

EXTRA STRUCTURE ON THE COHOMOLOGY OF CONFIGURATION SPACES OF CLOSED ORIENTABLE SURFACES

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Abstract

The rational homology of unordered configuration spaces of points on any surface was studied by Drummond-Cole and Knudsen. We compute the rational cohomology of configuration spaces on a closed orientable surface, keeping track of the mixed Hodge numbers and the action of the symplectic group on the cohomology. We find a series with coefficients in the Grothendieck ring of $\mathfrak{sp}(2g)$ that describes explicitly the decomposition of the cohomology into irreducible representations. From that we deduce the mixed Hodge numbers and the Betti numbers, obtaining a new formula without cancellations.

1. Introduction

The ordered configuration space of n points in a complex algebraic variety X is

$$\mathcal{F}_n(X) = \{(p_1, \dots, p_n) \in X^n \mid p_i \neq p_j\}.$$

We are interested in the unordered configuration space of X , that is

$$\mathcal{C}_n(X) = \{I \subset X \mid |I| = n\} = \mathcal{F}_n(X) / \mathfrak{S}_n.$$

We compute the rational cohomology of $\mathcal{C}_n(\Sigma_g)$ where Σ_g is a Riemann surface of genus g . Our computation is dual to the one by Drummond-Cole and Knudsen [DCK]: they used the Chevalley–Eilenberg complex to compute $H_*(\mathcal{C}_n(S); \mathbb{Q})$ for any topological surface S of finite type. Since the mixed Hodge structure is defined for any algebraic varieties, there is no evident motivation for which the manipulations in [DCK] and in [K1] are compatible with the Hodge structure. Our work is based on the previous one by Félix and Tanré [FT1] that uses the Cohen–Taylor spectral sequence to study the cohomology.

The Cohen–Taylor spectral sequence is a spectral sequence $E_*(X, n)$ that converges to the rational cohomology of $\mathcal{F}_n(X)$, as proven in [CT, pp. 117, 118]. Križ [K2] and Totaro [T] used the Fulton and MacPherson’s compactification [FM] to prove that for any X smooth projective variety the spectral sequence degenerates at the second page,

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i.e. $H(E_1^{\bullet,\bullet}(X, n), d_1) = \text{gr}_{\bullet}^W H^{\bullet}(\mathcal{F}_n(X))$, where W is the weight filtration defined by Deligne [D].

The symmetric group \mathfrak{S}_n acts on $E_1(X, n)$ and the \mathfrak{S}_n -invariant subalgebra computes the cohomology of the space $\mathcal{C}_n(X)$, indeed

$$H(E_1^{\bullet,\bullet}(X, n)^{\mathfrak{S}_n}, d_1) \simeq \text{gr}_{\bullet}^W H^{\bullet}(\mathcal{C}_n(X)).$$

Furthermore, this isomorphism holds for all closed oriented manifolds, see [FT1, Theorem 2]. There exists an isomorphism $\text{gr}_{\bullet}^W H^{\bullet}(\mathcal{C}_n(X)) \simeq H^{\bullet}(\mathcal{C}_n(X))$ both as algebras and as mixed Hodge structures, but in the case of Riemann surfaces $X = \Sigma_g$, this is not compatible with the action of the mapping class group.

Félix and Tanré [FT1] presented the differential graded algebra $E_1^{\bullet,\bullet}(X, n)^{\mathfrak{S}_n}$ as a bigraded vector space C_n with a differential d and a complex multiplication law \bullet that depends only on the cup product \smile of $H^{\bullet}(X)$. See [FT1] for the definition in the general case or see Definition 2.1 for the case $X = \Sigma_g$.

Using the result of Félix and Tanré, we construct an algebra B_g with a filtration $\{F_n B_g\}$ and surjections of differential algebras $\varphi_n: B_g \rightarrow C_n$ that restricts to isomorphisms $F_n B_g \xrightarrow{\sim} C_n$. The advantage of this method is that the algebra structure on B_g is easy since it is an exterior algebra and the filtration $F_n B$ is induced by another grading of B_g . On the other hand, $F_n B_g$ is not an algebra, so information about the ring structure of $H^{\bullet}(\mathcal{C}_n(\Sigma_g))$ is lost.

We write B as a shorthand for the algebra B_g . We find an acyclic ideal I of B , and then we define A as the quotient B/I . The algebra A is filtered by $F_n A := \text{im } F_n B$, the induced filtration from B . This filtration $F_n A$ is strictly compatible with the differential, and this allow us to simplify the differential d of A . Finally, we obtain $H^{\bullet}(F_n A) \simeq \text{gr}_{\bullet}^W H^{\bullet}(\mathcal{C}_n(\Sigma_g))$ as algebras and as representations of the symplectic group.

Although the action of the mapping class group $\Gamma_g = \text{MCG}(\Sigma_g)$ on $H^{\bullet}(\mathcal{C}_n(\Sigma_g))$ is not symplectic, it preserves the weight filtration W . Hence the induced action on $\text{gr}_{\bullet}^W H^{\bullet}(\mathcal{C}_n(\Sigma_g))$ is symplectic and we study it as a representation of the symplectic group (see also Remark 3.1).

The next step is the explicit computation of the cohomology using the action of the Lie algebra $\mathfrak{sp}(2g)$ on the model (A, d) . From this analysis, we find out a formal power series with coefficients in R_g , i.e. the Grothendieck ring of $\mathfrak{sp}(2g)$. For $g > 0$ the following equation in $R_g[[t, s, u]]$ is proved in Theorem 4.11:

$$\begin{aligned} & \sum_{i,j,n} [\text{gr}_{i+2j}^W H^{i+j}(\mathcal{C}_n(\Sigma_g))] t^i s^j u^n \\ &= \frac{1}{1-u} \left((1+t^2 s u^3)(1+t^2 u) + (1+t^2 s u^2) t^{2g} s u^{2(g+1)} + (1+t^2 s u^2) \right. \\ & \quad \left. (1+t^2 s u^3) \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1 + \omega_j}] t^{j+i} s^i u^{j+2i} (1+t^{2(g-j)} s u^{2(g-j+1)}) \right). \quad (1) \end{aligned}$$

Eq. (1) describes explicitly the decomposition of the associated graded module $\text{gr}_{\bullet}^W H^{\bullet}(\mathcal{C}_n(\Sigma_g))$ into irreducible representations. Moreover, by taking the dimension $\dim: R_g \rightarrow \mathbb{Z}$, we obtain the mixed Poincaré polynomial of $\mathcal{C}_n(\Sigma_g)$ as the coefficient of u^n in eq. (1). The dimension of the representations involved in our formula is

calculated in Lemma 3.10:

$$\dim V_{i\omega_1+\omega_j} = \binom{2g+i+1}{i,j} \frac{2g+2-2j}{2g+2+i-j} \frac{j}{i+j}.$$

The formula for the Betti numbers given in [DCK] is different from the one in this paper, which has no cancellations and a more geometric meaning, because each summand corresponds to a specific submodule of the cohomology with a weight and a description of the symplectic group action.

We can give some explicit information about the mixed Hodge numbers.

Corollary 1.1. *For $g > 1$ the weights h that appear in $H^k(\mathcal{C}_n(\Sigma_g))$ are in the range $\frac{3}{2}k - g - 1 \leq h \leq \frac{3}{2}k$. Moreover, in this range the dimension of $\mathrm{gr}_h^W H^k(\mathcal{C}_n(\Sigma_g))$ is polynomial in k of degree $2g - 1$.*

The polynomial growth of the Betti numbers was already established in [DCK, Corollary 4.9].

The cases of genera 0,1 have already been studied in [S4, S1, S3] and in [S2, MCF, P1], respectively. The Euler characteristic of the configuration spaces of any even-dimensional orientable closed manifold M was computed by Félix and Thomas in [FT2] and it is given by the formula:

$$\sum_{n=0}^{\infty} \chi(\mathcal{C}_n(M)) u^n = (1+u)^{\chi(M)}.$$

In the case of surfaces, this formula can be obtained from eq. (1) by setting $t = s = -1$ and taking the dimension of the representations.

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2. Models for $H^\bullet(\mathcal{C}_n)$

Given a graded vector space V we denote by sV the same vector space with the degree shifted by one, i.e. for any $v \in V$ of degree i , the element $sv \in sV$ has degree $i + 1$. We fix a symplectic base of $H^\bullet(\Sigma_g)$ whose elements are $1 \in H^0(\Sigma_g)$, $a_1, \dots, a_g, b_1, \dots, b_g \in H^1(\Sigma_g)$, $p \in H^2(\Sigma_g)$. The cup product \smile in $H^\bullet(\Sigma_g)$ is given by $a_i \smile b_j = 0$ for $i \neq j$ and $a_i \smile b_i = p$ for all $i = 1, \dots, g$.

Definition 2.1. Let \mathcal{C}_n be the bigraded vector space

$$\mathcal{C}_n \stackrel{\mathrm{def}}{=} \bigoplus_{r=0}^{\lfloor \frac{n}{2} \rfloor} \Lambda^{n-2r}(H(\Sigma_g)) \otimes \Lambda^r(sH(\Sigma_g))$$

where the bigrade of $H^k(\Sigma_g)$ is $(k, 0)$, the bigrade of $sH^k(\Sigma_g)$ is $(k, 1)$, and the graded-symmetric algebra is constructed with respect to the total degree. We endow

| | deg ₁ | deg ₂ | deg ₃ |
|--------------|------------------|------------------|------------------|
| a_i, b_i | 1 | 0 | 1 |
| p | 2 | 0 | 1 |
| $s1$ | 0 | 1 | 2 |
| sa_i, sb_i | 1 | 1 | 2 |
| sp | 2 | 1 | 2 |

Table 1: The degree of the generators of B_g .

C_n with the product:

$$(x_1 \wedge \cdots \wedge x_{n-2r} \otimes sy_1 \wedge \cdots \wedge sy_r) \bullet (z_1 \wedge \cdots \wedge z_{n-2t} \otimes sw_1 \wedge \cdots \wedge sw_t) \stackrel{\text{def}}{=} \frac{1}{(n-2r-2t)!} \sum_{\substack{\sigma \in \mathfrak{S}_{n-2r} \\ \tau \in \mathfrak{S}_{n-2t}}} \pm \alpha_{\sigma, \tau} \otimes (\beta_\sigma \wedge \gamma_\tau),$$

where:

$$\begin{aligned} \alpha_{\sigma, \tau} &= (x_{\sigma(1)} \smile z_{\tau(1)}) \wedge \cdots \wedge (x_{\sigma(n-2r-2t)} \smile z_{\tau(n-2r-2t)}) \\ \beta_\sigma &= s(x_{\sigma(n-2r-2t+1)} \smile x_{\sigma(n-2r-2t+2)} \smile w_1) \wedge \cdots \wedge s(x_{\sigma(n-2r-1)} \smile x_{\sigma(n-2r)} \smile w_t) \\ \gamma_\tau &= s(z_{\tau(n-2r-2t+1)} \smile z_{\tau(n-2r-2t+2)} \smile y_1) \wedge \cdots \wedge s(z_{\tau(n-2t-1)} \smile z_{\tau(n-2t)} \smile y_r), \end{aligned}$$

and the sign \pm is given by the Koszul rule. We consider the differential

$$d: \Lambda^{n-2r}(H(\Sigma_g)) \otimes \Lambda^r(sH(\Sigma_g)) \rightarrow \Lambda^{n-2r+2}(H(\Sigma_g)) \otimes \Lambda^{r-1}(sH(\Sigma_g))$$

of degree $(2, -1)$ defined on the generators by

$$\begin{aligned} d(1 \wedge \cdots \wedge 1 \otimes sy) &\stackrel{\text{def}}{=} \frac{1}{2} \left(1 \wedge \cdots \wedge 1 \wedge y \wedge p + 1 \wedge \cdots \wedge 1 \wedge (y \smile p) \right. \\ &\quad \left. - \sum_{i=1}^g 1 \wedge \cdots \wedge 1 \wedge (y \smile a_i) \wedge b_i + \sum_{i=1}^g 1 \wedge \cdots \wedge 1 \wedge (y \smile b_i) \wedge a_i \right). \end{aligned}$$

For the sake of notation, in the following we will write $x_1 \wedge \cdots \wedge x_k \otimes sy_1 \wedge \cdots \wedge sy_r$ for the element $x_1 \wedge \cdots \wedge x_k \wedge 1 \wedge \cdots \wedge 1 \otimes sy_1 \wedge \cdots \wedge sy_r$ where the number of 1 omitted is $n - 2r - k$.

Theorem 2.2 ([FT1, Theorems 1, 14]). *The triple (C_n, \bullet, d) is a differential graded algebra and it is isomorphic to the \mathfrak{S}_n -invariants of the first page E_1 of the Cohen-Taylor spectral sequence for $\mathcal{C}_n(\Sigma_g)$:*

$$(E_1, d_1)^{\mathfrak{S}_n} \cong (C_n, d).$$

Definition 2.3. Let $B = B_g = \Lambda^\bullet(\tilde{H}^\bullet(\Sigma_g) \oplus sH^\bullet(\Sigma_g))$ be the graded-symmetric algebra on the trigraded vector space $\tilde{H}^\bullet(\Sigma_g) \oplus sH^\bullet(\Sigma_g)$, where the grading is given in Table 1 and the total degree is $|x| = \text{deg}_1(x) + \text{deg}_2(x)$. We endow B_g with the

following differential:

$$\begin{aligned} d(x) &= 0 \quad \text{for } x \in \tilde{H}^*(\Sigma_g) \\ d(sp) &= p^2, \quad d(s1) = p - \sum_{i=1}^g a_i b_i \\ d(sa_i) &= a_i p, \quad d(sb_i) = b_i p \quad \text{for } i = 1, \dots, g. \end{aligned} \tag{2}$$

Remark 2.4. The elements of C_n (resp. of B) can be interpreted geometrically as follows. Generators a_i and b_i for $i = 1, \dots, g$ are an average over all particles (i.e. points of the configuration) of the motion of that particle along the curve a_i (resp. b_i). The element p is the average over all particles of the motion of that particle on the entire surface. The generator $s1$ is the average over all pairs of particles of the rotation of one particle around the other. Similar description holds for the other generators.

In order to describe cohomological classes we need to resolve the collision problems of the moving particle with the other particles in the configuration. This is possible only if its differential is zero.

The only difference between the generators in B and in C_n consists in the multiplication by a numerical coefficients, as shown in the following Lemma 2.5.

Notice that the differential d is compatible with \deg_1 and \deg_2 of degree $(2, -1)$, but not with the third grading since $d(s1) = p - \sum_{i=1}^g a_i b_i$. For all $n \in \mathbb{N}$ consider the morphisms $\varphi_n: B \rightarrow C_n$ defined by

$$\begin{aligned} \varphi_n(a_i) &= a_i, \quad \varphi_n(sa_i) = (n-1)!(n-1)(n-1+g)sa_i \quad \text{for } i = 1, \dots, g, \\ \varphi_n(b_i) &= b_i, \quad \varphi_n(sb_i) = (n-1)!(n-1)(n-1+g)sb_i \quad \text{for } i = 1, \dots, g, \\ \varphi_n(p) &= (n-1)!(n-1+g)p, \quad \varphi_n(sp) = 2(n-1)!^3(n-1)(n-1+g)^2sp, \\ \varphi_n(s1) &= (n-1)!(n-1)s1. \end{aligned}$$

Lemma 2.5. *The map $\varphi_n: B \rightarrow C_n$ is a morphism of differential graded algebras.*

Proof. We verify the compatibility between φ_n and the differential. We first compute products in C_n :

$$a_i \bullet b_i = \frac{1}{n!} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \tau \in \mathfrak{S}_n}} \alpha_{\sigma, \tau} = (n-1)!(n-1)a_i \wedge b_i + (n-1)!p,$$

analogously we have

$$\begin{aligned} a_i \bullet p &= (n-1)!(n-1)a_i \wedge p, \\ b_i \bullet p &= (n-1)!(n-1)b_i \wedge p, \\ p \bullet p &= (n-1)!(n-1)p \wedge p. \end{aligned}$$

Now we prove the equalities $d(\varphi_n(x)) = \varphi_n(d(x))$ for all the generators x : for

$x = s1$ we have

$$\begin{aligned}
d(\varphi_n(s1)) &= d((n-1)!(n-1)s1) \\
&= (n-1)!(n-1)(p - \sum_{i=1}^g a_i \wedge b_i) \\
&= (n-1)!(n-1)p - \sum_{i=1}^g a_i \bullet b_i - (n-1)!p \\
&= (n-1)!(n-1+g)p - \sum_{i=1}^g a_i \bullet b_i \\
&= \varphi_n(p - \sum_{i=1}^g a_i b_i) = \varphi_n(d(s1)),
\end{aligned}$$

for $x = sp$

$$\begin{aligned}
d(\varphi_n(sp)) &= d(2(n-1)!^3(n-1)(n-1+g)^2 sp) \\
&= 2(n-1)!^3(n-1)(n-1+g)^2 \frac{1}{2} p \wedge p \\
&= (n-1)!^2(n-1+g)^2 p \bullet p \\
&= \varphi_n(p^2) = \varphi_n(d(sp)),
\end{aligned}$$

for $x = sa_i, i = 1, \dots, g$

$$\begin{aligned}
d(\varphi_n(sa_i)) &= d((n-1)!^2(n-1)(n-1+g)sa_i) \\
&= (n-1)!^2(n-1)(n-1+g)a_i \wedge p \\
&= (n-1)!(n-1+g)a_i \bullet p \\
&= \varphi_n(a_i p) = \varphi_n(d(sa_i)),
\end{aligned}$$

and analogously for sb_i . For all other generators $x = a_i, b_i, p$ we have

$$d(\varphi_n(x)) = 0 = \varphi_n(d(x)).$$

□

Definition 2.6. Let $\{F_k B\}_{k \in \mathbb{N}}$ be the filtration of B defined by the third grading, i.e $F_k B = \bigoplus_{i \leq k} B^{\bullet, \bullet, i}$.

