such horn inclusions. In the case $n = 1$, this was noted by Rezk in [**18**, Section 11] using a combinatorial description of the simplices in $I\Delta^1$. Although the combinatorics of the $I\Delta^n$ for $n \geqslant 2$ is considerably more complicated, we can make use of a suitable alternative description of the nerve of a groupoid, given for example in [16]. Therefore, consider diagrams in $\mathbf{S}(C_m, I\Delta^n)$ depicted as follows:

$$\begin{array}{ccccc} & & c_0 & & \\ & \overset{h_1}{\swarrow} & \overset{h_2}{\swarrow} & \overset{h_m}{\searrow} & \\ c_1 & c_2 & \dots & c_m. & \end{array} \tag{8}$$

Every such conical diagram is uniquely determined by its sequence $(c_i \mid 0 \leqslant i \leqslant m)$ of objects in $[n]$. Indeed, we obtain a bijection

$$(I\Delta^n)_m \cong \mathbf{S}(C_m, I\Delta^n),$$

which sends an m-simplex $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_m} c_m$ to the diagram $(c_0 \xrightarrow{f_i \cdots f_1} c_i)_{i \leqslant m}$. Under this identification, the boundaries and degeneracies of an m-simplex in $I\Delta^n$ represented as a cone $(c_0 \xrightarrow{h_i} c_i)_{i \leqslant m} \in \mathbf{S}(C_m, I\Delta^n)$ are given as follows:

$$d_j((c_0 \xrightarrow{h_i} c_i)_{i \leqslant m}) = \begin{cases} (c_0 \xrightarrow{h_i} c_i)_{i \leqslant m, i \neq j} & \text{if } j > 0, \\ (c_1 \xrightarrow{h_i \dots h_2} c_i)_{1 < i \leqslant m} & \text{if } j = 0, \end{cases}$$

$$s_j((c_0 \xrightarrow{h_i} c_i)_{i \leqslant m}) = \begin{cases} (h_1, \dots, h_j, h_j, h_{j+1}, \dots, h_m) & \text{if } j > 0, \\ (1_{c_0}, h_1, \dots, h_m) & \text{if } j = 0. \end{cases}$$

Proposition 4.3. *The canonical inclusions $\{\Delta^n \to I\Delta^n \mid n \geqslant 1\}$ are cellular cofibrations with respect to the set $\{h_0^n \mid n \geqslant 2\}$.*

Proof. Let $n \geqslant 1$. The subobject $\Delta^n \subset I\Delta^n$ consists exactly of those conical diagrams (8) such that the sequence $(c_i \mid 0 \leqslant i \leqslant m)$ is monotone in the linear order $[n]$. We will add the non-degenerate simplices in its complement step by step via recursion. Therefore, we define a filtration $\{F^{(m)} \mid m \geqslant 1\}$ of $I\Delta^n$ with $F^{(1)} := \Delta^n$ such that the inclusions $F^{(m)} \to F^{(m+1)}$ are cellular for the left horn inclusions.

Note that every 1-simplex in $I\Delta^n$ of the form $0 \to c_1$ for $c_1 \in [n]$ is contained in $F^{(1)}$. Thus, we may assume that $F^{(m)} \subseteq I\Delta^n$ has been defined such that every diagram of the form $(0 \to c_i \mid 1 \leqslant i \leqslant m)$ is contained in $F^{(m)}$.

Then every non-degenerate non-monotone sequence $(c_i \mid 0 \leqslant i \leqslant m+1)$ in $[n]$ with $c_0 = 0$ determines a unique map $\Lambda_0^{m+1} \to I\Delta^n$. This map factors through the subobject $F^{(m)}$ since, for every $0 < i \leqslant m+1$, the boundary

$$d_i((c_i \mid 0 \leqslant i \leqslant m+1)) = (0, \dots, \hat{c}_i, \dots, c_{m+1})$$

is contained in $F^{(m)}$ by assumption.

Let T_{m+1} be the set of all non-degenerate non-monotone sequences $(c_i \mid 0 \leqslant i \leqslant m+1)$ with $c_0 = 0$, and let $F^{(m+1)}$ be the pushout

$$\begin{array}{ccc} \coprod_{c \in T_{m+1}} \Lambda_0^{m+1} & \xrightarrow{\coprod h_0^{m+1}} & \coprod_{c \in T_{m+1}} \Delta^{m+1} \\ {\scriptstyle c}\downarrow & \ulcorner & \downarrow \\ F^{(m)} & \hookrightarrow & F^{(m+1)}. \end{array}$$

We obtain a natural map $F^{(m+1)} \to I\Delta^n$ via the inclusion $F^{(m)} \subseteq I\Delta^n$ on the one hand, and the unique maps $c\colon \Delta^{m+1} \to I\Delta^n$ extending $c\colon \Lambda_0^{m+1} \to I\Delta^n$ for $c \in T_{m+1}$ on the other hand. This map $F^{(m+1)} \to I\Delta^n$ is a monomorphism as can be verified by case distinction. The subobject $F^{(m+1)} \subseteq I\Delta^n$ contains every sequence of the form $(0 \to c_i \mid 1 \leqslant i \leqslant m+1)$ in $[n]$ by construction.

Every inclusion $F^{(m)} \to F^{(m+1)}$ is cellular for the set $\{h_0^n \mid n \geqslant 2\}$, and so is the composite inclusion $\Delta^1 \to \bigcup_{1 \leqslant m} F^{(m)}$. We are left to show that every simplex in $I\Delta^n$ is contained in some $F^{(m)}$.

Therefore, let $(c_i \mid 0 \leqslant i \leqslant m)$ be a non-degenerate m-simplex in $I\Delta^n$. If the sequence is monotone or $c_0 = 0$, it is contained in $F^{(m)}$. Otherwise, $c_0 > 0$ and the sequence $(0, c_0, \ldots, c_m)$ is contained in the set T_{m+1}. It follows that $(0, c_0, \ldots, c_m)$ is an $(m+1)$-simplex in $F^{(m+1)}$, and so $d_0((0, c_0, \ldots, c_m)) = (c_0, \ldots, c_m)$ is contained in $F^{(m+1)}$ as well. This finishes the proof. □

Corollary 4.4. *Let A be a saturated class of morphisms in* **S** *with 3-for-2 for monomorphisms (i.e. with both the left and right cancellation property for monomorphisms). Then $B = \{c_n \mid n \geqslant 2\} \subseteq A$ implies $S = \{\mathrm{sp}_n \mid n \geqslant 2\} \subseteq A$.*

Proof. We show the cases $n = 2$ and $n > 2$ separately.

For $n = 2$, we denote the non-degenerate 2-cell in Δ^2 by $[0] \xrightarrow{f} [1] \xrightarrow{g} [2]$. Let $I[\mathrm{Sp}_2] \subset I\Delta^2$ and $I[C_2] \subset I\Delta^2$ be the respective pushouts $I\Delta^1 \cup_{\Delta^0} I\Delta^1 \subset I\Delta^2$ along the obvious pairs of boundaries.

Then the assignment $0 \mapsto 2$, $1 \mapsto 0$, $2 \mapsto 1$, $f \mapsto (gf)^{-1}$, $g \mapsto f$ generates an automorphism $a\colon I\Delta^2 \to I\Delta^2$ with $a[I[\mathrm{Sp}_2]] = I[C_2]$. We obtain a diagram of inclusions as follows:

$$\begin{array}{ccccccc}
\mathrm{Sp}_2 & \longrightarrow & I[\mathrm{Sp}_2] & \xrightarrow[\cong]{a} & I[C_2] & \longleftarrow & C_2 \\
\downarrow{\scriptstyle \mathrm{sp}_2} & & \downarrow{\scriptstyle I[\mathrm{sp}_2]} & & \downarrow{\scriptstyle I[c_2]} & & \downarrow{\scriptstyle c_2} \\
\Delta^2 & \longrightarrow & I\Delta^2 & \xrightarrow[\cong]{a} & I\Delta^2 & \longleftarrow & \Delta^2.
\end{array}$$

The bottom inclusions are contained in A by Lemma 4.1 and Proposition 4.3. The top inclusions are pushouts of the inclusion $\Delta^1 \to I\Delta^1$ and it follows via saturation that they are contained in A as well. The map c_2 is contained in A by assumption, and hence it follows by left and right cancellation that $\mathrm{sp}_2 \in A$.

