# On Prof. Yau's Problem


## by Zheyan Wan*



**Abstract.** In this note, we solve a problem proposed by Prof. Yau in his autobiography [2].


## 1. Introduction

On page 20 of Prof. Yau's autobiography [2], he mentions a problem that he came up with as a teenager.

**Problem 1.1.** Suppose you know the length of one side of a triangle, one angle, and the length of one angle bisector. Can you construct the corresponding triangle, using just a compass and ruler?

Note that the length of one side of a triangle, one angle, and the length of one angle bisector form a three-tuple in 27 ways. By symmetry, there are only 5 three-tuples that are essentially different. For a triangle $\triangle ABC$, the 5 three-tuples can be chosen as

$$\begin{cases} (BC, \angle A, \ell_a) \\ (BC, \angle A, \ell_b) \\ (BC, \angle B, \ell_a) \\ (BC, \angle B, \ell_b) \\ (BC, \angle B, \ell_c) \end{cases} \tag{1.1}$$

where $\ell_a$, $\ell_b$, and $\ell_c$ are the lengths of the angle bisector of $\angle A$, $\angle B$, and $\angle C$ respectively.

In order to solve this problem, we introduce the general theory of ruler and compass constructions in Section 2. Then we restate this problem and solve it in Section 3.


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## 2. General Theory of Ruler and Compass Constructions

In this section, we review the general theory of ruler and compass constructions discussed in [1].

Rules:

- Two points in the plane are given to start with. These points are constructed.
- If two points $p_0$, $p_1$ have been constructed, we may draw the line through them, or draw a circle with center at $p_0$ and passing through $p_1$, such lines and circles are then constructed.
- The points of intersection of constructed lines and circles are constructed.

Points, lines, and circles will be called constructible if they can be obtained in finitely many steps, using these rules.

Caveat: "We are not allowed to use the ruler for measurement."

**Construction 2.1.** Construct a line through a constructed point $p$ and perpendicular to a constructed line $\ell$. See Figure 1 and 2.

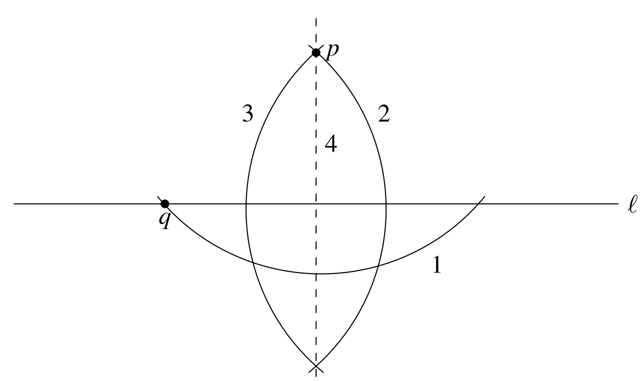


*Figure 1. Case 1:* $p \notin \ell$.

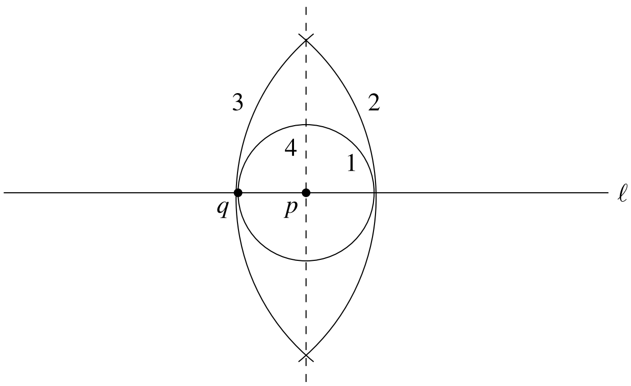


*Figure 2. Case 2: $p \in \ell$.*

**Construction 2.2.** Construct a line parallel to a constructed line $\ell$ and passing through a constructed point $p$. See Figure 3.

