

The Rees product of posets

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Dedicated to Dennis Stanton on the occasion of his 60th birthday

We determine how the flag f -vector of any graded poset changes under the Rees product with the chain, and more generally, any t -ary tree. As a corollary, the Möbius function of the Rees product of any graded poset with the chain, and more generally, the t -ary tree, is exactly the same as the Rees product of its dual with the chain, respectively, t -ary chain. We then study enumerative and homological properties of the Rees product of the cubical lattice with the chain. We give a bijective proof that the Möbius function of this poset can be expressed as n times a signed derangement number. From this we derive a new bijective proof of Jonsson's result that the Möbius function of the Rees product of the Boolean algebra with the chain is given by a derangement number. Using poset homology techniques we find an explicit basis for the reduced homology and determine a representation for the reduced homology of the order complex of the Rees product of the cubical lattice with the chain over the symmetric group.

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1. Introduction

Björner and Welker [2] initiated a study to generalize concepts from commutative algebra to the area of poset topology. Motivated by the ring-theoretic Rees algebra, one of the new poset operations they define is the Rees product.

Definition 1.1. For two graded posets P and Q with rank function ρ the *Rees product*, denoted $P * Q$, is the set of ordered pairs (p, q) in the Cartesian product $P \times Q$ with $\rho(p) \geq \rho(q)$. These pairs are partially ordered by $(p, q) \leq (p', q')$ if $p \leq_P p'$, $q \leq_Q q'$, and $\rho(p') - \rho(p) \geq \rho(q') - \rho(q)$.

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The rank of the resulting poset is $\rho(P * Q) = \rho(P)$. For more details concerning the Rees product and other poset products, see [2].

From the perspective of topological combinatorics, one of the most important results that Björner and Welker show in their paper is that the poset theoretic Rees product preserves the Cohen-Macaulay property; see [2].

Theorem 1.2 (Björner-Welker). *If P and Q are two Cohen-Macaulay posets then so is the Rees product $P * Q$.*

Very little is known about the Rees product of specific examples of Cohen-Macaulay posets. However, what has been studied has yielded rich combinatorial results. The first example in this vein is due to Jonsson [5], who settled an open question of Björner and Welker concerning the Rees product of the Boolean algebra with the chain. For brevity, throughout we will use the notation $\text{Rees}(P, Q)$ to denote the Rees product

$$\text{Rees}(P, Q) = ((P - \{\hat{0}\}) * Q) \cup \{\hat{0}, \hat{1}\}.$$

As usual, we will assume that P and Q are graded posets with P having unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$.

Theorem 1.3 (Jonsson). *The Möbius function of the Rees product of the Boolean algebra B_n on n elements with the n element chain C_n is given by the n th derangement number, that is,*

$$\mu(\text{Rees}(B_n, C_n)) = (-1)^{n+1} \cdot D_n.$$

Recall the n th derangement number D_n is the number of permutations in the symmetric group \mathfrak{S}_n on n elements having no fixed points. Classically $D_n = \lfloor \frac{n!}{e} \rfloor$ for $n \geq 1$ where $\lfloor \cdot \rfloor$ denotes the nearest integer function. Jonsson's original proof uses a non-acyclic element matching to show the Euler characteristic vanishes appropriately.

The paper is organized as follows. In the next section we begin by expressing the flag f -vector of the Rees product of any graded poset with a t -ary tree in terms of the flag f -vector of the original poset. We obtain the surprising conclusion that the Möbius function of the poset with the tree coincides with the Möbius function of its dual with the tree. We then study the signed version of Jonsson's results, that is, the Rees product of the rank $n+1$ cubical lattice \mathcal{C}_n , (i.e., the face lattice of the n -dimensional cube) with the $n+1$ element chain C_{n+1} . Using poset techniques, we give explicit formulas for the Möbius function of $\text{Rees}(\mathcal{C}_n, C_{n+1})$ and show its Möbius function equals $(-1)^n \cdot n \cdot D_{n-1}^\pm$. Here D_n^\pm is the signed derangement number with $D_n^\pm = \lceil \frac{2^{n-1}(n-1)!}{\sqrt{e}} \rceil$ for $n \geq 1$. As a corollary to our enumerative

results, we give an explicit bijective proof of Jonsson's theorem. We then find an explicit basis for the reduced homology of the order complex of $\text{Rees}(\mathcal{C}_n, C_{n+1})$ and determine a representation of the reduced homology of this order complex over the symmetric group. In the last section we end with further questions.

2. Rees product of graded posets with a tree

In this section we determine the flag vector of the Rees product of any graded poset P with a t -ary tree. As a consequence we show the Möbius function of the Rees products $\text{Rees}(P, T_{t,n+1})$ and $\text{Rees}(P^*, T_{t,n+1})$ coincide, although the posets are not isomorphic in general.

For nonnegative integers n and t , let $T_{t,n+1}$ be the poset corresponding to a t -ary tree of rank n , that is, the poset consisting of t^k elements of rank k for $0 \leq k \leq n$ with each nonleaf element covered by exactly t children. Observe that the 1-ary tree $T_{1,n+1}$ is precisely the $(n+1)$ -chain C_{n+1} . Recall for a graded poset P of rank $n+1$ and $S = \{s_1, \dots, s_k\} \subseteq \{1, \dots, n\}$ with $s_1 < \dots < s_k$, the *flag f -vector* $f_S = f_S(P)$ is the number of chains $\hat{0} < x_1 < \dots < x_k < \hat{1}$ with $\rho(x_i) = s_i$.

We now define two weight functions. Here we use the notation $[k]$ to denote the t -analogue of the nonnegative integer k , i.e., $[k] = 1 + t + \dots + t^{k-1}$.

Definition 2.1. For a nonempty subset $S = \{s_1 < \dots < s_k\} \subseteq \mathbb{P}$ define

$$w(S) = [s_1] \cdot [s_2 - s_1 + 1] \cdots [s_k - s_{k-1} + 1]$$

with $w(\emptyset) = 1$. For a nonempty subset $S = \{s_1 < \dots < s_k\} \subseteq \{1, \dots, n\}$ define

$$v(S) = w(S \cup \{n+1\}) - w(S) = t \cdot w(S) \cdot [(n+1) - s_k]$$

with $v(\emptyset) = t \cdot [n]$.

Lemma 2.2. For a graded poset P of rank $n+1$, let $R = \text{Rees}(P, T_{t,n+1})$. Then the flag f -vector of the poset R is given by

$$(2.1) \quad f_S(R) = w(S) \cdot f_S(P),$$

$$(2.2) \quad f_{S \cup \{n+1\}}(R) = w(S \cup \{n+1\}) \cdot f_S(P),$$

for $S \subseteq \{1, \dots, n\}$.

Proof. Consider first $S = \{s_1 < \dots < s_k\} \subseteq \{1, \dots, n\}$. Given an element x_1 of rank $\rho(x_1) = s_1$ from the poset P , there are $[s_1]$ copies of it in the Rees poset R . Each of these copies has $[s_2 - s_1 + 1]$ elements in R of rank s_2 which

are greater than it with respect to the partial order of the Rees poset R . In general, each rank s_i element in R has $[s_{i+1} - s_i + 1]$ elements greater than it in the Rees poset R . Hence relation (2.1) holds.

To show (2.2), note the maximal element $\hat{1}$ of P gets mapped to the $[n + 1]$ coatoms of the Rees poset R . In particular $[(n + 1) - s_k + 1]$ of these elements will cover a given element of rank s_k in R . Hence the result follows. \square

Lemma 2.3. *For a graded poset P of rank $n + 1$, let $R = \text{Rees}(P, T_{t,n+1})$. Then*

$$\mu(R) = \sum_{S \subseteq \{1, \dots, n\}} (-1)^{|S|} \cdot v(S) \cdot f_S(P).$$

Proof. By Philip Hall's theorem, we have

$$\begin{aligned} \mu(R) &= \sum_{S \subseteq \{1, \dots, n+1\}} (-1)^{|S|-1} f_S(R) \\ &= \sum_{S \subseteq \{1, \dots, n\}} (-1)^{|S|-1} f_S(R) + \sum_{S \subseteq \{1, \dots, n\}} (-1)^{|S|} f_{S \cup \{n+1\}}(R) \\ &= \sum_{S \subseteq \{1, \dots, n\}} (-1)^{|S|-1} w(S) \cdot f_S(P) \\ &\quad + \sum_{S \subseteq \{1, \dots, n\}} (-1)^{|S|} w(S \cup \{n+1\}) \cdot f_S(P), \end{aligned}$$

where we have expanded the flag f -vector of the poset R using Lemma 2.2. Combining the two sums proves the desired identity. \square

Theorem 2.4. *For a graded poset P of rank $n + 1$ we have*

$$\mu(\text{Rees}(P, T_{t,n+1})) = \mu(\text{Rees}(P^*, T_{t,n+1})),$$

where P^* is the dual of P . In particular, for the chain on $n + 1$ elements we have

$$\mu(\text{Rees}(P, C_{n+1})) = \mu(\text{Rees}(P^*, C_{n+1})).$$

Proof. Let $S = \{s_1 < \dots < s_k\} \subseteq \{1, \dots, n\}$. The result follows by noting that $v(S) = v(S^{\text{rev}})$, where the reverse of S is $S^{\text{rev}} = \{n + 1 - s_k, n + 1 - s_{k-1}, \dots, n + 1 - s_1\}$ and applying Lemma 2.3. \square

It is clear from the definition of the weight $v(S)$ that the Möbius function $\mu(\text{Rees}(P, T_{t,n+1}))$ is divisible by t . When the poset has odd rank we can say more.

Corollary 2.5. *For a graded poset P of odd rank $n+1$, the Möbius function $\mu(\text{Rees}(P, T_{t,n+1}))$ is divisible by $[2] = 1+t$. In particular, for a graded poset P of odd rank $n+1$, the Möbius function $\mu(\text{Rees}(P, C_{n+1}))$ is even.*

Proof. Observe that $1+t$ divides $[k]$ if and only if k is even. Hence $1+t$ does not divide $v(S)$ for a set $S = \{s_1 < \dots < s_k\}$ implies that s_1 is odd, s_i has the same parity as s_{i+1} and $n+1-s_k$ is odd. This implies that n is odd. Hence that n is even implies that the weight $v(S)$ is divisible by $1+t$ for all subsets S , including the empty set. Thus by Lemma 2.3 the Möbius function of $\text{Rees}(P, T_{t,n+1})$ is divisible by $1+t$. \square

3. Rees product of the cubical lattice with the chain

In this section we give an explicit formula for the Möbius function of the poset $\text{Rees}(\mathcal{C}_n, C_{n+1})$. After finding an R -labeling in Section 4, we relate the Möbius function with a class of permutations, that is, the double augmented barred signed permutations. These are in a one-to-one correspondence with certain skew diagrams. We will return to these when we consider homological questions for $\text{Rees}(\mathcal{C}_n, C_{n+1})$. In Section 6 we give a bijective proof of the Möbius function result expressed as a permanent of a certain matrix.

We represent an element $(x, i) \in \text{Rees}(\mathcal{C}_n, C_{n+1}) - \{\hat{0}, \hat{1}\}$ as an ordered pair where the n -tuple $x = (x_1, x_2, \dots, x_n) \in \{0, 1, *\}^n$ and $i \in \{1, \dots, n\}$. Observe that such an element (x, i) has rank k if there are exactly $k-1$ stars appearing in its first coordinate, $1 \leq i \leq k$.

For a graded poset P with minimal element $\hat{0}$ and maximal element $\hat{1}$, throughout we will use the shorthand $\mu(P)$ to denote the Möbius function $\mu_P([\hat{0}, \hat{1}])$.

Proposition 3.2 gives an explicit formula for the Möbius function of the poset $\text{Rees}(\mathcal{C}_n, C_{n+1})$. (See Table 1 for the Möbius values for $n = 0, \dots, 10$.) The proof will require the following lemma.

Lemma 3.1. *The following identity holds:*

$$1 + \sum_{k=0}^n \binom{n}{k} (-1)^{k+1} k! (n-k+1) = 0.$$

Proof. Define sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ by $a_n = (-1)^{n+1} n!$ and $b_n = n+1$. These sequences have exponential generating functions

$$A(x) = \sum_{n \geq 0} (-1)^{n+1} x^n = -\frac{1}{1+x}$$

Table 1: Table of Möbius values for the Rees product of the Boolean algebra with the chain and the Rees product of the cubical lattice with the chain

n	$(-1)^{n+1}\mu(\text{Rees}(B_n, C_n))$	$(-1)^n\mu(\text{Rees}(\mathcal{C}_n, C_{n+1}))$	$=$	Factorization
0	1	0	$=$	0
1	0	1	$=$	$1 \cdot 1$
2	1	2	$=$	$2 \cdot 1$
3	2	15	$=$	$3 \cdot 5$
4	9	116	$=$	$4 \cdot 29$
5	44	1165	$=$	$5 \cdot 233$
6	265	13974	$=$	$6 \cdot 2329$
7	1854	195643	$=$	$7 \cdot 27949$
8	14833	3130280	$=$	$8 \cdot 391285$
9	133496	56345049	$=$	$9 \cdot 6260561$
10	1334961	1126900970	$=$	$10 \cdot 112690097$

and

$$B(x) = \sum_{n \geq 0} (1+n) \frac{x^n}{n!} = (1+x)e^x.$$

Thus, $D(x) = A(x)B(x) = -e^x$. But

$$\begin{aligned} D(x) &= \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!} \\ &= \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} (-1)^{k+1} k! (n-k+1) \frac{x^n}{n!}, \end{aligned}$$

which proves the claim. \square

Proposition 3.2. *The Möbius function of the Rees product of the cubical lattice with the chain is given by*

$$\mu(\text{Rees}(\mathcal{C}_n, C_{n+1})) = -1 + \sum_{i=0}^n (-1)^{n-i} \cdot 2^{n-i} \binom{n}{i} (i+1)(n-i)!.$$

Proof. Let x be an element of corank k from $\text{Rees}(\mathcal{C}_n, C_{n+1}) - \{\hat{0}, \hat{1}\}$. First note that the number of elements of corank i in the half-open interval $[x, \hat{1})$ is $\binom{k-1}{i-1} \cdot (k-i+1)$. This follows from the fact that the element $x = (b, p)$ has $k-1$ non-stars appearing in b , so a corank i element $y = (c, q) \in [x, \hat{1})$ has $i-1$ more stars appearing in c and the second coordinate q satisfying $p \leq q \leq p+k-i+1$. Hence there are $\binom{k-1}{i-1} \cdot (k-i+1)$ such elements y .

Secondly, we claim that for a corank k element $x \in \text{Rees}(\mathcal{C}_n, C_{n+1}) - \{\hat{0}, \hat{1}\}$, we have

$$(3.1) \quad \mu([x, \hat{1}]) = (-1)^k \cdot (k-1)!.$$

We induct on the corank k . The case $k = 0$ is clear, as then x is a coatom. For the general case, we have

$$\begin{aligned} \mu([x, \hat{1}]) &= - \sum_{x < y \leq \hat{1}} \mu([y, \hat{1}]) \\ &= - \left(1 + \sum_{\substack{x < y \leq \hat{1}, \\ 1 \leq \text{corank}(y) \leq k-1}} \mu([y, \hat{1}]) \right) \\ &= - \left(1 + \sum_{i=1}^{k-1} (-1)^i \cdot (i-1)! \cdot \# \text{ elements of corank } i \text{ in } [x, \hat{1}] \right), \end{aligned}$$

where the third equality is applying the induction hypothesis. The number of corank i elements in the half-open interval $[x, \hat{1}]$ is $\binom{k-1}{i-1} \cdot (k-i+1)$, giving

$$\begin{aligned} \mu([x, \hat{1}]) &= - \left(1 + \sum_{i=1}^{k-1} (-1)^i \binom{k-1}{i-1} \cdot (i-1)! \cdot (k-i+1) \right) \\ &= (-1)^k \cdot (k-1)! \end{aligned}$$

by Lemma 3.1.

To finish the argument, there are $2^{n-k} \cdot \binom{n}{k} \cdot (k+1)$ elements of rank $k+1$, each having Möbius value $\mu(x, \hat{1}) = (-1)^{n-k+1} \cdot (n-k)!$. Hence the lemma follows the fact that for a poset P with $\hat{0}$ and $\hat{1}$, the identity $\mu_P(\hat{0}, \hat{1}) = - \sum_{\hat{0} < x \leq \hat{1}} \mu_P(x, \hat{1})$ holds. \square

4. Edge labeling

We begin by recalling some facts about R -labelings. For a complete overview, we refer the reader to Section 5 of Björner and Wachs' paper [1].

Given a poset P an *edge labeling* is a map $\lambda : E(P) \rightarrow \Lambda$, where $E(P)$ denotes the edges in the Hasse diagram of P and the labels are elements from a poset Λ . An edge labeling λ is said to be an *R -labeling* if in every interval $[x, y]$ of P there is a unique saturated chain $c : x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ whose labels are rising, that is, which satisfies $\lambda(x_0, x_1) <_{\Lambda} \lambda(x_1, x_2) <_{\Lambda} \cdots <_{\Lambda} \lambda(x_{k-1}, x_k)$. Given a maximal chain $m : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}$

in P , the *descent set* of m is the set $D(m) = \{i : \lambda(x_{i-1}, x_i) \not\prec_{\Lambda} \lambda(x_i, x_{i+1})\}$. Alternatively, when we view the labels of the maximal chain as the word $\lambda(m) = \lambda_1 \cdots \lambda_n$, where $\lambda_i = \lambda(x_{i-1}, x_i)$ and the rank of P is n , there is a descent in the i th position of $\lambda(m)$ if the labels λ_i and λ_{i+1} are either incomparable in the label poset Λ or satisfy $\lambda_i >_{\Lambda} \lambda_{i+1}$. In particular, a maximal chain m is said to be *rising* if its descent set satisfies $D(m) = \emptyset$ and *falling* if $D(m) = \{1, \dots, n\}$.

The usefulness of an R -labeling is that it gives an alternate way to compute the Möbius function μ of a poset. Variations of this result are due to Stanley in the case of admissible lattices, Björner for R -labelings and edge lexicographic labelings, and Björner–Wachs for non-pure posets with a CR -labeling. See [1] for historical details.

