Jürgen Jost

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Shing-Tung Yau as a Personality Creating and Shaping the Field of Geometric Analysis

Dedicated to Shing-Tung Yau

1. Introduction

In 1975, in a paper [11] with his schoolmate and friend Shiu-Yuen Cheng, Shing-Tung Yau wrote "Most of the problems in differential geometry can be reduced to problems in differential equations on Riemannian manifolds. Our main purpose here is to study these equations and their applications in geometry". This, in fact, summarizes the basic project of his research. In this contribution, I shall try to elaborate upon this project. This will be interwoven with the personal story of my collaboration with Shing-Tung Yau over almost two decades. During these years, I not only learned a vast range of mathematical techniques, but also benefitted in many other ways from his generous friendship.

2. Geometric Analysis

When I first met Shing-Tung Yau, or simply Yau, as he is called by his friends and colleagues, in the fall of 1980, he was already recognized as one of the world's leading mathematicians. In particular, he was seen as a leading contender for the Fields Medal, which he then duly received at the ICM 1983 in Warsaw. He was famous for his solution of the Calabi conjecture [131, 129], based on a profound analysis of complex Monge-Ampère equations, and the work with Shiu-Yuen Cheng on real Monge-Ampère equations [12, 13], as well as for the solution of the positive mass conjecture of general relativity (with his student Richard Schoen, [108, 109], the solution of the Frankel conjecture in complex geometry (with Yum-Tong Siu [120]), his novel approach to eigenvalue estimates on Riemannian manifolds (together with Peter Li [81]), his new ideas about the global theory of harmonic functions [127] and an impressive number of further fundamental contributions. And in this period, he was also working with Bill Meeks on a new approach to the topology of 3-manifolds using minimal surfaces [86, 87, 88] and with Peter Li on the Harnack inequality for the heat kernel of a Riemannian manifold [82] and several other projects.

All these works had a profound impact on the subsequent development of geometric analysis and several other mathematical disciplines, like Riemannian, Kähler, algebraic and complex geometry, geometric topology, nonlinear partial differential equations etc, but also in various fields of theoretical physics, like general relativity or quantum field theory. In particular, his solution of the Calabi conjecture identified the solutions of the compactification problem in string theory, that is, certain compact manifolds of six real dimensions that by this theorem carry a metric of vanishing Ricci curvature, and which are therefore called Calabi-Yau manifolds. They are, in fact, analogues of the K3-surfaces, certain complex manifolds of real dimension four, and those also carry Ricci flat metrics by his theorem, and this was the key for the analysis of their moduli space by Siu [115]. Conversely, Yau could also use concepts from theoretical physics to discover profound mathematical structural relations, as in his work with Strominger and Zaslow on mirror symmetry and T-duality [121]. And in collaborations with several theoretical physicist and mathematicians, he could also combine the study of mirror symmetry and Calabi-Yau manifolds, see e.g. [47, 48, 76].

The new ideas with Peter Li [81, 82] on eigenvalue bounds and heat kernels on Riemannian manifolds also inspired analogous work on graphs of Yau with Fan Chung and Alexander Grigor'yan, see [19, 20, 17, 21, 22, 18], culminating in the Li-Yau type Harnack inequality on graphs achieved in [5] (see also [93]).

The work with Bill Meeks [86, 87, 88], enriched by geometric measure theory tools in collaboration with Meeks and Leon Simon [85], opened up the geometric analysis approach to the geometric topology of 3-manifolds, and in particular to the Poincaré conjecture. This approach showed that potential counterexamples to that conjecture had tangible geometric properties; in fact, they had to carry certain embedded minimal surfaces. This was further explored in, for instance, [101, 102, 37] and lead to significant advances in three-dimensional topology. Although ultimately another approach, the Ricci flow introduced by Yau's friend Richard Hamilton [42] lead to the solution of the Poincaré conjecture, the vision of Meeks and Yau opened the door for the geometric analysis attack on that problem. In fact, the Poincaré conjecture was finally solved by Grigory Perel'man [98, 100, 99] who was able to continue the flow through the singularities that inevitably arise when the flow does not start with a metric of positive Ricci curvature. As Perel'man, however, only sketched some crucial details, it required effort to complete all those details. Yau was very happy that the first complete proof was achieved by his former student Huai-Dong Cao and his friend Xiping Zhu [8, 9]. (Some controversy ensued, but this is not the occasion to enter into any details, and so let it suffice to also mention [75, 92].) In this direction, let us also mention the monographs [15, 16].

In [127], Yau had shown that on a complete manifold with non-negative Ricci curvature, any positive harmonic function is constant. In [11], this result was improved to show that on such a manifold, non-constant harmonic functions must grow at least linearly. Subsequently, in [126], he conjectured that on an open manifold with nonnegative Ricci curvature the space of harmonic functions with polynomial growth of a fixed rate is finite dimensional. This famous polynomial growth conjecture was finally proved by Colding-Minicozzi [23], see also [79, 80]. In fact, this is a more general structural phenomenon that also holds on graphs, see [49]. In [128], Yau had shown that L^q harmonic functions ($1 < q < \infty$) on complete Riemannian manifolds are constant, without any further condition, a very remarkable and surprising result. Again, this is a general structural property that also holds on graphs, see for instance [50].

This brief sketch could, of course, discuss only some selected contributions of Yau and their impact. I could not even mention all papers by Yau himself, let alone those of others that built upon his work. In the next section, I shall enter in more detail into a particular topic, that of harmonic maps, where I also had the opportunity to collaborate with Yau and be inspired by him.

3. Harmonic Maps as a Tool in Geometric Analysis

In fact, his paper [106] (with Richard Schoen) had been an important inspiration for my thesis (see [51], and as background also the local estimates of Heinz [43, 44] as well as [61] where those methods are further elaborated, as well as the subsequent [63, 27]), and therefore my advisor Stefan Hildebrandt had suggested that I visit him as a postdoc to learn the new methods in geometric analysis first hand. When I arrived at the Institute for Advanced Study in Princeton in September 1980, Yau suggested that to study the paper [114] by Yum-Tong Siu about a new approach to rigidity problems in complex geometry via harmonic mappings, an approach clearly also inspired by Yau. In fact, in the introduction of [114], Siu wrote "Yau conjectured that … two compact Kähler manifolds of complex dimension ≥ 2 with negative sectional curvature are biholomorphic or conjugate biholomorphic if they are of the same homotopy type". Siu then showed that harmonic mappings into a Kähler manifold with so-called strongly negative curvature (a curvature condition stronger than negative sectional curvature, but satisfied by important classes of Kähler manifolds, in particular by Hermitian symmetric spaces of non-compact type) had to be holomorphic or antiholomorphic, and in fact under appropriate conditions biholomorphic or conjugate biholomorphic diffeomorphisms. An important intermediate step in Siu's analysis is to show that harmonic maps in such contexts are in fact pluriharmonic, that is, their restriction to any subvariety is harmonic again. Such a property holds, of course, automatically for holomorphic maps, but not in general for harmonic maps. Further results of Siu in this direction can be found in [116, 117].

Siu's result showed the rigidity of such spaces in the category of complex manifolds. Following his suggestion to study harmonic maps between Kähler manifolds, I then came up with a surprising computation that showed that even in situations of negative target curvature where the harmonic maps could not be holomorphic, their level sets nevertheless are holomorphic subvarieties of the domain [65]. This applies for instance when the target is negatively curved higher genus Riemann surfaces, whose complex structure can be deformed, as studied in Teichmüller theory. In discussions with Siu, important consequences gradually emerged, for instance in [119] that it is a topological invariant of a Kähler manifold to admit a holomorphic map to some Riemann surface.

