Open Problems

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Readers are invited to propose solutions to the following problems. Solutions should be sent to liu@math.ucla.edu, and posted on arXiv. The correct solutions will be announced, and tokens of appreciation will be presented to the solvers. — *the Editors*

Problem 2013001 (Geometry) *Proposed by Shing-Tung Yau, Harvard University, USA.*

Find a good way to conformally embed an elliptic curve E defined over $\mathbb{Q}$ on which the rational points are dense, into the Euclidean three space $\mathbb{R}^3$ so that the rational points in E have most uniform distribution measured by the induced metric.

There is a natural embedding of compact surfaces into three dimensional Euclidean space by minimizing the L^2-norm of its mean curvature. While such embeddings may not be unique, it is finite dimensional as it satisfies elliptic equations. If we choose an elliptic curve with dense rational points, can we exhibit them in such canonical embedding? Of course, we do not need to restrict ourselves to elliptic curve, but also to curves of higher genus.

Problem 2013002 (Arithmetic Geometry) *Proposed by Ching-Li Chai, Academia Sinica, Taipei/ University of Pennsylvania, USA.*

Let A be an abelian variety over a finite field $\kappa \supset \mathbb{F}_p$. Let K be a CM field with $[K : \mathbb{Q} = 2\dim(A)]$ such that the ring of integers $\mathscr{O}_K$ of K operates on A via a ring homomorphism $\alpha : \mathscr{O}_K \hookrightarrow \mathrm{End}(A/\kappa)$. Either prove that there exists a local domain R of generic characteristic 0 with residue field κ and a CM lifting $(\mathscr{A}, \beta : K \to \mathrm{End}(\mathscr{A}) \otimes_{\mathbb{Z}} \mathbb{Q})$ over R whose closed fiber is compatible with (A, α), or construct a counter-example.

Remark The above Strong CM lifting(sCML) is an open problem in a recent manuscript by Ching-Li Chai, Brian Conrad and Frans Oort on CM liftings. It is expected that sCML is *false.*

Problem 2013003 (Algebraic Geometry) *Proposed by Baohua Fu, Chinese Academy of Sciences, China.*

Classify K-equivalent birational maps $\phi : X \dashrightarrow X'$ between two smooth projective varieties which can be resolved by single smooth blowups, namely there exist smooth irreducible subvarieties $Z \subset X$ and $Z' \subset X'$ such that the induced birational map $\tilde{\phi} : \mathrm{Bl}_Z(X) \dashrightarrow \mathrm{Bl}_{Z'}(X')$ is an isomorphism.

Examples of such maps include Mukai flops and standard flops. Are there any more examples? Note that if one drops the condition "K-equivalent", then there are many more such examples.

Problem 2013004 (Arithmetic Geometry) *Proposed by Wei Zhang, Columbia University, USA.*

Consider two smooth projective curves C_1, C_2 defined over $\overline{\mathbb{Q}}$ with algebraic points $x_i, y_i \in C_i(\overline{\mathbb{Q}})$. Prove that the zero cycle $(x_1, y_1) + (x_2, y_2) - (x_1, y_2) - (x_2, y_1)$ is torsion in the Chow group of the surface $C_1 \times C_2$.

Remark This is a very special case of a conjecture of Beilinson and Bloch on the filtration of Chow group of varieties over number fields. If we replace $\overline{\mathbb{Q}}$ by $\mathbb{C}$ the same conclusion does not hold in general.

Problem 2013005 (PDEs) *Proposed by Xu-Jia Wang, Australia National University, Australia.*

The affine Bernstein problem: prove that a convex solution to the affine maximal surface equation in the entire Euclidean space $\mathbb{R}^n$ is a quadratic polynomial when the dimension $n < 10$; and find a smooth counterexample when the dimension $n \geq 10$.

Problem 2013006 (Anderson Model, 1958) *Proposed by Horng-Tzer Yau, Harvard University, USA.*

Consider the random Schrödinger operator

$$H = -\Delta + \lambda V_\omega$$

where V_ω is a random potential on $\mathbb{Z}^d$ and Δ is the discrete Laplacian operator. Prove that for λ small and $d \geq 3$, the eigenvectors in the bulk of the spectrum are extended. Consider also the finite volume version when $\mathbb{Z}^d$ is replaced by a finite lattice. Prove that the eigenvalue statistics in the bulk of the spectrum is asymptotically equal to the statistics of Gaussian orthogonal ensembles.

