Open Problems

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Note. This column is edited by Pengfei Guan (McGill University), Yizhao Hou (Caltech), Jun Li (Stanford University), Kefeng Liu (UCLA), Zhouping Xin (Chinese University of Hong Kong), Hao Xu (Zhejiang University; Managing editor), Shing-Tung Yau (Harvard University), Zhiwei Yun (MIT), and Wei Zhang (MIT). The readers are welcome to propose the solutions. The authors may send their solutions to Hao Xu (mathxuhao@gmail.com) and post the solutions in MathSciDoc (http://archive.ymsc.tsinghua.edu.cn/). The correct solutions will be announced and some souvenirs will be awarded.

—*The Editors*

Problem 2019001 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

In my paper with Chi [1], we studied the pseudonormed space over an n-dimensional complex manifold M. In particular, let mK_M be the m-th tensor product of the canonical line bundle of M, the space of holomorphic sections of mK_M admits a pseudonorm defined by the integral of the $2/m$ power of norm of the section. If $m=2$, we have L^1-spaces.

Now the Teichmüller metric is defined by the dual of such an L^1-norm. Can we isometrically embed the moduli space of curves equipped with this L^1-norm to the L^1-space defined by some algebraic manifold described above.

[1] C.-Y. Chi and S.-T. Yau, *A geometric approach to problems in birational geometry*. Proc. Natl. Acad. Sci. USA **105** (2008), no. 48, 18696–18701.

Problem 2019002 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Given a Lipschitz manifold defined by an elliptic variational method, is it Lipschitz homeomorphic to a smooth manifold? In particular, this should apply to minimal submanifolds.

Problem 2019003 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Gekhtman and Markovic [1] conjectured that for a holomorphic disk in the Teichmüller space, in order for the Kobayahi metric and the Carathéodory metric to be equal, it is necessary that the associated holomorphic quadratic differential has zeros of even order only. The sufficiency was known to be true by the works of Kra [2] and McMullen through applying Schwarz lemma to composite of the map of the disc into Teichmüller space with the period map into Siegel upper half space. See also [3].

[1] D. Gekhtman, Vladimir Markovic, *Classifying complex geodesics for the Carathéodory metric on low-dimensional Teichmüller spaces*, arXiv:1711.04722.

[2] I. Kra, *The Carathéodory metric on abelian Teichmüller disks*. J. Anal. Math. **40** (1981), 129–143.

[3] F. Gardiner, *Carathéodory's and Kobayashi's metrics on Teichmüller space*, arXiv:1711.00035.

Problem 2019004 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Consider the $\bar{\partial}$ operator operating on the smooth sections on a holomorphic line bundle L over a complex manifold. With Hermitian metrics in the line bundle and the manifold, we can form the Hodge Laplacian acting on the space of smooth sections of the line bundle. The operator has discrete spectra and the zero set of the eigen-sections has (real) codimenson-two measure. Prove that the measure of this set can be estimated from above and below by the square of the eigenvalue. Would it be possible to relate the measure of the zeros of sections of powers of L?

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Problem 2019005 (Differential Geometry). *Proposed by Shing-Tung Yau, Harvard University.*

Given a compact manifold M with boundary, we can look at the Laplacian with boundary condition given by a linear combination of Dirichlet and Neumann conditions. Hence we have a way to join the Laplacian with Dirichlet condition to Laplacian with Neumann condition and back to Dirichlet condition. The eigenvalues and the eigenfunctions move accordingly. During this motion, we shall arrive at eigenvalues with multiplicity. Can one describe the dynamics of such a movement?