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Distinguishing number and adjacency properties

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Dedicated to the memories of Roland Fraïssé and Michael O. Albertson

The distinguishing number of countably infinite graphs and relational structures satisfying a simple adjacency property is shown to be 2. This result generalizes both a result of Imrich et al. on the distinguishing number of the infinite random graph, and a result of Laflamme et al. on homogeneous relational structures whose age satisfies the free amalgamation property.

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1. Introduction

One of the most widely studied infinite graphs is the *Rado* or *infinite random graph*, written R. A graph satisfies the *existentially closed* or *e.c.* adjacency property if for all finite disjoint sets of vertices A and B (one of which may be empty), there is a vertex $z \notin A \cup B$ joined to all of A and to no vertex of B. By a back-and-forth argument, R is the unique isomorphism type of countably infinite graphs that is e.c. Further, R is *homogeneous*: every isomorphism between finite induced subgraphs extends to an automorphism. For a survey of these and other results on R, see [3].

The *distinguishing number* of a graph G, written $D(G)$, is the least positive integer n such that there exists an n-colouring of $V(G)$ (not necessarily proper) so that no non-trivial automorphism preserves the colours. Rigid graphs (which possess no non-trivial automorphisms) have distinguishing number 1, and $D(G)$ may be viewed as the minimum number of colours needed to make G rigid. The parameter $D(G)$ was introduced by Albertson and Collins [1].

The distinguishing number of graphs generalizes in a straightforward fashion to relational structures. A *relation* on a set X is a set of n-tuples from

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X, where $n > 0$ is its *arity*. A *signature* μ is a (possibly infinite) sequence $(\mu_i : i \in I)$ of positive integers. A *relational structure* S *with signature* μ consists of a non-empty vertex set $V(S)$, and a set of relations R_i on $V(S)$ for $i \in I$ of arity μ_i. Isomorphisms, induced subgraphs, distinguishing number, and many other notions from graph theory generalize naturally to relational structures. For background on relational structures, we refer the reader to [4]. All graphs we consider are simple.

While most research on the distinguishing number has focused on the finite case, recent work considers infinite structures as well. Imrich, Klavžar, and Trofimov [5] recently proved (among other things) that $D(R) = 2$. Laflamme, Nguyen Van Thé, and Sauer [7] generalized this fact by showing that a homogeneous relational structure with minimal arity 2, whose *age* (that is, set of isomorphism types of induced finite substructures) satisfies the free amalgamation property has distinguishing number 2. In [8], $D(T)$ is determined for infinite, locally finite trees T.

In this short note, we introduce an adjacency property called weak-e.c. for countable relational structures (generalizing the e.c. property) which is a sufficient condition to have distinguishing number at most 2; see Theorem 1.2. As a consequence of this fact, in Corollary 1.4 we show that homogeneous structures whose age has free amalgamation have distinguishing number 2. Our results generalize the results of [5; 7] stated in the previous paragraph. Further, they supply a large class of relational structures with distinguishing number 2. For example, there are $2^{\aleph_0}$ many non-isomorphic countable graphs with the weak-e.c. property; see [2].

Definition 1.1. *A graph G that is not a clique is* ***weak-e.c.*** *if for each pair u, v of (possibly equal) non-joined vertices and a finite set T of vertices containing neither u nor v, there is a vertex z joined to u and v but not joined nor equal to a vertex in T.*

The graph R has the weak-e.c. property, as does the universal homogeneous triangle-free graph, although the latter graph is not e.c. Note that the weak-e.c. property implies that the graph has diameter 2, and has no vertex of finite degree.

If S is a relational structure, then the *(Gaifman) graph of* S, written $G(S)$, has vertices those of S with two vertices x and y joined if $x \neq y$ and only if they appear together in some tuple in a relation of S. Note that an automorphism of S induces an automorphism of $G(S)$.

A relational structure S is *weak-e.c.* if $G(S)$ is weak-e.c. Our main result is the following.

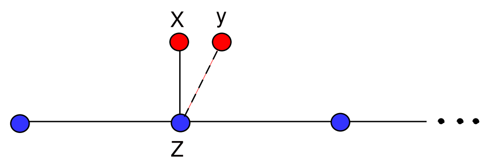

Figure 1: Fixing x and y in Claim 1.

Theorem 1.2. *If the countable relational structure S satisfies the weak-e.c. property, then $D(S) \leq 2$.*

Proof. Let $G = G(S)$. We prove first that $D(G(S)) \leq 2$. We actually prove that a weak-e.c. graph satisfies another adjacency property, which in turn implies a distinguishing number at most 2. A graph G satisfies ($\clubsuit$) if there is an induced ray Z (that is, an infinite one-way path) in G such that for all pairs of distinct vertices x and y not in Z, there is a vertex in Z joined to exactly one of either x or y.

Claim 1. Property ($\clubsuit$) implies that $D(G) \leq 2$.

To see this, let $\mathbf{B}$—the *blue vertices*—be the vertices of the induced ray Z, and let $\mathbf{R}$, the *red vertices*, be the vertices in $V(G)\backslash\mathbf{B}$. It is straightforward to see that no automorphism f of G preserving the colour sets can move an element of $\mathbf{B}$. We claim that f restricted to $\mathbf{R}$ is the identity. To see this, let us suppose that $f(x) = y$ for some distinct red vertices x and y. By ($\clubsuit$), there is a blue vertex z joined to (say) x but not y. See Figure 1. But this is a contradiction as f fixes z. The proof of the Claim 1 follows.

The fact that $D(G) \leq 2$ is implied by the following claim.

Claim 2. The weak-e.c. property implies ($\clubsuit$).

For the proof of Claim 2, enumerate all unordered pairs of distinct vertices of G as

$$R_{-1} = \{\{x_i, y_i\} : i \in \mathbb{N}\}.$$

We inductively process pairs of vertices from R_{-1}. Each pair will be labeled *processed* or *unprocessed*; at the beginning of the base step, all pairs are unprocessed.

By the weak-e.c. property, there is a vertex z_0 joined to x_0 that is neither joined nor equal to y_0 (in the notation of the definition of weak-e.c., we are setting $u = v = x_0$, and $T = \{y_0\}$). Delete all pairs $\{x_j, y_j\}$ from R_{-1}

containing z_0 to form the set of pairs R_0. Label $\{x_0, y_0\}$ as processed. For ease of notation, we relabel the remaining pairs of R_{-1} so that

$$R_0 = \{\{x_i, y_i\} : i \in \mathbb{N}\}.$$

For $n \geq 0$ fixed, assume that we have found a finite set of distinct vertices Z_n and a set R_n of pairs from $V(G)$ with the following properties. For simplicity, we assume the pairs of R_n have been relabeled so that

$$R_n = \{\{x_i, y_i\} : i \in \mathbb{N}\}.$$

For each R_n and $k \geq 0$, define its *k-initial segment* $R_n[k]$ to consist of the set

$$\{\{x_0, y_0\}, \{x_1, y_1\}, ..., \{x_k, y_k\}\}.$$

We require that $R_{n+1}[n] = R_n[n]$. Indices of the x_i and y_i in (1) to (3) below refer to the enumeration of pairs in R_n.

1. For each $0 \leq i \leq n$, there is a vertex $z_i \in Z_n$ that is distinct from x_i and y_i, and is joined to exactly one of x_i or y_i. The vertex z_i is not equal to any x_j nor y_j, where $0 \leq j \leq i - 1$.
2. The set Z_n induces in G an n-path with terminal vertices z_0 and z_n.
3. For all $z \in Z_n$, the vertex z is not in a pair in R_n. Each of the pairs $\{x_i, y_i\}$, where $0 \leq i \leq n$, are labeled as processed.

To complete the inductive step, we note that the vertex z_n may or may not be joined to the vertices x_{n+1} or y_{n+1}. We do know for certain that z_n is not equal to either x_{n+1} or y_{n+1} by item (3) of the induction hypothesis. By the weak-e.c. property, we may find a vertex z' joined to z_n but not joined nor equal to any vertex in

$$T'' = (Z_n \setminus \{z_n\}) \cup \{x_0, \ldots, x_{n+1}\} \cup \{y_0, \ldots, y_{n+1}\}.$$

