Lattices of paths:
Representation theory and valuations

Luca Ferrari and Emanuele Munarini

We study some distributive lattices arising in the combinatorics of lattice paths. In particular, for the Dyck, Motzkin and Schröder lattices we describe the spectrum and we determine explicitly the Euler characteristic in terms of natural parameters of lattice paths.

1. Introduction

The set of all lattice paths of equal length with steps of a prescribed kind (usually starting from the origin and ending on the x-axis with respect to a fixed Cartesian coordinate system) can always be ordered by containment. More precisely, we say that $\gamma_1 \leq \gamma_2$ when γ_1 lies weakly below γ_2. In several cases, the resulting poset has the structure of a distributive lattice, as for *Dyck*, *Motzkin* and *Schröder paths*. Some conditions that guarantee to have a distributive lattice are given in [13], where there is also a first attempt to provide a systematic classification of these posets of paths. Some of these lattices turn out to be isomorphic with the lattices arising from other well known structures [3, 4, 12]. For instance, the lattice generated by all Dyck paths of length n is isomorphic to the dual of the Young lattice associated with the staircase partition $(n, n-1, \ldots, 2, 1)$, as proved in [24] or in [22] where a different combinatorial interpretation is provided. The language of lattice paths, however, gives a geometric flavor to the subject which allows to express several properties in a more fascinating way. Moreover, the study of lattices in terms of paths is a growing area of research, as witnessed by the investigation of a generalization of Dyck lattices appearing in [19] or by the deep study of the properties of Dyck lattices developed in [20].

In the present paper we consider essentially two kinds of questions concerning lattices of paths. The first topic concerns the representation theory of lattices of *Dyck-like paths*, i.e. paths defined as the ordinary Dyck paths except for the fact that they use up steps of the form $(1, a)$ and down steps

of the form $(1, -b)$, where a and b are two assigned positive integers. Specifically, we obtain a representation theorem for these lattices describing explicitly their spectrum. We also prove that these lattices can be described as dual of Young lattices associated to a given partition. Then we review some results scattered in the literature and, in particular, for the Dyck lattices we observe some elementary properties which will be extensively used in the rest of the paper.

The second topic concerns the *Euler characteristic* of lattices of paths. The Euler characteristic is a classical invariant measure which plays an important role, for instance, in *combinatorial geometry* [16] and in *geometric probability* [15]. The combinatorial interest of the Euler characteristic lies in its deep relation with the Möbius function [18] (see also [17]). In the present paper, we give a general technique to determine the Euler characteristic of *Dyck-like lattices*, which generalize in a natural way the ordinary Dyck lattices. In particular, in the case of Dyck and Schröder lattices, we obtain a combinatorial interpretation for the Euler characteristic in terms of the number of *tunnels* of a path [10, 11]. The case of Motzkin lattices cannot be dealt with using the machinery developed for Dyck-like lattices. So, we provide an *ad hoc* argument which allows to obtain a (not obvious) combinatorial interpretation of the Euler characteristic also for these lattices.

2. Basic definitions and properties

As usual, $\mathbb{N}$ denotes the set of all non-negative integers, $\mathbb{Z}$ denotes the set of all integers, $[n] = \{1, 2, \ldots, n\}$ and $[0, n] = \{0, 1, \ldots, n\}$. For any real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x, whereas $\lceil x \rceil$ is the least integer greater than or equal to x. In the rest of this section, we will recall some basic definitions concerning lattice paths and the theory of partially ordered sets [1, 5, 8].

2.1. Lattice paths

Given a finite subset $\Gamma \subseteq \mathbb{Z}$, a Γ-*path* of length n is a function $f : [0, n] \to \mathbb{N}$ such that $f(0) = f(n) = 0$ and $f(k+1) - f(k) \in \Gamma$, for every $k \in [0, n-1]$. Equivalently, a Γ-path of length n is a sequence of n steps $(1, k)$, with $k \in \Gamma$, starting from the origin, ending on the x-axis and never going below that axis.

Considered as functions, Γ-paths of length n can be ordered coordinate-wise setting $f \leq g$ whenever $f(k) \leq g(k)$ for every $k \in [0, n]$. In this way, we obtain a poset $\mathcal{C}_n^{\Gamma}$ which, under suitable conditions [13], turns out to be a lattice. In particular, $\mathcal{C}_n^{\Gamma}$ is a distributive lattice when $\Gamma = \{-b, a\}$, with

$a, b \in \mathbb{N}$. Here $\{-b, a\}$-paths will be called *Dyck-like paths of type* (a, b), since they are a natural generalization of ordinary Dyck paths (corresponding to the case $a = b = 1$). Dyck-like paths have already been considered in [9], where several results are proved and the case $(a, b) = (3, 2)$ is examined in great detail, and also in [2], where they are used as a source of examples.

We will use U, H and D to denote up steps, horizontal steps and down steps. Given a path γ having precisely one type of up step and one type of down step, a *peak* of γ is just a sequence formed by an up step and a down step. An *elevated Dyck path* is a Dyck path touching the x-axis just at its starting and ending points.

In the sequel, a path will be denoted by a Latin letter when considered as an element of a lattice and by a Greek letter in all other cases.

2.2. Partially ordered sets

In a poset P, an element x is *covered* by an element y when $x \leq u \leq y$ implies $x = u$ or $u = y$. A poset P is *ranked* when it admits a *rank function*, that is a function $r : P \to \mathbb{N}$ such that $r(y) = r(x) + 1$ whenever x is covered by y. For finite posets the function r is usually chosen so that the minimal elements have rank 0. The *height* of P is its maximum rank.