For $n > 2$, let $\Delta^{\{i,j,k\}} \subset \Delta^n$ denote the subobject generated by the 2-cell $[i] \to [j] \to [k]$ whenever $i \leqslant j \leqslant k$. Let $(\mathrm{SpC})_n \subset \Delta^n$ be the subobject given by the union

$$(\mathrm{SpC})_n := \bigcup_{0<i<n} \Delta^{\{0,i,i+1\}}.$$

It contains all essential and all initial edges of Δ^n, so both the spine and left cone inclusions factor through $(\mathrm{SpC})_n$:

$$\begin{array}{ccccc}
\mathrm{Sp}_n & \searrow & & \swarrow & C_n \\
& & (\mathrm{SpC})_n & & \\
{\scriptstyle \mathrm{sp}_n}\searrow & & \downarrow & & \swarrow{\scriptstyle c_n} \\
& & \Delta^n. & &
\end{array} \tag{9}$$

The inclusion $\mathrm{Sp}_n \to (\mathrm{SpC})_n$ is a finite composition of pushouts of $\mathrm{sp}_2\colon \mathrm{Sp}_2 \to \Delta^2$, and, likewise, the inclusion $C_n \to (\mathrm{SpC})_n$ is a finite composition of pushouts of $c_2\colon C_2 \to \Delta^2$. It follows that both these inclusions are contained in A, and so does $c_n\colon C_n \to \Delta^n$ by assumption. It follows that $(\mathrm{SpC})_n \to \Delta^n$ lies in A by right cancellation, and hence so does the composition $\mathrm{Sp}_n \to (\mathrm{SpC})_n \to \Delta^n$. □

Theorem 4.5. *Every Bousfield–Segal space is a Segal space. In particular, the model structures* $(s\mathbf{S}, B)$ *and* $\mathcal{L}_B(s\mathbf{S}, S)$ *coincide.*

Proof. Let X be a Bousfield–Segal space and consider the class

$$A := \{f \in \mathbf{S} \mid f \text{ is a monomorphism and } f \setminus X \text{ is an acyclic fibration}\}.$$

The class A is saturated, because the class of acyclic fibrations in $(\mathbf{S}, \mathrm{Kan})$ is closed under pullbacks, retracts and sequential limits. It satisfies 3-for-2 for monomorphisms, because $\_ \setminus X$ maps all monomorphisms to fibrations and the class of weak equivalences in $(\mathbf{S}, \mathrm{Kan})$ satisfies 3-for-2. We have $B \subseteq A$ by assumption and hence $S \subseteq A$ by Corollary 4.4. Therefore, X is a Segal space.

In other words, every fibrant object in $(s\mathbf{S}, B)$ is also fibrant in $\mathcal{L}_B(s\mathbf{S}, S)$. This implies that the left Bousfield localizations $(s\mathbf{S}, B)$ and $\mathcal{L}_B(s\mathbf{S}, S)$ have the same class of fibrant objects, and hence coincide. □

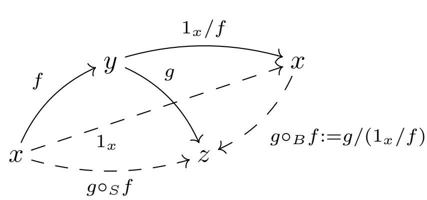
Remark 4.6. One also can show Theorem 4.5 in a more algebraic fashion. The algebraic laws of the fraction operation described in Lemma 3.3 hold in a homotopy coherent manner and can be used to construct a homotopy inverse to the Segal map ξ_2 of any given Bousfield–Segal space. Acyclicity of the Segal maps ξ_n for $n > 2$ can then be proven as in Corollary 4.4, or by induction similar to Lemma 4.1.

Theorem 4.5 implies that the constructions for Segal spaces from [**18**] apply to the class of Bousfield–Segal spaces. For example, every Bousfield–Segal space X comes equipped with a homotopy category $\mathrm{Ho}(X)$ as constructed in [**18**, 5.5]. This induces a notion of Dwyer–Kan equivalences ([**18**, 7.4]) between Bousfield–Segal spaces X and Y, which are exactly those maps between X and Y to be inverted in the model structure for complete Segal spaces ([**18**, Theorem 7.7]).

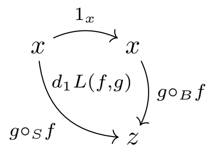
Recall the groupoid $\mathrm{Ho}_B(X)$ associated to X in Corollary 3.4.

Corollary 4.7. *For any Bousfield–Segal space* X*, the categories* $\mathrm{Ho}(X)$ *and* $\mathrm{Ho}_B(X)$ *coincide. In particular,* $\mathrm{Ho}(X)$ *is a groupoid.*

Proof. Let X be a Bousfield–Segal space. The families $\mathrm{Ho}_B(X)$ and $\mathrm{Ho}(X)$ of sets coincide and have the same identity, so we have to show that the two corresponding compositions induced by the Bousfield and Segal maps coincide, too. Let η_2 be a section to $\xi_2\colon X_2 \xrightarrow{\sim} X_1 \times^S_{X_0} X_1$, such that $\_ \circ_S \_ := d_1\eta_2$ is a composition for the Segal space X. For any two morphisms $f \in X(x, y)$ and $g \in X(y, z)$, the associated inner 3-horn $\eta_2(f, g) \cup \mu_2(1_x, f) \cup \mu_2(g, 1_x/f)\colon \Lambda^3_1 \to X_{\bullet 0}$ of the form



has a lift $L(f,g)\colon \Delta^3 \to X_{\bullet 0}$. Both the simplex



and $s_0(g \circ_S f)$ lie in the contractible fibre $\beta_2^{-1}(g \circ_S f, 1_x)$. In particular, $d_1L(f,g)$ and $s_0(g \circ_S f)$ lie in the same connected component of X_2. By naturality of d_0, we obtain

$$[g \circ_B f] = [d_0 d_1 L(f,g)] = [d_0 s_0(g \circ_S f)] = [g \circ_S f]$$

in $\pi_0 X_1(x,z) = \mathrm{Ho}(X)(x,z) = \mathrm{Ho}_B(X)(x,z)$. □

5. Further characterizations

In this section we prove that $(s\mathbf{S}, B)$ can be obtained as the left Bousfield localization of $(s\mathbf{S}, S)$ at one single map which exhibits Bousfield–Segal spaces as exactly those Segal spaces whose homotopy category is a groupoid. In algebraic terms, this means that every homotopy-coherent fraction operation on a space X_0 induces a unique invertible homotopy-coherent composition operation on X_0 and vice versa. Furthermore, we construct a right adjoint to the inclusion of Bousfield–Segal spaces into Segal spaces, and thereby define the *core* of a Segal space.

Proposition 5.1. *A Segal space X is a Bousfield–Segal space if and only if the Bousfield map*

$$\beta_2 = c_2 \setminus X \colon X_2 \to X_1 \times^B_{X_0} X_1$$

is an acyclic fibration. In particular, the model structures $\mathcal{L}_{c_2}(s\mathbf{S}, S)$ and $(s\mathbf{S}, B)$ coincide.