Theorem 4.1. *Let P be a graded poset of rank n with an R -labeling. Then with respect to this R -labeling the Möbius function is given by*

$$\mu(\hat{0}, \hat{1}) = (-1)^n \cdot \text{number of falling maximal chains in } P.$$

Let $\lambda : E(\text{Rees}(\mathcal{C}_n, C_{n+1})) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm n, n+1\} \times \{0, 1\}$ be a labeling of the edges of the Hasse diagram of $\text{Rees}(\mathcal{C}_n, C_{n+1})$ defined by

Edge		Condition	$\lambda(E)$	Notation
(x, i)	\prec	(y, i) $x_a = 1, y_a = *$	$(a, 0)$	a
(x, i)	\prec	(y, i) $x_a = 0, y_a = *$	$(-a, 0)$	$-a$
(x, i)	\prec	$(y, i+1)$ $x_a = 1, y_a = *$	$(a, 1)$	\bar{a}
(x, i)	\prec	$(y, i+1)$ $x_a = 0, y_a = *$	$(-a, 1)$	$\overline{-a}$
$\hat{0}$	\prec	$(x, 1)$	$(0, 0)$	0
(x, i)	\prec	$\hat{1}$	$(n+1, 0)$	$n+1$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. The elements $\{0, \pm 1, \dots, \pm n, n+1\} \times \{0, 1\}$ are partially ordered with the product order, that is $(x, i) \leq (y, j)$ if $x \leq y$ and $i \leq j$.

Proposition 4.2. *The labeling λ is an R -labeling of $\text{Rees}(\mathcal{C}_n, C_{n+1})$.*

Proof. Let $I = [(x, i), (y, j)]$ be an interval in $\text{Rees}(\mathcal{C}_n, C_{n+1}) - \{\hat{0}, \hat{1}\}$ of length m with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. We wish to find a saturated chain $c : (x, i) = (z_0, p_0) \prec (z_1, p_1) \prec \cdots \prec (z_m, p_m) = (y, j)$ in the interval I with increasing edge labels.

Let $S_0 = \{k : x_k = 0 \text{ and } y_k = *\}$ and $S_1 = \{k : x_k = 1 \text{ and } y_k = *\}$. Let $s = j - i$ and $t = |S_0|$. Without loss of generality, we may assume $S_0 = \{i_1, \dots, i_t\}$ and $S_1 = \{i_{t+1}, \dots, i_m\}$ with $i_1 > \cdots > i_t$ and $i_{t+1} < \cdots < i_m$.

Set $(z_0, p_0) = (x, i)$. For $1 \leq k \leq m$, let $(z_k, p_k) = ((z_{1,k}, \dots, z_{n,k}), p_k)$ where

$$z_{i,k} = \begin{cases} * & \text{if } i = i_k, \\ z_{i,k-1} & \text{otherwise,} \end{cases}$$

and

$$p_k = \begin{cases} p_{k-1} & \text{if } 1 \leq k \leq m-s, \\ p_{k-1} + 1 & \text{otherwise.} \end{cases}$$

The first coordinate of the edge labels of the chain c form the strictly increasing sequence $-i_1 < \dots < -i_t < i_{t+1} < \dots < i_m$ as the i_j 's are all positive, while the second coordinate of the edge labels form the weakly increasing sequence $0 \leq \dots \leq 0 \leq 1 \leq \dots \leq 1$. Hence the chain c constructed is increasing.

We also claim that the chain c is the unique such chain that is increasing in the interval I . For any maximal chain in this interval, each $i \in S_0$ appears as the first coordinate in an edge label with a negative sign and every $i \in S_1$ must appear with a positive sign. Hence there is exactly one way to linearly order these m values. The second coordinate of the labels of any maximal chain in I is a permutation of the multiset $\{0^{m-s}, 1^s\}$. Again, there is exactly one way to order these m values in a weakly increasing fashion. Hence the increasing chain c is unique.

For the case when the interval is $[\hat{0}, (y, j)] \in \text{Rees}(\mathcal{C}_n, C_{n+1})$ with $(y, j) \neq \hat{1}$, the first edge label in any saturated chain is always $(0, 0)$. Hence the first coordinate of the labels in any increasing chain in this interval must all be non-negative, implying an increasing chain must pass through the atom $(a, 1) = ((1, \dots, 1), 1)$. The remainder of the increasing chain is given by the unique increasing maximal chain in the interval $[(a, 1), (y, j)]$.

For an interval of the form $[(x, i), \hat{1}]$, since the last edge label of any saturated chain has label $(n+1, 0)$, this forces all the elements of such a chain to be of the form (y, i) with $x \leq_{\mathcal{C}_n} y$. In particular, the rank n element of such a chain is precisely the element $(b, i) = ((*, \dots, *), i)$. Hence the increasing maximal chain in $[(x, i), \hat{1}]$ is given by the increasing maximal chain guaranteed in $[(x, i), (b, i)]$ concatenated with the element $\hat{1}$. \square

5. Falling chains

Define the set of (*double augmented*) *barred signed permutations* $\overline{\mathfrak{S}}_n^\pm$ to be those permutations $\pi = \pi_0 \pi_1 \dots \pi_{n+1}$ satisfying (i) $\pi_0 = 0$ and $\pi_{n+1} = n+1$, (ii) for $1 \leq i \leq n$, π_i is equal to one of a_i , $-a_i$, $\overline{a_i}$ or $-\overline{a_i}$ for some $a_i \in \{1, \dots, n\}$, and (iii) $a_1 \dots a_n$ is a permutation in the symmetric group \mathfrak{S}_n on n elements. Given a double augmented barred signed permutation $\pi =$

$\pi_0\pi_1\cdots\pi_{n+1}$, a *descent* at position i occurs when $|\pi_i| > |\pi_{i+1}|$, where $|\pi_j|$ denotes the element π_j with its (possible) bar removed and sign preserved.

Proposition 5.1. *With respect to the R -labeling λ of the poset $\text{Rees}(\mathcal{C}_n, C_{n+1})$, the falling chains are described as the set of double augmented barred signed permutations $\pi = \pi_0\pi_1\cdots\pi_{n+1} \in \overline{\mathfrak{S}}_n^\pm$ satisfying*

1. *if π_i is unbarred then there must be a descent at the i th position.*
2. *if π_i is barred, then either (i) π_{i+1} is unbarred or (ii) π_{i+1} is barred and there is a descent at the i th position.*

Example 5.2. The permutation $(0, -3, \overline{-4}, 2, \overline{-1}, 5) \in \overline{\mathfrak{S}}_4^\pm$ corresponds to the falling chain

$$\hat{0} \prec (0100, 1) \prec (01 * 0, 1) \prec (01 ** , 2) \prec (0 *** , 2) \prec (* *** , 3) \prec \hat{1}$$

in the poset $\text{Rees}(\mathcal{C}_4, C_5)$.

Proof of Proposition 5.1. Given a barred signed permutation satisfying the conditions of the proposition, we wish to find a falling chain $c : \hat{0} \prec (x_1, i_1) \prec \cdots \prec (x_n, i_n) \prec \hat{1}$ in $\text{Rees}(\mathcal{C}_n, C_{n+1})$. For $1 \leq k \leq n$, if $\pi_k < 0$ then set $x_{1,k} = 1$; otherwise set $x_{1,k} = 0$. To find (x_k, i_k) recursively, set $i_1 = 0$, let $x_{w_k,k} = *$, and set

$$i_k = \begin{cases} i_{k-1} + 1 & \text{if } \pi_k \text{ is barred,} \\ i_{k-1} & \text{if } \pi_k \text{ is not barred.} \end{cases}$$

Observe that c is a falling chain. The labels on the barred signed permutation correspond to the labels on the falling chain. Note that if the unbarred signed permutation does not have a descent at some position k , then π_k is barred and π_{k+1} is not, implying the second coordinate in the labeling $\lambda((x_k, i_k), (x_{k+1}, i_{k+1}))$ is 1, while the second coordinate in the labeling $\lambda((x_{k+1}, i_{k+1}), (x_{k+2}, i_{k+2}))$ is 0. Hence, the chain is not rising in the k th position. Otherwise, the unbarred permutation has a descent and hence the first coordinate in the labeling $\lambda((x_k, i_k), (x_{k+1}, i_{k+1}))$ is greater than the first coordinate in the labeling $\lambda((x_{k+1}, i_{k+1}), (x_{k+2}, i_{k+2}))$ and hence the chain is not rising. \square

Throughout we will use \mathcal{F}_n to denote the set of all the falling double augmented barred signed permutations in $\overline{\mathfrak{S}}_n^\pm$.

Theorem 5.3. *The Möbius function of the Rees product $\text{Rees}(\mathcal{C}_n, C_{n+1})$ is given by*

$$\mu(\text{Rees}(\mathcal{C}_n, C_{n+1})) = (-1)^n \cdot \sum_c 2^{n-c_1} \binom{n}{c_1, \dots, c_k} \cdot c_1 \cdot \prod_{i=2}^k (c_i - 1),$$

where the sum is over all compositions $c = (c_1, \dots, c_k)$ of n and $1 \leq k \leq n$.