A map $f : P \to Q$ between two posets P and Q is *order-preserving* when $x \leq y$ implies $f(x) \leq f(y)$ for every $x, y \in P$. In particular, it is a *poset isomorphism* when it is an order-preserving bijection, whose inverse is order-preserving, too. In this paper, all isomorphisms between posets are always understood to be poset isomorphisms.

A *join-semilattice* (*meet-semilattice*) is a poset in which there exists the supremum (the infimum) of any two elements. A lattice is a poset in which there exists the supremum and the infimum of any two elements.

A poset P has a *minimum* $\widehat{0}$ (*maximum* $\widehat{1}$) when it has only one minimal (maximal) element. In a poset P with minimum (maximum), an *atom* (*coatom*) is an element covering the minimum (covered by the maximum). In a (finite) lattice L, the *socle* is the join of all atoms and the *radical* is the meet of all coatoms.

An *order-ideal* of a poset P is a subset I such that $x \in I$ and $u \leq x$ imply $u \in I$. The *principal ideal* $\downarrow x$ generated by an element $x \in P$ is the set of all elements $u \in P$ such that $u \leq x$. Similarly, the *principal filter* $\uparrow x$ generated by an element $x \in P$ is the set of all elements $u \in P$ such that $u \geq x$.

A *join-irreducible element* of a distributive lattice $\mathcal{D}$ is any element $x \neq \widehat{0}$ with the property that if $x = u \vee v$ then $x = u$ or $x = v$. The set $\mathcal{J}(P)$ of all order-ideals of P, ordered by inclusion, is a distributive lattice. Conversely,

by Birkhoff's representation theorem, every finite distributive lattice $\mathcal{D}$ is isomorphic to the lattice $\mathcal{J}(P)$ where $P = \mathbf{Spec}(\mathcal{D})$, the *spectrum* of $\mathcal{D}$, is defined as the poset of all join-irreducibles of $\mathcal{D}$.

A *valuation* on a distributive lattice $\mathcal{D}$ with values in $\mathbb{R}$ is a function $\nu : \mathcal{D} \to \mathbb{R}$ such that $\nu(\widehat{0}) = 0$ and $\nu(x \vee y) + \nu(x \wedge y) = \nu(x) + \nu(y)$ for every $x, y \in \mathcal{D}$. A valuation on a finite distributive lattice $\mathcal{D}$ is uniquely determined by the values it takes on the set of join-irreducibles of $\mathcal{D}$, and these values can be arbitrarily assigned [18]. Every valuation ν satisfies the following generalized form of the *principle of inclusion-exclusion*:

$$(1) \qquad \nu(x_1 \vee \cdots \vee x_n) = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|-1} \, \nu\left(\bigwedge_{i \in S} x_i\right).$$

The (*Euler*) *characteristic* of $\mathcal{D}$ is defined as the unique valuation χ such that $\chi(\widehat{0}) = 0$ and $\chi(x) = 1$ for every join-irreducible x of $\mathcal{D}$. In particular, $\chi(D) = \chi(\widehat{1})$.

A *partition* of a non-negative integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The λ_i's are called the *parts* of λ, and the sum of all parts of λ will be denoted by $|\lambda|$. The (*Ferrers*) *diagram* of λ is a left-justified array of squares (or dots) with exactly λ_i squares in the i-th row. Partitions can be ordered by magnitude of parts [1]: if $\alpha = (a_1, \ldots, a_h)$ and $\beta = (b_1, \ldots, b_k)$, then $\alpha \leq \beta$ whenever $h \leq k$ and $a_i \leq b_i$ for every $i = 1, 2, \ldots, h$. If $\alpha \leq \beta$ the diagram of α is contained in the diagram of β. The resulting poset is an infinite distributive lattice, called *Young lattice*. In particular, the *Young lattice* $\mathcal{Y}_\lambda$ generated by a partition λ is the set of all integer partitions α such that $\alpha \leq \lambda$, i.e. $\mathcal{Y}_\lambda = {\downarrow}\, \lambda$ in $\mathcal{Y}$.

3. Dyck-like paths

3.1. Representation of Dyck-like paths

To study Dyck-like paths of type (a, b) we can always suppose, without loss of generality, that $a \geq b$. We begin our study noticing that the length of a Dyck-like path of type (a, b) strictly depends on a and b, as stated in the following proposition essentially due to Duchon [9].

Proposition 3.1. *Every Dyck-like path of type (a, b) has $\frac{b}{\gcd(a,b)} \cdot n$ up steps and $\frac{a}{\gcd(a,b)} \cdot n$ down steps, for some $n \in \mathbb{N}^*$, and it ends at the point $(n \cdot \ell(a, b), 0)$, where $\ell(a, b) = \frac{a+b}{\gcd(a,b)}$.*
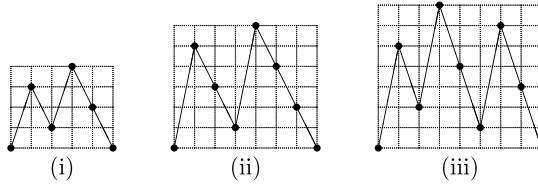
Figure 1: The minimum element in $\mathcal{D}_1^{(a,b)}$ for (i) $a = 3$, $b = 2$, (ii) $a = 5$, $b = 2$ and (iii) $a = 5$, $b = 3$.

Proof. Let h, k be the number of up steps and down steps of the path respectively. