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THE MARGOLIS HOMOLOGY OF THE COHOMOLOGY RESTRICTION FROM AN EXTRA-SPECIAL GROUP TO ITS MAXIMAL ELEMENTARY ABELIAN SUBGROUPS

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Abstract

Let p be an odd prime and let M_n be the extra-special p-group of order $p^{2n+1}(n \geqslant 1)$ and exponent p^2. We completely compute the mod p Margolis homology of the image $\mathrm{Im}\mathrm{Res}(A, M_n)$ for every maximal elementary abelian p-subgroup A of M_n.

Dedicated to Professor Nguyễn H. V. Hưng on the occasion of his 70th birthday.

1. Introduction

Let G be a p-group and let p be an odd prime. G is called an extra-special p-group if it satisfies the following condition

$$[G, G] = \Phi(G) = Z(G) = \mathbb{Z}/p.$$

Here, $[G, G], Z(G), \Phi(G) = G^p[G, G]$ denote the commutator subgroup, the center and the Frattini group of G respectively. (For details of extra-special p-groups see [1, §5.5].)

As well known, $V = G/Z(G)$ is a vector space of finite dimension over $\mathbb{Z}/p$ and the dimension of V is even. If $\dim V = 2$, then G is isomorphic to one of the following groups

$$E = \langle a, b \mid a^p = b^p = [a, b]^p = [a, [a, b]] = [b, [a, b]] = 1\rangle,$$
$$M = \langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p}\rangle.$$

Generally, if $\dim V = 2n, n \geqslant 1$, then G is isomorphic to one of the following central products

$$E_n = E \cdot \cdots \cdot E, \ (n \ \text{ times}),$$
$$M_n = E_{n-1} \cdot M.$$

Let A be a maximal elementary abelian p-subgroup of an extra-special p-group G.

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The inclusion $A \hookrightarrow G$ induces the restriction homomorphism

$$\mathrm{Res}(A,G)\colon H^*(G) \to H^*(A)^{W_G(A)},$$

where $W_G(A) = N_G(A)/C_G(A)$, the quotient of the normalizer by the centralizer of A in G. In [9], Minh proved that any maximal elementary abelian p-subgroup A of G is of rank $n+1$, i.e. $A \cong (\mathbb{Z}/p)^{n+1}$. Then, A is identified with the vector space $(\mathbb{Z}/p)^{n+1}$ and

$$W_G(A) \cong \left\{ \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & 0 & \ddots & & \\ * & * & \dots & * & 1 \end{pmatrix} \in GL(n+1, \mathbb{Z}/p) \right\}.$$

Let $x_1, \dots, x_{n+1} \in H^1(A)$ be the dual of $c_1, \dots, c_{n+1}$. Here, $(c_1, \dots, c_{n+1})$ is a basis of A. Let $y_i = \beta x_i$, where β denotes the Bockstein operator. As it is well known, we have

$$H^*(A) = E(x_1, \dots, x_{n+1}) \otimes P(y_1, \dots, y_{n+1}),$$

where $E(x_1, \dots, x_{n+1})$ (resp. $P(y_1, \dots, y_{n+1})$) denotes the exterior (resp. polynomial) algebra of generators $x_1, \dots, x_{n+1}$ (resp. $y_1, \dots, y_{n+1}$) of order 1 (resp. 2) over $\mathbb{Z}/p$.

In this article, following the method of [2], [3], [4], [5], and [6], we completely compute the mod p Margolis homology of the image $\mathrm{ImRes}(A, M_n)$ for every maximal elementary abelian p-subgroup A of the extra-special p-group M_n of order p^{2n+1} $(n \geqslant 1)$. The Margolis homology of the image $\mathrm{ImRes}(A,G)$ is, by definition, the homology of $\mathrm{ImRes}(A,G)$ with the differential to be the Milnor operation Q_j defined as follows.

Let $\mathcal{A}$ be the mod p Steenrod algebra. We denote P^i the i-th Steenrod power. Let Q_j be the Milnor operation (see [8]) of degree $2p^j - 1$ inductively defined for $j \geqslant 0$ as follows

$$Q_0 = \beta, Q_{j+1} = P^{p^j} Q_j - Q_j P^{p^j}.$$

Then Q_j is a differential, i.e. $Q_j^2 = 0$ for every j.