Proof. By Theorem 4.1, to determine the Möbius function of the poset $\text{Rees}(\mathcal{C}_n, C_{n+1})$ it is enough to count the number of falling chains in the poset. Proposition 5.1 allows one to separate the double augmented barred signed permutations corresponding to falling chains into substrings which consist of a sequence of unbarred elements followed by a sequence of barred elements.

By Proposition 5.1, the element 0 will always be part of the first substring and the last substring will consist only of the element $n+1$. Determining the size of each substring is equivalent to taking a composition $c = (c_1, c_2, \dots, c_k)$ of n . Note that the first substring will be of size $c_1 + 1$ to account for the element 0 and the $(k+1)$ st substring will consist only of the element $n+1$.

In each substring there is a sequence of elements without bars followed by a sequence of elements with bars. Given the size of each substring we determine at what place the barred elements begin. In the first substring we can begin the bars at any place, so there are c_1 ways. For all the other substrings the first element cannot be barred, for otherwise it would belong to the previous substring. Thus, we can begin the sequence of barred elements in $c_i - 1$ ways for $i = 2, \dots, k$. The total number of ways to place bars over elements is $c_1 \cdot \prod_{i=2}^k (c_i - 1)$.

Next, we choose the elements that will be in each substring. This is done in $\binom{n}{c_1, c_2, \dots, c_k}$ ways. Now we must sign these elements. Note that the elements in each substring must be arranged in decreasing order. Once we have chosen the signs, this can be done in exactly one way. Furthermore, all of the elements in the first block must be negative because the falling double augmented signed permutation begins with the element 0. This leaves 2^{n-c_1} ways to sign the remaining elements. \square

6. Signed derangement numbers, skew diagrams and a bijective proof

Recall that the derangement number D_n can be expressed as the permanent of an $n \times n$ matrix having 0's on the diagonal and 1's everywhere else. Motivated by this, define the *signed derangement number* D_n^\pm by

$$D_n^\pm = \text{per} \begin{bmatrix} 1 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 1 \end{bmatrix},$$

that is, the permanent of an $n \times n$ matrix having 1's on the diagonal and 2's everywhere else. It is straightforward to see that this permanent enumerates signed permutations $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n^\pm$ having no fixed points, that is, no index i satisfying $\pi_i = i$. See [3, 4] for details.

Lemma 6.1. *For $n \geq 0$, D_n^\pm is the nearest integer to $\frac{2^n \cdot n!}{\sqrt{e}}$.*

Proof. This follows directly from the generating function $\sum_{n \geq 0} D_n^\pm \frac{x^n}{n!} = \frac{e^{-x}}{1-2x}$. \square

In this section we give a bijective proof of the following theorem.

Theorem 6.2. *The Möbius function of the Rees product of the cubical lattice with the chain is given by*

$$\mu(\text{Rees}(\mathcal{C}_n, C_{n+1})) = (-1)^n \cdot n \cdot D_{n-1}^\pm.$$

As a corollary to Theorem 6.2, we can slightly modify our proofs to give a bijective proof of Jonsson's result (Theorem 1.3).

Corollary 6.3. *There is an explicit bijection implying that*

$$\mu(\text{Rees}(B_n, C_n)) = (-1)^{n+1} \cdot D_n.$$

In order to prove Theorem 6.2, we will work with skew diagrams associated to falling double augmented barred and signed permutations. In Section 7 we will use these skew diagrams to describe $\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1}))$, the order complex of the Rees product of the cubical lattice with the chain, in the spirit of Wachs' work with the d -divisible partition lattice [10]. We will also use these diagrams to construct an explicit basis for the homology of $\text{Rees}(\mathcal{C}_n, C_{n+1})$.

Besides the interest in the bijection itself to prove Theorem 6.2, these diagrams allow us to find explicit bases for the integer homology $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})), \mathbb{Z})$ indexed by the falling augmented signed barred permutations.

We begin by recalling some objects from combinatorial representation theory. For background material in this area, we refer to Sagan's book [6]. Let $(\lambda_1, \dots, \lambda_k) \vdash n$ be a partition of the integer n with $\lambda_1 \leq \dots \leq \lambda_k$. Recall the Ferrers diagram of λ consists of n boxes where row i has λ_i boxes for $i = 1, \dots, k$ and all the rows are left-justified. Given two Ferrers diagrams $\mu \subseteq \lambda$, the skew diagram λ/μ is the set of all boxes $\lambda/\mu = \{b : b \in \lambda \text{ and } b \notin \mu\}$.

For us, a *hook* is a skew diagram of the form λ/μ where $\lambda = ((h+1)^v)$ and $\mu = (h^{(v-1)})$. We will be interested in skew diagrams consisting of a disjoint union of hooks. More precisely, let $c = (c_1, \dots, c_k)$ be a composition

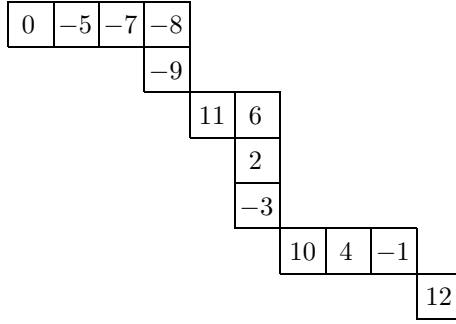


Figure 1: The skew diagram corresponding to the falling double augmented barred signed permutation $\pi = 0 \ -5 \ -7 \ \overline{-8} \ \overline{-9} \ 11 \ \overline{6} \ \overline{2} \ \overline{-3} \ 10 \ 4 \ \overline{-1} \ 12$ in \mathfrak{S}_{11}^{\pm} .

of n with $c_i = u_i + b_i$, for $i = 1, \dots, k$ where $u_1 \geq 0$, $u_i > 0$ for $i = 2, \dots, k$, and $b_i > 0$ for $i = 1, \dots, k$. Form the partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_k)$ where $\lambda_i = (u_1 + \dots + u_i + i)^{b_i}$ for $1 \leq i \leq k$, $\mu_i = ((u_1 + \dots + u_i + i - 1)^{b_i - 1}, u_1 + \dots + u_i + i)$ for $1 \leq i \leq k - 1$, and $\mu_k = (u_1 + \dots + u_k + k - 1)^{b_k - 1}$. The skew diagram λ/μ is then a union of k hooks where the southeast corner of the last box of the i th hook touches the northwest corner of the first box of the $(i + 1)$ st hook. We call such a diagram an *unsigned barred permutation skew diagram*. We call a filling of the n boxes with the elements $\{1, \dots, n\}$ *standard* if the rows are decreasing when read from left to right and the columns are decreasing when read from top to bottom. If we insert a box labelled 0 in front of the first horizontal row and add a box labelled $n + 1$ as the new last hook, then we call such a filled diagram a *standard double augmented unsigned barred skew diagram*. Given a double augmented unsigned barred permutation that is falling, recall that it consists of strings of unbarred and barred elements concatenated together. Given such a falling permutation, one forms the standard skew diagram by representing the first string of unbarred elements as the first horizontal string of boxes in the first hook concatenated with the same number of vertical boxes as the number of barred elements in the first string of the permutation. Note that the i th hook has $u_i + 1$ horizontal boxes, where u_i is the number of unbarred elements in the first string of the permutation. See Figure 1.

Theorem 6.4. *There exists an explicit bijection between the set of all fixed point free permutations in the symmetric group on n elements and the set*

of all standard skew diagrams λ/μ having n boxes and hooks of size greater than 1.

Proof. We describe an algorithm to move between these two sets. The idea is to first break a cycle at the end of each of its descent runs to form blocks. Each of these blocks will become a hook in the resulting skew diagram. The next step is to use the first element of each block (for the first block, use the second element) to determine which elements will be barred in a given block. The third step is to reverse the order of the blocks. The fact that the original first block contained the smallest element in the given cycle will enable us to recover the complete cycle decomposition of a permutation from its skew diagram in the general case when a permutation has more than one cycle.

We first consider the case where $\pi = (\pi_1, \dots, \pi_n) \in \mathfrak{S}_n$ consists of a single cycle of length n with $\pi_1 = 1$, that is, the smallest element of the set $\{\pi_1, \dots, \pi_n\}$.

1. Identify the descents within the cycle. For each run of consecutive descents, say $[i, j] = i, i+1, \dots, j$, break the permutation in front of the last descent in the run, that is, the $(j-1)$ st position provided this does not create a first block having size one.
2. Suppose reading from left to right the first element in the i th block is $m_{i,j}$, where the elements in this block have the linear order $m_{i_1} < m_{i_2} < \dots$. (For the case of the first block, let $m_{1,j}$ be the second element in this block when reading from left to write and where the block elements have linear order $m_{1,1} < m_{1,2} < \dots$.) Rewrite the elements in the block in decreasing order and place bars over each of the last $j-1$ elements.
3. Reverse the order of the blocks, that is, if $B_1|B_2|\dots|B_k$ is the original block decomposition, reverse this to $B_k|B_{k-1}|\dots|B_1$. Finally, remove the vertical block separators.

This yields the union of unsigned hooks, where a hook consists of the run of unbarred elements followed by the run of barred elements.

