Geometry of space, physics and analysis*

by Shing-Tung Yau†

> *We must admit with humility that, while number is purely a product of our minds, space has a reality outside our minds, so that we cannot completely prescribe its properties a priori.*
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> *C. F. Gauss (Letter to Bessel, 1830)*

The concept of space has gone through many stages of evolution. Many of them are related to the advancement of our understanding of nature.

From Ancient to Modern Geometry

In the days of the Greek geometers, plane and Euclidean geometry were reasonably adequate to describe most observations. The Greek scholars were convinced that the earth is round and were able to measure the diameter of the earth and its distance to the sun based on plane geometry. The Chinese scholars also measured the distance of the earth to the heaven using similar ideas.

But even then, Archimedes had already started to investigate infinite processes to understand geometric figures that could not be described by Euclidean geometry alone. Calculations of volumes of many different geometric figures started the investigation of integral calculus, as was also proposed by ancient Chinese mathematicians.

Not long afterwards, in the process of measuring the movement of celestial bodies, the ancient Indians and Chinese discovered the concept of differentiation of trigonometric functions.

Copernicus proposed that the sun, instead of the earth, is the center of the universe. The work of Kepler and Galileo that supported Copernicanism appeared a half century later. It was widely accepted only after Newton formulated the universal law of gravitation.

When Isaac Newton merged the development of differential and integral calculus together and verified Kepler's laws of motion of planets, mathematics reached a new plateau, largely because calculus gave us a most powerful tool to understand and calculate the geometry of curved objects that appear in nature.

Some historians said that Leibniz independently discovered calculus. Many of the mathematical notations we used today are in fact due to him. However, without Newton's important application of calculus to mechanics, few people would have paid much attention to the development of calculus.

Soon afterwards, many great mathematicians developed new disciplines of mathematics based on this new tool of calculus. The most notable was the development of differential geometry and the introduction of calculus of variations. Fermat, Euler and Monge are among the most notable contributors. Some of their works are still being investigated today. It is amazing that even then, calculus of variations was used extensively by differential geometers.

In his famous paper *On the Hypotheses Which Lie at the Foundation of Geometry (1854)*, Riemann wrote: "The theorems of geometry cannot be deduced from the general notion of magnitude alone, but only from those properties which distinguished space from other conceivable entities, and these properties can only be found experimentally … This takes us into the realm of another science – physics."

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The discovery of the concept of the intrinsic Gauss curvature is certainly a landmark for the modern development of differential geometry. It should have been a major inspiration for the birth of Riemannian geometry.

In the beginning of his famous address in Göttingen in 1854, Riemann says that the foundations had already been laid by Gauss' investigations of curved surfaces. A couple of years later, when he introduced the concept of Riemannian curvature tensor in an essay for a competition given by the French Academy of Sciences, he mentioned that the calculation was similar to the one given by Gauss.

It should be noted that Riemann was interested in his geometry because of his strong desire to find an adequate concept of space to understand the laws of nature. He was interested in the laws of electromagnetic field. In his collected works published posthumously in 1876, he discusses electrodynamics and apparently anticipates some of Maxwell's work. Riemann had originally submitted the paper in 1858, but then withdrew it because of an error in a calculation. Maxwell independently published his equations between 1861 and 1862.

In his famous address, Riemann asked how to come up with a concept of geometry that can incorporate the space of the enormously large and the space of the extremely small. He even raised the possibility that a discrete space can play an important role.

It is a curiosity whether Riemann would have discovered general relativity if he had lived forty years longer. (He died on July 20, 1866, less than forty years old.) In any case, Einstein's theory of general relativity was based on Riemannian geometry, as was introduced to him by Grossmann in 1910. General relativity is one of the most spectacular achievement of physics in the twentieth century. I shall comment later on how the theory of general relativity influenced the modern development of differential geometry.

At the same time when Riemann introduced his geometry, he was developing the geometric aspect of complex analysis and in particular he studied the concept of a Riemann surface – one of the most fundamental tool in modern mathematics and physics. Not only did he give deep insights into complex analysis from the point of view of geometry, but also he initiated the full development of modern topology. He studied the connectivity of a space and introduced several measurements of high connectivity. They were later called Betti numbers of those spaces.