It is well known that, the Milnor operation is the first non-trivial differential, $Q_j = d_{2p^j-1}$, in the Atiyah-Hirzebruch spectral sequence for computing $K(j)^*(X)$, the Morava K-theory of a space X. (See [12, §2].) Particularly, the E_{2p^j}-page in the Atiyah-Hirzebruch spectral sequence for $K(j)^*(G)$ maps to $H_*(\mathrm{ImRes}(A,G); Q_j)$, where as usual $K(j)^*(G)$ denotes the $K(j)^*$-theory of the classifying space BG of the group G. This is why the Margolis homology of the image $\mathrm{ImRes}(A,G)$ is interesting.

In [11], Quillen showed a method to determine the cohomology of a finite group G is to apply the Quillen restriction from this cohomology to the cohomologies of all maximal elementary abelian subgroups of G. Following Quillen's strategy, the calculation should provide a way to better understand the restriction map

$$\mathrm{Res}_K\colon K(j)^*(G) \to \prod_{A \subset G} K(j)^*(A),$$

where the product runs over all maximal elementary abelian subgroups of G.

Let $V_s (1 \leqslant s \leqslant n+1)$ be the Mùi invariant (see [10]) defined by:

$$V_s = \prod_{\lambda_i \in \mathbb{Z}/p} (\lambda_1 y_1 + \dots + \lambda_{s-1} y_{s-1} + y_s).$$

Theorem 1.1 (Minh [9, Theorem 3.1, p. 360])**.** *Let p be an odd prime. Then, for every maximal elementary abelian p-subgroup A of the extra-special p-group M_n,*

$$\mathrm{ImRes}(A, M_n) = E(x_1, \ldots, x_n) \otimes P(y_1, \ldots, y_n, V_{n+1}).$$

Let $\mathrm{ImRes}(A, M_n)^{ex}$ be the ideal of $\mathrm{ImRes}(A, M_n)$ generated by $x_1, \ldots, x_n$.

Theorem 1.2. *Let p be an odd prime. For every maximal elementary abelian p-subgroup A of M_n and $j \geqslant 0$,*

$$\mathrm{Ker}\, Q_j \cap \mathrm{ImRes}(A, M_n)^{ex} = \mathrm{Im}\, Q_j \cap \mathrm{ImRes}(A, M_n)^{ex}.$$

The following is the main result of our article. This is a consequence of the preceding one.

Theorem 1.3. *Let p be an odd prime. For every maximal elementary abelian p-subgroup A of M_n, the j-th Margolis homology of* $\mathrm{ImRes}(A, M_n)$ *is given by*

$$H_*(\mathrm{ImRes}(A, M_n); Q_j) \cong \frac{P(y_1, \ldots, y_n, V_{n+1})}{(y_1^{p^j}, \ldots, y_n^{p^j})}.$$

The paper is divided into 2 sections. In Section 2, we compute the j-th Margolis homology of $\mathrm{ImRes}(A, M_n)$ for every maximal elementary abelian p-subgroup A of M_n.

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2. Proof of the main results

In [7], Nguyễn H. V. Hưng and Nguyễn Đ. Ngà computed the action of Q_j on V_i, that is Lemma 2.1(b), for $p = 2$.

Lemma 2.1. *Let p be an odd prime. For $1 \leqslant i \leqslant n+1$ and $j \geqslant 0$,*

(a) $Q_j(x_i) = y_i^{p^j}, Q_j(y_i) = 0$.

(b) $Q_j(V_i) = 0$.

Proof. (a) Since $\beta(x_i) = y_i$, $P^k(x_i) = 0$ for any k, and $\beta(y_i) = 0$, $P^k(y_i^{p^j}) = y_i^{p^{j+1}}$ for $k = p^j$, we get $Q_j(x_i) = y_i^{p^j}$, and $Q_j(y_i) = 0$.