Example 6.5. As an example, let $\pi = (135764928) \in \mathfrak{S}_9$. We have

$$\begin{aligned}
 \pi &\rightarrow 1357|64|928 \\
 &\rightarrow 1357|46|298 \\
 &\rightarrow 753\bar{1}|6\bar{4}|9\bar{8}\bar{2} \\
 &\rightarrow 9\bar{8}2\bar{6}4753\bar{1}
 \end{aligned}$$

If a permutation consists of more than one cycle, without loss of generality we may assume the permutation is written in standard cycle notation, that is, each cycle is written so that it begins with the smallest element in its cycle and the cycles are then ordered in increasing order by the smallest element in each cycle. Given such a permutation, apply the algorithm to each individual cycle. Concatenate the resulting barred words using the original order of the cycles.

We can reverse this process beginning with a standard unsigned skew diagram.

1. Given a standard unsigned skew diagram, we will separate it into cycles based on the minimal element. Break the diagram after the hook containing the element 1. Next, break the diagram after the hook containing the smallest element occurring to the right of the first break. Then break after the hook containing the smallest element to the right of the second break. Continue this process until there is a break at the end of the diagram. These breaks now correspond to individual cycles in the final permutation.
2. Within each of these breaks, put parentheses around the elements of each hook and reverse the order of the hooks, that is, if break i has hooks $h_{i,1}h_{i,2}\cdots h_{i,j}$ then reverse these to $h_{i,j}h_{i,j-1}\cdots h_{i,1}$.
3. In each parenthetical piece, remove the bars and reorder the elements by the following rule. The now unbarred elements in each parenthesis can be linearly ordered, say $m_{i_1} < m_{i_2} < \cdots < m_{i_k}$. If there were bars over j numbers in this piece, reorder the elements as $m_{i_1}m_{i_{j+1}}m_{i_2}\cdots m_{i_k}$ if $j \neq k$ and $m_{i_1}m_{i_k}m_{i_2}\cdots m_{i_{k-1}}$ if $j = k$.
4. Within each cycle, leave the vertical bars fixed for the moment and switch the first two numbers of all the parenthetical pieces except the first piece which begins the cycle. Remove the inner parentheses and concatenate the pieces within each vertically barred piece into one cycle.

These processes we have described are the inverse of each other. Thus we have a bijection. \square

Example 6.6. Let $8\bar{7}\bar{2}6\bar{1}9\bar{5}4\bar{3}$ be a falling barred permutation. The algorithm gives:

$$\begin{aligned}
 8\bar{7}\bar{2}6\bar{1}9\bar{5}4\bar{3} &\rightarrow 8\bar{7}\bar{2}6\bar{1}|9\bar{5}4\bar{3}| \\
 &\rightarrow (6\bar{1})(8\bar{7}\bar{2})|(4\bar{3})(9\bar{5}) \\
 &\rightarrow (16)(287)|(34)(59) \\
 &\rightarrow (16827)(3495)
 \end{aligned}$$

Let $F \subseteq [n-1]$ be the set of fixed points for a permutation $\pi \in \mathfrak{S}_{n-1}$. We will build n ordered pairs, (F_i, τ) where $i = 1, \dots, n$ and τ is a partial permutation on $n - |F| - 1$ elements from the set $[n]$. Set

$$F_i = \begin{cases} F \cup \{i\} & \text{if } i \notin F, \\ F \cup \{n\} & \text{if } i \in F, \end{cases}$$

where $i = 1, \dots, n$. To define τ , consider the partial permutation $\hat{\pi}$ consisting of the cycles of π with sizes greater than 1. The elements in these cycles can be linearly ordered as $m_{i_1} < m_{i_2} < \dots < m_{i_{n-|F|-1}}$. The elements of $[n] - F_i$ also can be linearly ordered as $l_{i_1} < \dots < l_{i_{n-|F|-1}}$. Define a map Ψ which sends $m_{i_j} \mapsto l_{i_j}$. Set $\tau = \Psi(\hat{\pi})$. Let $F_\pi = \{(F_i, \tau) : i = 1, \dots, n\}$ so that $|F_\pi| = n$.

Proposition 6.7. *There exists a bijection between $\{F_\pi : \pi \in \mathfrak{S}_n\}$ and the set of standard unsigned skew diagrams where each hook except the first has size greater than one.*

Proof. Given a permutation π with fixed point set F and one ordered pair (F_i, τ) , we will define a map which sends F_i to the first hook of the diagram and which sends τ to the rest of the diagram. To create the first part of the map, write the elements of F_i in decreasing order. To place the bars, consider two cases.

1. If $i \notin F$ place bars over the element i and every element less than i .
2. If $i \in F$ we use the linear total order on F , say $f_1 < \dots < f_{|F|}$. We have $i = f_j$ for some $j = 1, \dots, |F|$. Place bars over the smallest j elements.

This map can be reversed given the first piece of some unsigned skew diagram.

To determine the rest of the diagram, we use τ , a partial permutation on an $n - |F| - 1$ element subset of $[n]$. There is a bijection between all such partial permutations and the set of fixed point free permutations in $\mathfrak{S}_{n-|F|-1}$. Use the linear order on the elements of τ , that is, these elements can be written $m_{i_1} < \dots < m_{i_k}$. Let Φ be a map between these two sets where $\Phi(m_{i_j}) = j$. Note that because the partial permutation τ can be written as a product of cycles with no one-cycles, then $\Phi(\tau)$ is also a fixed point free product of cycles. Composing Φ with the algorithm above, we can go from a partial permutation τ to the rest of the diagram having hook sizes greater than 1. \square

To prove Theorem 6.2, we sign the first hook (which consists of the horizontal piece 0 concatenated with the vertical piece) in one way, that is,

with all negative signs, and then reorder the elements in decreasing order. For the remaining hooks, we can sign these remaining elements in $2^{n-|F|-1}$ ways and within each hook reorder them in a decreasing manner in one way.

As a corollary, we can slightly modify our proofs to give a bijective proof of Jonsson's result (Theorem 1.3) for the Möbius function of the Rees product of the Boolean algebra with the chain.

Proof of Theorem 1.3. It is enough to observe that $\text{Rees}(B_n, C_n)$ is isomorphic to the upper order ideal generated by any atom of $\text{Rees}(\mathcal{C}_n, C_{n+1})$. Hence $\text{Rees}(B_n, C_n)$ inherits the R -labeling of $\text{Rees}(\mathcal{C}_n, C_{n+1})$. The maximal chains in $\text{Rees}(B_n, C_n)$ are described by augmented barred permutations, that is, permutations of the form $\pi = \pi_1 \cdots \pi_n \pi_{n+1}$ with $\pi_{n+1} = n+1$, $|\pi| = |\pi_1| \cdots |\pi_n| \in \mathfrak{S}_n$ (unlike before, here $|\pi_j|$ denotes removing any bar and negative sign occurring in π_j), π_1 not barred and each of the elements π_2, \dots, π_n may be barred. The falling chains correspond to unsigned labeled skew diagrams having hooks of size greater than or equal to 2 which are augmented at the end with a block containing the element $n+1$. Theorem 6.4 now applies to prove the result. \square

Shareshian and Wachs [7, Theorem 6.2] have proved a dual version of Proposition 3.2 where they instead work with a doubly-truncated face lattice of the crosspolytope \mathcal{O}_n . To state their results we use $\text{Rees}^-(P, Q)$ to indicate the maximal element is removed from P before taking the Rees product of two graded posets P and Q , that is, $\text{Rees}^-(P, Q) = \text{Rees}(P - \{\hat{1}\}, Q)$.

Theorem 6.8 (Shareshian–Wachs). *For all n ,*

$$\dim \tilde{H}_{n-1}(\Delta(\text{Rees}^-(\mathcal{O}_n, C_n))) = D_n^\pm.$$

Shareshian and Wachs' original proof follows from the Björner–Welker Theorem 1.2 and from the fact that the reduced homology of a Cohen–Macaulay poset vanishes everywhere except the top dimension, where the dimension is given by the Möbius function of the poset. One can also give a bijective proof along the lines of Theorem 6.4 using the standard R -labeling of the cross-polytope. For q -analogues of Theorems 1.3 and 6.8, see [7, Theorem 2.1.6 and Theorem 2.4.5].

7. A basis for the homology

Let P be a graded poset of rank n with minimal element $\hat{0}$ and maximal element $\hat{1}$. The *order complex* (or *chain complex*) of P , denoted $\Delta(P)$, is the simplicial complex with vertices given by the elements of $P - \{\hat{0}, \hat{1}\}$

and $(i - 1)$ -dimensional faces are given by chains of i elements $x_1 < x_2 < \cdots < x_i$ in the subposet $P - \{\hat{0}, \hat{1}\}$. See [11] for further details. In this section we consider homological questions for the order complex of the poset $\text{Rees}(\mathcal{C}_n, C_{n+1})$. A similar analysis for the d -divisible partition lattice was done by Wachs [10].

Proposition 7.1. *The order complex $\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1}))$ is Cohen-Macaulay and has vanishing homology groups in every dimension except for the top dimension. This is given by*

$$\dim \tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1}))) = n \cdot D_{n-1}^{\pm}.$$

This follows by a result of Björner and Welker [2] that the Rees product of any two Cohen-Macaulay posets is also Cohen-Macaulay. Furthermore, the absolute value of the Möbius function of the poset $\text{Rees}(\mathcal{C}_n, C_{n+1})$ gives the dimension of the top homology group of $\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1}))$.