Notice that the inclusion $d(F_k B) \subseteq F_k B$ holds since it holds for all generators of B .

Lemma 2.7. Let $n-1+g \neq 0$ and $k+2r \leq n$. For all $x_1, \dots, x_k, y_1, \dots, y_r \in H(\Sigma_g)$ we have in C_n the following equality

$$\begin{aligned}
1 \wedge \dots \wedge 1 \wedge x_1 \wedge \dots \wedge x_k \otimes sy_1 \wedge \dots \wedge sy_r \\
= \lambda \varphi_n(x_1) \bullet \dots \bullet \varphi_n(x_k) \bullet \varphi_n(sy_1) \bullet \dots \bullet \varphi_n(sy_r) + z
\end{aligned}$$

for some $\lambda \in \mathbb{Q}^*$ and some $z \in \varphi_n(F_{k+2r-1} B)$. In particular $\varphi_n: F_n B \rightarrow C_n$ is surjective.

Proof. We first consider the case $k = 0$: we prove the statement by induction on r . The base step for $r = 1$ follows from the definition of φ_n and from $n - 1 + g \neq 0$. Suppose $r \geq 2$ and consider the product $sy_1 \wedge \cdots \wedge sy_{r-1} \bullet sy_r$ in C_n :

$$\begin{aligned} sy_1 \wedge \cdots \wedge sy_{r-1} \bullet sy_r &= \frac{1}{(n-2r)!} \sum_{\substack{\sigma \in \mathfrak{S}_{n-2r+2} \\ \tau \in \mathfrak{S}_{n-2}}} \beta_{\sigma, \tau} \wedge \gamma_{\sigma, \tau} \\ &= \frac{(n-2r+2)!(n-2)!}{(n-2r)!} sy_1 \wedge \cdots \wedge sy_{r-1} \wedge sy_r. \end{aligned}$$

By inductive hypothesis we have $sy_1 \wedge \cdots \wedge sy_{r-1} = \lambda' \varphi_n(sy_1) \bullet \cdots \bullet \varphi_n(sy_{r-1})$, so $sy_1 \wedge \cdots \wedge sy_r = \lambda \varphi_n(sy_1) \bullet \cdots \bullet \varphi_n(sy_r)$ for some $\lambda \in \mathbb{Q}^*$.

Now we proceed by induction on k , the base step $k = 0$ is already been proved. Suppose $k > 0$, we have

$$\begin{aligned} 1 \wedge \cdots \wedge 1 \wedge x_1 \wedge \cdots \wedge x_{k-1} \otimes sy_1 \wedge \cdots \wedge sy_r \bullet x_k &= \frac{1}{(n-2r)!} \sum_{\substack{\sigma \in \mathfrak{S}_{n-2r} \\ \tau \in \mathfrak{S}_n}} \alpha_{\sigma, \tau} \otimes \beta_{\sigma, \tau} \\ &= (n-1)! \sum_{i=1}^{k-1} x_1 \wedge \cdots \wedge (x_i \cup x_k) \wedge \cdots \wedge x_{k-1} \otimes sy_1 \wedge \cdots \wedge sy_r \\ &\quad + 2(n-1)! \sum_{j=1}^r x_1 \wedge \cdots \wedge x_{k-1} \otimes sy_1 \wedge \cdots \wedge s(y_j \cup x_k) \wedge \cdots \wedge sy_r \\ &\quad + (n-1)!(n-k-2r+1)x_1 \wedge \cdots \wedge x_k \otimes sy_1 \wedge \cdots \wedge sy_r. \end{aligned}$$

The first two sums belong to $\varphi_n(F_{k+2r-1}B)$ by inductive hypothesis. We also have

$$\begin{aligned} x_1 \wedge \cdots \wedge x_{k-1} \otimes sy_1 \wedge \cdots \wedge sy_r \\ = \lambda'' \varphi_n(x_1) \bullet \cdots \bullet \varphi_n(x_{k-1}) \bullet \varphi_n(sy_1) \bullet \cdots \bullet \varphi_n(sy_r) + z \end{aligned}$$

for some $\lambda'' \in \mathbb{Q}^*$ and $z \in \varphi_n(F_{k+2r-2}B)$. Using that x_k is a multiple of $\varphi_n(x_k)$ and that $z \bullet \varphi_n(x_k) \in \varphi_n(F_{k+2r-1}B)$, we obtain the claimed equality. \square

Lemma 2.8. *The restricted chain map $\varphi_n: F_n B \rightarrow C_n$ is an isomorphism.*

Proof. Lemma 2.5 ensures that φ_n is a homomorphism of chain complexes and Lemma 2.7 gives the surjectivity. We complete the proof with a dimensional argument.

Let $S(t)$ be the formal power series $\frac{(1+t)^{2g}}{(1-t)^2}$ and $R(t) = \frac{(1+t)^2}{(1-t)^{2g}}$, they are the Poincaré series of $\Lambda^\bullet(H(\Sigma_g))$ and of $\Lambda^\bullet(sH(\Sigma_g))$ with respect to the cohomological degree. For any power series $K(t)$, we denote the coefficient of t^k by $[t^k]K(t)$. The dimension of C_n is

$$\dim C_n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} [t^{n-2r}]S(t) \cdot [t^r]R(t) = [t^n](S(t)R(t^2)).$$

By definition of $F_n B$, we have

$$\dim F_n B = \sum_{i=0}^n \dim B^{\bullet \bullet \bullet i} = \sum_{i=0}^n [t^i] \left(\frac{(1+t)^{2g}(1+t^2)^2}{(1-t)(1-t^2)^{2g}} \right).$$

Let $K(t)$ be the power series $\frac{(1+t)^{2g}(1+t^2)^2}{(1-t)(1-t^2)^{2g}}$. The equality

$$\begin{aligned} [t^n](S(t)R(t^2)) &= [t^n] \left(\frac{(1+t)^{2g}(1+t^2)^2}{(1-t)^2(1-t^2)^{2g}} \right) \\ &= \sum_{i=1}^n [t^{n-i}] \left(\frac{1}{1-t} \right) [t^i]K(t) \\ &= \sum_{i=1}^n [t^i]K(t) \end{aligned}$$

completes the proof. \square

Consider the ideal I of B generated by sp and p^2 and define A as the quotient B/I . Since $d(I) \subseteq I$, A is a differential graded algebra. The filtration $\{F_n B\}$ induces two filtrations $\{F_n I\}$ and $\{F_n A\}$ on I and A , respectively.

Lemma 2.9. *The ideal I is acyclic, i.e. $H(I, d) = 0$. Moreover the chain complexes $(F_n I, d)$ are acyclic for all n , i.e. $H(F_n I, d) = 0$.*

Proof. First, notice that the filtration $F_n I$ is induced by the third grading \deg_3 of B and that p and sp are homogeneous elements of degrees $\deg_3(p) = 1$ and $\deg_3(sp) = 2$, hence:

$$F_n I := F_n B \cap I = F_{n-2} p^2 + F_{n-2} sp. \quad (3)$$

Since B is an exterior algebra, we have that $(p^2) \cap (sp) = (p^2 sp)$ and, keeping track of the gradation, $(p^2) \cap (sp) \cap F_n B = p^2 sp F_{n-4} B$.

Consider a generic element $xp^2 + ysp$ of $F_n I$ homogeneous with respect to \deg_1 and \deg_2 , and suppose that $xp^2 + ysp$ belongs to $\ker d$, in which case we will prove that $xp^2 + ysp \in d(F_n I)$. Since d preserves the first two degrees, we can assume x and y to be homogeneous with respect to \deg_1 and \deg_2 . By eq. (3) we can suppose that x and y are in $F_{n-2} B$. We use the hypothesis $d(xp^2 + ysp) = 0$:

$$d(xp^2 + ysp) = d(x)p^2 + (-1)^{|y|} yp^2 + d(y)sp = 0.$$

The element $d(x)p^2 + (-1)^{|y|} yp^2$ belongs to $(p^2) \cap (sp) \cap F_n B$, hence there exists $z \in F_{n-4} B$ such that $d(x)p^2 + (-1)^{|y|} yp^2 = zp^2 sp$. We use again the fact that B is an exterior algebra to obtain $d(x) + (-1)^{|y|} y = zsp$. Therefore, we have

$$xp^2 + ysp = xp^2 - (-1)^{|y|} d(x)sp = p^2 x - sp d(x) = d(sp x).$$

Since $x \in F_{n-2} B$, then $sp x \in F_n I$. We have proven that $H(F_n I, d) = 0$, the vanishing of $H(I, d)$ follows from $I = \cup_n F_n I$. \square

Notice that A is isomorphic to $\Lambda^\bullet \left(\tilde{H}(\Sigma_g) \oplus sH^{\leq 1}(\Sigma_g) \right) / (p^2)$.