Problem 2013007 (Geometry) *Proposed by Feng Luo, Rutgers University, USA.*

Suppose $f : X \to Y$ is a diffeomorphism between two strictly convex smooth closed surfaces in the Euclidean 3-space $\mathbf{E}^3$ so that f preserves the second fundamental form. Is f induced by a rigid motion of $\mathbf{E}^3$?

Remark This is the smooth version of a conjecture of Stoker for compact convex polytopes in $\mathbf{E}^3$.

Problem 2013008 (Nonlinear Elliptic PDEs) *Proposed by Jun-cheng Wei, The Chinese University of Hong Kong, China.*

Consider the following Ginzburg-Landau equation

$$\Delta u + u - |u|^2 u = 0 \quad \text{in} \quad \mathbb{R}, \quad u : \mathbb{R}^2 \to \mathbb{R}^2. \tag{1}$$

It is known that there exist radially symmetric solutions of the following type

$$U_0 = S(|x|)e^{id\theta} \tag{2}$$

where d is called the degree of u.
Question: Is this solution U_0 non-degenerate? Namely, if we consider the linearized operator of (1) at U_0, the only kernels are $\frac{\partial U_0}{\partial x}$, $\frac{\partial U_0}{\partial y}$, iU_0.

Remark $d = 1$ has been proven.

Problem 2013009 (PDEs) *Proposed by Tong Yang, City University of Hong Kong, China.*

How to prove the existence of the time periodic solutions to the Boltzmann equation with time periodic external forcing and/or time periodic boundary condition?

Problem 2013010 (Fractal) *Proposed by Jiaxin Hu, Tsinghua University, China.*

How to obtain Li-Yau estimate of the heat kernel on the Sierpiski gasket?

Problem 2013011 (The spectrum, Differential Geometry) *Proposed by Qing-Ming Cheng, Fukuoka University, Japan.*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in an n-dimensional Euclidian space $\mathbb{R}^n$. Assume that Γ_i is the ith eigenvalue of the clamped plate problem:

$$\begin{cases} \Delta^2 u = \Gamma u & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplacian in $\mathbb{R}^n$ and ν denotes the outward unit normal. Find a method to prove the following universal inequality:

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \le \frac{8}{n} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)\Gamma_i.$$

Remark If one can prove that the above universal inequality holds, by making use of the asymptotic formula of eigenvalues for the clamped plate problem and the recursion formula of Q.-M. Cheng and H. C. Yang (Bounds on eigenvalues of Dirichlet Laplacian, Math. Ann., 337 (2007), 159–175), one can obtain an upper bound and a lower bound for eigenvalues, which is sharp in the sense of the order of k. For partial results, see Q.-M. Cheng and H. C. Yang (Trans. Amer. Math. Soc., 358 (2005), 2625–2635) and Q. L. Wang, C. Y. Xia (J. Funct. Anal. 245 (2007), 334–352).

Problem 2013012 (Metrics on Teichmüller spaces) *Proposed by Lizhen Ji, University of Michigan, USA.*

Let $T_{g,n}$ be the Teichmüller space of Riemann surfaces of genus g with n punctures. Then $T_{g,n}$ is diffeomorphic to $\mathbb{R}^{6g-6+2n}$, and the corresponding mapping class group $Mod_{g,n}$ acts properly on $T_{g,n}$, whose quotient is the moduli space $M_{g,n}$ of Riemann surfaces of genus g with n punctures. It is known that $T_{g,n}$ admits several natural complete Riemannian (and Finsler) metrics which are invariant under Mod_g, but they are not nonpositively curved. The negatively curved Weil-Petersson metric is not complete.

A folklore open problem is to show that $T_{g,n}$ does not admit any nonpositively curved complete Riemannian metric (or more generally complete CAT(0)-metric) which is invariant under $Mod_{g,n}$.

Remark An analogue of $T_{g,n}$ is the outer space X_n of marked metric graph (or tropical Teichmüller space) on which $Out(F_n)$ acts properly. ($Mod_{g,n}$ and $Out(F_n)$ are two of the most important groups in geometric group theory.)

A natural conjecture is that X_n does not admit any complete CAT(0)-metric which is invariant under $Out(F_n)$. See the reference: J. Brock, B. Farb, Curvature and rank of Teichmüller space. Amer. J. Math. 128 (2006), no. 1, 1–22.