We next give an explicit basis for the homology $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$, \mathbb{Z} indexed by the falling augmented signed barred permutations. Recall that \mathcal{F}_n denotes the set of falling augmented signed barred permutations from \mathfrak{S}_n^{\pm} . For each $\sigma \in \mathcal{F}_n$ we define a subposet \mathcal{C}_σ of $\text{Rees}(\mathcal{C}_n, C_{n+1})$ as follows. Let $m_\sigma = m_{\sigma,0} \prec m_{\sigma,1} \prec \cdots \prec m_{\sigma,n}$ be the chain in $\text{Rees}(\mathcal{C}_n, C_{n+1}) - \{\hat{0}, \hat{1}\}$ labeled by $\sigma \in \mathcal{F}_n$. For example, for the double augmented barred signed permutation $\sigma = \sigma_0 \cdots \sigma_6 = 0 \ -1 \ \overline{-3} \ 5 \ \overline{2} \ \overline{-4} \ 6$, we have $m_\sigma = (01001, 1) \prec (*1001, 1) \prec (*1*01, 2) \prec (*1*0*, 2) \prec (**0*, 3) \prec (****, 4)$.

We define the elements of \mathcal{C}_σ recursively. The rank 0 elements of \mathcal{C}_σ are of the form $(x, 1)$, where x is a 0-dimensional face of the n -cube. For $1 \leq i \leq n - 1$, the rank i elements of \mathcal{C}_σ are of the form (x, j) , where x is an i -dimensional face of the n -cube and the second coordinate j is determined according to the following rules:

- i. If σ_{i-1} is not barred, σ_i is not barred, and σ_{i+1} is either barred or unbarred, then $j = k$ where (y, k) is any rank $i - 1$ element of \mathcal{C}_σ .
- ii. If σ_{i-1} is either barred or unbarred, and both σ_i and σ_{i+1} are barred, then $j = k + 1$ where (y, k) is any rank $i - 1$ element of \mathcal{C}_σ .
- iii. If σ_{i-1} is either barred or unbarred, σ_i is barred and σ_{i+1} is not barred, then $j = k$ where (y, k) is a rank $i - 1$ element of \mathcal{C}_σ . The exception to this rule is for the i -dimensional element x occurring in the chain m_σ , that is, $m_{\sigma,i} = (x, r)$. In this case, $m_{\sigma,i}$ becomes the element $(x, k + 1)$ in \mathcal{C}_σ .
- iv. If σ_{i-1} is barred, σ_i is not barred, and σ_{i+1} is either barred or unbarred, then $j = k + 1$ where (y, k) is any rank $i - 1$ element of \mathcal{C}_σ different

from $m_{\sigma,i-1}$. Notice that both $m_{\sigma,i-1}$ and $m_{\sigma,i}$ have the same second coordinate, namely $k + 1$.

Finally, there are two rank n elements $(*\cdots*, k)$ and $(*\cdots*, k + 1)$, where k is the second coordinate of any rank $n - 1$ element of \mathcal{C}_σ .

Define $\widetilde{\mathcal{C}}_n$ to be the poset $\mathcal{C}_n - \{\hat{0}\} \cup \{\hat{1}'\}$, that is, the face lattice of the n -dimensional cube with its minimal element removed and adjoined with a second maximal element $\hat{1}'$ which also covers all the coatoms in $\mathcal{C}_n - \{\hat{0}\}$.

Theorem 7.2. *For $\sigma \in \mathcal{F}_n$, the order complex $\Delta(\mathcal{C}_\sigma)$ is isomorphic to the suspension of the barycentric subdivision of the boundary of the n -cube.*

Proof. It is enough to show the posets \mathcal{C}_σ and $\widetilde{\mathcal{C}}_n$ are isomorphic. Define the “forgetful” map $f : \mathcal{C}_\sigma \rightarrow \widetilde{\mathcal{C}}_n$ which sends an element $(x, k) \in \mathcal{C}_\sigma$ to the element x for elements of ranks 1 through $n - 1$ in \mathcal{C}_{n-1} . For the two rank n elements, let $f(*\cdots*, j_n) = \hat{1}$ and $f(*\cdots*, j_n + 1) = \hat{1}'$. Clearly the map f is a bijection from the elements of \mathcal{C}_σ to those of $\widetilde{\mathcal{C}}_n$. Additionally, f is order-preserving since for $(y, k) \prec (x, j)$ in \mathcal{C}_σ , one has $y \prec x$ in the cubical lattice \mathcal{C}_n .

To define the inverse map f^{-1} , one follows the described scheme to determine the second coordinate as above. Note that for elements x and y with $y \prec x$ in \mathcal{C}_n and $\rho(y) < n$, the inverse map satisfies $f^{-1}(y) = (y, k) \prec f^{-1}(x) = (x, j)$ since $k \leq j$ by construction. The two maximal elements of $\widetilde{\mathcal{C}}_n$ are easily seen to be mapped to the two maximal elements of \mathcal{C}_σ , so the bijection is order-preserving as desired. \square

Corollary 7.3. *For $\sigma \in \mathcal{F}_n$, the order complex $\Delta(\mathcal{C}_\sigma)$ is homotopy equivalent to the suspension of the $(n - 1)$ -dimensional sphere S^{n-1} .*

Proof. The order complex of \mathcal{C}_n is the barycentric subdivision of the boundary of the n -cube. The boundary of the n -cube is homotopic to S^{n-1} . The poset $\widetilde{\mathcal{C}}_n$ differs from $\mathcal{C}_n - \{\hat{0}, \hat{1}\}$ by the addition of two maximal elements $\hat{1}$ and $\hat{1}'$. Therefore, the order complex of $\widetilde{\mathcal{C}}_n$ is found from $\Delta(\mathcal{C}_n)$ by forming two $(k + 1)$ -dimensional faces on the vertices $V(\psi) \cup \{\hat{1}\}$ and $V(\psi) \cup \{\hat{1}'\}$, where $V(\psi)$ are the vertices of a k -face ψ in $\Delta(\mathcal{C}_n)$. This is a suspension over the barycentric subdivision of the boundary of the n -cube which is homotopic to the suspension of S^{n-1} . Thus, by Theorem 7.2 we then have $\Delta(\mathcal{C}_\sigma) \cong \Delta(\widetilde{\mathcal{C}}_n)$ and we have proven the corollary. \square

The suspension of S^{n-1} is homotopic to S^n , and as a result $\Delta(\mathcal{C}_\sigma)$ is a triangulation of the n -sphere. Let ρ_σ denote a fundamental cycle of the spherical complex $\Delta(\mathcal{C}_\sigma)$. To show that the set $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ forms a basis for $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$, we first place a total order on \mathcal{F}_n . Let

$\sigma = 0\sigma_1 \cdots \sigma_n n + 1$ and $\tau = 0\tau_1 \cdots \tau_n n + 1$ be two permutations from \mathcal{F}_n . If the entries $\sigma_1, \dots, \sigma_{i-1}$ and $\tau_1, \dots, \tau_{i-1}$ are unbarred, σ_i is barred and τ_i is unbarred, then we say $\sigma > \tau$. Otherwise, if σ and τ are barred and unbarred at exactly the same places and the permutation σ without the bars is lexicographically greater than the permutation τ without its bars, then we say $\sigma > \tau$. We then have the following result:

Lemma 7.4. *If m_τ is a maximal chain in \mathcal{C}_σ then $\tau \leq \sigma$.*

Proof. Let σ and τ be permutations in \mathcal{F}_n with $\tau > \sigma$. We want to show m_τ is not a chain in \mathcal{C}_σ . There are two cases to consider.

First suppose σ and τ are barred at precisely the same locations and that the unbarred permutation τ is lexicographically greater than the unbarred permutation σ . Let i be the least index where σ_i is barred and σ_{i+1} is not barred. If such an i does not exist, then each permutation corresponds to a diagram consisting of one hook and as such has a labeling $-1 \cdots -n$, implying $\sigma = \tau$, a contradiction. So we may assume such an i satisfying $1 \leq i < n$ exists. We see the first i elements in the chain m_τ are elements in \mathcal{C}_σ . However, the i th element $m_{\tau,i} = (x, r_{\sigma,i} + 1)$ where x is the unique rank i element in $\mathcal{C}_n - \{\hat{0}\}$ given by the unbarred permutation τ and $r_{\sigma,i}$ is the number of bars over elements $\sigma_1, \sigma_2, \dots, \sigma_i$, will not be an element in \mathcal{C}_σ . (Note, the second coordinate $r_{\sigma,i} + 1$ in $m_{\sigma,i}$ is the same as the i th second coordinate $r_{\tau,i} + 1$ in $m_{\tau,i}$ for all i .) We can see this by observing that the only element in \mathcal{C}_σ with first coordinate a rank i element in $\mathcal{C}_n - \{\hat{0}\}$ and with second coordinate $r_{\sigma,i} + 1 = r_{\tau,i} + 1$ corresponds to the unique element given by the unbarred σ . Thus, since the unbarred σ is not equal to the unbarred τ , m_τ is not a chain in \mathcal{C}_σ .