The concept of conformal geometry and conformal invariance have been major tools for string theory. And its higher dimensional generalizations to complex geometry have been fundamental in understanding complex manifolds.

Geometry and Topology of Manifolds

While Newton proposed the space to be static, Einstein demanded the space to be dynamic, integrating special relativity with Newtonian gravity. The idea of Minkowski to interpret special relativity using geometry of four-dimensional spacetime was essential for Einstein's development of general relativity. This led Einstein and Grossmann to use the work of Riemann and Ricci.

When Einstein developed the theory of general relativity, he consulted, besides Grossmann, Levi-Civita and David Hilbert. It was clear that Hilbert had given tremendous help to Einstein and Hilbert had even derived the action for general relativity before Einstein.

After the theory of general relativity was accomplished, a group of scientists were interested in generalizing the theory to incorporate other fields. Levi-Civita was the one who developed the concept of differentiation on a curved manifold, which he called the theory of connections. In order for the connection to behave like differentiation in Euclidean space, he demands the connection to be symmetric. If parallel transportation preserves the metric, the connection would be uniquely determined by the metric. Such a connection is called Levi-Civita connection.

Levi-Civita observed that if the connection is not assumed to be symmetric, the skew-symmetric part of the connection can be used as a free variable to be a new field to be considered in physics. This inspired both Élie Cartan and Hermann Weyl to develop the theory much further.

Weyl studied connections on bundles and hence initiated the fundamental concept of gauge theory with abelian gauge groups, and this was later generalized by Yang and Mills in 1954 to include nonabelian gauge groups. But it was Weyl who understood Maxwell's equations as a gauge theory and hence gave topological meaning to the theory of electricity and magnetism. Later in 1970, 't Hooft, using ideas of DeWitt and Faddeev-Popov, succeeded in quantizing Yang-Mills gauge theory. This then led to the current fundamental understanding of particle physics based on nonabelian gauge theory in which the Standard Model was built.

All these spectacular works in modern physics are related to parallel developments from the strong desire of topologists and geometers to understand the structure of manifolds. The basic idea is to simplify complicated spaces by cutting them into simple building blocks.

The simplest building blocks are cells, contractible spaces or handle bodies. In the process of attaching simple spaces together, we often need to twist before gluing them together. Hence, this led to the notion of a fiber space.

The concept of a fiber bundle was introduced in early days, first by Whitney in 1935 for sphere bundles over any spaces. (Seifert considered 3-manifolds which can be written as circle bundles in 1932, but the point of view is different.) The theory was studied by Stiefel, Hopf, Ehresmann, Pontryagin, Chern, Steenrod, Leray and Serre.

The characteristic classes introduced by the founders of the theory of fiber bundles had deep influences in modern geometry and physics. These are cohomology classes with integer values. After Chern introduced his Chern classes, A. Weil made a remark that Chern classes should be used as a tool to quantize fundamental physical quantities. This is indeed the case and shows how remarkable the connection is between geometry and physics.

When topologists and algebraic geometers were working hard to understand homology and cohomology of manifolds, Poincaré, E. Cartan, Weyl, de Rham, Hodge, Kodaira and Morse realized the importance of analysis on manifolds.

Analysis is a subject that studies how functions and their derivatives behave. Most laws of physics are expressed in terms of differential equations. One of the goals of analysis is to describe solutions of these equations, both qualitatively and quantitatively.

Differential geometers soon found that conversely we can use the solutions of the equations to understand the geometry of the underlying manifolds. In the case of Morse, he can use any reasonable good function to study the topology of a manifold. Take an example, look at the donut as a two-dimensional surface. A natural function is the height of each point above the plane.

The gradient of this function on the surface defines a vector field tangential to the surface. There are some special points on the surface: the points where the gradient vector field vanish. In the present case, there are four such points shown in the picture. They are called critical points.