Proof. Since both model structures are left Bousfield localizations of the same Reedy structure, we only have to compare their fibrant objects. Every Bousfield–Segal space is c_2-local by definition and fibrant in $(s\mathbf{S}, S)$ by Theorem 4.5. Thus, one direction is immediate. Vice versa, we have to show that fibrant objects in $\mathcal{L}_{c_2}(s\mathbf{S}, S)$ are $p_1^* c_n$-local for all $n \geqslant 2$. Therefore, consider the class

$$A := \{f \in \mathbf{S} \mid f \text{ is a monomorphism and } p_1^* f \text{ is an acyclic cofibration in } \mathcal{L}_{c_2}(s\mathbf{S}, S)\}.$$

Since the fibrant objects in $\mathcal{L}_{c_2}(s\mathbf{S}, S)$ are f-local for all $f \in A$, we want to show that $(c_n \mid n \geqslant 2) \subseteq A$. The class A is saturated and has the right cancellation property for monomorphisms. By construction $S \cup \{c_2\}$ is a subset of A. Via the common factorization of c_n and sp_n for $n \geqslant 2$ given in Diagram (9), it follows that the left cone inclusions c_n for $n > 2$ are contained in A as well (in the same way the proof of Corollary 4.4 showed that $B \cup \{\mathrm{sp}_2\} \subset A$ implies $\mathrm{sp}_n \in A$ for all $n \geqslant 2$). □

Remark 5.2. One can define both Segal objects and Bousfield–Segal objects in every model category $\mathbb{M}$, via imposing acyclicity of associated Segal and Bousfield maps. The functor $\_ \setminus X \colon \mathbf{S}^{op} \to \mathbb{M}$ can be defined accordingly for every Reedy fibrant simplicial object $X \in \mathbb{M}^{\Delta^{op}}$. Then the proofs of Theorem 4.5 and Proposition 5.1 in fact show that Bousfield–Segal objects in $\mathbb{M}$ are exactly the Segal objects with invertible Bousfield map β_2.

Let $X_{\text{hoequiv}} \subseteq X_1$ denote the full simplicial subset of *homotopy equivalences* in X, generated by those $f \in (X_1)_0$ which become isomorphisms in $\mathrm{Ho}(X)$.

Corollary 5.3. *A Segal space X is a Bousfield–Segal space if and only if its associated homotopy category* $\mathrm{Ho}(X)$ *is a groupoid.*

Proof. One direction was shown in Corollary 4.7. For the other direction, let X be a Segal space and assume $\mathrm{Ho}(X)$ is a groupoid. That means that $X_{\text{hoequiv}} = X_1$, and so it follows from [**18**, Lemma 11.6] that the Bousfield map $\beta_2 \colon X_2 \twoheadrightarrow X_1 \times_{X_0} X_1$ is a weak equivalence. Hence X is a Bousfield–Segal space by Proposition 5.1. □

Whenever X is a Segal space, the fibration $I\Delta^1 \setminus X \twoheadrightarrow X_1$ induced by the canonical inclusion $e \colon \Delta^1 \to I\Delta^1$ factors through the subspace X_{hoequiv}, and Rezk showed in [**18**, Theorem 6.2] that the resulting fibration $e \setminus X \colon I\Delta^1 \setminus X \to X_{\text{hoequiv}}$ is acyclic.

Corollary 5.4. *A Segal space X is a Bousfield–Segal space if and only if the fibration $I\Delta^1 \setminus X \twoheadrightarrow X_1$ is acyclic. In particular, the model structures $(s\mathbf{S}, B)$ and $\mathcal{L}_{p_1^*(e)}(s\mathbf{S}, S)$ coincide.*

Proof. If X is a Bousfield–Segal space, the fibration $I\Delta^1 \setminus X \twoheadrightarrow X_1$ is acyclic by Proposition 4.3. Vice versa, if X is a Segal space and the fibration $I\Delta^1 \setminus X \twoheadrightarrow X_1$ is acyclic, it is surjective as well. Since it factors through the subobject X_{hoequiv}, this implies $X_{\text{hoequiv}} = X_1$. It follows that $\mathrm{Ho}(X)$ is a groupoid and hence X is a Bousfield–Segal space by Corollary 4.5. □

Remark 5.5. Corollary 5.4 justifies the motivating analogies stated in the Introduction, which related the construction of Bousfield–Segal spaces from Segal spaces to both the construction of **Gpd** from **Cat** and the construction of $(\mathbf{S}, \mathrm{Kan})$ from $(\mathbf{S}, \mathrm{QCat})$. In a similar vein, Corollary 5.3 yields an analogy to Joyal's criterion for a quasi-category to be Kan complex via its homotopy category (for instance as presented in [**15**, Proposition 1.2.4.3 and 1.2.5.1]).

It follows that a Segal space X is a Bousfield–Segal space if and only if its rows are Kan complexes. That means Bousfield–Segal spaces are exactly the Segal spaces horizontally fibrant in the projective model structure over $(\mathbf{S}, \mathrm{Kan})$.

At this point we have seen in various ways that the inclusion $(s\mathbf{S}, B) \to (s\mathbf{S}, S)$ is part of a left Bousfield localization, and as such it is the right adjoint in a homotopy localization in the sense of [**13**, Definition 7.16]. We conclude this section with the construction of a further right adjoint to this inclusion, given by a core construction that associates to a Segal space X the largest Bousfield–Segal space $\mathrm{Core}(X)$ contained in X. We proceed in the following two steps.

First, we construct a functor "Core" as the strict right adjoint to the inclusion

$$\iota \colon \text{Bousfield–Segal} \hookrightarrow \text{Segal} \tag{10}$$

of the full subcategory of Bousfield–Segal spaces into the full subcategory of Segal spaces in $s\mathbf{S}$. We do so by an elementary procedure which takes the subcollection

$X_{\text{hoequiv}} \subseteq X_1$ of equivalences in a Segal space X and assigns it the largest sub-bisimplicial set $\text{Core}(X) \subseteq X$ generated from it.

Second, we show that the colocalization (ι, Core) can be extended to a homotopy colocalization of the form

$$(L, R)\colon (s\mathbf{S}, \mathrm{B}) \to (s\mathbf{S}, \mathrm{S}).$$

This means (L, R) is a Quillen pair such that $L(X) \simeq \iota(X)$ in $(s\mathbf{S}, \mathrm{S})$ whenever X is a Bousfield–Segal space, $R(X) \simeq \text{Core}(X)$ in $(s\mathbf{S}, \mathrm{B})$ whenever X is a Segal space, and such that the derived unit of the adjunction is a natural weak equivalence in $(s\mathbf{S}, \mathrm{B})$. The first step is given by the following lemma.

Lemma 5.6. *The inclusion (10) has a right adjoint.*

Proof. Let X be a Segal space and let $X_{\text{hoequiv}} \subseteq X_1$ be the space of equivalences in X. We can define a bisimplicial set $\text{Core}(X)$ recursively as follows. Let $\text{Core}(X)_0 = X_0$, let

$$\text{Core}(X)_1 = X_{\text{hoequiv}} \subseteq X_1,$$

and let the corresponding two boundaries and the degeneracy between these two objects be induced directly from those of X by restriction.