(b) From part (a) of this lemma and the fact that

$$V_i = \prod_{\lambda_k \in \mathbb{F}_p} (\lambda_1 y_1 + \cdots + \lambda_{i-1} y_{i-1} + y_i)$$

(see [10]), we obtain $Q_j(V_i) = 0$. ∎

Definition 2.2. Let $s_1, \ldots, s_k$ be pairwise distinct, with $0 < s_1, \ldots, s_k < n+1$. The s-th partial derivation $\partial_s \colon \operatorname{ImRes}(A, M_n) \to \operatorname{ImRes}(A, M_n)$ is the morphism defined for $0 < s < n+1$ by

$$\partial_s(x_{s_1} \cdots x_{s_k} Z) = \begin{cases} (-1)^{i+1} x_{s_1} \cdots \hat{x_{s_i}} \cdots x_{s_k} y_s^{p^j} Z, & s = s_i, \\ 0, & \text{otherwise} \end{cases}$$

for $Z \in P(y_1, \ldots, y_n, V_{n+1})$. Here $\hat{x_{s_i}}$ means x_{s_i} being omitted

Obviously, $\operatorname{Im} \partial_s \subset y_s^{p^j} \operatorname{ImRes}(A, M_n)$.

Lemma 2.3. *Let $s_1, \ldots, s_k$ be pairwise distinct, with $0 < s_1, \ldots, s_k < n+1$. Then*

$$Q_j(x_{s_1} \cdots x_{s_k} Z) = \sum_{s=1}^{n} \partial_s(x_{s_1} \cdots x_{s_k} Z)$$

for $Z \in P(y_1, \ldots, y_n, V_{n+1})$.

Proof. It is straightforward from Lemma 2.1 and Definition 2.2. □

Definition 2.4. The integral on r-th direction

$$I_r \colon y_r^{p^j} \operatorname{ImRes}(A, M_n) \to \operatorname{ImRes}(A, M_n)$$

for $0 < r < n+1$, is the morphism given by:

$$I_r(x_{s_1} \cdots x_{s_k} y_r^{p^j} Z) = \begin{cases} x_r x_{s_1} \cdots x_{s_k} Z, & r \neq s_1, \ldots, s_k, \\ 0, & \text{otherwise,} \end{cases}$$

where $s_1, \ldots, s_k$ be pairwise distinct, with $0 < s_1, \ldots, s_k < n+1$, $0 \leqslant k$, as well as $Z \in P(y_1, \ldots, y_n, V_{n+1})$.

Lemma 2.5. *Let $s_1, \ldots, s_k$ be pairwise distinct, with $0 < s_1, \ldots, s_k < n+1$, while $0 < s < n+1$, and $Z \in P(y_1, \ldots, y_n, V_{n+1})$. Then*

(i) $I_s \partial_s(x_{s_1} \cdots x_{s_k} Z) = \begin{cases} x_{s_1} \cdots x_{s_k} Z, & s \in \{s_1, \ldots, s_k\}, \\ 0, & \textit{otherwise.} \end{cases}$

(ii) $\partial_s I_s(x_{s_1} \cdots x_{s_k} y_s^{p^j} Z) = \begin{cases} x_{s_1} \cdots x_{s_k} y_s^{p^j} Z, & s \notin \{s_1, \ldots, s_k\}, \\ 0, & \textit{otherwise.} \end{cases}$

Proof. (i) Using Definition 2.2 and Definition 2.4, we have

$$\begin{aligned} I_s \partial_s(x_{s_1} \cdots x_{s_k} Z) &= \begin{cases} (-1)^{i+1} I_s(x_{s_1} \cdots \hat{x_{s_i}} \cdots x_{s_k} y_s^{p^j} Z), & s = s_i, \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} (-1)^{i+1} x_{s_i} x_{s_1} \cdots \hat{x_{s_i}} \cdots x_{s_k} Z, & s = s_i, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Interchanging x_s with each of $x_{s_1}, \ldots, x_{s_{i-1}}$, we get

$$I_s \partial_s(x_{s_1} \cdots x_{s_k} Z) = \begin{cases} x_{s_1} \cdots x_{s_k} Z, & s \in \{s_1, \ldots, s_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

(ii)

$$\partial_s I_s(x_{s_1} \cdots x_{s_k} y_s^{p^j} Z) = \begin{cases} \partial_s(x_s x_{s_1} \cdots x_{s_k} Z), & s \notin \{s_1, \ldots, s_k\}, \\ 0, & \text{otherwise}, \end{cases}$$
$$= \begin{cases} x_{s_1} \cdots x_{s_k} y_s^{p^j} Z, & s \notin \{s_1, \ldots, s_k\}, \\ 0, & \text{otherwise}. \end{cases} \qquad \square$$

Note that $\operatorname{ImRes}(A, M_n)^{ex}$ is not a Q_j-submodule of $\operatorname{ImRes}(A, M_n)$, but

$$\operatorname{Im} Q_j \cap \operatorname{ImRes}(A, M_n)^{ex} \quad \text{and} \quad \operatorname{Ker} Q_j \cap \operatorname{ImRes}(A, M_n)^{ex}$$

are, since Q_j vanishes on these modules. The evident equality

$$\operatorname{ImRes}(A, M_n) = P(y_1, \ldots, y_n, V_{n+1}) \oplus \operatorname{ImRes}(A, M_n)^{ex}$$

implies

$$\operatorname{Ker} Q_j = P(y_1, \ldots, y_n, V_{n+1}) \oplus (\operatorname{Ker} Q_j \cap \operatorname{ImRes}(A, M_n)^{ex}),$$
$$\operatorname{Im} Q_j = (\operatorname{Im} Q_j \cap P(y_1, \ldots, y_n, V_{n+1})) \oplus (\operatorname{Im} Q_j \cap \operatorname{ImRes}(A, M_n)^{ex}).$$

Theorem 2.6. *Let p be an odd prime. For every maximal elementary abelian p-subgroup A of M_n and $j \geqslant 0$,*

$$\operatorname{Ker} Q_j \cap \operatorname{ImRes}(A, M_n)^{ex} = \operatorname{Im} Q_j \cap \operatorname{ImRes}(A, M_n)^{ex}.$$

Proof. Suppose $P \in \operatorname{ImRes}(A, M_n)^{ex}$ and $Q_j(P) = 0$. Let

$$P = \epsilon x_{t_1} \cdots x_{t_l} Z + \text{others},$$

where $0 \neq \epsilon \in \mathbb{F}_p$, $t_1, \ldots, t_l$ are pairwise distinct, $0 < t_1, \ldots, t_l < n+1, l > 0$, with $Z \in P(y_1, \ldots, y_n, V_{n+1})$; and "others" means a sum of terms of the form $\lambda x_{u_1} \cdots x_{u_m} S$, with $0 \neq \lambda \in \mathbb{F}_p$, $u_1, \ldots, u_m$ are pairwise distinct, $0 < u_1, \ldots, u_m < n+1, m > 0$, with $S \in P(y_1, \ldots, y_n, V_{n+1})$, $\{t_1, \ldots, t_l\} \neq \{u_1, \ldots, u_m\}$. A term of the form $x_{t_1} \cdots x_{t_l} Z$ satisfying the conditions listed above is referred to as a generalized monomial in $\operatorname{ImRes}(A, M_n)$. By Lemma 2.3,

$$Q_j(P) = Q_j(\epsilon x_{t_1} \cdots x_{t_l} Z) + Q_j \text{ (others)} = \sum_{s=1}^{n} \epsilon \partial_s(x_{t_1} \cdots x_{t_l} Z) + Q_j \text{ (others)},$$

where $\partial_s(x_{t_1} \cdots x_{t_l} Z) \in y_s^{p^j} \operatorname{ImRes}(A, M_n)$. Since $Q_j(P)$ vanishes, $Q_j(P)$ should satisfy the following property: For each term $\partial_s(x_{t_1} \cdots x_{t_l} Z)$ of it, there exist pairwise distinct indices $r_1, \ldots, r_u$ with $0 < r_1, \ldots, r_u \neq s < n+1$, and $0 \neq \epsilon_1, \ldots, \epsilon_u \in \mathbb{F}_p$ such that $\partial_s(x_{t_1} \cdots x_{t_l} Z)$ is not only the image by ∂_s of the generalized monomial $x_{t_1} \cdots x_{t_l} Z$ in P, but also the image by respectively $\partial_{r_1}, \ldots, \partial_{r_u}$ of some other generalized monomials in P with corresponding coefficients $\epsilon_1, \ldots, \epsilon_u$, where

$$\epsilon + \epsilon_1 + \cdots + \epsilon_u = 0 \in \mathbb{F}_p.$$

So there is a decomposition $Z = T + \text{others}, P = \epsilon x_{t_1} \cdots x_{t_l} T + \text{others}$, such that

$$\epsilon \partial_s(x_{t_1} \cdots x_{t_l}) T \in \operatorname{Im} \partial_s \cap \operatorname{Im} \partial_{r_1} \cap \cdots \cap \operatorname{Im} \partial_{r_u}.$$

Since $y_s^{p^j}$ and $y_r^{p^j}$ are coprime for $0 < r \neq s < n+1$, we get

$$\operatorname{Im} \partial_s \cap \operatorname{Im} \partial_{r_1} \cap \cdots \cap \operatorname{Im} \partial_{r_u} \subset y_s^{p^j} y_{r_1}^{p^j} \cdots y_{r_u}^{p^j} \operatorname{ImRes}(A, M_n).$$

That is $T = y_{r_1}^{p^j} \cdots y_{r_u}^{p^j} U$ for $U \in P(y_1, \dots, y_n, V_{n+1})$.