There is a neighborhood at each critical point where the gradient vector field has well defined directions of pointing upward or downward. Such neighborhoods allow Morse to decompose the manifold into pieces according to the distribution of these critical points. He also showed how to build the manifold by gluing these pieces together.

The work of Morse had deep influence on the works of Smale and Bott on handle body decomposition of manifolds and homotopy groups of Lie groups. It also led to the work of Witten in quantum field theory.

An entity called differential form is used very frequently in modern geometry. They can be considered as dual objects of vector fields. A very important operation in the theory of differential form is called exterior differentiation. This is a concept which generalizes the divergence and curl in multivariable calculus.

In the early twentieth century, Poincaré, Cartan and de Rham uncovered the topological meaning of the differential forms and their exterior derivatives, which inspired Hodge and Kodaira to develop the theory of harmonic forms; some of their ideas can be traced back to the classical theory of fluid dynamics and electricity and magnetism.

Hodge theory was soon generalized to handle twisted differential forms, namely forms with coefficients in a bundle. A fundamental breakthrough was the vanishing theorem of Bochner-Kodaira based on the positivity of the bundle which can be described by the curvature of the bundle. The vanishing theorem allows us to compute the dimension of solutions of linear differential operators and prove the existence of power series solutions (the unobstructedness theorem). The Riemann-Roch formula and the Atiyah-Singer index theorem play fundamental roles in such calculations.

As a result, Hodge theory is also an important tool to study the moduli space of geometric structures. The first important geometric structure was due to Riemann who studied conformal structures on two dimensional surfaces where he counted the dimension of the moduli space to be $6g-6$. Here g is the number of handles on the two dimensional surface and the dimension of the moduli space can be interpreted as the degree of freedom to change the conformal structures.

Riemann's famous work was extended by Poincaré, Schottky, Teichmüller and others in the early 20th century.

After Riemann, a very important concept was introduced by Felix Klein. His famous Erlangen Program proposed to study geometry according to the group of symmetries. Many important branches of geometry were studied. For example, affine geometry studies quantities on a surface that are invariant when we move the surface by a linear transformation in the ambient Euclidean space. In contrast to Euclidean geometry, the length is no more invariant in affine geometry. Similarly, projective differential geometry studies quantities invariant under projective transformations.

Sophus Lie (who is a close friend of Klein) created the theory of continuous groups and observed that they could be better understood through their linearized version – Lie algebra.

While Klein used global symmetries to create and classify new types of geometries, the concept of internal symmetry took over in the 20th century. Classification of geometries was dictated by the holonomy group associated to the connections. Holonomy group is a group that describes the behavior of parallel transports of vectors along closed loops in the manifold that is governed by the connection.

The Italian algebraic geometers, most of them were followers of Riemann, had studied the subject of classification of algebraic varieties. Many of the concepts they introduced were later understood to be expressible in terms of the cohomology of forms with coefficients in bundles. Kähler geometry and the theory of complex structures became popular after the works of Hirzebruch, Chern, Kodaira and Spencer.

Note that while algebraic varieties are the zero locus of a set of polynomials, complex manifolds are more general and may be considered as a generalization of conformal structures on surfaces to higher dimensional spaces modeled after complex Euclidean spaces.

Kähler geometry was introduced by Erich Kähler. It includes algebraic manifolds as special case. It gives the most coherent way to put complex structure and Riemannian geometry together. An important feature of Kähler manifold is that it has internal symmetry associated to the unitary group. A very fundamental structure in Kähler geometry was introduced by Hodge. He observed that the complex coordinates give rise in a natural way to write differential k-forms as a sum of forms with p holomorphic part and q anti-holomorphic part where $k = p+q$. Assuming the manifold is Kähler, he was able to decompose the de Rham cohomology into a sum of cohomologies of (p,q)-type. This decomposition gives rise to the Hodge structure in the cohomology of Kähler manifolds. This is remarkable as it imposes beautiful constraints on the topology of such manifolds.

The way that the cohomology of a manifold is split into a sum of (p,q)-types gives rise to the concept of Hodge structure. It is believed that this structure determines the complex structure of the complex manifold. It provides therefore a fundamental tool to study the moduli space of Kähler manifolds.