3. Some facts of representation theory

Let Σ_g be a Riemann surface, Γ_g be its mapping class group, and $\text{Tor}(\Sigma_g)$ the Torelli subgroup. Recall the short exact sequence

$$0 \rightarrow \text{Tor}(\Sigma_g) \rightarrow \Gamma_g \rightarrow \text{Sp}(2g; \mathbb{Z}) \rightarrow 0.$$

Consider the weight filtration W_\bullet of the cohomology of the algebraic variety $\mathcal{C}_n(\Sigma_g)$,

definition and properties of this filtration can be found in [D]. We define the module $\text{gr}_i^W H^\bullet(\mathcal{C}_n(\Sigma_g))$ as the quotient $W_i H^\bullet(\mathcal{C}_n(\Sigma_g))/W_{i-1} H^\bullet(\mathcal{C}_n(\Sigma_g))$ and their direct sum $\text{gr}_\bullet^W H^\bullet(\mathcal{C}_n(\Sigma_g)) = \bigoplus_i \text{gr}_i^W H^\bullet(\mathcal{C}_n(\Sigma_g))$ is a bigraded ring.

The natural action of the subgroup $\text{Tor}(\Sigma_g)$ on $H^i(\mathcal{C}_n(\Sigma_g))$ may be non-trivial, but the induced action on $\text{gr}_\bullet^W H^\bullet(\mathcal{C}_n(\Sigma_g))$ is trivial. Indeed, since Σ_g is a compact algebraic variety, Totaro proved in [T, Theorem 3] that the Leray filtration for the inclusion $\mathcal{F}_n(\Sigma_g) \hookrightarrow \Sigma_g^n$ coincides with the weight filtration on $H^\bullet(\mathcal{F}_n(\Sigma_g))$. Therefore, each homeomorphism of the pair $(\mathcal{F}_n(\Sigma_g), \Sigma_g^n)$ preserves the Leray filtration, and thus the weight filtration W_\bullet . In particular this applies to each element in Γ_g acting on the pair $(\mathcal{F}_n(\Sigma_g), \Sigma_g^n)$. Indeed, $\text{gr}_\bullet^W H^\bullet(\mathcal{C}_n(\Sigma_g)) \simeq H(E_1^{\mathfrak{S}_n}, d_1)$ functorially, hence the isomorphism is Γ_g -equivariant. The action of Γ_g on the algebra $E_1^{\mathfrak{S}_n}$ is clearly symplectic thus $\text{Tor}(\Sigma_g)$ acts trivially on $\text{gr}_\bullet^W H^\bullet(\mathcal{C}_n(\Sigma_g))$. The action of the Torelli group is studied in [B] in the case of once punctured surfaces and it is non-trivial on $H_2(\mathcal{C}_n(\Sigma_g \setminus \{*\}))$; the case of compact surfaces is similar.

Remark 3.1. From Theorem 4.11, we deduced that the filtration W_\bullet is trivial in cohomological degrees 0, 1, 2 and also in degree 3 if $g = 2$, since the graded module $\text{gr}_\bullet^W H^i(\mathcal{C}_n(\Sigma_g))$ is concentrated in a unique degree for $i = 0, 1, 2$ (and $i = 3, 4$ if $g = 2$). Thus in these cases the action of the mapping class group is symplectic.

Looijenga in [L2] proves that the action of $\text{Tor}(\Sigma_g)$ on $H^3(\mathcal{C}_3(\Sigma_g))$ is non-trivial for $g \geq 3$ and so in these cases the action is not symplectic.

We consider $\text{gr}_\bullet^W H^\bullet(\mathcal{C}_n(\Sigma_g); \mathbb{C})$ as a representation of the Lie algebra $\mathfrak{sp}(2g)$ associated to the complex symplectic group. If we denote the fundamental weights of $\mathfrak{sp}(2g)$ by $\omega_1, \dots, \omega_g$, the irreducible representations of $\mathfrak{sp}(2g)$ are the highest weight representation V_λ for all dominant weights $\lambda = \sum_{i=1}^g \lambda_i \omega_i$, $\lambda_i \in \mathbb{N}$. The cohomology of Σ_g in degree one is given by the standard representation, i.e. $H^1(\Sigma_g) = V_{\omega_1}$.

Let V be the $\mathfrak{sp}(2g)$ -representation $= H^1(\Sigma_g) \simeq V_{\omega_1}$. Before computing the cohomology of (A, d) we need to know the cohomology of $(\Lambda^\bullet V \otimes S^\bullet V, \tilde{d})$, where the differential \tilde{d} is defined by $\tilde{d}(1 \otimes v) = v \otimes 1$ and $\tilde{d}(v \otimes 1) = 0$ for all $v \in V$. The standard action of $\mathfrak{sl}(2g)$ on V induces an action on $(\Lambda^\bullet V \otimes S^\bullet V, \tilde{d})$, since the differential \tilde{d} is $\mathfrak{sl}(2g)$ -equivariant.

We will call $\omega_1, \dots, \omega_{2g-1}$ the fundamental weights of $\mathfrak{sl}(2g)$ and W_μ its irreducible representations associated to a dominant weight $\mu = \sum_{i=1}^{2g-1} m_i \omega_i$, $m_i \in \mathbb{N}$.

Lemma 3.2. *The $\mathfrak{sl}(2g)$ -representation $\Lambda^j V \otimes S^i V$ decomposes, for $j \leq 2g$, as*

$$\Lambda^j V \otimes S^i V = W_{i\omega_1 + \omega_j} \oplus W_{(i-1)\omega_1 + \omega_{j+1}}.$$

Proof. It is known that $S^i W_{\omega_1} = W_{i\omega_1}$ and $\Lambda^j W_{\omega_1} = W_{\omega_j}$. Let $\sigma = (1, j+1) \in \mathfrak{S}_{2g}$ be an element of the Weyl group of $\mathfrak{sl}(2g)$. The element $i\omega_1 + \sigma(\omega_j) = (i-1)\omega_1 + \omega_{j+1}$ is a dominant weight for $i > 0$. By the Parthasarathy–Ranga-Rao–Varadarajan conjecture (see [L1, K3]) $W_{i\omega_1 + \omega_j}$ and $W_{(i-1)\omega_1 + \omega_{j+1}}$ are contained in the tensor product $W_{i\omega_1} \otimes W_{\omega_j}$. Use the Weyl dimension formula to find

$$\dim W_{i\omega_1 + \omega_j} = \binom{i+j-1}{i} \binom{i+2g}{i+j}.$$

The equality $\binom{i+j-1}{i} \binom{i+2g}{i+j} + \binom{i+j-1}{i-1} \binom{i+2g-1}{i+j} = \binom{2g}{j} \binom{i+2g-1}{i}$ completes the proof. \square

Lemma 3.3. *The differential complex $(\Lambda^\bullet V \otimes S^\bullet V, \tilde{d})$ is exact in positive degree.*

Proof. The differential

$$\tilde{d}: W_{i\omega_1+\omega_j} \oplus W_{(i-1)\omega_1+\omega_{j+1}} \rightarrow W_{(i-1)\omega_1+\omega_{j+1}} \oplus W_{(i-2)\omega_1+\omega_{j+2}}$$

is a non-zero morphism of representations. Therefore, we have $W_{i\omega_1+\omega_j} = (\ker \tilde{d})^{j,i}$ and $W_{(i-1)\omega_1+\omega_{j+1}} = (\operatorname{im} \tilde{d})^{j+1,i-1}$ for $j \geq 0$. Obviously $(\operatorname{im} \tilde{d})^{0,0} = 0$, so the equality

$$H^\bullet(\Lambda^\bullet V \otimes S^\bullet V, \tilde{d}) = \mathbb{C}$$

completes the proof. \square

Remark 3.4. The complex $(\Lambda^\bullet V \otimes S^\bullet V, \tilde{d})$ is the Koszul resolution of the trivial $S^\bullet V$ -module \mathbb{C} , hence it is an exact complex.

Since the Lie algebra $\mathfrak{sl}(2g)$ does not act on A_g , we need to present a branching rule for $\mathfrak{sp}(2g) \subset \mathfrak{sl}(2g)$. For the sake of an uniform notation, we define $V_\lambda = 0$ if λ is not a dominant weight.

Lemma 3.5 (Branching rule). *The $\mathfrak{sl}(2g)$ -module $W_{i\omega_1+\omega_j}$ decomposes as an $\mathfrak{sp}(2g)$ -module in the following ways:*

$$W_{i\omega_1+\omega_j} = \bigoplus_{k=0}^{\lfloor \frac{j-1}{2} \rfloor} V_{i\omega_1+\omega_{j-2k}} \oplus \bigoplus_{k=0}^{\lfloor \frac{j-2}{2} \rfloor} V_{(i-1)\omega_1+\omega_{j-2k-1}} \quad \text{if } 2 \leq j \leq g,$$

$$W_{i\omega_1+\omega_j} = \bigoplus_{k=0}^{\lfloor \frac{2g-j-1}{2} \rfloor} V_{i\omega_1+\omega_{2g-j-2k}} \oplus \bigoplus_{k=0}^{\lfloor \frac{2g-j-2}{2} \rfloor} V_{(i-1)\omega_1+\omega_{2g-j-2k-1}} \quad \text{if } j \geq g.$$

Proof. We apply the result of [ST, Theorem 1]. The diagram associated to $i\omega_1 + \omega_j$ has a hook shape with row length $i + 1$ and column length j . Fill each box with labels in the ordered set $\{1 < \dots < g < \bar{g} < \dots < \bar{1}\}$, such that it becomes a semi-standard Young tableau (SSYT) i.e. the rows are non-decreasing and columns are increasing. The word $w(T)$ – associated to a SSYT T – is the word obtained by reading the tableaux from right to left and from top to bottom. By convention, $e_{\bar{a}} = -e_a$. A word $w(T) = a_1 a_2 \dots a_k$ is admissible if for each $r \leq k$ the element $\sum_{s=1}^r e_{a_s}$ is a dominant weight for $\mathfrak{sp}(2g)$. The decomposition of $W_{i\omega_1+\omega_j}$ into $\mathfrak{sp}(2g)$ -representations is given by

$$W_{i\omega_1+\omega_j} = \bigoplus_{w(T) \text{ admissible}} V_{\lambda(T)},$$

where $\lambda(T) = \sum_{s=1}^{|w(T)|} e_{a_s}$.