The vertex z' will not be our choice for z_{n+1}, but plays an intermediary role in finding such a vertex. Define T' to be the set of vertices in $\{x_0, \ldots, x_n\} \cup \{y_0, \ldots, y_n\}$ not equal to either x_{n+1} or y_{n+1}. (We note that since we are enumerating unordered pairs in R_n, either of the vertices x_{n+1} or y_{n+1} may be equal to some x_i or y_i for some $1 \leq i \leq n$.) By the weak e.c. property with $u = z'$ and $v = x_{n+1}$, there is a vertex z_{n+1} joined to z' and x_{n+1}, but not joined nor equal to a vertex in

$$T = Z_n \cup T' \cup \{y_{n+1}\}.$$

In particular, z_{n+1} is distinct from, and joined to exactly one of x_{n+1} or y_{n+1} as required in item (1). Set $Z_{n+1} = Z_n \cup \{z', z_{n+1}\}$, and note that the subgraph induced by Z_{n+1} is a path with terminal vertices z_0 and z_{n+1}. Form R_{n+1} by deleting any pairs in R_n containing z' or z_{n+1}, and then relabeling the pairs so that $R_{n+1} = \{\{x_i, y_i\} : i \in \mathbb{N}\}$. Note that this deletion preserves the property that $R_{n+1}[n] = R_n[n]$, since $\{x_{n+1}, y_{n+1}\}$ will not be deleted as z_{n+1} and z' were chosen to be distinct from these two vertices. Hence, properties (1), (2), and (3) are satisfied with this choice of z_{n+1}, R_{n+1}, and Z_{n+1}.

Set

$$Z = \bigcup_{n \in \mathbb{N}} Z_n,$$

and let P be the vertices in $V(G)\backslash Z$. The subgraph induced by Z is a ray. Note that each distinct pair of vertices $\{x, y\}$ in P is processed in the above induction as some pair $\{x_i, y_i\}$. In particular, there is a vertex in Z joined to exactly one of x or y. Hence, Claim 2 follows.

Now, let $\mathrm{Aut}(S, \mathbf{B}, \mathbf{R})$ be the automorphism group of the relational structure S with two additional unary predicates, $\mathbf{B}$ and $\mathbf{R}$, identified with the colour sets $\mathbf{B}$ and $\mathbf{R}$, respectively. The property that $D(X) \leq 2$ is equivalent to $\mathrm{Aut}(S, \mathbf{B}, \mathbf{R})$ being the trivial group. The proof now follows from Claims 1 and 2 since $\mathrm{Aut}(S, \mathbf{B}, \mathbf{R})$ is isomorphic to a subgroup of $\mathrm{Aut}(G(S), \mathbf{B}, \mathbf{R})$. $\square$

Theorem 3.1 of Imrich et al. [5] follows directly from Theorem 1.2 as a corollary, since R is weak-e.c. We point out that the property (♣) introduced in Theorem 1.2 is a more general sufficient condition for having distinguishing number at most 2 than the weak-e.c. property. For example, the infinite random bipartite graph R_B satisfies (♣) and hence, has distinguishing number 2 by Claim 1 in the proof of Theorem 1.2, but is not weak-e.c. since its diameter is not 2. (The proof that R_B satisfies (♣) is similar to the proof of Claim 2, and so is omitted. The additional detail in the inductive step is to consider cases of the colours of x_{n+1} and y_{n+1}.)

The high degree of symmetry exhibited by R may be formalized in a notion which applies to many other relational structures. A structure is *homogeneous* if each isomorphism between finite induced substructures extends to an automorphism. Fix $\mathcal{K}$ a class of structures of the same signature that is closed under isomorphisms. An *amalgam* is a 5-tuple (A, f, B, g, C) such that A, B, and C are structures in $\mathcal{K}$, and $f : A \to B, g : A \to C$ are *embeddings* (that is, isomorphisms onto their images). Then $\mathcal{K}$ has the *amalgamation property*, written (AP), if for every amalgam (A, f, B, g, C), there exist both a structure $D \in \mathcal{K}$ and embeddings $f' : B \to D, g' : C \to D$ such that $f' \circ f = g' \circ g$. The connection between classes with (AP) and homogeneous structures is made transparent by Fraïssé's theorem, which we restate as Theorem 1.3 below. A structure G is *universal* in $\mathcal{K}$ if each member $\mathcal{K}$ is isomorphic to an induced substructure of G. The class $\mathcal{K}$ has the *joint embedding property* or (*JEP*) if for every pair B and C in $\mathcal{K}$, there is a $D \in \mathcal{K}$ so that B and C are isomorphic to induced substructures of D. (If we allow empty structures, then (JEP) is a special case of (AP). Since we only consider non-empty structures, we will not use this convention.)