Siu's result can be viewed as a generalization of Mostow-Margulis rigidity in the case of Hermitian symmetric spaces, that is, when the symmetric space carries a Kähler structure. The challenge to carry the harmonic map approach over to the general situation without a Kähler structure then lead to substantial further mathematical developments, culminating in the theory of Higgs bundles and non-abelian Hodge theory, that is, the investigation of representations of fundamental groups. The idea is that for a representation in a linear algebraic group G of the fundamental group of a compact Riemannian manifold M, one constructs an equivariant harmonic map from the universal cover $\tilde{M}$ into the symmetric space G/K on which G operates. The properties of that harmonic map then have to be explored. This scheme is described in [68].

Since I have described the resulting developments already in an earlier article [58] that was also dedicated to Yau, I can be brief here. In particular, detailed references can be found in that paper.

Siu's analysis was based on a new Bochner type identity for harmonic mappings. In general, the idea of such an identity is to take some expression involving first order derivatives of the harmonic map in question, like the pointwise squared norm of those derivatives, and compute its Laplacian. Because of the differential equation satisfied by the harmonic map, third order terms drop out, after commuting some covariant derivative operators, and the latter then leads to curvature terms from the target and domain to show up in the formula. When those curvatures have suitable signs, typically some non-positivity assumption on the target curvature, all terms on the right hand side have the same sign, that is, the expression of which the Laplacian has been computed, is subharmonic. Therefore, on a compact manifold, it has to be constant.

The challenge then is to find expressions that have a rich geometric content on which this scheme works and to identify the necessary curvature conditions. The next successful steps in this direction were taken by Sampson [105] and Corlette [26]. For a general theory of representations as pioneered in the superrigidity theory of Margulis [84], one also needs to study representations in linear algebraic groups over non-Archimedean fields. The associated symmetric spaces are Euclidean Bruhat-Tits buildings. A Euclidean Bruhat-Tits is a certain simplicial metric space with nonpositive curvature in the sense of Alexandrov, that is, no longer a Riemannian manifold, but having similar global properties as Riemannian manifolds with nonpositive sectional curvature. Gromov and Schoen [40] then constructed harmonic maps from Riemannian manifolds into such spaces and investigated their properties. Subsequently, general theories of harmonic maps with values in metric spaces with nonpositive Alexandrov curvature were developed by Korevaar-Schoen [77, 78] and Jost [54, 55, 56], and the latter theory even applies to target spaces that are not necessarily locally compact. This is important for a general theory of representations in groups over arbitrary fields. Also, that approach even works under the more general condition of nonpositive Busemann curvature. See [57] for a systematic presentation. In fact, the approach of [54, 55, 56] raised the problem to a higher level of abstraction, as a minimization problem for convex functionals on (typically non-locally compact) spaces of generalized non-positive curvature.

Using such harmonic map tools in a systematic manner, the most general superrigidity results for harmonic maps were obtained by Jost and Yau [71] and Mok, Siu and Yeung [91]. Let us collect the results stemming from the harmonic map approach to superrigidity in the following

Theorem 1. *We assume that*

- $\tilde{M} = G/K$ *is an irreducible symmetric space of noncompact type, different from* $SO_0(p,1)/SO(p)\times SO(1), SU(p,1)/S(U(p)\times U(1))$*,*
- Γ *is a cocompact lattice in* G*,*
- Y *is a complete simply connected Riemannian manifold of nonpositive curvature operator with isometry group* $I(Y)$*,*
- $\rho : \Gamma \to I(Y)$ *is a homomorphism for which* $\rho(\Gamma)$ *either does not have a fixpoint on the sphere at* ∞ *of* Y*, or if it does, it centralizes a totally geodesic flat subspace.*

Then there exists a totally geodesic ρ*-equivariant map,*

$$u : \tilde{M} \to Y.$$

In particular, if H *is a semisimple noncompact Lie group with trivial center and* $\rho : \Gamma \to H$ *a homomorphism with Zariski dense image, then* ρ *can be extended to a homomorphism from* G *onto* H*.*

And if $\rho : \Gamma \to Sl(n,\mathbb{Q}_p)$ *(where* $\mathbb{Q}_p$ *are the* p*-adic numbers) is a homomorphism, for some* $n \in N$ *and some prime* p*, then* $\rho(\Gamma)$ *sits in a compact subgroup of* $Sl(n,\mathbb{Q}_p)$*.*

These results generalize Margulis superrigidity and include earlier results by Margulis (for rank $(G/K) \geq 2$), by Corlette [26] (for $Sp(p,1)/Sp(p)\times Sp(1)$ and the hyperbolic Cayley plane), by Gromov and Schoen [40] (for quaternionic hyperbolic space) and Mok [90, 89] (for Hermitian symmetric spaces). For a survey of some aspects of the theory, see also [96].