For $n \geqslant 2$ define $\text{Core}(X)_n \subseteq X_n$ to be the subspace generated by the vertices of X_{n0} whose horizontal 1-boundaries lie in X_{hoequiv}. The boundaries and degeneracies of $\text{Core}(X)$ are directly obtained by restriction of the respective boundaries and degeneracies of X, so we obtain a bisimplicial subset $\text{Core}(X) \subseteq X$. By construction, we obtain cartesian squares between the matching maps for all $n \geqslant 2$ as follows:

$$\begin{array}{ccc} \text{Core}(X)_n & \hookrightarrow & X_n \\ \downarrow \partial & \lrcorner & \downarrow \partial \\ \partial\Delta^n \setminus \text{Core}(X) & \hookrightarrow & \partial\Delta^n \setminus X. \end{array} \tag{11}$$

Since $X_{\text{hoequiv}} \subseteq X_1$ is closed under composition, we furthermore obtain cartesian squares between the respective Segal maps as well:

$$\begin{array}{ccc} \text{Core}(X)_n & \hookrightarrow & X_n \\ \downarrow \xi_n & \lrcorner & \downarrow \xi_n \\ \text{Core}(X)_1 \times^S_{X_0} \cdots \times^S_{X_0} \text{Core}(X)_1 & \hookrightarrow & X_1 \times^S_{X_0} \cdots \times^S_{X_0} X_1. \end{array} \tag{12}$$

It follows that $\text{Core}(X)$ is Reedy fibrant by (11) and satisfies the Segal conditions by (12). Furthermore, by construction we have

$$\text{Core}(X)_{\text{hoequiv}} = X_{\text{hoequiv}} = \text{Core}(X)_1,$$

so $\text{Core}(X)$ is a Bousfield–Segal space by Corollary 5.3. It is straightforward to show that this construction extends to a functor

$$\text{Core}\colon \text{Segal} \to \text{Bousfield–Segal},$$

which is right adjoint to the inclusion Bousfield–Segal $\subset$ Segal. $\square$

Towards the second step, note that for any Segal space X the first row $\mathrm{Core}(X)_{\bullet 0}$ is the largest Kan complex $J(X_{\bullet 0})$ contained in the quasi-category $X_{\bullet 0}$ as constructed in [**13**, Proposition 1.16]. So the given right adjoint to the inclusion (10) is a direct generalization of the right adjoint to the inclusion of Kan complexes into quasi-categories. Joyal also shows in [**12**] that the functor $J\colon \mathrm{QCat} \to \mathrm{Kan}$ can be extended to the Quillen right adjoint $k^!$ of a homotopy colocalization $(k_!, k^!)\colon (\mathbf{S}, \mathrm{QCat}) \to (\mathbf{S}, \mathrm{Kan})$. Dual to the notion of homotopy localization, a Quillen pair is a homotopy colocalization if the left derived of its left adjoint is fully faithful on the underlying homotopy categories.

Analogously, one can extend the core construction of a Segal space to a homotopy colocalization between the respective model structures. Therefore, consider the functor $k \,\square\, \mathrm{id}\colon \Delta \times \Delta \to s\mathbf{S}$ for $k([n]) = I\Delta^n$, together with its left Kan extension $(k \,\square\, \mathrm{id})_!\colon s\mathbf{S} \to s\mathbf{S}$. It has a right adjoint $(k \,\square\, \mathrm{id})^!$, given by

$$(k \,\square\, \mathrm{id})^!(X)_{nm} = s\mathbf{S}(I\Delta^n \,\square\, \Delta^m, X).$$

The right adjoint $(k \,\square\, \mathrm{id})^!$ is a simplicially enriched functor with respect to the hom-spaces Hom_2 given in Proposition 2.2, and preserves the associated tensors (described in [**13**, Proposition 2.4]) as well. Hence the pair $((k \,\square\, \mathrm{id})_!, (k \,\square\, \mathrm{id})^!)$ gives rise to a simplicially enriched adjoint pair on $s\mathbf{S}$.

Proposition 5.7. *The functor $k \,\square\, \mathrm{id}\colon \Delta \times \Delta \to s\mathbf{S}$ induces a Quillen pair*

$$(k \,\square\, \mathrm{id})_!\colon (s\mathbf{S}, B) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (s\mathbf{S}, S)\colon (k \,\square\, \mathrm{id})^!. \tag{13}$$

It comes together with a natural transformation $\beta\colon (k \,\square\, \mathrm{id})^! \to \mathrm{id}$ which factors through a vertical Reedy weak equivalence $\beta_X\colon (k \,\square\, \mathrm{id})^!(X) \to \mathrm{Core}(X)$ whenever X is a Segal space.

Proof. The inclusions $[n] \to I[n]$ induce a natural transformation $\beta\colon (k \,\square\, \mathrm{id})^! \to \mathrm{id}$ by precomposition. It factors through $\mathrm{Core}(X)$ whenever X is a Segal space, because $(\beta_X)_1$ is exactly the fibration $e \setminus X\colon I\Delta^1 \setminus X \twoheadrightarrow X_1$. Generally, the maps

$$(\beta_X)_n\colon (k \,\square\, \mathrm{id})^!(X)_n \to \mathrm{Core}(X)_n$$

are isomorphic to the fibrations $I\Delta^n \setminus \mathrm{Core}(X) \to \Delta^n \setminus \mathrm{Core}(X)$ and hence are acyclic by Proposition 4.3. Thus the natural transformation β yields a levelwise acyclic fibration $\beta_X\colon (k \,\square\, \mathrm{id})^!(X) \to \mathrm{Core}(X)$ whenever X is a Segal space. This shows that the right adjoint $(k \,\square\, \mathrm{id})^!$ is an extension of the core construction.

The left adjoint $(k \,\square\, \mathrm{id})_!$ furthermore preserves Reedy cofibrations and Reedy acyclic cofibrations as can be verified on the generating sets $\mathcal{I}_v$ and $\mathcal{J}_v$. Thus, the two functors yield a Quillen pair from the vertical Reedy model structure to itself. In order for them to descend to a Quillen pair between the two localizations as in (13) we have to show that the right adjoint $(k \,\square\, \mathrm{id})^!$ maps every Segal space X to a Bousfield–Segal space. But we have seen that for every Segal space X, the map $\beta_X\colon (k \,\square\, \mathrm{id})^!(X) \to \mathrm{Core}(X)$ is a Reedy weak equivalence. We know that $\mathrm{Core}(X)$ is a Bousfield–Segal space and that the domain $(k \,\square\, \mathrm{id})^!(X)$ is Reedy fibrant, so $(k \,\square\, \mathrm{id})^!(X)$ is a Bousfield–Segal space as well. □

Theorem 5.8. *The pair (13) is a homotopy colocalization. It furthermore comes together with a natural transformation $\alpha\colon \mathrm{id} \to (k \,\square\, \mathrm{id})_!$ such that α_X is a weak equivalence in $(s\mathbf{S}, \mathrm{S})$ whenever X is a Bousfield–Segal space.*

Proof. We divide the proof into four parts. First, consider the natural transformation

$$\alpha\colon \mathrm{id} \to (k \,\Box\, \mathrm{id})_!$$

induced by the inclusions $[n] \to I[n]$, dual to the transformation β from Proposition 5.7. Its restriction $\bar{\alpha}\colon y \to (k \,\Box\, \mathrm{id})$ in the functor category $(s\mathbf{S})^{\Delta\times\Delta}$ is a pointwise weak equivalence in $(s\mathbf{S}, \mathrm{B})$. That is because for every pair $n, m \geqslant 0$ and every Bousfield–Segal space X, the map $\mathrm{Hom}_2(\alpha([n],[m]), X)$ is isomorphic to the fibration $((\Delta^n \to I\Delta^n) \setminus X)^{\Delta^m}$ and hence acyclic by Proposition 4.3. It follows that the natural transformation $\alpha\colon y_! \to (k \,\Box\, \mathrm{id})_!$ between left Kan extensions is also a pointwise weak equivalence in $(s\mathbf{S}, \mathrm{B})$. This can be seen for example using that every bisimplicial set is the "canonical" homotopy colimit of representables ([**9**, Proposition 2.9]) and that the left Quillen functor $(k \,\Box\, \mathrm{id})_!$ preserves homotopy colimits.