For the second case, suppose that σ_j is barred if and only if τ_j is barred for $j = 1, \dots, i-1$ and τ_i is barred while σ_i is not barred. We claim the i th element $m_{\tau,i}$ in m_τ is not an element of \mathcal{C}_σ . Note the second coordinate $r_{\tau,i} + 1$ in $m_{\tau,i}$ is the same as the second coordinate $r_{\sigma,i} + 1$ in $m_{\sigma,i}$ where $j = 1, \dots, i-1$ because the pattern of bars coincides for the first $i-1$ terms in the permutations. However, in \mathcal{C}_σ all rank i elements $\mathcal{C}_n - \{\hat{0}\}$ have second coordinate r_{i-1} . As there is no bar over σ_i , the second coordinate does not increase. Since the element τ_i is barred, the element $(x, r_{\sigma,i-1} + 1)$ is an element in the chain m_τ but not in the poset \mathcal{C}_σ . \square

Theorem 7.5. *The set $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ forms a basis for the integer homology of $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$.*

Proof. To show that $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ are linearly independent, let $\sum_{\sigma \in \mathcal{F}_n} a_\sigma \rho_\sigma = 0$ where $a_\sigma \in \mathbf{k}$. With respect to the total order we have described above, suppose τ is the greatest element of \mathcal{F}_n for which $a_\tau \neq 0$. We apply Lemma 7.4

to derive a contradiction. We have for a maximal chain m_τ in C_σ

$$0 = \sum_{\sigma \in \mathcal{F}_n} a_\sigma \rho_\sigma|_{m_\tau} = \sum_{\sigma \in \mathcal{F}_n, \sigma \leq \tau} a_\sigma \rho_\sigma|_{m_\tau} = a_\tau \rho_\tau|_{m_\tau} = \pm a_\tau,$$

since the fundamental cycle evaluated at a facet has coefficient ± 1 . However, this gives a contradiction. Since the rank of $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ is equal to $|\mathcal{F}_n|$, we have proven the basis result when \mathbf{k} is a field.

When $\mathbf{k} = \mathbb{Z}$, linear independence of $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ implies this set is also linearly independent over the rationals \mathbb{Q} and hence that it spans $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ over \mathbb{Q} . Let $\rho \in \tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$. Then $\rho = \sum_{\sigma \in \mathcal{F}_n} c_\sigma \rho_\sigma$ for $c_\sigma \in \mathbb{Q}$. We will show $c_\sigma \in \mathbb{Z}$ for all $\sigma \in \mathcal{F}_n$. Suppose τ is the greatest element of \mathcal{F}_n for which $c_\tau \neq 0$. Then by Lemma 7.4

$$\rho|_{m_\tau} = \sum_{\sigma \in \mathcal{F}_n, \sigma \leq \tau} c_\sigma \rho_\sigma|_{m_\tau} = c_\tau \rho_\tau|_{m_\tau} = \pm c_\tau.$$

Since $\rho \in \tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$, we have $\rho|_{m_\tau} \in \mathbb{Z}$. Thus $c_\tau \in \mathbb{Z}$ and $\rho - c_\tau \rho_\tau \in \tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$. Repeat this argument for $\rho - c_\tau \rho_\tau$ to conclude $c_\nu \in \mathbb{Z}$ and $\rho - c_\tau \rho_\tau - c_\nu \rho_\nu \in \tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ for ν the next to the last element in the total order on \mathcal{F}_n for which $c_\nu \neq 0$. Since there are finitely-many elements in \mathcal{F}_n , we may conclude that $c_\sigma \in \mathbb{Z}$ for all $\sigma \in \mathcal{F}_n$. Hence $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ spans $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ over \mathbb{Z} and thus is a basis for $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ over \mathbb{Z} . \square

8. Representation over \mathfrak{S}_n

In this section we develop a representation of $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ over the symmetric group. This can be done using a set of skew Specht modules.

The homology of the order complex of $\text{Rees}(\mathcal{C}_n, C_{n+1})$ is an \mathfrak{S}_n -module in the following manner. A signed permutation $\pi \in \overline{\mathfrak{S}}_n^\pm$ corresponds to a labeled maximal chain of the poset $\text{Rees}(\mathcal{C}_n, C_{n+1})$. A permutation $\tau \in \mathfrak{S}_n$ acts on the chains of $\text{Rees}(\mathcal{C}_n, C_{n+1})$ by sending the maximal chain labeled with π to the maximal chain whose labels are $\tau\pi$. Note that under the action of τ the placement of the bars is fixed and the signs remain attached to the same numbers. This action induces an action on the faces of $\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1}))$ and even further on the homology group itself whose basis is indexed by a subset of chains in $\text{Rees}(\mathcal{C}_n, C_{n+1})$. We have $\tau\rho_\pi = \rho_{\tau\pi}$ for any basis element $\rho_\pi \in \tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$.

Theorem 8.1. *There exists an \mathfrak{S}_n -module isomorphism between*

$$\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1}))) \text{ and } \bigoplus 2^{n-|\lambda_1|} S^\lambda,$$

where the direct sum is over all partitions λ with each λ_i shaped into hooks as described in Section 7 taken with multiplicity $2^{n-|\lambda_1|}$.

To prove this result, we will need some tools from combinatorial representation theory. For more details and background information, see [6].

Recall that two tableaux t_1 and t_2 of shape λ are *row equivalent*, written $t_1 \sim t_2$, if the entries in each row of t_1 are a permutation of the entries in the corresponding row of t_2 . A *tabloid of shape λ* (λ -*tabloid* or *tabloid*, for short) is then an equivalence class $\{t\}$. For a fixed partition λ we denote by M^λ the k -vector space having λ -tabloids as a basis. In the usual way a permutation $\sigma \in \mathfrak{S}_n$ acts on a λ -tableau by replacing each entry by its image under σ . Thus σ acts on a λ -tabloid $\{t\}$ by $\sigma\{t\} = \{\sigma t\}$. For a tableau t of shape λ , the *polytabloid* corresponding to t is

$$e_t = \sum_{\sigma \in C_t} \text{sgn}(\sigma) \{\sigma t\},$$

where the sum is over all permutations belonging to the column stabilizer C_t of t .

The *Specht module* S^λ is the submodule of M^λ spanned by the polytabloids e_t , where t has shape λ . The Specht module S^λ is an \mathfrak{S}_n -module in the following manner. A permutation $\tau \in \mathfrak{S}_n$ acts linearly on the elements of S^λ by permuting the entries of t , that is, $\tau e_t = e_{\tau t}$.

Specht modules were developed to construct all irreducible representations of the symmetric group over \mathbb{C} . We will use these modules to give a representation of $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ over \mathfrak{S}_n .

Recall that a tableau t is said to be *standard* if the entries are increasing in each row and column of t . The following theorem is originally due to Young, though not in this form. It is also due to Specht.

Theorem 8.2 (Specht, Young). *The set*

$$\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$$

is a basis for S^λ .

With these definitions in mind, we can begin the proof of Theorem 8.1. Let $\nu = \lambda - \mu$ be a skew diagram consisting of the union of k hooks, as

described in Section 4, where the i th hook has size $|\nu_i|$. We consider the case where $\nu = \nu_1 \cdots \nu_k$ is fixed.

Define a new set $\mathcal{F}_n^- = \{-\sigma : \sigma = \sigma_1 \cdots \sigma_n \in \mathcal{F}_n\}$ where $-\sigma = -\sigma_1 - \sigma_2 \cdots -\sigma_n$. It is easily noted there exists a bijection between \mathcal{F}_n^- and \mathcal{F}_n . We use this bijection to move between basis elements of $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ which correspond to decreasing labeled skew shapes and standard tableaux which have increasing labels.

Consider the usual unsigned Specht module S^λ in the case λ is composed of hooks of size at least two and is augmented at the end with a block containing the element $n + 1$. It is generated by polytabloids which are indexed by standard labelings of λ . Define an \mathfrak{S}_n -module homomorphism

$$\theta : S^\lambda \rightarrow \tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$$

where $e_t \mapsto \rho_{-\sigma}$ for t a standard λ -tableau and $\sigma \in \mathcal{F}_n^-$ is found by writing the labels on t from left to right and by placing bars over all elements which occur in the rightmost columns of a hook.

We wish to extend this map over “signings” of S^λ . Given a standard polytabloid $e_t \in S^\lambda$ where t is a standard tableau, we sign the elements occurring in the last $k - 1$ hooks of t , that is, sign the labels on $\lambda_2, \dots, \lambda_k$. This can be written as a subset $A \subset [n] - \{\lambda_1\}$ where A corresponds to the elements in t labeled with a negative sign. For each e_t there are $\sum_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j}$ such signings, or equivalently, such subsets A . We let e_t^A denote the signing by A of the polytabloid e_t . Using the binomial theorem, there is an isomorphism

$$\bigoplus_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j} S^\lambda \cong 2^{n-|\lambda_1|} S^\lambda.$$

This is an $\mathbb{C}\mathfrak{S}_n$ -module with action $\pi e_t^A = e_{\pi t}^A$ where $\pi \in \mathfrak{S}_n$ permutes the labels of the tableau t .

To show each Specht module S^λ occurs with multiplicity $2^{n-|\lambda_1|}$ in the top homology group, we extend the map θ to $\theta : \sum_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j} S^\lambda \rightarrow \tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ where basis elements are mapped by $e_t^A \mapsto \rho_{-\sigma}$. The permutation σ is found by attaching negative signs to the labels in t which are also in A . Then the labels in each hook are written in increasing order. As before, we form the permutation σ by writing down the labels reading from left to right with bars placed over the rightmost element in every row. Note that when A is empty, we are back in the usual unsigned case.