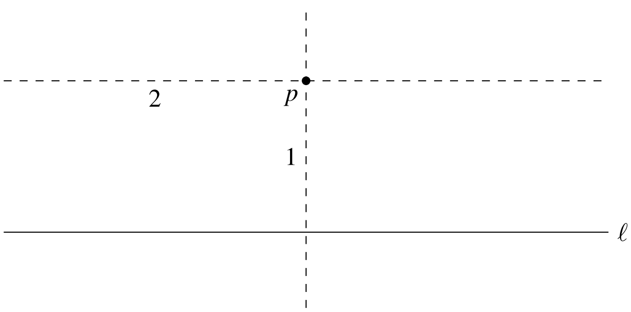


*Figure 3. Apply Case 1 and 2 above.*

**Construction 2.3.** Mark off a length defined by two points onto a constructed line $\ell$, with endpoint $p$. See Figure 4.

We introduce Cartesian coordinates into the plane so that the points that are given at the start have coordinates $(0,0)$ and $(1,0)$. We call a real number $a$ constructible if the point $(a,0)$ is constructible. Since we can construct perpendiculars, this is the same thing as saying that $a$ is the $x$-coordinate of a constructible point. And since we can mark off lengths, a positive real number $a$ is constructible if and only if there is a pair of $p$, $q$ of constructible points whose distance apart is $a$.

**Lemma 2.4** (Lemma 15.5.11 of [1]). 1. The constructible numbers form a subfield of $\mathbb{R}$.

2. If $a$ is a positive constructible number, then so is $\sqrt{a}$.

*Proof.* 1. We must show that if $a$ and $b$ are positive constructible numbers, then $a+b$, $-a$, $ab$ and $a^{-1}$ (if $a \neq 0$) are also constructible. The closure in case $a$ or $b$ is negative follows easily. Addition and subtraction are done by marking lengths on a line. For multiplication and division, we use similar right triangles. See Figure 5.
Given one triangle and one side of a second triangle, the second triangle can be constructed by parallels. To construct the product $ab$, we take $r=1$, $s=a$, and $r'=b$. Then $s'=ab$. To construct $a^{-1}$, we take $r=a$, $s=1$, and $r'=1$. Then $s'=a^{-1}$.

2. We use similar triangles again. We must construct them so that $r=a$, $r'=s$, and $s'=1$. Then $s=\sqrt{a}$. We can use inscribed triangles in a circle. A triangle inscribed into a circle with a diameter as its hypotenuse, is a right triangle. So we construct a circle whose diameter is $1+a$ and proceed as in Figure 6. □

We call an angle $\alpha$ is constructible if the real number $\cos\alpha$ is constructible.

## 3. Statement and Solution of Prof. Yau's Problem

Now we can state the 5 cases (1.1) of our problem as:

**Problem 3.1.** Given two constructible real numbers $a$, $\ell_a$, a constructible angle $\alpha$. Can we construct a triangle $\triangle ABC$ such that $a$ is the length of $BC$, $\alpha$ is the angle $\angle A$, and $\ell_a$ is the length of the angle bisector of $\angle A$?

**Problem 3.2.** Given two constructible real numbers $a$, $\ell_b$, a constructible angle $\alpha$. Can we construct a triangle

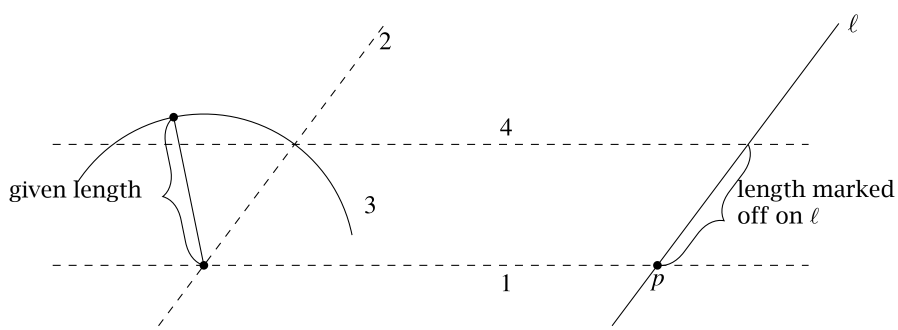


*Figure 4. Use the construction of parallels.*

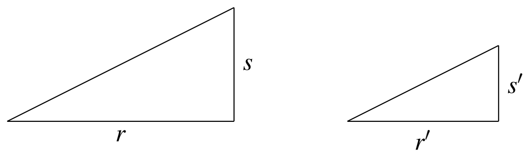


*Figure 5.*

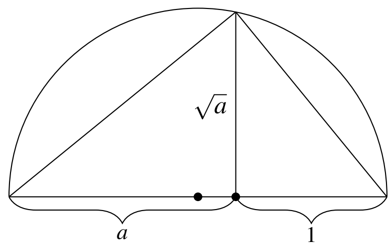


*Figure 6.*

$\triangle ABC$ such that $a$ is the length of $BC$, $\alpha$ is the angle $\angle A$, and $\ell_b$ is the length of the angle bisector of $\angle B$?

**Problem 3.3.** Given two constructible real numbers $a$, $\ell_a$, a constructible angle $\beta$. Can we construct a triangle $\triangle ABC$ such that $a$ is the length of $BC$, $\beta$ is the angle $\angle B$, and $\ell_a$ is the length of the angle bisector of $\angle A$?

**Problem 3.4.** Given two constructible real numbers $a$, $\ell_b$, a constructible angle $\beta$. Can we construct a triangle $\triangle ABC$ such that $a$ is the length of $BC$, $\beta$ is the angle $\angle B$, and $\ell_b$ is the length of the angle bisector of $\angle B$?

**Problem 3.5.** Given two constructible real numbers $a$, $\ell_c$, a constructible angle $\beta$. Can we construct a triangle $\triangle ABC$ such that $a$ is the length of $BC$, $\beta$ is the angle $\angle B$, and $\ell_c$ is the length of the angle bisector of $\angle C$?

*Solution of Problem 3.1.* First, by the sine theorem, $\frac{BC}{\sin\angle A} = 2R$ where $R$ is the radius of the circumcircle of $\triangle ABC$. So by Lemma 2.4, $2R = \frac{a}{\sin\alpha}$ is constructible. Therefore, the center $O$ of the circumcircle of $\triangle ABC$ is constructible.

First draw $BC$ with length $a$ and the center $O$ of the circumcircle of $\triangle ABC$ (there are two choices for $O$ which are symmetric, we choose either one of them), then draw the diameter perpendicular to $BC$ with foot $M$, choose one of the two endpoints and denote it by $P$ (if $\alpha < \frac{\pi}{2}$, choose $P$ such that $PM < R$; if $\alpha > \frac{\pi}{2}$, choose $P$ such that $PM > R$; if $\alpha = \frac{\pi}{2}$, choose either of the two). Denote the other endpoint by $Q$. See Figure 7.

Since $A$ is on the circle centered at $O$ with radius $R$. If it were constructible, denote the intersection point of $AP$ and $BC$ by $N$. Then $AN$ is the angle bisector of $\angle A$. Let $PM = h$, $PN = s$, then $h$ is constructible. Since $BN \cdot NC = AN \cdot PN$, we have

(3.1)

$$s \cdot \ell_a = (\frac{a}{2} + MN)(\frac{a}{2} - MN) = \frac{a^2}{4} - MN^2 = \frac{a^2}{4} - (s^2 - h^2).$$