There are still some open problems in the harmonic map approach to rigidity. It has not yet been able to derive Mostow's rigidity theorem for quotients of real hyperbolic space. Also, not all cases of spaces that are of finite volume but not compact (i.e. for nonuniform lattices) have been dealt with, while Margulis' results also hold for such noncompact cases. In the Hermitian symmetric case, this problem was solved by Jost and Zuo [73], who proved the existence and achieved the control of harmonic maps of possibly infinite energy. (For the finite energy case, see [66, 67].)

In a slightly different direction, Carlson-Toledo [10] and Jost-Yau [68] could show that lattices in $SO(n,1)$ for $n \geq 3$ cannot be Kähler groups. This means that the topologies of real hyperbolic space forms of dimension $n \geq 3$ and those of Kähler manifolds are completely different. This is different from $n = 2$ where such a space form is simultaneously a hyperbolic Riemann surface and a Kähler manifold. For further results in this direction, see [133, 45, 125, 64]. The result of [65] that harmonic maps into nonpositively curved spaces, while not necessary holomorphic themselves, yield holomorphic foliations, was also seminal for the subsequent of the theory of Higgs bundles and non-abelian Hodge theory. The result says that for a harmonic map $u : M \to N$ between Kähler manifolds, N being nonpositively curved, we obtain a factorization through a holomorphic map. The level sets $u^{-1}(z)$ for $z \in N$ are analytic subvarieties of M. Dividing by this foliation yields a holomorphic map $h : M \to S$ into some Kähler manifold, together with a harmonic map $v : S \to N$, such that $u = v \circ h$.

A theory of representations of $\pi_1(M)$ for a Kähler manifold M was developed in [46, 24, 25, 111, 112, 134, 113, 135]. This embodied an abstract version of such a factorization. More precisely, a Higgs bundle (E,θ) over M consists of a holomorphic vector bundle E and

$$\theta : E \to E \otimes \Omega^1(M) \text{ with } \theta \wedge \theta = 0. \tag{1}$$

For a stable Higgs bundle, we can then apply the fundamental results of Narasimhan-Seshadri [94], Donaldson [31, 32], Uhlenbeck-Yau [123] and Simpson [111] to obtain a Hermitian Yang-Mills connection D, and $D + \theta$ is flat if all $c_i(E) = 0$. Such a D then defines a harmonic metric on E, i.e., a harmonic map into a symmetric space G/K, as above. Conversely, from a reductive representation

$$\rho : \pi_1(M) \to G, \quad G \text{ a linear algebraic group}, \tag{2}$$

one obtains a ρ-equivariant harmonic map

$$h : \tilde{M} \to G/K, \tag{3}$$

and this defines a Higgs bundle (E,θ) with $\theta = dh$. This harmonic h turns out to be pluriharmonic which implies $\theta \wedge \theta = 0$, the condition 1 for a Higgs bundle.

This closes the circle and identifies the moduli space of stable Higgs bundles over M with that of reduction representations of $\pi_1(M)$ (although the complex structures on these two spaces are different [46]).

Harmonic maps also provide powerful tools for investigating families of Riemann surfaces. For this, we need to recall the solutions of the Shafarevitch and Mordell problems over function fields. The first was obtained by Parshin [97], with

Theorem 2. *Let C be a compact smooth holomorphic curve and let $g \in \mathbb{N}, g \geq 2$. Then there exist at most finitely many algebraic surfaces B fibered over C with smooth fibres of genus g that are not isotrivial, that is, not finitely covered by a product.*

and Arakelov [3] who showed this more generally when the fibers may have singularities over some fixed finite subset S of C.