In the other direction, Hodge was very interested in knowing which homology class (over real number) can be represented by algebraic sums of cycles defined by algebraic subvarieties. The famous conjecture of Hodge says that the only condition is that the homology class is (p,p)-type.

The Hodge conjecture remained to be one of the most important questions relating topology, geometry to algebraic and arithmetic geometry.

The only major contributions to the Hodge conjecture so far are the case of $p = 1$ due to Lefschetz and the theorem of Chern that Chern classes of any holomorphic bundle over a compact algebraic manifold are represented by algebraic cycles.

Chern's theorem gives a strong link between the Hodge conjecture and the subject of K-theory, which is about the space of vector bundles over a manifold. It provides important information on the manifold.

The strong desire to unify several fields of physics with gravity brought about many important advances in both physics and geometry. A very important development was due to Kaluza and Klein. It was proposed in 1919 by Kaluza, who was a mathematician. They found that if we reduce a five-dimensional vacuum Einstein equation by a circle of isometry, one finds a unification of Einstein gravity with Maxwell equation.

Einstein was excited by this beautiful theory. It was abandoned later as it contains a scalar field which was not observed in nature. The Kaluza-Klein idea was introduced again several times later in physics, with its most notable accomplishment being the introduction of compactification in string theory, where the circle is replaced by the Calabi-Yau manifold.

Calabi-Yau Manifolds

Calabi-Yau manifolds are those Kähler manifolds whose Ricci curvature is identically zero. In particular, they are Riemannian manifolds that satisfy the vacuum Einstein equation. The important point here is that it has a parallel spinor which makes the Kaluza-Klein model "supersymmetric."

The spinor is a concept introduced by E. Cartan, but was used by physicists to describe particles. It continues to play a mysterious but fundamental role in geometry.

Supersymmetry is a concept introduced by particle physicists who believe that there is a correspondence between Bosons and Fermions. In order to define such an operation on a curved spacetime, one needs to use a spinor to make the transformation. But in order for the transformation to be consistent, the spinor has to be constant. (This means that the spinor is independent of the choice of the path if we parallel transport it along different paths.)

String theory is built on the idea that particles are strings vibrating in spacetime that is supersymmetric. In order for the theory to be consistent, the spacetime has to be ten-dimensional. The idea of Candelas, Horowitz, Strominger, and Witten [3] was to consider the ten-dimensional spacetime to be the product of the standard flat Minkowski spacetime and another six-dimensional compact space.

In order for the ten-dimensional spacetime to be supersymmetric and vacuum, this six-dimensional space must be a Calabi-Yau, which are Kähler manifolds with vanishing first Chern class.

Particles are described by the spectrum of Dirac operators over the ten-dimensional manifold. The eigenfucntions come from the product of eigenfucntions. From here, one derives that when the Calabi-Yau space is small, the light particles will be created by the harmonic spinors on the Calabi-Yau manifolds. They can be described by harmonic forms and hence topology of the Calabi-Yau manifolds.

There are other models of string vacua, and other types of string theories. A most notable one is the heterotic string theory where Hermitian Yang-Mills equation play an important role. We shall focus on the development of Calabi-Yau space and Hermitian Yang-Mills equations. Both of them are building blocks of string theory – Susskind referred to the Calabi-Yau manifold as the DNA of the multiverse.

Let me now describe how to construct those structures that physicists want to use:

The basic question in geometry is to build and classify geometric structures over a manifold. In the case of Calabi-Yau manifolds, we fix the topology of a manifold and want to know whether it supports a complex structure which admits a Kähler metric. Then we would like to know whether such a Kähler metric can be deformed to one with zero Ricci curvature. Hence, there are two important steps we need to investigate.

The first step is to find out which manifolds can be made into complex manifolds. This is a very tough question. Geometers try an easier problem. If we linearize the concept of complex structure, we find a linear operator J with $J^2 = -1$ which acts as endomorphism on the tangent space of each point of the even dimensional manifold. We call it an almost complex structure.