To kill $Q_j(\epsilon x_{t_1} \cdots x_{t_l} T)$ so that $Q_j(P) = 0$, the "polynomial" P should contain not only $\epsilon x_{t_1} \cdots x_{t_l} T$ but also the sum

$$\sum_{v=1}^{u} \epsilon_v I_{r_v} Q_j(\epsilon x_{t_1} \cdots x_{t_l}) T = \sum_{v=1}^{u} \epsilon_v \sum_{s=1}^{n} I_{r_v} \partial_s(\epsilon x_{t_1} \cdots x_{t_l}) T,$$

with $\epsilon + \epsilon_1 + \cdots + \epsilon_u = 0 \in \mathbb{F}_p$, where $r_1, \dots, r_u$ satisfy the 2 conditions:

(a) $T = y_{r_1}^{p^j} \cdots y_{r_u}^{p^j} U$ for $U \in P(y_1, \dots, y_n, V_{n+1})$.

(b) If $\partial_s(\epsilon x_{t_1} \cdots x_{t_l}) T \neq 0$, then

$$I_r \partial_s(\epsilon x_{t_1} \cdots x_{t_l}) T \neq \epsilon x_{t_1} \cdots x_{t_l} T,$$

for every s and $r \in \{r_1, \dots, r_u\}$. (See Definitions 2.2 and 2.4.) Hence the indices $r_1, \dots, r_u, t_1, \dots, t_l$ are pairwise distinct. Indeed, if $\partial_s(\epsilon x_{t_1} \cdots x_{t_l}) T \neq 0$, then s should be one of the indices $t_1, \dots, t_l$ (by Definition 2.2). The inequality $I_r \partial_s(\epsilon x_{t_1} \cdots x_{t_l}) T \neq \epsilon x_{t_1} \cdots x_{t_l} T$ for every s means that $r \neq t_1, \dots, t_l$ (by Lemma 2.5).

Suppose $r \notin \{t_1, \dots, t_l\}$. By Lemma 2.1 and Definition 2.4, we have

$$\begin{aligned} I_r Q_j(x_{t_1} \cdots x_{t_l}) T &= I_r \left\{ \sum_{i=1}^{l} (-1)^{i+1} x_{t_1} \cdots \hat{x_{t_i}} \cdots x_{t_l} y_{t_i}^{p^j} T \right\} \\ &= \sum_{i=1}^{l} (-1)^{i+1} x_r x_{t_1} \cdots \hat{x_{t_i}} \cdots x_{t_l} y_{t_i}^{p^j} \frac{T}{y_r^{p^j}}. \end{aligned}$$

The starting term $\epsilon x_{t_1} \cdots x_{t_l} T$ combines with $\sum_{v=1}^{u} \epsilon_v I_{r_v} Q_j(\epsilon x_{t_1} \cdots x_{t_l}) T$ to give

$$\begin{aligned} &(\epsilon id + \epsilon_1 I_{r_1} Q_j + \cdots + \epsilon_u I_{r_u} Q_j)(x_{t_1} \cdots x_{t_l} T) \\ &\quad = \sum_{v=1}^{u} -\epsilon_v (id - I_{r_v} Q_j)(x_{t_1} \cdots x_{t_l} T) \\ &\quad = \sum_{v=1}^{u} -\epsilon_v \left\{ x_{t_1} \cdots x_{t_l} y_{r_v}^{p^j} \frac{T}{y_{r_v}^{p^j}} - \left\{ \sum_{i=1}^{l} (-1)^{i+1} x_{r_v} x_{t_1} \cdots \hat{x_{t_i}} \cdots x_{t_l} y_{t_i}^{p^j} \frac{T}{y_{r_v}^{p^j}} \right\} \right\} \\ &\quad = \sum_{v=1}^{u} -\epsilon_v Q_j(x_{r_v} x_{t_1} \cdots x_{t_l}) \frac{T}{y_{r_v}^{p^j}}. \end{aligned}$$