The second problem was solved by Manin [83], with another proof by Grauert [38],

Theorem 3. *Let $f : B \to C$ be a nontrivial fibering as in Theorem 2. Then there exist at most finitely many holomorphic sections $s : C \to B$.*

In [70] (see also [72] for a survey), a harmonic map approach to these results was developed, based on the geometry of the moduli space M_g of holomorphic curves of genus $g (\geq 2)$. While M_g has certain quotient singularities, a finite cover of it is a manifold with a natural Kähler metric, the Weil-Petersson metric g_{WP}. Therefore, for our discussion, we simply treat M_g as a Kähler manifold. It is not compact, but admits a natural compactification $\bar{M}_g$ constructed by Mumford and Deligne [30]. Tromba [122] proved that g_{WP} has negative sectional curvature, and its holomorphic sectional curvature even has a negative upper bound $k < 0$. Different proofs of these results were found in [124, 118, 53, 62]. Equipping C with a metric of constant curvature κ, we may apply the Schwarz Lemma of Yau [130] and Royden [104] for a holomorphic $h : C \to M_g$ to obtain

$$\|dh(z)\|^2 \leq \frac{\kappa}{k} \text{ for all } z \in C, \tag{4}$$

unless h is constant. If the genus of C is 0, 1, then $\kappa \geq 0$, and so, in this case, it follows that any fibering $f : B \to C$ by curves of genus ≥ 2 is isotrivial, because $k < 0$ in 4.

In general, by 4, all holomorphic maps $h : C \to M_g$ are equicontinuous. A subtle technical argument is needed at the boundary of compactified moduli space, but we may therefore conclude that at most finitely many homotopy classes of maps from C to M_g can contain holomorphic maps. This the boundedness part of the proof of Theorem 2 (for a related argument, see [39, 95]). For the finiteness part, one has to show that any nontrivial homotopy class of maps from C' to M'_g can contain at most one holomorphic map. This can again be achieved by using the negativity of the curvature of M'_g. For the proof of Theorem 3, we utilize the holomorphic fibering

$$\psi : \mathcal{M}'_g \to M'_g \tag{5}$$

with the fiber of $q \in M'_g$ being the holomorphic curve defined by q. $\mathcal{M}'_g$ also carries a Weil-Petersson metric with the same negativity properties as the one of M'_g. A holomorphic section s as in Theorem 3 then induces a holomorphic map $k : C' \to \mathcal{M}'_g$. The boundedness proof works as before, and the finiteness can also be concluded on the basis of the negativity of the sectional curvature.

Let me also mention that his paper [107] with Richard Schoen also inspired my solution [52] of the Plateau-Douglas problem [33, 34, 35, 36, 28, 29]; [107] used that the collar lemma [74, 103, 7, 41, 6] to control families of degenerating Riemann surfaces, and this technique then was also useful for degenerating minimal surfaces, when combined with the Courant-Lebesgue lemma.

Finally, let me mention [69] where we have introduced a variant of the harmonic map system that is adapted to Hermitian manifolds that need not be Kähler. In contrast to the usual harmonic map, the resulting system is no longer in divergence form (unless the domain manifold is Kähler, in which case it reduces to the standard harmonic map system) and therefore not of variational origin. Hence, the analysis becomes much harder. It enables us to obtain rigidity results when the Kähler condition is weakened to what we have called astheno-Kähler (after the Greek word for "weak"). It turned out that analogous systems could also be constructed and analyzed in other non-Riemannian contexts, in particular on affine manifolds [59, 60]. Remarkably, this makes contact with another fundamental paper by Cheng and Yau [14] where the analyzed affine flat structures as real analogues of Kähler manifolds. Instead of a strictly plurisubharmonic potential as in Kähler theory, here one works with a strictly convex potential. This was developed further in [110], and it constitutes a differential geometric foundation of information geometry [2, 1, 4], the differential geometric treatment of families of probability distributions, that is, the geometric foundation of statistics.