Second, given a bisimplicial set X, let

$$r\colon (k \,\Box\, \mathrm{id})_!(X) \to X'$$

be a fibrant replacement in $(s\mathbf{S}, \mathrm{S})$. Note that by the definition of the left adjoint $(k \,\Box\, \mathrm{id})_!$, every $f \in (k \,\Box\, \mathrm{id})_!(X)_1$ is mapped to a homotopy equivalence in the Segal space X', and so the map r factors through $\mathrm{Core}(X')$. The canonical inclusion $\iota\colon \mathrm{Core}(X') \hookrightarrow X'$ is a Reedy fibration between Segal spaces by its definition via Diagram (11) and the fact that the inclusion $X_{\mathrm{hoequiv}} \to X_1$ is a Kan fibration. As such it is a fibration in $(s\mathbf{S}, \mathrm{S})$ as well. Hence, we obtain a lift to the square

$$\begin{array}{ccc}
(k \,\Box\, \mathrm{id})_!(X) & \xrightarrow{\ r\ } & \mathrm{Core}(X') \\
{\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\
X' & = & X',
\end{array}$$

which shows that $\mathrm{Core}(X') = X'$. This means that X' is a Bousfield–Segal space.

Third, in a similar way to the proof of [**12**, Proposition 6.27], let X be a bisimplicial set and consider the following commutative diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{\ \eta\ } & (k \,\Box\, \mathrm{id})^!(k \,\Box\, \mathrm{id})_!(X) & \xrightarrow{(k\Box\mathrm{id})^!(r)} & (k \,\Box\, \mathrm{id})^!(X') \\
& {\scriptstyle \alpha_X}\searrow & \big\downarrow{\scriptstyle \beta_{(k\Box\mathrm{id})_!(X)}} & & \big\downarrow{\scriptstyle \beta_{X'}} \\
& & (k \,\Box\, \mathrm{id})_!(X) & \xrightarrow[\ r\ ]{} & X'.
\end{array}$$

To prove that the pair (13) is a homotopy colocalization, we need to show that the top composition of arrows is a weak equivalence in $(s\mathbf{S}, \mathrm{B})$. In the first step of the proof, we have seen that the map α_X is a weak equivalence in $(s\mathbf{S}, \mathrm{B})$. The same goes for the bottom map r since $(s\mathbf{S}, \mathrm{B})$ is a left Bousfield localization of $(s\mathbf{S}, \mathrm{S})$. The vertical map $\beta_{X'}$ is a Reedy weak equivalence by Proposition 5.7, since $X' = \mathrm{Core}(X')$ as shown in the second step. Thus, the top horizontal map is a weak equivalence in $(s\mathbf{S}, \mathrm{B})$ by 3-for-2.

Lastly, the fact that α_X is a weak equivalence in $(s\mathbf{S}, \mathrm{S})$ whenever X is a Bousfield–Segal space follows from a 3-for-2 argument as well. Indeed, the composite

$$X \xrightarrow{\alpha_X} (k \,\Box\, \mathrm{id})_!(X) \xrightarrow{r} X'$$

is a weak equivalence in $(\mathbf{S}, \mathrm{B})$ since both of its components are. When X is a Bousfield–Segal space, this composite is a weak equivalence between B-local objects and hence a weak equivalence in $(s\mathbf{S}, \mathrm{S})$ as well. Since the map r is a weak equivalence in $(s\mathbf{S}, \mathrm{S})$ by assumption, so is α_X. This finishes the proof. □

6. Complete Bousfield–Segal spaces

In this section we study the model structure for Bousfield–Segal spaces which satisfy Rezk's completeness condition. We show that they are exactly the Reedy fibrant bisimplicial sets which are homotopically constant, and hence appear in various forms in the existing literature. We will deduce that the model structure $(s\mathbf{S}, \mathrm{CB})$ for complete Bousfield–Segal spaces is Quillen equivalent to the standard Quillen model structure for Kan complexes.

The model structure $(s\mathbf{S}, \mathrm{S})$ for Segal spaces can be localized at the set

$$\mathrm{C} := \{p_1^*(\{0\})\colon p_1^*(\Delta^0) \to p_1^*(I\Delta^1)\}.$$

This defines the model structure $(s\mathbf{S}, \mathrm{CS}) := \mathcal{L}_{\mathrm{C}}(s\mathbf{S}, \mathrm{S})$ which originally was presented in [**18**] and is further studied in [**13**, Section 4]. Its fibrant objects are the *complete* Segal spaces, i.e. the Segal spaces X such that the map

$$\{0\} \setminus X \colon I\Delta^1 \setminus X \twoheadrightarrow X_0$$

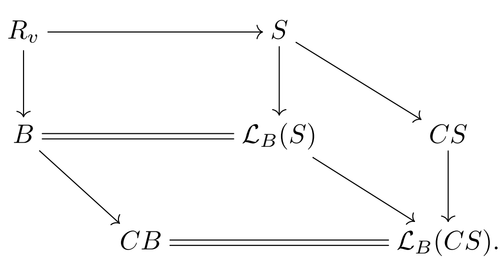
is an acyclic fibration. In other words, a Segal space X is complete whenever we may identify vertical homotopies in X_0 with horizontal equivalences in X. This defining identification implies that $(s\mathbf{S}, \mathrm{CS})$ is also the localization of $(s\mathbf{S}, \mathrm{S})$ at the class of homotopically fully faithful and essentially surjective maps (the *Dwyer Kan equivalences*). This is [**19**, Corollary 7.9]. Joyal and Tierney show in [**13**] that the homotopy theory of complete Segal spaces is equivalent to the homotopy theory of $(\infty, 1)$-categories. This justifies to think of complete Segal spaces as the $(\infty, 1)$-category objects in the $(\infty, 1)$-category of spaces.

Analogously, localizing $(s\mathbf{S}, \mathrm{B})$ at the set C yields the simplicial, left-proper and combinatorial model category

$$(s\mathbf{S}, \mathrm{CB}) := \mathcal{L}_C(s\mathbf{S}, \mathrm{B}) = \mathcal{L}_{\mathrm{B}}(s\mathbf{S}, \mathrm{CS}).$$

Definition 6.1. We say that $X \in s\mathbf{S}$ is a complete Bousfield–Segal space if X is a B-local complete Segal space. In other words, complete Bousfield–Segal spaces are exactly the fibrant objects in the model category $(s\mathbf{S}, \mathrm{CB})$.

Remark 6.2. The genealogy of the model structures on $s\mathbf{S}$ that we have considered can be depicted by the following diagram, where the arrows indicate the direction of the left Quillen functor:



The horizontal equality in the middle of the diagram was shown in Theorem 4.5. The bottom equality is an immediate corollary. The reader may note however that there are direct and concise proofs of the bottom equality which do not require an understanding of the model structure for (non-complete) Bousfield–Segal spaces. For example, it can be deduced from the next lemma.

By [**13**, Proposition 2.8], every v-fibrant bisimplicial set X is *categorically constant* in that every map $f\colon [m] \to [n]$ in Δ induces a categorical equivalence $X_{\bullet f}\colon X_{\bullet n} \to X_{\bullet m}$ between the two rows. In the following we show that a v-fibrant X is a complete Bousfield–Segal space if and only if it is homotopically constant in the vertical direction as well.

Definition 6.3. A bisimplicial set X is *homotopically constant* if the map $X(f)\colon X_n \to X_m$ is a weak homotopy equivalence for every function $f\colon [m] \to [n]$ in Δ.