Any complex manifold admits an almost complex structure. So it is natural to ask which smooth manifolds admit almost complex structures. This turns out to be manageable and is reducible to a question in homotopy theory in algebraic topology. However, whether an almost complex manifold admits complex structure is a very difficult problem.

We can derive from the Hirzebruch-Riemann-Roch formula that for four-dimensional manifolds, there are many almost complex manifolds that cannot admit an integrable complex structure. However, the formula is not powerful enough to give topological constraints for manifolds with dimension greater than 4.

Many years ago, I conjectured that there is in fact no obstructions in higher dimensions. And there is no contradiction to my conjecture so far. Once we know the existence of a complex structure, it is a deep question to know whether it also admits a Kähler structure.

It is well known that the existence of Kähler metrics gives a lot of constraints in the homology (as was provided by Hodge theory) and the homotopy type of the complex manifold (the rational homotopy type is determined by the minimal model of the de Rham complex). However, there is no known criterion for a complex manifold to admit a Kähler structure.

At one point, there was a conjecture that every Kähler manifold can be deformed to an algebraic manifold. But this was proven to be wrong by Voisin [9].

A very important feature of Kähler manifold is that the differential forms are very special relative to the fundamental $\bar{\partial}$ operator. There is the $\partial\bar{\partial}$-Lemma which plays a very powerful tool: *Any (p,p)-form that is exact can be written as $\partial\bar{\partial}$ of some $(p-1,p-1)$-form.*

The $\partial\bar{\partial}$-Lemma is equivalent to that Fröhlicher spectral sequence degenerates at E_1 and a Hodge structure exists on the cohomology groups. It was used by Deligne, Griffiths, Morgan and Sullivan [4] to give constraints on homotopy type of Kähler manifolds.

I propose the following conjecture: *Every complex manifold that satisfies the $\partial\bar{\partial}$-Lemma can be deformed to a complex variety that is birational to a Kähler manifold.*

Hopefully, this may get us closer to determining whether a complex manifold can admit a Kähler structure or not.

Calabi Conjecture

Once we know the manifold is Kähler, we may ask when will the manifold admit some canonical Kähler metric. The most natural canonical metric is the Kähler-Einstein metric. A necessary condition for its existence is the first Chern class must either be positive, zero or negative. Calabi-Yau manifolds are those with zero first Chern class. This was the center of a conjecture made by Calabi [2] forty years ago.

Calabi actually posted the problem in a more general form: *Given a smooth volume element on a Kähler manifold, can we deform a Kähler metric within its Kähler class so that, up to a new constant, the new Kähler metric has the same volume element as the given one.*

This problem can be reduced to a complex Monge-Ampère equation:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) = \exp(F)\det(g_{i\bar{j}})$$

Recently I looked up the paper by Kähler in 1932. It was the first paper in Kähler geometry. It turns out that this equation was already constructed by him in the process of understanding Kähler-Einstein metrics.

The simplicity of this equation comes from the fact that the Ricci tensor for a Kähler metric can be expressed as a complex Hessian of the logarithm of the volume element of the Kähler metric. Since the Kähler metric, when we fix the cohomology class that the metric represents, is determined by the complex Hessian of a real valued function, the Einstein equation of Ricci flatness can be expressed in terms of some form of complex Monge-Ampère equation mentioned above.

Calabi told me that when he wrote this equation, he thought that the existence is straightforward. He showed to André Weil who expressed great interest in the canonical Ricci-flat metric. However, Weil warned Calabi that the theory of nonlinear elliptic equations may not be mature enough to settle this question. Despite that, Calabi was able to prove uniqueness of the solution if it exists.

Indeed, when I was a graduate student in 1970, I was extremely excited by the potential importance of this equation. However, I found that no literature was devoted to the study of fully nonlinear elliptic equations on manifolds with dimension greater than three. I considered my major task is to build a foundation to solve this equation.

As a first step, I spent quite a bit of my efforts on understanding the real Monge-Ampère equation with my friend S.-Y. Cheng. I learned a lot about the theory of nonlinear elliptic equations from Charles Morrey during my graduate study, and later from Leon Simon and Rick Schoen at Stanford.