4. A Remarkable and Unique Personality

The achievements of Shing-Tung Yau are not confined to geometric analysis and related fields. He has made seminal contributions to a wide range of mathematical fields, as well as to other domains like theoretical physics. This is amply documented in other contributions to this volume. But his achievements also transcend scientific research; some of them are listed below.

- When he was asked to become the editor-in-chief of the *Journal of Differential Geometry*, within a short period, he transformed this journal into the leading topical journal of the field. And the field of differential geometry was conceived in a wide sense, including in particular geometric analysis, complex geometry and the relations to theoretical physics. It published several seminal papers, like Witten's novel approach to Morse theory originating from the supersymmetric sigma model of quantum field theory.
- Being angry and upset about the prices for journals and books charged by established publishers, he founded his own publishing company *International Press*. This publishing house not only took over the established *Journal of Differential Geometry*, but he also succeeded in launching several new journals in many fields of mathematics. These journals are generally very successful and quickly publish mathematical papers of high quality. *International Press* also took over other well-established journals like *Acta Mathematica*. By now, this publishing house is firmly established as a high quality and efficient scientific publisher. Yau himself is on the editorial board of most of its journals and book series, which certainly contributes to their success.
- Being very patriotic and wanting to raise mathematics in China to the highest international level, he undertook several initiatives. He founded new research institutes in China, Hong Kong, and Taiwan. These institutes organize international conferences in mathematics and theoretical physics, offer visiting positions for excellent young Chinese mathematicians, and attract international visitors. He also founded a new conference center, the *Tsinghua International Mathematics Forum, TSIMF*, at Sanya on the tropical island of

Hainan (the southernmost province of China from where its navy controls the South China Sea and which is the most popular vacation destination of middle class Chinese). At this conference center, Oberwolfach style mathematical workshops can be held, but also larger conferences, because the facilities can accommodate up to 150 participants in a beautiful resort with an excellent swimming pool and tennis courts and very good library access.

- And being very patriotic and wanting to raise the level of mathematics in China, he also became openly critical of several practices among academics in China which he held as corrupt. This lead to public disputes and created many enemies, but in the end served to raise the awareness of the importance of basing academic decisions exclusively on scientific achievement and merit.
- And being very patriotic and wanting to raise the level of mathematics in China, he inaugurated the *International Congress of Chinese Mathematicians* that congegrates every three years at some location in China (which includes Hong Kong and Taiwan) to present the most important achievements of mathematicians of Chinese origin and awards medals to young Chinese mathematicians of outstanding talent. And since his Chinese patriotism is not just political, but based on the culture, history and language of China, which in turn are the proud achievement of the Chinese people, the selection criteria of that congress are based on ethnic criteria, in contrast to comparable structures in western countries. Likewise, at the TSIMF, he insists that a certain number of young Chinese postdocs and students be invited for every conference.
- Being aware of the need of generous funding for fundamental mathematical research, he succeeded in convincing some rich Hong Kong business people to donate generously both to his research centers in China, in particular the *Morningside Institute* in Beijing, as well as to his alma mater, Harvard University. In fact, he acquired the highest single donation in the entire history of that university.
- Realizing the importance of the support of political leaders for basic science, he did a lot of political lobbying in China, and also publicly advocated that China take the lead in building the next generation particle accelerator to probe hitherto experimentally unaccessible, but theoretically deeply explored realms of elementary particle physics.
- Understanding the importance of mathematical tradition, he organized a project on the history of the Mathematics Department of Harvard University; like his other endeavors, this was very well received.
- Being aware of the importance of communicating fundamental research to the general scientifically interested public, he wrote a popular book [132] explaining his vision of geometry and physics in non-technical terms.
- More generally, as a dynamic leader, he fulfilled the role of chairman of that department with much success, several new scientific initiatives, and a clear vision for the future.
- And, of course, being both an international leading mathematician and a very open and generous personality, he educated a huge number of graduate students and postdocs, many of which became highly successful scientists themselves.

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