Lemma 6.4. *A v-fibrant bisimplicial set X is a complete Bousfield–Segal space if and only if X is homotopically constant.*

Proof. Clearly, X is homotopically constant if and only if all boundary and degeneracy maps of X are weak homotopy equivalences. This in turn holds if and only if all boundary maps of X are weak homotopy equivalences since the degeneracies are sections of the boundaries.

If X is homotopically constant, all the pullbacks $X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1$ are contractible over X_0 via the projection to the vertex [0] (using right properness of $(\mathbf{S}, \mathrm{Kan})$). The same holds for the X_n, and so the Bousfield maps are weak homotopy equivalences over X_0 by 3-for-2. It follows that $(\{0\}\colon \Delta^0 \to I\Delta^1) \setminus X \simeq d_1$, which is a weak homotopy equivalence by assumption as well. Thus X is complete.

Vice versa, if X is a complete Bousfield–Segal space, the acyclic fibration $I\Delta^1 \setminus X \twoheadrightarrow X_0$ factors through the fibrations $I\Delta^1 \setminus X \twoheadrightarrow X_1 \twoheadrightarrow X_0$. The first of these two is acyclic by Corollary 5.4, and so is the boundary $d_1\colon X_1 \twoheadrightarrow X_0$ by 3-for-2. It follows that $d_0\colon X_1 \twoheadrightarrow X_0$ is a weak homotopy equivalence, too, because s_0 is a mutual section. Consequently all the pullbacks $X_1 \times^B_{X_0} \cdots \times^B_{X_0} X_1$ are homotopy equivalent to one another via the projections. Since the Bousfield maps are weak homotopy equivalences by assumption, it follows that the boundaries of X are weak homotopy equivalences again by a repeated application of 3-for-2. □

Corollary 6.5. *The model structure $(s\mathbf{S}, \mathrm{CB})$ is cartesian closed, i.e. it is a monoidal model category with respect to the cartesian product.*

Proof. By [**18**, Proposition 9.2] it suffices to show that the exponential $X^{p_1^*\Delta^1}$ is a complete Bousfield–Segal space whenever X is such. Since $X^{p_1^*\Delta^1}$ is again Reedy fibrant, by Lemma 6.4 this means that we have to show that $X^{p_1^*\Delta^1}$ is homotopically constant. So let $d^i\colon [n] \to [n+1]$ be a coboundary inclusion and let X be a complete Bousfield–Segal space. We want show that the boundary $X^{p_1^*\Delta^1}(d^i)$ is a weak equivalence. There are natural isomorphisms

$$(X^{p_1^*(\Delta^1)})_n \cong \Delta^n \setminus X^{p_1^*(\Delta^1)} \cong (\Delta^1 \times \Delta^n) \setminus X$$

and $X^{p_1^*(\Delta^1)}(d^i) \cong (\mathrm{id}_{\Delta^1} \times d^i) \setminus X$. By assumption X is homotopically constant and it follows that $\_ \setminus X$ takes all anodyne maps to trivial fibrations by [**13**, Lemma 3.7]. Thus, $(\mathrm{id}_{\Delta^1} \times d^i) \setminus X$ is an acyclic fibration. □

From Lemma 6.4 it follows that the model structure $(s\mathbf{S}, \mathrm{CB})$ is contained in various classes of well understood model structures studied in the literature.

In [20] for instance, given a model category $\mathbb{M}$, the model structure on $\mathbb{M}^{\Delta^{op}}$ whose fibrant objects are exactly the homotopically constant Reedy fibrant simplicial objects is called the *canonical model structure* on $\mathbb{M}^{\Delta^{op}}$ (whenever it exists). In the case $\mathbb{M} = (\mathbf{S}, \mathrm{Kan})$, this implies that the projection $\iota_2^*\colon (s\mathbf{S}, \mathrm{CB}) \to (\mathbf{S}, \mathrm{Kan})$ onto the first column is part of a Quillen equivalence by [20, Theorem 3.6]. The fact that the projection ι_2^* is part of a Quillen equivalence was stated in Bergner's paper ([1, Theorem 6.12]) as well, however its proof was omitted.

By a symmetry inherent to the model structure $(s\mathbf{S}, \mathrm{CB})$, this implies that the "perpendicular" projection $\iota_1^*\colon (s\mathbf{S}, \mathrm{CB}) \to (\mathbf{S}, \mathrm{Kan})$ onto the first row is a Quillen equivalence as well, as the following theorem shows.

Theorem 6.6. *The pair*

$$(p_1^*, \iota_1^*)\colon (\mathbf{S}, \mathrm{Kan}) \to (s\mathbf{S}, \mathrm{CB})$$

is a Quillen equivalence.

Proof. Recall the involution $\sigma^*\colon s\mathbf{S} \to s\mathbf{S}$ induced by the permutation $\sigma\colon \Delta \times \Delta \to \Delta \times \Delta$ which swaps the components $([n], [m]) \mapsto ([m], [n])$. Following the notation from Section 2, note that $\sigma^*[\mathcal{W}_v] = \mathcal{W}_h$, $\sigma^*[\mathcal{C}_v] = \mathcal{C}_h = \mathcal{C}$ and even $\sigma^*[\mathcal{I}_v] = \mathcal{I}_h$ and $\sigma^*[\mathcal{J}_v] = \mathcal{J}_h$ since σ^* preserves colimits. Furthermore, for all objects $X, Y \in s\mathbf{S}$ the involution satisfies the following identities.

$$\mathrm{Hom}_2(\sigma^* X, \sigma^* Y) := \iota_2^*((\sigma^* Y)^{\sigma^* X}) = \iota_2^* \sigma^*(Y^X) = \iota_1^*(Y^X) =: \mathrm{Hom}_1(X, Y). \quad (14)$$

In analogy to Proposition 2.2, the functor Hom_1 turns $(s\mathbf{S}, R_h)$ into a simplicial model category. Let

$$CB^{\perp} := \{p_2^*(\{0\})\} \cup \{p_2^*(c_n) \mid n \geqslant 2\},$$

so we can construct the Bousfield localization $\mathcal{L}_{CB^{\perp}}(s\mathbf{S}, R_h)$. Via (14) one computes that a bisimplicial set X is h-fibrant and $\mathrm{CB}^{\perp}$-local (with respect to the Hom_1-enrichment) if and only if it is v-fibrant and $\mathrm{C} \cup \mathrm{B}$-local (with respect to the Hom_2-enrichment). Then the model structures $(s\mathbf{S}, \mathrm{CB}) = \mathcal{L}_{C \cup B}(s\mathbf{S}, \mathrm{R}_v)$ and $\mathcal{L}_{\mathrm{CB}^{\perp}}(s\mathbf{S}, R_h)$ coincide since they have the same cofibrations and the same fibrant objects. Hence, the involution (σ^*, σ^*) is a Quillen equivalence from $(s\mathbf{S}, \mathrm{CB})$ to itself. By [20, Theorem 3.6], the projection $\iota_2^*\colon (s\mathbf{S}, \mathrm{CB}) \to (\mathbf{S}, \mathrm{Kan})$ onto the first column is a Quillen equivalence, and so it follows that the first row projection $\iota_1^* = \iota_2^* \circ \sigma^*$ is part of a Quillen equivalence as well. □

One also can show Theorem 6.6 in another way, using that the pair (p_1^*, ι_1^*) gives rise to a Quillen equivalence between $(\mathbf{S}, \mathrm{QCat})$ and $(s\mathbf{S}, \mathrm{S})$ as shown in [13, Theorem 4.11]. Localizing both sides at the left horn inclusions yields the same result (see [23, Theorem 5.1.14]).