Set $E^\lambda = \{e_t^A\}$ where t ranges over all standard Young tableaux of shape λ and A ranges over all subsets of $[n] - \{\lambda_1\}$.

Proposition 8.3. *The map*

$$\theta : E^\lambda \longrightarrow \{\rho_\sigma | \sigma \in \mathcal{F}_n \text{ and } sh(\sigma) = \lambda\}$$

is a bijection.

Proof. Let θ' be a map from $\{\rho_\sigma : \sigma \in \mathcal{F}_n \text{ and } sh(\sigma) = \lambda\}$ to E^λ . Given $\sigma \in \mathcal{F}_n$ with shape λ , we will define $\theta'(\rho_\sigma) = e_t^A$ such that $\theta(e_t^A) = \rho_\sigma$.

Set $\theta'(\rho_\sigma) = e_t^A$ by labeling λ from left to right with the elements of $-\sigma$. Then in each hook, rearrange the labels so the absolute value of these labels is increasing. Call this labeling t' . The subset A is determined by the negatively-labeled elements in t' and t is given by the absolute value of t' .

One can check $\theta(\theta'(\rho_\sigma)) = \rho_\sigma$ and $\theta'(\theta(e_t^A)) = e_t^A$. \square

Proposition 8.3 can be extended by linearity to a vector space isomorphism between the two spaces.

We sum over all possible partitions and signings of λ to get a bijection between basis elements of $\sum_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j} S^\lambda$ and the basis elements of $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$ to conclude the following corollary.

Corollary 8.4. *The map*

$$\theta : \{E^\lambda\}_\lambda \longrightarrow \{\rho_\sigma | \sigma \in \mathcal{F}_n\}$$

is a bijection where λ ranges over all skew diagrams which are finite unions of hooks of size at least two augmented at the end by a block containing the element $n+1$.

Again we extend by linearity to a vector space isomorphism between these two spaces. It is left to prove the module isomorphism properties in order to prove Theorem 8.1. First, we look at which elements of \mathfrak{S}_n fix basis elements of $\sum_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j} S^\lambda$ and $\tilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$.

For a given tableau t , define $S_t = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ where the λ_i are subsets of $[n]$ corresponding to the labels of the i th hook t .

Claim 8.5. *The polytabloid e_t^A satisfies $e_t^A = e_{\pi t}^A$ for all $\pi \in S_t$.*

Proof. Given such a permutation $\pi \in S_t$, it acts on t by permuting labels only within individual hooks of t . If the labels within a row are permuted, the polytabloid is fixed because each tabloid is a row equivalence class. If the labels within a column are permuted, π acts as an element of the column stabilizer C_t . For such an element π we have $e_t = e_{\pi t}$. Lastly, if an element in a column is moved out of its column but within its row because of the

equivalence class, we can rewrite the tabloid with that element occurring at the end of the row, leaving π to act as an element of C_t . \square

Claim 8.6. *The fundamental cycle ρ_σ satisfies $\rho_\sigma = \rho_{\pi\sigma}$ for all $\pi \in S_t$.*

Proof. It is enough to show the posets \mathcal{C}_σ and $\mathcal{C}_{\pi\sigma}$ are isomorphic to prove the equality of the fundamental cycles of their order complexes ρ_σ and $\rho_{\pi\sigma}$. The elements of the posets \mathcal{C}_σ and $\mathcal{C}_{\pi\sigma}$ have bars in the same places and negative signs with the same numbers, so it is left to consider the ranks in the poset where one element of rank i for some i has a different second coordinate from all other elements of that rank. If the set of ranks with this property is the same in \mathcal{C}_σ and $\mathcal{C}_{\pi\sigma}$, the two posets are isomorphic. In σ or $\pi\sigma$ an element having rank i must correspond to a label at the end of a piece j for some j . In \mathcal{C}_σ , this element will have stars in positions corresponding to labels in the first j places in the permutation. This is the same in $\mathcal{C}_{\pi\sigma}$ because π only permutes elements within individual pieces. The fixed negative signs assure the non-starred elements are the same in both. Thus, we have $\mathcal{C}_\sigma = \mathcal{C}_{\pi\sigma}$. \square

We now prove Theorem 8.1. For $\pi \in S_t$, we have $\pi\theta(e_t^A) = \theta(\pi e_t^A)$. That is,

$$\pi\theta(e_t^A) = \pi\rho_\sigma = \rho_{\pi\sigma} = \rho_\sigma = \theta(e_t^A) = \theta(e_{\pi t}^A) = \theta(\pi e_t^A).$$

It is left to show this relationship holds for $\tau \in S_n - S_t$. In fact, it is enough to show $\theta(\tau e_t^A) = \pi\tau\rho_\sigma$ for some $\pi \in S_t$.

Consider $\theta(\tau e_t^A)$ and $\tau\rho_{-\sigma}$ for some $\tau \in S_n - S_t$ and some e_t^A such that $\theta(e_t^A) = \rho_\sigma$. The permutation τ acts on t by permuting the labels. The polytabloid $e_{\tau t}^A$ is a sum of tabloids under action by the column stabilizer C_t . Hence, we are only concerned with cycles of τ which move labels from one hook of t to another hook of t . Let θ take $e_{\tau t}^A$ onto $\rho_{-\hat{\sigma}}$. We know $\hat{\sigma}$ is found by attaching the signs from A to t and reordering so each piece is decreasing, and τ acts on σ also by permuting the labels. There is no guarantee that $\tau\sigma$ will have hooks each of which having labels in decreasing order. However, we can find a permutation $\pi \in S_t$ such that $\pi\tau\sigma$ will have hooks whose labels are in decreasing order. Since there is only one way to write a set of integers in decreasing order, it is left to show the labels on each hook of $\hat{\sigma}$ are the same as the labels on the corresponding hook of $\tau\sigma$. (Hooks of $\sigma \in \overline{\mathfrak{S}}_n^\pm$ correspond to the hooks in the λ associated with σ .) If label l is in a different hook in $\hat{\sigma}$ than in $\tau\sigma$, then τt mapped l to a different hook than $\tau\sigma$. This is a contradiction because labels in t are in the same corresponding hooks as labels in σ . Hence, $\hat{\sigma} = \pi\tau\sigma = \tau\sigma$ and $\theta(\tau e_t^A) = \rho_{-\tau\sigma}$.

The isomorphism $\bigoplus_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j} S^\lambda \cong 2^{n-|\lambda_1|} S^\lambda$ induces the desired module isomorphism $2^{n-|\lambda|} S^\lambda \cong \widetilde{H}_n(\Delta(\text{Rees}(\mathcal{C}_n, C_{n+1})))$. Thus, we have proved Theorem 8.1.

9. Concluding remarks

What poset P would have its Möbius function related to the permanent of a matrix having s 's occur on the diagonal and r 's in the off-diagonal entries? The case when $s = r - 1$ is the Rees product of the r -cubical lattice with the chain, the case $(r, s) = (1, 0)$ corresponds to the Rees product of the Boolean algebra with the chain, and $(r, s) = (2, 1)$ to the Rees product of the cubical lattice with the chain.

The derangement numbers occur as the local h -vector of the barycentric subdivision of the n -simplex [9]. Is there a relation between the local h -vector and the Rees product?

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References

- [1] A. Björner and M. Wachs, Shellable nonpure complexes and posets. I, *Trans. Amer. Math. Soc.* **348** (1996), no. 4, 1299–1327. [MR1333388](#)
- [2] A. Björner and V. Welker, Segre and Rees products of posets, with ring-theoretic applications, *J. Pure Appl. Algebra* **198** (2005), 43–55. [MR2132872](#)
- [3] W. Chen and J. Zhang, The skew and relative derangements of type B , *Electron. J. Combin.* **14** (2007), #N24. [MR2365968](#)
- [4] C.-O. Chow, On derangement polynomials of type B , *Sém. Lothar. Combin.* **55** (2006), Article B55b. [MR2223025](#)
- [5] J. Jonsson, The Rees product of a Boolean algebra and a chain, preprint 2008.
- [6] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Springer-Verlag, New York, Inc., 2001. [MR1824028](#)

- [7] J. Shareshian and M. Wachs, Poset homology of Rees products, and q -Eulerian polynomials, *Electron. J. Combin.* **16**(2) (2009), #R20. [MR2576383](#)
- [8] R. P. Stanley, *Enumerative Combinatorics, Vol. I*, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986. [MR0847717](#)
- [9] R. P. Stanley, Subdivisions and local h -vectors, *J. Amer. Math. Soc.* **5** (1992), 805–851. [MR1157293](#)
- [10] M. Wachs, A basis for the homology of d -divisible partition lattices, *Adv. Math.* **117** (1996), 294–318. [MR1371655](#)
- [11] M. Wachs, Poset topology: Tools and applications, in *Geometric Combinatorics* (E. Miller, V. Reiner, B. Sturmfels, eds.), *IAS/Park City Math Series, 13*, Amer. Math. Soc., Providence, RI, 2007, 497–615. [MR2383132](#)

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