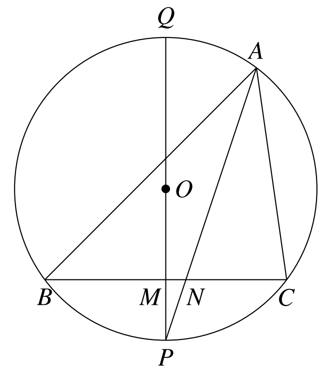


*Figure 7.*

Since $PM \cdot QM = BM \cdot MC$, we have

$$h \cdot (2R - h) = \frac{a}{2} \cdot \frac{a}{2}. \tag{3.2}$$

So $s^2 + s\ell_a = \frac{a^2}{4} + h^2 = 2Rh$. Therefore, $s = \frac{-\ell_a \pm \sqrt{\ell_a^2 + 8Rh}}{2}$. Since $s > 0$, we have

$$s = \frac{\sqrt{\ell_a^2 + 8Rh} - \ell_a}{2} \tag{3.3}$$

which is constructible by Lemma 2.4.

Then we draw the circle centered at $P$ with radius $s$.

**Case 1.** If $s > h$, we get two intersection points of the circle and $BC$. Draw the two lines passing through $P$ and the two intersection points, we get two intersection points of the circumcircle and the two lines, these two intersection points are the two possibilities of $A$.

**Case 2.** If $s = h$, we get one intersection point of the circle and $BC$. Draw the line passing through $P$ and the intersection point, we get one intersection point of the circumcircle and the line, this intersection point is the only possibility of $A$.

**Case 3.** If $s < h$, we get no intersection point of the circle and $BC$. Such a triangle can not be constructed.

Note that

$$\frac{s}{h} = \frac{4R}{\sqrt{\ell_a^2 + 8Rh} + \ell_a} \begin{cases} > 1 & \text{if } \ell_a < 2R - h, \\ = 1 & \text{if } \ell_a = 2R - h, \\ < 1 & \text{if } \ell_a > 2R - h. \end{cases} \tag{3.4}$$

□

*Solution of Problem 3.2.* Similarly as before, first draw $BC$ with length $a$ and the center $O$ of the circumcircle of $\triangle ABC$. If $\cos\angle B$ were constructible, draw a circle centered at $B$ with radius 1. Then mark the length $\cos\angle B$ on $BC$ with endpoint $B$, namely $BD = \cos\angle B$. Then draw the line passing through $D$

and perpendicular to $BC$, denote the intersection point of the circle centered at $B$ with radius 1 and the line by $E$ (there are two choices for $E$, we can determine it by comparing $\alpha$ with $\frac{\pi}{2}$). Then the intersection point of the line $BE$ and the circumcircle of $\triangle ABC$ is the point $A$, thus such a triangle is constructible. Conversely, if such a triangle were constructible, mark the length 1 on $BA$ with endpoint $B$, namely $BF = 1$. Then draw a line passing through $F$ and perpendicular to $BC$ with foot $G$, then $BG = \cos\angle B$ is constructible.

Therefore, in this case, such a triangle is constructible if and only if $\cos\angle B$ is constructible, or equivalently $\cos\frac{\angle B}{2}$ is constructible by Lemma 2.4.

See Figure 8.

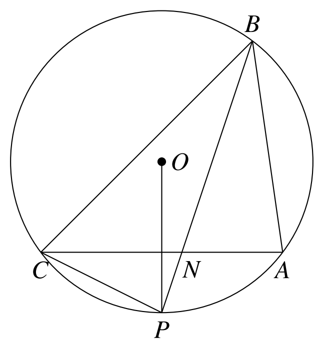


*Figure 8.*

Since $\triangle ABN \sim \triangle PBC$, we have

$$\frac{AB}{BP} = \frac{BN}{BC}. \tag{3.5}$$

Since $\triangle PNC \sim \triangle PCB$, we have

$$\frac{PN}{PC} = \frac{PC}{BP}. \tag{3.6}$$

Note that $CP = 2R\sin\frac{\angle B}{2}$ and $AB = 2R\sin\angle C = 2R\sin(\angle A + \angle B)$. Since $BN = \ell_b$, let $PN = s$, then we have