Theorem 6.7. *The diagonal $d^*\colon s\mathbf{S} \to \mathbf{S}$ is part of a Quillen equivalence*

$$(d^*, d_*)\colon (s\mathbf{S}, \mathrm{CB}) \to (\mathbf{S}, \mathrm{Kan}).$$

Proof. The statement can be shown along the lines of [13, Theorem 4.12], using Theorem 6.6 and a 3-for-2 argument. □

The fact that the diagonal induces an equivalence on the homotopy categories of the two model structures is exactly the unpointed version of [**3**, Theorem 3.1] for very special bisimplicial sets of type $n = 0$.

Remark 6.8. We have seen in the proof of Theorem 6.6 that $(s\mathbf{S}, \mathrm{CB})$ is also a left Bousfield localization of the horizontal Reedy structure $(s\mathbf{S}, \mathrm{R}_h)$. It thus follows that every complete Bousfield–Segal space is both v-fibrant and h-fibrant. Vice versa, every h-fibrant bisimplicial set is homotopically constant by [**13**, Proposition 2.8] (or, more precisely, by its horizontal dual). Thus, by Lemma 6.4 it follows that every bisimplicial set which is simultaneously v-fibrant and h-fibrant is a complete Bousfield–Segal space. In this sense, the property of being a complete Bousfield–Segal space is the conjunction of two Reedy fibrancy conditions. We will use this in Theorem 7.3 to give a direct proof of right properness of $(s\mathbf{S}, \mathrm{CB})$.

The model structure $(s\mathbf{S}, \mathrm{CB})$ furthermore coincides with Dugger's *realization* or *hocolim* model structure on $s\mathbf{S}$ ([**8**]) and with Cisinski's model structure for "locally constant presheaves" on Δ ([**6**]). This implies that $(s\mathbf{S}, \mathrm{CB})$ yields a model for univalent type theory as discussed in the next section.

7. Right properness and semantics of univalent type theory

In this last section, we prove right properness of $(s\mathbf{S}, \mathrm{CB})$ and record that the model structure furthermore is "type theoretic" in the sense that it is a model for Homotopy Type Theory with univalent universes as treated in the HoTT-Book [**17**].

Consequently, complete Bousfield–Segal spaces and their fibrations can be studied formally in the syntax of their underlying univalent type theory. By soundness of the calculus, all results proven synthetically in this type theory automatically follow to hold in their associated model category $(s\mathbf{S}, \mathrm{CB})$ as well.

The interpretation of HoTT in $(s\mathbf{S}, \mathrm{CB})$ can be constructed from a combination of existing results in the literature. Indeed, we can take the results of Shulman in [**21**] as a given to reduce the construction of a semantics for HoTT to the verification of a few model categorical properties. More precisely, the model structure CB is a cofibrantly generated model structure on the presheaf category $s\mathbf{S}$ whose cofibrations are exactly the monomorphisms. This means that the simplicial model category $(s\mathbf{S}, \mathrm{CB})$ defines a *Cisinski model category*. Therefore, by [**21**, Theorem 5.1], in order for $(s\mathbf{S}, \mathrm{CB})$ to support a univalent type theoretical interpretation, we only have to show that $(s\mathbf{S}, \mathrm{CB})$ is right proper and that it comes equipped with an infinite sequence of univalent universal fibrations.

On the one hand, both properties follow from abstract results of Cisinski (Observation 1). On the other hand, one obtains the universal fibrations by more concrete means via a result of Rezk, Schwede and Shipley (Observation 2). The fact that $(s\mathbf{S}, \mathrm{CB})$ is right proper can be proven directly by results from the previous section (Observation 3).

Observation 1 In [**6**, 1], Cisinski introduces the *locally constant model structure* $([\mathcal{A}^{op}, \mathbf{S}], \mathrm{lc})$ on the category of simplicial presheaves on any elegant Reedy category $\mathcal{A}$. It is a left Bousfield localization of the Reedy model structure, its fibrant objects are exactly the homotopically constant Reedy fibrant objects $X \in [\mathcal{A}^{op}, \mathbf{S}]$. These are the Reedy fibrant objects X such that the f-action $X(f)\colon X(b)\to X(a)$ is a weak homotopy equivalence for all maps $f\colon a\to b$ in $\mathcal{A}$. Hence, Lemma 6.4 shows that $(s\mathbf{S},\mathrm{lc})=(s\mathbf{S},\mathrm{CB})$. In [**5**], he shows that $([\mathcal{A}^{op},\mathbf{S}],\mathrm{lc})$ is always right proper, drawing back on general observations about fundamental localizers. In [**6**, Proposition 1.1] he shows that this model category contains a fibrant univalent universe classifying κ-small maps for every inaccessible cardinal κ large enough.

Observation 2 More concretely, we obtain a sequence of univalent universes for $(s\mathbf{S},\mathrm{CB})$ from [**21**, Theorem 3.2] and [**23**, Corollary 2.5.9] if we can simply show that there is a set of generating acyclic cofibrations for $(s\mathbf{S},\mathrm{CB})$ with representable codomain. Since we obtained the model structure $(s\mathbf{S},\mathrm{CB})$ by left Bousfield localization, a priori it is very hard to present a well behaved set of generating acyclic cofibrations. But the authors of [**20**] show that the fibrations in the canonical model structure $(s\mathbf{S},\mathrm{CB})$ are exactly the *equi-fibred Reedy fibrations.* For such, a set of generating acyclic cofibrations is given in [**20**, Proposition 8.5] by $\mathcal{J}_{CB}=\mathcal{J}_h\cup\mathcal{J}''$ for

$$\mathcal{J}_h=\{h_i^n\square'\delta_m\colon(\Delta^n\square\partial\Delta^m)\cup_{\Lambda_i^n\square\partial\Delta^m}(\Lambda_i^n\square\Delta^m)\to(\Delta^n\square\Delta^m)\mid 0\leqslant i\leqslant m,n\},$$

$$\mathcal{J}'':=\{\delta_n\square' d_i^m\colon(\Delta^n\square\Delta^{m-1})\cup_{\partial\Delta^n\square\Delta^{m-1}}(\partial\Delta^n\square\Delta^m)\to(\Delta^n\square\Delta^m)\mid n\geqslant 0,m\geqslant i\geqslant 0\}.$$

The box products $\Delta^n\square\Delta^m$ are exactly the representables in $s\mathbf{S}$, thus a set of generating acyclic cofibrations with representable codomain does indeed exist.

Observation 3 We can give an elementary proof of right properness of $(s\mathbf{S},\mathrm{CB})$ using the symmetry of the model structure observed in the last section. More precisely, we can use that complete Bousfield–Segal spaces are exactly the objects which are simultaneously v-fibrant and h-fibrant as noted in Remark 6.8.

Recall that a model category $\mathbb{M}$ is right proper if and only if the pullback of any acyclic cofibration with fibrant codomain along fibrations is a weak equivalence. This is shown in [**4**, Lemma 9.4] for example. Furthermore, recall the generating sets $\mathcal{J}_v$ and $\mathcal{J}_h$ for the acyclic cofibrations in $(s\mathbf{S},R_v)$ and in $(s\mathbf{S},R_h)$, respectively, from Section 2. In the following, we denote the class of weak equivalences in $(s\mathbf{S},\mathrm{CB})$ by $\mathcal{W}_{CB}$, and the class of its fibrations by $\mathcal{F}_{CB}$. Given a class S of arrows in $s\mathbf{S}$, we denote the class of arrows with the left lifting property (right lifting property) against all arrows in S by ${}^{\pitchfork}S$ (by $S^{\pitchfork}$).