When a complex Monge-Ampère equation is invariant under the group of translation on the imaginary components, it becomes a real Monge-Ampère equation. Hence, my first step was to understand the real Monge-Ampère equation. Such equation appeared in classical differential geometry when one wants to solve the famous Minkowski problem. The two-dimensional Minkowski problem was solved by Pogorelov and Nirenberg in the early 1950s, completing the work of Hans Lewy in the real analytic case. The n-dimensional version was solved in the early 1970s independently by Pogorelov and Cheng-Yau.

While the methods of real Monge-Ampère equations inspired the way to analyze the complex Monge-Ampère equation, it takes many nontrivial steps to find third order estimate for the complex Monge-Ampère equation. And it takes much more nontrivial efforts to find second and zero order estimates. Once the Calabi conjecture was proved, we have an effective way, based on algebraic geometry, to parametrize all Kähler Ricci-flat metrics on a compact Kähler manifold.

I published this paper [10] entitled "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I." Many people asked me where is II. Well, II is supposed to cover the case when the manifold is singular and noncompact. I published them with S.-Y. Cheng and Tian in later years. (I gave a survey on those works in my talk [11] in the ICM 1978, Helsinki.) While a great deal is now known in classifying complete noncompact Kähler-Einstein manifolds, the task has not been accomplished. I believe a complete classification would be important for both algebraic geometry and math physics.

Mirror Symmetry in String Theory

The theory of Calabi-Yau manifolds has been useful in algebraic geometry: giving structure theorem for algebraic manifolds with zero first Chern class, positivity of second chern form, Torelli type theorems, etc.

But the impact is much greater after the arrival of the theory of superstring in physics. Some of the ideas initiated by physical considerations are rather spectacular. The concept of mirror symmetry introduced in the late 1980s was totally unexpected. Physicists also introduced the concept of quantum cohomology, Gromov-Witten invariants and studied their properties. Candelas et al. were able to calculate such invariants. It opened a new horizon for mathematicians to study such manifolds.

We started to realize that in order to study such manifolds, it is more effective to study a pair of them. Based on the duality between two different string theories on these two manifolds, one can study the properties of one manifold by looking at the dual theory of another manifold. This is very similar to the uncertainty principle where position and momentum are dual to each other.

Kontsevich [7] was the first one to interpret mirror symmetry in terms of the theory of derived category (called homological mirror symmetry). At the same time, Fukaya also developed a theory in symplectic geometry which may be recognized as a theory dual to the works of Kontsevich. The geometric explanation based on brane theory was introduced by Strominger-Yau-Zaslow [8] in 1996.

SYZ interpreted mirror symmetry in the following way: A Calabi-Yau manifold (that has a mirror partner) should have a singular foliation by a special Lagrangian torus whose quotient space is a real three-dimensional manifold where outside a codimension-two set, the leaves are nonsingular. The mirror manifold is supposed to be constructed by replacing each real torus by its dual. Special Lagrangian tori are those submanifolds where the restriction of the Kähler form is zero and the restriction of the holomorphic three-form is constant.

The picture described in this way has been accurate so far. And it has inspired many important works such as the algebraic approach by Gross-Seibert and Conan Leung, Chan and Lau. Despite its successes, there is no concrete example of the SYZ picture except in the case of the K3 surface. In that case, Greene-Shapere-Vafa-Yau [6] wrote down an explicit Ricci-flat Kähler metric on K3 surface which has an elliptic fibration over a sphere, but the metric is singular along singular fibers.

It is argued that a nonsingular Calabi-Yau metric is obtained by perturbing this GSVY metric by using information from open strings (holomorphic disks whose boundary are topologically nontrivial curves on the Lagrangian torus). This approach was attempted by Gross-Wilson. But unfortunately, their argument depends on my proof of the Calabi conjecture and the perturbation is not refined enough to read out the instanton corrections.

Up to now, an explicit perturbation series of Calabi-Yau metric from the GSVY metric with instanton sum has not been found, despite claims by many authors that it can be done.