Theorem 1.3 (Fraïssé, [4])**.** *Let $\mathcal{K}$ be a class of finite structures with the same signature closed under isomorphisms. Then the following are equivalent.*

1. *The class $\mathcal{K}$ has* (AP), (JEP)*, and is closed under taking induced substructures.*
2. *There is a countable universal and homogeneous structure S whose age is $\mathcal{K}$, and which is a limit of a chain of structures from $\mathcal{K}$.*

The structure S in Theorem 1.3 (2) is called the *Fraïssé limit* of $\mathcal{K}$. For example, R is the Fraïssé limit of the class finite graphs. Note that S has the following useful property. Suppose that A, B are structures in the age of S, with A an induced substructure of both B and S. Then there is an isomorphism β from B to an induced substructure of S so that β is the identity on A. We say that B *amalgamates into S over A*.

Given relational structures S_1 and S_2 of the same signature their *union* (or *free amalgam*) $S_1 \cup S_2$ has vertices the union of the vertex sets of S_1 and S_2, and whose relations are the union of the relations of S_1 and S_2. Note that S_1 and S_2 may not in general have disjoint vertex sets; in which case we say that the union $S_1 \cup S_2$ is formed with intersection $V(S_1) \cap V(S_2)$. If $V(S_1) \cap V(S_2)$ is empty, then $S_1 \cup S_2$ is simply their disjoint union. A class of finite relational structures with fixed signature so that $\mathcal{K}$ closed under isomorphism has *free amalgamation* if it is closed under taking unions of structures; that is, if S_1 and S_2 are in $\mathcal{K}$, then $S_1 \cup S_2 \in \mathcal{K}$.

The following corollary gives a short and elementary proof of Theorem 3.1 of LaFlamme et al. [7]. To avoid degenerate cases in the following theorem, we only consider *non-null* structures; that is, structures S where $G(S)$ contains edges.

Corollary 1.4. *Let S be a countable, homogeneous, non-null structure with minimal arity of at least two whose age has free amalgamation. Then $D(S) = 2$.*

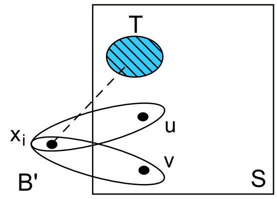

Figure 2: Amalgamating B' into S over A.

Proof. We first show that S satisfies the weak-e.c. property. We may then apply Theorem 1.2 to prove that $D(S) \leq 2$. By homogeneity, S is not rigid so $D(S) = 2$.

Now fix u, v and a finite set T in $V(S)$ so that u, v are not joined in $G(S)$, and $u, v \notin T$. Fix a k-tuple $\overline{x} = (x_1, \ldots, x_k)$ in some relation of S, where $k > 1$, and at least two vertices in $\overline{x}$ are distinct; say these two vertices are x_i and x_j (this is possible as S is non-null). Consider the substructure X of S induced by the vertices in $\overline{x}$. As the age of S contains X, is closed under isomorphism, and has free amalgamation, the age of S contains the structure B' formed by the union of two isomorphic copies of X with intersection $\{x_i\}$. Label the two distinct copies of x_j in B' as x_{j1} and x_{j2}. As the minimum arity of a relation is at least 2, there is exactly one isomorphism type of structure in the age of S with one vertex. Hence, we identify x_{j1} and x_{j2} with u and v, respectively. See Figure 2.

Let A_1 be the substructure of S induced by $\{u, v\}$, and let A be the substructure of S induced by $\{u, v\} \cup T$. Let B be the union of B' and A over A_1. As S is homogeneous, we may amalgamate B into S over A. Hence, there is a vertex $z \in V(S)$ (corresponding to the isomorphic image of x_i) joined in $G(S)$ to both u and v but not T. $\square$

Not all relational structures with distinguishing number 2 are weak-e.c. (for example, consider the infinite binary tree). An open problem is to determine a necessary and sufficient condition for a countably infinite relational structure to have distinguishing number 2.

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