Suppose $w(T)$ is admissible, then the first row of T is labelled only by ones. For $j \leq g$, all possible labels of the first column of T , from top to bottom, are the following:

- $1, 2, \dots, j - k, \overline{j - k}, \overline{j - k - 1}, \dots, \overline{j - 2k + 1}$, where k is an integer such that $0 \leq 2k \leq j - 1$
- $1, 2, \dots, j - k - 1, \overline{j - k - 1}, \dots, \overline{j - 2k}, \bar{1}$, where k here is an integer such that $0 \leq 2k \leq j - 2$ and $i > 0$.

Our decomposition follows, the case $j > g$ being analogous. \square

Let ω be the element $\sum_{i=1}^g a_i \wedge b_i \in \Lambda^\bullet H^1(\Sigma_g) \subset A$. The differential d of A involves the multiplication by ω (see eq. (2)). Thus we need to study the operator

$$L_\omega: \Lambda^j V \otimes S^i V \rightarrow \Lambda^{j+2} V \otimes S^i V$$

defined by left multiplication by ω .

Lemma 3.6 ([FH, Theorem 17.5]). *The $\mathfrak{sp}(2g)$ -representation $\Lambda^j V$ is isomorphic to $\Lambda^{2g-j} V$ and decomposes, for $j \leq g$, as*

$$\Lambda^j V = W_{\omega_j} = \bigoplus_{k=0}^{\lfloor \frac{j}{2} \rfloor} V_{\omega_j - 2k}.$$

Moreover, $(\ker L_\omega)^{2g-j} = V_{\omega_j} \subset \Lambda^{2g-j} V$ and $(\operatorname{coker} L_\omega)^j = V_{\omega_j} \subset \Lambda^j V$.

We denote by R_g the Grothendieck ring of $\mathfrak{sp}(2g)$, i.e. R_g is the free \mathbb{Z} -module with basis the irreducible (finite dimensional) representations of $\mathfrak{sp}(2g)$. The ring structure on R_g is induced by the tensor product of representations, however we do not need the multiplicative structure.

Lemma 3.7. *For $i > 0$ and $1 \leq j \leq g$, we have*

$$V_{\omega_j} \otimes S^i V = V_{i\omega_1 + \omega_j} \oplus V_{(i-1)\omega_1 + \omega_{j+1}} \oplus V_{(i-1)\omega_1 + \omega_{j-1}} \oplus V_{(i-2)\omega_1 + \omega_j}. \quad (4)$$

Proof. We use Lemmas 3.2, 3.5 and 3.6:

$$\begin{aligned} V_{\omega_j} \otimes S^i V &= \Lambda^j V \otimes S^i V \ominus \Lambda^{j-2} V \otimes S^i V \\ &= W_{i\omega_1 + \omega_j} \oplus W_{(i-1)\omega_1 + \omega_{j+1}} \ominus W_{i\omega_1 + \omega_{j-2}} \ominus W_{(i-1)\omega_1 + \omega_{j-1}} \\ &= V_{i\omega_1 + \omega_j} \oplus V_{(i-1)\omega_1 + \omega_{j+1}} \oplus V_{(i-1)\omega_1 + \omega_{j-1}} \oplus V_{(i-2)\omega_1 + \omega_j}, \end{aligned}$$

where the symbol \ominus is the negation in the Grothendieck ring R_g . \square

Definition 3.8. Let $W = \bigoplus_{i,j} W^{i,j}$ be a bigraded representation of the symplectic Lie algebra $\mathfrak{sp}(2g)$. The Hilbert–Poincaré series of W is the formal power series

$$P_W(t, s) = \sum_{i,j} [W^{i,j}] t^i s^j \in R_g[[t, s]].$$

Recall that the bidegree of $v \otimes 1$ is $(1, 0)$ and the one of $1 \otimes v$ is $(1, 1)$, for all $v \in V$.

Corollary 3.9. *The Hilbert–Poincaré series of the representation $\Lambda^\bullet V \otimes S^\bullet V$ is*

$$\begin{aligned} P_{\Lambda V \otimes S V}(t, s) &= \frac{t^{2(g+1)} - 1}{t^2 - 1} + t^2 s \frac{t^{2g} - 1}{t^2 - 1} + \\ &\quad + (1 + s)(1 + t^2 s) \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1 + \omega_j}] \frac{t^{2(g-j+1)} - 1}{t^2 - 1} t^{i+j} s^i. \end{aligned}$$

Proof. We first notice that Lemma 3.6 implies

$$\sum_{j=0}^{2g} [\Lambda^j V] t^j = \sum_{j=0}^g \sum_{k=0}^{g-j} [V_{\omega_j}] t^{j+2k} = \sum_{j=0}^g [V_{\omega_j}] \frac{t^{2(g-j+1)} - 1}{t^2 - 1} t^j,$$

where we set $[V_{\omega_0}] = 1 \in R_g$. For $1 \leq j \leq g$, Lemma 3.7 implies that

$$\begin{aligned} \sum_{i \geq 0} [V_{\omega_j} \otimes S^i V](ts)^i &= \sum_{i \geq 0} [V_{i\omega_1 + \omega_j}](ts)^i (1 + t^2 s^2) + \sum_{i \geq 0} [V_{i\omega_1 + \omega_{j+1}}](ts)^{i+1} + \\ &\quad + \sum_{i \geq 0} [V_{i\omega_1 + \omega_{j-1}}](ts)^{i+1}. \end{aligned}$$

We want to compute $P_{\Lambda V \otimes S} v(t, s)$:

$$P_{\Lambda V \otimes S} v(t, s) = \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [\Lambda^j V \otimes S^i V] t^{j+i} s^i = \sum_{\substack{0 \leq j \leq g \\ i \geq 0}} [V_{\omega_j} \otimes S^i V] \frac{t^{2(g-j+1)} - 1}{t^2 - 1} t^{j+i} s^i. \quad (5)$$

In this last sum, the addendum for $j = 0$ is:

$$\sum_{i \geq 0} [V_{i\omega_1}] \frac{t^{2(g+1)} - 1}{t^2 - 1} t^i s^i = \frac{t^{2(g+1)} - 1}{t^2 - 1} + \sum_{i \geq 0} [V_{i\omega_1 + \omega_1}] \frac{t^{2(g+1)} - 1}{t^2 - 1} t^{i+1} s^{i+1}$$

and the addenda for $j > 0$ are:

$$\begin{aligned} &\sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1 + \omega_j}] \frac{t^{2(g-j+1)} - 1}{t^2 - 1} t^{j+i} s^i (1 + t^2 s^2) \\ &+ \sum_{\substack{1 \leq j \leq g-1 \\ i \geq 0}} [V_{i\omega_1 + \omega_{j+1}}] \frac{t^{2(g-j+1)} - 1}{t^2 - 1} t^{j+i+1} s^{i+1} \\ &+ \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1 + \omega_{j-1}}] \frac{t^{2(g-j+1)} - 1}{t^2 - 1} t^{j+i+1} s^{i+1}. \end{aligned}$$

Thus, eq. (5) is equal to

$$\begin{aligned} &\frac{t^{2(g+1)} - 1}{t^2 - 1} + \frac{t^{2g} - 1}{t^2 - 1} t^2 s + \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1 + \omega_j}] \left(\frac{t^{2(g-j+1)} - 1}{t^2 - 1} t^{j+i} s^i (1 + t^2 s^2) + \right. \\ &\quad \left. + \frac{t^{2(g-j)} - 1}{t^2 - 1} t^{j+i+2} s^{i+1} + \frac{t^{2(g-j+2)} - 1}{t^2 - 1} t^{j+i} s^{i+1} \right). \end{aligned}$$

Since the last factor is equal to $(1+s)(1+t^2s) \frac{t^{2(g-j+1)} - 1}{t^2 - 1} t^{i+j} s^i$, we have proven the claimed equality. \square

Lemma 3.10. For $i \geq 0$ and $1 \leq j \leq g$, we have

$$\dim V_{i\omega_1 + \omega_j} = \binom{2g+i+1}{i, j} \frac{2g+2-2j}{2g+2+i-j} \frac{j}{i+j}. \quad (6)$$