For three-dimensional Calabi-Yau manifolds, it will be highly desirable to find a similar GSVY metric and a perturbation series of CY metric based on this GSVY metric. The theory of Calabi-Yau manifolds has been extremely rich and there are still much to be learned about them. But in the string equations, there are also fields such as Yang-Mills field and scalar fields. They are all interesting to be studied in connection with geometry.

Yang-Mills Field

As was mentioned earlier, gauge theory played a fundamental role in both modern geometry and particle physics. For four-dimensional manifolds, there is a concept of self-dual or anti-self-dual connections first studied by particle physicists. They were finally classified by Atiyah-Drinfeld-Hitchin-Manin [1]. They are important for four-dimensional topology and particle physics.

They correspond to Yang-Mills fields that saturate some topological bounds and are sometimes called BPS states. Because of this, they are considered to be stable configurations.

One of the most natural generalization of this BPS state to higher-dimensional manifolds are Hermitian Yang-Mills connections over Kähler manifolds. These connections can be proved to be supersymmetric. I proposed to Witten to use it to build models for heterotic string theory.

In fact, Strominger observed that the ten-dimensional spacetime can be a warped product of a compact six-dimensional manifold with the Minkowski spacetime. The six-dimensional manifold is required to admit a complex structure and a Hermitian Yang-Mills connection coupled with some Hermitian metric.

The Hermitian metric is not necessarily Kähler in general, but satisfies some anomaly equations linked with the Hermitian Yang-Mills field. In this case, there is a scalar field coming from the warped factor and is related to the flux of the theory. Fu and I [5] were able to find nontrivial solutions for this complicated system of equations. But a lot more study is still needed.

Calabi-Yau manifolds and Hermitian Yang-Mills equations provide candidates to build ground states for string theory. The known number of such configurations are huge, but so far only a finite number of moduli spaces of them have been found. Thirty years ago, I proposed that they are indeed finite [12, p. 621]. But this question has not been settled.

Parallel to string theory, there is M-theory which requires the spacetime to be 11-dimensional and so the compactified manifold is seven-dimensional with holonomy group equal to G_2. While special construction modeled after the Kummer construction was done by Joyce, it is far from enough to understand the moduli space of such manifolds. This may be a key problem to apply geometry to study M-theory. We really have no idea how large is the class of G_2 manifolds and the duality theory of these manifolds are much less understood, especially those with isolated conical singularities.

So far, we have been talking about spacetime with dimension greater than four. The classical four-dimensional spacetime is of course equally rich. However, the tools to study classical general relativity are largely based on geometric analysis and less algebraic.

Singularities in General Relativity

One major difficulty is the natural appearance of singularities in classical GR. The most notable singularity appears in the classical Schwarzschild and Kerr solutions. When Penrose and Hawking proved that such singularities cannot be perturbed away, we have to accept singularities as part of the fate of GR unless quantum effects can cure such problems. It is a fundamental question in classical GR on classifying the structure of singularities of spacetime that satisfy the dominant energy condition. If naked singularity is allowed, we shall have difficulty to predict the future based on the original initial data set.

The subject of general relativity has been recognized to be correct by experiments, largely based on several exact solutions of the Einstein equation.

On the other hand, it is amazing that global solutions that describe the dynamical nature of spacetime are still not well understood. This is indeed troublesome as the most important test on general relativity is based on finding gravitational radiation.

Many physicists think that classical relativity is well understood and the only interesting theory is quantum gravity. This is really far from the case. The dynamics governed by Einstein equation seems to lead to singularities of black hole types. But this is far from being proven. Roger Penrose called the principle (that generic spacetime can have only black hole singularities) as cosmic censorship.

This is probably the most challenging problem in nonlinear evolution equation. While spacetime at large distances is not that well understood, it is even more tough when the distances are very tiny. Whether quantum effects can cure singularities is a major question. But I believe string theory will be part of a grand theory that will help us build a geometry that can describe nature more accurately. Such a geometry should be some form of quantum geometry. It would demand close collaborations between physicists and geometers.

Modern geometers and physicists are still struggling to learn how to build a good quantum geometry to explain the physics of very large and very small. It may take next fifty years to accomplish such a goal.

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