$$\ell_b \cdot (s + \ell_b) = a \cdot 2R\sin(\angle A + \angle B) \tag{3.7}$$

and

$$s \cdot (s + \ell_b) = 4R^2 \sin^2 \frac{\angle B}{2}. \tag{3.8}$$

Therefore,

$$\begin{aligned}(s+\ell_b)^2 &= 4R^2 \cdot \frac{1-\cos\angle B}{2} + a\cdot 2R\sin(\angle A+\angle B)\\ &= 2R^2(1-\cos\angle B) + 4R^2\sin\angle A\sin(\angle A+\angle B)\\ &= 2R^2(1-\cos\angle B) + 2R^2(\cos\angle B - \cos(2\angle A+\angle B))\\ &= 4R^2\sin^2(\angle A + \frac{\angle B}{2}).\end{aligned} \tag{3.9}$$

Hence $s + \ell_b = 2R\sin(\angle A + \frac{\angle B}{2})$. Therefore,

$$\begin{aligned}\ell_b &= \frac{a\cdot 2R\sin(\angle A+\angle B)}{2R\sin(\angle A + \frac{\angle B}{2})}\\ &= \frac{a\sin(\angle A+\angle B)}{\sin(\angle A+\frac{\angle B}{2})}.\end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned}\frac{\ell_b}{a} &= \frac{\sin\alpha\cos\angle B + \cos\alpha\sin\angle B}{\sin\alpha\cos\frac{\angle B}{2} + \cos\alpha\sin\frac{\angle B}{2}}\\ &= 2\cos\frac{\angle B}{2} - \frac{\sin\alpha}{\sin\alpha\cos\frac{\angle B}{2}+\cos\alpha\sin\frac{\angle B}{2}}.\end{aligned} \tag{3.11}$$

Let $\cos\frac{\angle B}{2} = x$, then

$$\sin\alpha \cdot (\frac{1}{2x - \frac{\ell_b}{a}} - x) = \cos\alpha\cdot\sqrt{1-x^2}. \tag{3.12}$$

Square and expand the equation, we get

$$\begin{aligned}&4x^4 - 4\frac{\ell_b}{a}x^3 + (\frac{\ell_b^2}{a^2} - 4)x^2 + 2\frac{\ell_b}{a}(1+\cos^2\alpha)x\\ &+\sin^2\alpha - \cos^2\alpha\cdot\frac{\ell_b^2}{a^2} = 0.\end{aligned} \tag{3.13}$$

Let $y = x - \frac{1}{4}\frac{\ell_b}{a}$, then $x$ is constructible if and only if $y$ is constructible, and

$$\begin{aligned}&y^4 + (-\frac{1}{8}\frac{\ell_b^2}{a^2} - 1)y^2 + \frac{1}{2}\frac{\ell_b}{a}\cos^2\alpha\cdot y\\ &+\frac{1}{256}\frac{\ell_b^4}{a^4} + \frac{1}{16}\frac{\ell_b^2}{a^2} - \frac{1}{8}\frac{\ell_b^2}{a^2}\cos^2\alpha + \frac{1}{4}\sin^2\alpha = 0.\end{aligned} \tag{3.14}$$

By Theorem 15.5.6 of [1], a real number is constructible if and only if it is in an iterated quadratic extension of $\mathbb{Q}$, namely it is an algebraic number with degree a power of 2. So we hope to decompose the quartic polynomial in (3.14) as the product of two quadratic polynomials with constructible coefficients.

In general, for a quartic polynomial $y^4 + py^2 + qy + r$ with constructible coefficients, we introduce an arbitrary parameter $t$ such that

$$\begin{aligned}y^4 + py^2 + qy + r &= (y^2 + \frac{p}{2} + t)^2\\ &- [2t(y^2 + \frac{p}{2}) + t^2 - qy + \frac{p^2}{4} - r].\end{aligned} \tag{3.15}$$