The above argument shows that, if $P \in \operatorname{Ker} Q_j \cap \operatorname{ImRes}(A, M_n)^{ex}$ and

$$P = \epsilon x_{t_1} \cdots x_{t_l} Z + \text{others},$$

then P is associated with a decomposition

$$Z = \sum_{r_1, \dots, r_u} T_{r_1, \dots, r_u},$$

where $r_1, \dots, r_u, t_1, \dots, t_l$ are pairwise distinct and $T_{r_1, \dots, r_u} = y_{r_1}^{p^j} \cdots y_{r_u}^{p^j} U_{r_1, \dots, r_u}$ for $U_{r_1, \dots, r_u} \in P(y_1, \dots, y_n, V_{n+1})$, some $U_{r_1, \dots, r_u}$'s would be zero. Here the number of indices u may change when the terms runs in the sum. The association of P with the preceding decomposition of Z means that P contains not only the term $\epsilon x_{t_1} \cdots x_{t_l} Z$ but also the sum containing this term

$$\sum_{v=1}^{u} -\epsilon_v Q_j(x_{r_v} x_{t_1} \cdots x_{t_l}) \frac{T}{y_{r_v}^{p^j}},$$

where $\epsilon + \epsilon_1 + \cdots + \epsilon_u = 0 \in \mathbb{F}_p$.

We conclude that if $P \in \operatorname{Ker} Q_j \cap \mathrm{ImRes}(A, M_n)^{ex}$, then

$$P = P_1 + \text{others},$$

where $P_1 \in \operatorname{Im} Q_j \cap \mathrm{ImRes}(A, M_n)^{ex}$. If others $\neq 0$, then the above process is repeated for the other terms. Since $P \in \operatorname{Ker} Q_j \cap \mathrm{ImRes}(A, M_n)^{ex}$ is a sum of finitely many terms, this process should finish in finite steps. So

$$\operatorname{Ker} Q_j \cap \mathrm{ImRes}(A, M_n)^{ex} = \operatorname{Im} Q_j \cap \mathrm{ImRes}(A, M_n)^{ex}. \qquad \square$$

Theorem 2.7. *Let p be an odd prime. For every maximal elementary abelian p-subgroup A of M_n, the j-th Margolis homology of* $\mathrm{ImRes}(A, M_n)$ *is given by*

$$H_*(\mathrm{ImRes}(A, M_n); Q_j) \cong \frac{P(y_1, \ldots, y_n, V_{n+1})}{(y_1^{p^j}, \ldots, y_n^{p^j})}.$$

Proof. From Theorem 2.6, we get

$$\begin{aligned} H_*(\mathrm{ImRes}(A, M_n); Q_j) &:= \operatorname{Ker} Q_j / \operatorname{Im} Q_j \\ &\cong \frac{\operatorname{Ker} Q_j \cap P(y_1, \ldots, y_n, V_{n+1})}{\operatorname{Im} Q_j \cap P(y_1, \ldots, y_n, V_{n+1})} \oplus \frac{\operatorname{Ker} Q_j \cap \mathrm{ImRes}(A, M_n)^{ex}}{\operatorname{Im} Q_j \cap \mathrm{ImRes}(A, M_n)^{ex}} \\ &= \frac{P(y_1, \ldots, y_n, V_{n+1})}{\operatorname{Im} Q_j \cap P(y_1, \ldots, y_n, V_{n+1})}. \end{aligned}$$

From Lemma 2.1, $\operatorname{Im} Q_j \cap P(y_1, \ldots, y_n, V_{n+1})$ is spanned by

$$Q_j(x_1 Z), \ldots, Q_j(x_n Z)$$

for $Z \in P(y_1, \ldots, y_n, V_{n+1})$. In others words, as ideals of $P(y_1, \ldots, y_n, V_{n+1})$,

$$\operatorname{Im} Q_j \cap P(y_1, \ldots, y_n, V_{n+1}) = (Q_j(x_1), \ldots, Q_j(x_n)) P(y_1, \ldots, y_n, V_{n+1}).$$

Therefore

$$H_*(\mathrm{ImRes}(A, M_n); Q_j) \cong \frac{P(y_1, \ldots, y_n, V_{n+1})}{(y_1^{p^j}, \ldots, y_n^{p^j})}. \qquad \square$$

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