Lemma 7.1. *The class of acyclic cofibrations with fibrant codomain in* $(s\mathbf{S},\mathrm{CB})$ *is exactly the class of maps in the saturation of* $\mathcal{J}_v\cup\mathcal{J}_h$ *with fibrant codomain, i.e.*

$$(\mathcal{W}_{CB}\cap\mathcal{C})/\textit{C. B.-S. spaces}={}^{\pitchfork}((\mathcal{J}_v\cup\mathcal{J}_h)^{\pitchfork})/\textit{C. B.-S. spaces}.$$

Proof. As $(s\mathbf{S},\mathrm{CB})$ is a left Bousfield localization of both $(s\mathbf{S},R_v)$ and $(s\mathbf{S},R_h)$, we have

$$\mathcal{J}_v\cup\mathcal{J}_h\subseteq\mathcal{W}_{CB}\cap\mathcal{C},$$

so one direction is clear. Vice versa, let $j\colon X\hookrightarrow Y$ be a weak equivalence in $(s\mathbf{S},\mathrm{CB})$ with Y a complete Bousfield–Segal space. Note that $(\mathcal{J}_v\cup\mathcal{J}_h)^{\pitchfork}$ is the intersection of the set $\mathcal{F}_v$ of v-fibrations and the set $\mathcal{F}_h$ of h-fibrations, and hence the pair $({}^{\pitchfork}((\mathcal{J}_v\cup\mathcal{J}_h)^{\pitchfork}),\mathcal{F}_v\cap\mathcal{F}_h)$ is a weak factorization system on $s\mathbf{S}$ by general abstract non-sense. Pick a factorization $X \xrightarrow{k} Z \xrightarrow{q} Y$ of j with $k \in {}^{\pitchfork}((\mathcal{J}_v \cup \mathcal{J}_h)^{\pitchfork})$ and $q \in \mathcal{F}_v \cap \mathcal{F}_h$,

$$\begin{array}{ccc} X & \xrightarrow{k} & Z \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle q} \\ Y & = & Y. \end{array} \tag{15}$$

Since Y is a complete Bousfield–Segal space, Z is now both v-fibrant and h-fibrant, hence a complete Bousfield–Segal space, too. But a map between complete Bousfield–Segal spaces is a fibration in $(s\mathbf{S}, \mathrm{CB})$ if and only if it is a v-fibration. This in turn holds if and only if it is an h-fibration as can be seen by [**13**, Proposition 7.21]. Hence, we obtain a lift for the square (15) which exhibits j as retract of k. Therefore, $j \in {}^{\pitchfork}((\mathcal{J}_v \cup \mathcal{J}_h)^{\pitchfork})$. □

Lemma 7.2. *Let S and $\mathcal{F}$ be classes of morphisms in $s\mathbf{S}$ such that S is a set and $\mathcal{F}$ is closed under pullbacks. Suppose for every map $f\colon X \to Y$ in S and every map $p\colon Z \to Y$ in $\mathcal{F}$ the pullback $p^*f\colon p^*X \to Z$ is in the saturation ${}^{\pitchfork}((S)^{\pitchfork})$ of S. Then the pullback of every map in the saturation ${}^{\pitchfork}((S)^{\pitchfork})$ along a map in $\mathcal{F}$ is again contained in ${}^{\pitchfork}((S)^{\pitchfork})$ as well.*

Proof. In the language of [**21**, 3], this holds in virtue of the "exactness" properties of Grothendieck toposes, i.e. pullbacks in $s\mathbf{S}$ commute with pushouts, transfinite compositions and retracts in such a way that the proof becomes a straightforward induction. □

Theorem 7.3. *The model category $(s\mathbf{S}, \mathrm{CB})$ is right proper.*

Proof. The class $\mathcal{F}_{\mathrm{CB}}$ is closed under pullbacks, retracts and sequential limits. Hence, by Lemma 7.1 and Lemma 7.2 it remains to check that a pullback square of the form

$$\begin{array}{ccc} P & \longrightarrow & Y \\ {\scriptstyle p^*j}\downarrow & \lrcorner & \downarrow{\scriptstyle j} \\ X & \overset{}{\underset{p}{\twoheadrightarrow}} & \Delta^n \Box \Delta^m, \end{array}$$

with a fibration p in $(s\mathbf{S}, \mathrm{CB})$ and $j \in \mathcal{J}_v \cup \mathcal{J}_h$ exhibits the arrow p^*j to be a weak equivalence in $(s\mathbf{S}, \mathrm{CB})$. But $\mathcal{F}_{CB}$ is a subclass of $\mathcal{F}_v \cap \mathcal{F}_h$, so p is both a v-fibration and an h-fibration. Both Reedy structures $(s\mathbf{S}, R_v)$ and $(s\mathbf{S}, R_h)$ are right proper due to the right properness of $(\mathbf{S}, \mathrm{Kan})$. Therefore, $p^*j \in \mathcal{W}_v \cup \mathcal{W}_h$. But both $\mathcal{W}_v$ and $\mathcal{W}_h$ are contained in $\mathcal{W}_{CB}$, since the model structure CB is a left Bousfield localization of both. This finishes the proof. □

In [**23**, Lemma 7.3.9] it is shown that the left Bousfield localization of a right proper model category is right proper again if and only if the localization is semi-left exact ([**23**, Definition 7.3.1]). That means, if the left derived of the localization preserves homotopy pullbacks along maps between local objects. It therefore follows from Theorem 7.3 that the localization $(s\mathbf{S}, \mathrm{R}_v) \to (s\mathbf{S}, \mathrm{CB})$ is semi-left exact.

Furthermore, in virtue of the Quillen equivalence to $(\mathbf{S}, \mathrm{Kan})$, the model category $(s\mathbf{S}, \mathrm{CB})$ is a model topos in the sense of [**19**]. One may therefore ask whether the semi-left exact localization $(s\mathbf{S}, \mathrm{R}_v) \to (s\mathbf{S}, \mathrm{CB})$ is in fact left exact. Or in other words, whether the model topos $(s\mathbf{S}, \mathrm{CB})$ is a subtopos of the presheaf model topos $(s\mathbf{S}, \mathrm{R}_v)$. Therefore, recall that a left Bousfield localization is left exact if the left derived of the localization preserves all homotopy pullbacks.

Proposition 7.4. *The localization* $(s\mathbf{S}, R_v) \to (s\mathbf{S}, \mathrm{CB})$ *is not left exact.*

Proof. Since every map between (discrete simplicial) sets is a Kan fibration, every map $S \to T$ of simplicial sets induces a Reedy fibration $p_1^*S \to p_1^*T$ of bisimplicial sets. Let

$$\begin{array}{ccc} P & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & B \end{array}$$

be a cartesian square in $\mathbf{S}$ such that $C \to B$ is a weak homotopy equivalence and its pullback $P \to A$ is not. Then

$$\begin{array}{ccc} p_1^*P & \longrightarrow & p_1^*C \\ \downarrow & \lrcorner & \downarrow \\ p_1^*A & \twoheadrightarrow & p_1^*B \end{array}$$

is cartesian in $s\mathbf{S}$, $p_1^*C \to p_1^*B$ is a weak equivalence in $(s\mathbf{S}, \mathrm{CB})$ and $p_1^*A \to p_1^*B$ is a Reedy fibration (although $A \to B$ is not a Kan fibration). Then $p_1^*P \to p_1^*A$ cannot be a weak equivalence in $(s\mathbf{S}, \mathrm{CB})$, because p_1^* is the left adjoint of a Quillen equivalence and hence reflects weak equivalences between cofibrant objects. In particular, the square cannot be homotopy cartesian in $(s\mathbf{S}, \mathrm{CB})$. But it is homotopy cartesian in $(s\mathbf{S}, R_v)$, because $p_1^*A \to p_1^*B$ is a Reedy fibration and the Reedy model structure is right proper. □

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