We aim to find $t$ such that the second term in (3.15) is a perfect square. The logic behind this is that if the second term is a perfect square, then (3.15) becomes the difference of two squares which is the product of two quadratic polynomials in $y$. Since the second term in (3.15) is a quadratic polynomial in $y$, we know that it is a perfect square if and only if its discriminant is zero:

$$\Delta = q^2 - 8t(tp + t^2 + \frac{p^2}{4} - r)$$

$$(3.16)\qquad = -8t^3 - 8pt^2 - 8(\frac{p^2}{4} - r)t + q^2 = 0.$$

If (3.16) has a constructible solution, then (3.15) becomes

(3.17)

$$y^4 + py^2 + qy + r$$
$$= (y^2 + \frac{p}{2} + t + \sqrt{2t}y - \frac{q}{\sqrt{8t}})(y^2 + \frac{p}{2} + t - \sqrt{2t}y + \frac{q}{\sqrt{8t}}).$$

And then $y$ is constructible. Substitute in the values of $p, q, r$, (3.16) becomes

(3.18)

$$-8t^3 + (\frac{\ell_b^2}{a^2} + 8)t^2 - (\frac{\ell_b^2}{a^2} + 2)\cos^2\alpha \cdot t + \frac{1}{4}\frac{\ell_b^2}{a^2}\cos^4\alpha = 0.$$

Because (3.18) is a cubic polynomial in $t$ with constructible coefficients, in general, $t$ is not a constructible number. Therefore, in general, $y$ is not constructible. But if the coefficients of (3.18) satisfy certain condition, then (3.18) has a constructible solution. For example, if $\frac{3}{32}\frac{\ell_b^2}{a^2} = \frac{1}{4} - \sin^2\alpha$, then $t = \frac{1}{8}\frac{\ell_b^2}{a^2} + \sin^2\alpha$ is a constructible solution of (3.18). □

*Solution of Problem 3.3.* First draw $BC$ with length $a$ and a circle centered at $B$ with radius 1. Then mark the length $\cos\beta$ on $BC$ with endpoint $B$, namely $BD = \cos\beta$. Then draw the line passing through $D$ and perpendicular to $BC$, denote the intersection point of the circle centered at $B$ with radius 1 and the line by $E$ (there are two choices for $E$ which are symmetric, we choose either one of them). Then $A$ lies on the line $BE$.

Similarly as before, if $\cos\angle A$ were constructible, then $2R = \frac{a}{\sin\angle A}$ is constructible. Draw the center $O$ of the circumcircle of $\triangle ABC$ (there are two choices for $O$, we can determine it by comparing $\cos\angle A$ with 0). Then the intersection point of the line $BE$ and the circumcircle of $\triangle ABC$ is the point $A$, thus such a triangle is constructible. Conversely, if such a triangle were constructible, then $\cos\angle A$ is constructible.

Therefore, in this case, such a triangle is constructible if and only if $\cos\angle A$ is constructible, or equivalently $\sin\frac{\angle A}{2}$ is constructible by Lemma 2.4.

See Figure 9.

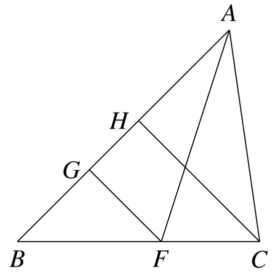


*Figure 9.*

If $AF$ is the angle bisector of $\angle A$, draw $FG \perp AB$ and $CH \perp AB$. Then $FG = \ell_a \sin\frac{\angle A}{2}$, $CH = a\sin\beta$. Since $\triangle BFG \sim \triangle BCH$, we have

$$(3.19)\qquad \frac{FG}{CH} = \frac{BF}{BC}.$$

On the other hand, we have

$$(3.20)\qquad \frac{AB}{AC} = \frac{BF}{CF}.$$

So we get

$$\frac{\ell_a \sin\frac{\angle A}{2}}{a\sin\beta} = \frac{AB}{AC+AB} = \frac{\sin\angle C}{\sin\beta + \sin\angle C}$$
$$(3.21)\qquad = \frac{\sin(\angle A + \beta)}{2\sin\frac{\beta+\angle C}{2}\cos\frac{\beta-\angle C}{2}} = \frac{\sin(\angle A+\beta)}{2\cos\frac{\angle A}{2}\sin(\beta + \frac{\angle A}{2})}.$$