Proof. Recall that the positive roots of the Lie algebra $\mathfrak{sp}(2g)$ are

$$e_k \pm e_h \text{ for } 1 \leq k < h \leq g \quad \text{and} \quad 2e_k \text{ for } 1 \leq k \leq g.$$

Moreover, the half-sum of the positive roots is $\rho = \sum_{k=1}^g (g+1-k)e_k$. Now, we apply the Weyl dimension formula:

$$\begin{aligned} \prod_{k < h} \frac{\langle i\omega_1 + \omega_j + \rho, e_k - e_h \rangle}{\langle \rho, e_k - e_h \rangle} &= \frac{(g+i)!}{i!(g-j)!(j-1)!(i+j)} \\ \prod_{k < h} \frac{\langle i\omega_1 + \omega_j + \rho, e_k + e_h \rangle}{\langle \rho, e_k + e_h \rangle} &= \frac{(2g+i+1)!(g+1-j)!(2g+2-2j)}{(g+i+1)!(2g+1-j)!(2g+i+2-j)} \\ \prod_{k=1}^g \frac{\langle i\omega_1 + \omega_j + \rho, 2e_k \rangle}{\langle \rho, 2e_k \rangle} &= \frac{g+1+i}{g+1-j}. \end{aligned}$$

We obtain eq. (6) by multiplying the right hand sides of the above identities. \square

4. The cohomology of configuration spaces

The case of the sphere ($g = 0$) is essentially different from the case $g > 0$ and our approach is useless since $\mathfrak{sp}(2g)$ is trivial for $g = 0$. We refer to [S4] for the following theorem.

Theorem 4.1. *The rational homology of $\mathcal{C}_n(S^2)$ is:*

$$\begin{aligned} H^0(\mathcal{C}_n(S^2); \mathbb{Q}) &= \mathbb{Q} && \text{for all } n, \\ H^2(\mathcal{C}_1(S^2); \mathbb{Q}) &= \mathbb{Q} \\ H^3(\mathcal{C}_n(S^2); \mathbb{Q}) &= \mathbb{Q} && \text{for } n \geq 3, \\ H^k(\mathcal{C}_n(S^2); \mathbb{Q}) &= 0 && \text{otherwise.} \end{aligned}$$

From now on we assume $g > 0$ and so $\omega = \sum_{i=1}^g a_i b_i \neq 0$.

Lemma 4.2. *For $g > 0$ the filtration $F_n A$ is strictly compatible with the differential. Therefore, $\text{gr}^{F_\bullet} H^\bullet(A, d) \simeq H^\bullet(\text{gr}^{F_\bullet} A, \text{gr}^{F_\bullet} d)$.*

Proof. We need to prove that $\text{im } d \cap F_n A \subseteq d(F_n A)$ for all $n \geq 0$. Consider a generic element $xs1 + yps1 + z + wp$ in $F_n A$ with $x, y, z, w \in \Lambda V \otimes S V$. Since the filtration $F_n A$ is induced by deg_3 , we can assume that $x \in F_{n-2} A$, $y \in F_{n-3} A$, $z \in F_n A$, and $w \in F_{n-1} A$. Suppose that $d(xs1 + yps1 + z + wp) \in F_{n-1} A$, then we have

$$d(xs1 + yps1 + z + wp) = \tilde{d}(x)ps1 + (-1)^{|x|}x(p - \omega) - (-1)^{|y|}yp\omega + \tilde{d}(z)p.$$

By looking at the third degree of the element in the right hand side, it follows that $\tilde{d}(x) \in F_{n-4} A$, $\tilde{d}(z) - \omega y \in F_{n-2} A$ and $\omega x \in F_{n-1} A$. Since on $\Lambda^\bullet V \otimes S^\bullet V$ the third grading coincides with the total degree (i.e. $\text{deg}_3(x) = |x|$), we can suppose x, y, z being homogeneous of total degree $n-2$, $n-3$, and n respectively. So we have

$$\tilde{d}(x) = 0, \quad \tilde{d}(z) - \omega y = 0, \quad \text{and} \quad \omega x = 0.$$

From $\omega x = 0$ and $\tilde{d}(x) = 0$ we deduce that $\text{deg}(x) > 0$ and $x = \tilde{d}(x')$ for some $x' \in F_{n-1} A$.

total degree $n - 1$. It follows that

$$d(xs1 + yps1 + z + wp) = d(x')$$

for $x' \in F_{n-1}A$ and so $\text{im } d \cap F_{n-1}A \subseteq d(F_{n-1}A)$. \square

From now on we will work in $\text{gr}_F A$ with the differential $\text{gr}_F d$. The only difference between d and $\text{gr}_F d$ is that $(\text{gr}_F d)(s1) = -\omega$. By an abuse of notation we denote the differential of $\text{gr}_F A = A$ by d .

For any bigraded vector space W we denote by $W[a, b]$ the same vector space with the bigraded shifted by (a, b) .

Lemma 4.3. *The kernel of the differential d is the direct sum of the following vector spaces:*

1. $(\ker \tilde{d} \cap \ker L_\omega)[0, 1]$,
2. $(\text{im } \tilde{d} \cap \ker L_\omega)[2, 2] \oplus \ker \tilde{d}[2, 1]$,
3. $\ker \tilde{d}$,
4. $\Lambda^\bullet V \otimes S^\bullet V[2, 0]$.

Proof. Consider a generic element $xs1 + yps1 + z + wp$ with $x, y, z, v \in \Lambda^\bullet V \otimes S^\bullet V$: its differential is

$$d(xs1 + yps1 + z + vp) = \tilde{d}(x)ps1 - (-1)^{|x|}x\omega - (-1)^{|y|}yp\omega + \tilde{d}(z)p. \quad (7)$$

Therefore $d(xs1 + yps1 + z + wp) = 0$ if and only if $\omega x = 0 = \tilde{d}(x)$ and $(-1)^{|y|}y\omega = \tilde{d}(z)$. The equations $\omega x = 0 = \tilde{d}(x)$ imply that $x \in (\ker \tilde{d} \cap \ker L_\omega)$. The condition $y\omega \in \text{im } \tilde{d}$ is equivalent to $\tilde{d}(y\omega) = \tilde{d}(y)\omega = 0$, thus

$$y \in \ker \tilde{d} \oplus (\text{im } \tilde{d} \cap \ker L_\omega)[0, 1].$$

Let $z' = z'(y)$ be a fixed element such that $\omega y = \tilde{d}(z')$: then z is of the form $z' + z''$ for some $z'' \in \ker \tilde{d}$ and v can be any element in $\Lambda^\bullet V \otimes S^\bullet V$. \square

Lemma 4.4. *The image of the differential d is the direct sum of the following vector spaces:*

1. 0 ,
2. $\text{im } \tilde{d}[2, 1]$,
3. $\omega \ker \tilde{d}$,
4. $\text{im } L_\omega[2, 0] + \text{im } \tilde{d}[2, 0]$.

Proof. Eq. (7) implies that the image of d has trivial intersection with the submodule $s1 \Lambda^\bullet V \otimes S^\bullet V$. Consider x such that $\tilde{d}(x) \neq 0$, then the element $(-1)^{|x|}x\omega + \tilde{d}(x)ps1$ gives the addendum $\text{im } \tilde{d}[2, 1]$. Now suppose $\tilde{d}(x) = 0$ and $x\omega \neq 0$, then $x\omega$ is in the image and generates a submodule isomorphic to $\omega \ker \tilde{d}$.

Finally,

$$\text{im } d \cap p \Lambda^\bullet V \otimes S^\bullet V \quad \text{coincides with} \quad \text{im } L_\omega[2, 0] + \text{im } \tilde{d}[2, 0]$$

(in general this is not a direct sum). \square

Let $\bar{\omega} \in \Lambda^1 V \otimes S^1 V$ be the unique $\mathfrak{sp}(2g)$ -invariant element such that $\tilde{d}(\bar{\omega}) = \omega$. The following lemma is an immediate consequence of Lemmas 4.3 and 4.4.

Lemma 4.5. *The cohomology $H^\bullet(A, \tilde{d})$ is generated by:*

1. $xs1$ for $x \in \ker \tilde{d} \cap \ker L_\omega$,
- 2.1. $ps1 + \bar{\omega}$,
- 2.2. $yps1 + x$ if $\tilde{d}y \in \ker \tilde{d} \cap \ker L_\omega$ and $(-1)^{|y|}y\omega = \tilde{d}x$,
3. y for $y \in \ker \tilde{d}/\omega \ker \tilde{d}$,
4. yp for $y \in \Lambda^\bullet V \otimes S^\bullet V / (\text{im } \tilde{d} + \text{im } L_\omega)$.

Lemma 4.6. *The cohomologies of $\ker L_\omega$, $\text{im } L_\omega$, and $\text{coker } L_\omega$ with respect to the differential \tilde{d} are given by:*

$$H^{0,0}(\text{coker } L_\omega) = \langle 1 \rangle, \quad (8)$$

$$H^{1,1}(\text{coker } L_\omega) = \langle \bar{\omega} \rangle, \quad (9)$$

$$H^{2,0}(\text{im } L_\omega) = \langle \omega \rangle, \quad (10)$$

$$H^{g,i}(\text{coker } L_\omega) = H^{g+1,i-1}(\text{im } L_\omega) = H^{g,i-2}(\ker L_\omega) \simeq V_{(i-2)\omega_1+\omega_g}, \quad (11)$$

$$H^{j,i}(\text{coker } L_\omega) = H^{j+1,i-1}(\text{im } L_\omega) = H^{j,i-2}(\ker L_\omega) = 0. \quad (12)$$

Proof. Consider the two short exact sequences

$$\begin{aligned} 0 \rightarrow \ker L_\omega \rightarrow \Lambda^\bullet V \otimes S^\bullet V \rightarrow \text{im } L_\omega[2, 0] \rightarrow 0 \\ 0 \rightarrow \text{im } L_\omega \rightarrow \Lambda^\bullet V \otimes S^\bullet V \rightarrow \text{coker } L_\omega \rightarrow 0. \end{aligned}$$

By Lemma 3.3

$$\begin{aligned} H^{j,i}(\text{coker } L_\omega) &\simeq H^{j+1,i-1}(\text{im } L_\omega) && \text{for } (j, i) \neq (0, 0) \\ H^{j,i}(\text{im } L_\omega) &\simeq H^{j-1,i-1}(\ker L_\omega) && \text{for } (j, i) \neq (2, 0). \end{aligned}$$

Eq. (8), (9) and (10) follow immediately from the long exact sequence in cohomology. Since $(\ker L_\omega)^{j,i} = 0$ for $j < g$ and $(\text{coker } L_\omega)^{j,i} = 0$ for $j > g$, we deduce eq. (12). The only representation that can appear in

$$H^{g,i}(\text{coker } L_\omega) \simeq H^{g,i-2}(\ker L_\omega)$$

is $V_{(i-2)\omega_1+\omega_g}$. It is easy to see that the subspace $V_{i\omega_1+\omega_g} \subset V_{\omega_g} \otimes S^i V$ is contained in $\ker L_\omega \cap \ker \tilde{d}$, but cannot lie in $\tilde{d}(\ker L_\omega)$ since $(\ker L_\omega)^{g-1,i+1} = 0$. This proves eq. (11). \square

Lemma 4.7. *The Hilbert–Poincaré series of $\ker \tilde{d} \cap \ker L_\omega$ is*

$$P_{\ker \tilde{d} \cap \ker L_\omega}(t, s) = t^{2g} + (1 + t^2s) \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1+\omega_j}] t^{2g-j+i} s^i.$$

Proof. Notice that $\ker \tilde{d} \cap \ker L_\omega = \ker(\tilde{d}|_{\ker L_\omega})$ and

$$\begin{aligned} P_{\ker L_\omega}(t, s) &= \sum_{\substack{1 \leq j < g \\ i \geq 0}} [V_{i\omega_1+\omega_j}] t^{2g-j+i} s^i (1 + s)(1 + t^2s) + \\ &\quad + t^{2g} + t^{2g}s + \sum_{i \geq 0} [V_{i\omega_1+\omega_g}] t^{g+i} s^i (1 + t^2s + t^2s^2), \\ P_{H(\ker L_\omega)}(t, s) &= \sum_{i \geq 0} [V_{i\omega_1+\omega_g}] t^{g+i} s^i. \end{aligned}$$

Using the formula $(s+1)P_{\ker(\tilde{d}|_{\ker L_\omega})}(t, s) = P_{\ker L_\omega}(t, s) + sP_{H(\ker L_\omega)}(t, s)$ we obtain the claimed equality. \square

Lemma 4.8. *The Hilbert–Poincaré series of $\ker \tilde{d}/\omega \ker \tilde{d}$ is*

$$P_{\ker \tilde{d}/\omega \ker \tilde{d}}(t, s) = 1 + (1 + t^2 s) \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1 + \omega_j}] t^{j+i} s^i.$$

Proof. Consider the exact sequence

$$0 \rightarrow \ker \tilde{d} \cap \ker L_\omega \rightarrow \ker \tilde{d} \xrightarrow{L_\omega} \ker \tilde{d} \rightarrow \ker \tilde{d} / \omega \ker \tilde{d} \rightarrow 0.$$

We have

$$P_{\ker \tilde{d}/\omega \ker \tilde{d}}(t, s) = (1 - t^2)P_{\ker \tilde{d}}(t, s) + t^2 P_{\ker \tilde{d} \cap \ker L_\omega}(t, s),$$

and from Lemma 4.7 we obtain the claimed equality. \square

Lemma 4.9. *The Hilbert–Poincaré series of $\Lambda^\bullet V \otimes S^\bullet V / \text{im } L_\omega + \text{im } \tilde{d}$ is*

$$P_{\Lambda V \otimes S V / \text{im } L_\omega + \text{im } \tilde{d}}(t, s) = (1 + t^2 s) \left(1 + s \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1 + \omega_j}] t^{j+i} s^i \right).$$

Proof. Let K be the quotient $\Lambda^\bullet V \otimes S^\bullet V / \text{im } L_\omega + \text{im } \tilde{d}$. Consider the exact sequence

$$0 \rightarrow \text{im } \tilde{d} \cap \text{im } L_\omega \rightarrow \text{im } \tilde{d} \oplus \text{im } L_\omega \rightarrow \Lambda^\bullet V \otimes S^\bullet V \rightarrow K \rightarrow 0$$

and observe that $\text{im } \tilde{d} \cap \text{im } L_\omega = \ker \tilde{d} \cap \text{im } L_\omega = \ker(\tilde{d}|_{\text{im } L_\omega})$. We compute the series $P_{\ker(\tilde{d}|_{\text{im } L_\omega})}$ using the formula

$$(1 + s)P_{\ker(\tilde{d}|_{\text{im } L_\omega})}(t, s) = P_{\text{im } L_\omega}(t, s) + sP_{H(\text{im } L_\omega)}(t, s)$$

applied to the bigraded complex $(\text{im } L_\omega, \tilde{d}|_{\text{im } L_\omega})$. Notice that

$$P_{\Lambda V \otimes S V}(t, s) = (1 + s)P_{\text{im } \tilde{d}} + 1$$

by Lemma 3.3, so we obtain

$$\begin{aligned} (1 + s)P_K &= (1 + s)(P_{\Lambda V \otimes S V} - P_{\text{im } \tilde{d}} - P_{\text{im } L_\omega} + P_{\ker(\tilde{d}|_{\text{im } L_\omega})}) \\ &= sP_{\Lambda V \otimes S V} + 1 - sP_{\text{im } L_\omega} + sP_{H(\text{im } L_\omega)} \\ &= sP_{\text{coker } L_\omega} + 1 + sP_{H(\text{im } L_\omega)}. \end{aligned}$$

The equalities

$$\begin{aligned} P_{H(\text{im } L_\omega)}(t, s) &= t^2 + t^2 s \sum_{i \geq 0} [V_{i\omega_1 + \omega_g}] t^{g+i} s^i, \\ P_{\text{coker } L_\omega} &= 1 + t^2 s + (1 + s)(1 + t^2 s) \sum_{\substack{1 \leq j < g \\ i \geq 0}} [V_{i\omega_1 + \omega_j}] t^{j+i} s^i + \\ &\quad + (1 + s + t^2 s^2) \sum_{i \geq 0} [V_{i\omega_1 + \omega_g}] t^{g+i} s^i \end{aligned}$$

complete the proof. \square

Theorem 4.10. *The Hilbert–Poincaré series $P_{H(A)}(t, s) \in R_g[[t, s]]$ of $H(A_g, \mathfrak{d})$ is*

$$(1 + t^2s)(1 + t^2 + t^{2g}s) + (1 + t^2s)^2 \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1 + \omega_j}] t^{j+i} s^i (1 + t^{2(g-j)}s).$$

Proof. By Lemma 4.5:

$$P_{H(A)} = (s + t^2s^2)P_{\ker \bar{\mathfrak{d}} \cap \ker L_\omega} + t^2s + P_{\ker \bar{\mathfrak{d}}/\omega \ker \bar{\mathfrak{d}}} + t^2P_{\Lambda V \otimes S V / \text{im } L_\omega + \text{im } \bar{\mathfrak{d}}}.$$

The computations of Lemmas 4.7 to 4.9 complete the proof. \square

Let $Q_g(t, s, u)$ be the following series in the Grothendieck ring of $\mathfrak{sp}(2g)$:

$$Q_g(t, s, u) \stackrel{\text{def}}{=} \sum_{i, j, n} [\text{gr}_{i+2j}^W H^{i+j}(\mathcal{C}_n(\Sigma_g))] t^i s^j u^n.$$

Theorem 4.11. *If $g > 0$, the polynomial $Q_g(t, s, u) \in R_g[[t, s, u]]$ is equal to*

$$\begin{aligned} Q_g(t, s, u) = & \frac{1}{1-u} \left((1 + t^2su^3)(1 + t^2u) + (1 + t^2su^2)t^{2g}su^{2(g+1)} + \right. \\ & \left. + (1 + t^2su^2)(1 + t^2su^3) \sum_{\substack{1 \leq j \leq g \\ i \geq 0}} [V_{i\omega_1 + \omega_j}] t^{j+i} s^i u^{j+2i} (1 + t^{2(g-j)}su^{2(g-j+1)}) \right). \end{aligned}$$