Therefore,

$$\frac{\ell_a}{a} = \frac{\sin(\angle A+\beta)\sin\beta}{\sin(\beta+\frac{\angle A}{2})\sin\angle A}$$
$$(3.22)\qquad = \frac{(\sin\angle A\cos\beta + \cos\angle A\sin\beta)\sin\beta}{(\sin\beta\cos\frac{\angle A}{2} + \cos\beta\sin\frac{\angle A}{2})\sin\angle A}.$$

Let $\cos\frac{\angle A}{2} = x$, $\sin\frac{\angle A}{2} = y$, then $x^2 + y^2 = 1$, and

$$\frac{\ell_a}{a} = \frac{2xy\cos\beta\sin\beta + (x^2 - y^2)\sin^2\beta}{(x\sin\beta + y\cos\beta)2xy}$$
$$(3.23)\qquad = \frac{(2\cos\beta + (\frac{x}{y} - \frac{y}{x})\sin\beta)\sin\beta}{2(x\sin\beta + y\cos\beta)}.$$

Therefore,

$$(3.24)\qquad \frac{1}{\frac{a\sin\beta}{\ell_a} - y} - \frac{1}{y} + 2y = \frac{2x\cos\beta}{\sin\beta}.$$

Square and expand the equation, we get

$$(3.25)\quad 4y(\frac{a\sin\beta}{\ell_a} - y)(1-y^2)(1 + \frac{y(\frac{a\sin\beta}{\ell_a} - y)}{\sin^2\beta}) = \frac{a^2\sin^2\beta}{\ell_a^2}.$$

By Theorem 15.5.6 of [1], a real number is constructible if and only if it is in an iterated quadratic extension of $\mathbb{Q}$, namely it is an algebraic number with degree a power of 2. But (3.25) is a sextic polynomial in $y$ with constructible coefficients. Therefore, in general, $y$ is not constructible. But if the coefficients of (3.25) satisfy certain condition, then (3.25) has a constructible solution. For example, if $(1 - \frac{1}{4}\frac{a^2\sin^2\beta}{\ell_a^2})(1 + \frac{1}{4}\frac{a^2}{\ell_a^2}) = 1$, then $y = \frac{1}{2}\frac{a\sin\beta}{\ell_a}$ is a constructible solution of (3.25). □

*Solution of Problem 3.4.* Similarly as before, first draw $BC$ with length $a$ and a line $BE$ such that $\angle EBC = \beta$. Draw the angle bisector of $\angle B$ and a circle centered at $B$ with radius $\ell_b$, denote their intersection point by $F$ (we choose $F$ such that $F$ lies in the angle $\angle EBC$). Then the intersection point of the lines $BE$

and $CF$ is the point $A$. So such a triangle can always be constructed. □

*Solution of Problem 3.5.* Similarly as before, first draw $BC$ with length $a$ and a line $BE$ such that $\angle EBC = \beta$. Then draw a circle centered at $C$ with radius $\ell_c$.

**Case 4.** If $\ell_c > a\sin\beta$, then we get two intersection points $F_1$, $F_2$ of the circle and the line $BE$. Draw the reflection points $G_1$, $G_2$ of $B$ about the two lines $CF_1$ and $CF_2$ respectively. Then the intersection points of the lines $BE$ and $CG_1$ ($CG_2$) are the two possibilities of $A$.

**Case 5.** If $\ell_c = a\sin\beta$, then we get one intersection point $F$ of the circle and the line $BE$. Draw the reflection point $G$ of $B$ about the line $CF$. Then the intersection point of the lines $BE$ and $CG$ is the only possibility of $A$ ($G = A$).

**Case 6.** If $\ell_c < a\sin\beta$, then we get no intersection point of the circle and the line $BE$. Such a triangle can not be constructed. □


## Acknowledgements

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## References


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