Proof. Use Lemma 4.2 and notice that $Q_K(t, s, u) = P_K(tu, su)$ for any sub-quotient K of $\Lambda^\bullet V \otimes S^\bullet V$, thus:

$$\begin{aligned} (1-u)Q_g = & su^2P_{\ker \bar{\mathfrak{d}} \cap \ker L_\omega} + t^2su^3 + t^2s^2u^4P_{\ker \bar{\mathfrak{d}} \cap \ker L_\omega} + \\ & + P_{\ker \bar{\mathfrak{d}}/\omega \ker \bar{\mathfrak{d}}} + t^2uP_{\Lambda V \otimes S V / \text{im } L_\omega + \text{im } \bar{\mathfrak{d}}}. \end{aligned} \quad (13)$$

Lemmas 4.7 to 4.9 complete the proof. \square

Theorem 4.11 and Lemma 3.10 give a formula for the mixed Hodge numbers and for the Betti numbers of $\mathcal{C}_n(\Sigma_g)$. We use this formula to give a different proof of the result in [DCK, Corollary 4.9] about the polynomial growth of the Betti numbers, and to extend it to the mixed Hodge numbers.

Corollary 4.12. *For $n > k \gg 0$, the weights h that appear in $H^k(\mathcal{C}_n(\Sigma_g))$ are in the range*

$$\min\{1, g-1\} \leq 3k-2h \leq \max\{g+2, 2g\} \text{ and } h \geq k.$$

Moreover, in this range the dimension of $\text{gr}_h^W H^k(\mathcal{C}_n(\Sigma_g))$ is polynomial in k of degree $2g-1$.

Proof. For $i \gg 0$ the dimension of $V_{i\omega_1 + \omega_j}$ is polynomial in i of degree $2g-1$ (see Lemma 3.10). Theorem 4.11 implies $\text{gr}_h^W H^k(\mathcal{C}_n(\Sigma_g)) = 0$ for $h < k$ or outside the range $\min\{1, g-1\} \leq 3k-2h \leq \max\{g+2, 2g\}$. For such h and k , $\text{gr}_h^W H^k(\mathcal{C}_n(\Sigma_g))$ is the direct sum of at most 8 irreducible representations $V_{i\omega_1 + \omega_j}$ for some i, j such that $i = k + \mathcal{O}(1)$. It follows that $\dim(\text{gr}_h^W H^k(\mathcal{C}_n(\Sigma_g))) \sim i^{2g-1} \sim k^{2g-1}$ has polynomial growth in k . Notice that, for any fixed k the weights h that appear in $H^k(\mathcal{C}_n(\Sigma_g))$ are at most $g+1$. The claim about the Betti numbers follows since they are the sum of $g+1$ positive numbers (i.e. $\dim(\text{gr}_h^W H^k(\mathcal{C}_n(\Sigma_g)))$) that grow polynomially in k of degree $2g-1$. \square

Notice that the growth of Betti numbers of the unordered configuration space of the torus ($g = 1$) is polynomial in n of degree $2k - 2$. Indeed in [P2] it is proven that $\beta_k(\mathcal{F}_n(\Sigma_1)) \sim \binom{n}{2k-2}$.

The same techniques can be applied to compute the invariants of configuration spaces of algebraic surfaces with zero irregularity.

Comparison with [DCK]

Our work is dual to the previous one by Drummond-Cole and Knudsen, we briefly compare the two articles. The Chevalley–Eilenberg complex $CE(\mathfrak{g}_{\Sigma_g})$ ([DCK, Definition 2.1]) is dual to our differential algebra B_g . Indeed, Lemma 2.8 and [FT1, Theorems 1.14] imply that $H^*(F_n B_g, d) \simeq H^*(\mathcal{C}_n(\Sigma_g))$. Dually, the main result of [K1, Theorem 1.1] asserts that for $M = \Sigma_g$

$$\bigoplus_{n \geq 0} H_*(\mathcal{C}_n(\Sigma_g)) = H_*(CE(\mathfrak{g}_{\Sigma_g})).$$

In [DCK, Lemma 5.1] is proven that a complex $\mathcal{Z} \oplus v\mathcal{Z}$ is a deformation retract of $CE(\mathfrak{g}_{\Sigma_g})$; the dual of $\mathcal{Z} \oplus v\mathcal{Z}$ is the algebra A_g and the analogous statement is given by Lemma 2.9. The submodule \mathcal{K}_g of [DCK] is dual to our module $P_{\Lambda V \otimes S V / \text{im } L_\omega + \text{im } \tilde{d}}$ (see Lemma 4.3). The last part of the proofs, in which the Poincaré polynomial of \mathcal{K}_g (resp. of $\Lambda V \otimes S V / \text{im } L_\omega + \text{im } \tilde{d}$) is computed, are essentially different: ours uses the representation theory of the symplectic group while theirs uses homotopy and auxiliary spaces $\mathcal{V}(g, n)$.

Drummond-Cole and Knudsen identified a stable range for $H_i(\mathcal{C}_n(\Sigma_g))$ for $n > i$ and it is known that $H_i(\mathcal{C}_n(\Sigma_g)) = 0$ for $n < i - 1$. Hence the unstable (non-trivial) range is for $n = i$ and $n = i - 1$. We observe the same phenomenon: in eq. (1) the total degree in t and s differs from the degree in u by at most one. More precisely, the unstable polynomial $\mathcal{P}_1(t)$ of [DCK, Theorem 4.2] is the Poincaré polynomial of the summand $t^2 u \Lambda V \otimes S V / \text{im } L_\omega + \text{im } \tilde{d}$ of eq. (13) (corresponding to $n = i - 1$). The other unstable polynomial $\mathcal{P}_0(t)$ of [DCK, Theorem 4.2] is the Poincaré polynomial of the summands

$$t^2 s u^3 + t^2 s^2 u^4 P_{\ker \tilde{d} \cap \ker L_\omega} + P_{\ker \tilde{d} / \omega \ker \tilde{d}} + t^2 u P_{\Lambda V \otimes S V / \text{im } L_\omega + \text{im } \tilde{d}}$$

of eq. (13) (corresponding to $n = i$).

Notice that their work provides a uniform treatment for any genus $g \geq 0$, but ours excludes the case $g = 0$ because we need $\omega \neq 0$.

Taking the dimension of both sides of our main eq. (1), we obtain a new formula for the Poincaré polynomial that is essentially different from the one given by cases in [DCK, Corollary 4.5, 4.6 and 4.7].

We do not have a direct proof that our formula coincides with the one given in [DCK], but both provide the Betti numbers of configuration spaces on Σ_g , and a computational check shows that they agree in a very big range. The code is available on request. The following example shows in few cases that our formula for Betti numbers agrees with the exceptional value given in [DCK, Corollary 4.5, 4.6, 4.7].

Example 4.13. Consider $g = 2$ i.e. the case of genus two surface, the first terms (with

respect the total degree in t and s) of eq. (1) are:

$$\frac{1}{1-u} \left(1 + [V_{\omega_1}]tu + t^2u + [V_{\omega_2}]t^2u^2 + t^2su^3 + [V_{2\omega_1}]t^2su^3 + [V_{\omega_2}]t^2su^4 + [V_{\omega_1}]t^3su^3 + [V_{\omega_1}]t^3su^4 + [V_{\omega_1+\omega_2}]t^3su^4 + [V_{\omega_1}]t^3su^5 + \dots \right). \quad (14)$$

The dimension of the \mathfrak{sp}_4 -representations involved can be computed using eq. (6) and they are:

$$\begin{aligned} \dim V_{\omega_1} &= 4, \\ \dim V_{\omega_2} &= 5, \\ \dim V_{2\omega_1} &= 10, \\ \dim V_{\omega_1+\omega_2} &= 16. \end{aligned}$$

We deduce $\dim H^i(\mathcal{C}_n(\Sigma_2))$ from eq. (14) by setting $s = t$ and by considering the dimension of the coefficient of $t^i u^n$:

- $\dim H^1(\mathcal{C}_n(\Sigma_2)) = 4$ for $n \geq 1$,
- $\dim H^2(\mathcal{C}_1(\Sigma_2)) = 1$ and $\dim H^2(\mathcal{C}_n(\Sigma_2)) = 6$ for $n \geq 2$,
- $\dim H^3(\mathcal{C}_2(\Sigma_2)) = 0$, $\dim H^3(\mathcal{C}_3(\Sigma_2)) = 11$, and $\dim H^3(\mathcal{C}_n(\Sigma_2)) = 16$ for $n \geq 4$,
- $\dim H^4(\mathcal{C}_3(\Sigma_2)) = 4$, $\dim H^4(\mathcal{C}_4(\Sigma_2)) = 24$, and $\dim H^4(\mathcal{C}_n(\Sigma_2)) = 28$ for $n \geq 5$.

These numbers coincide with the one provided in [DCK, Corollary 4.5, 4.6, 4.7].

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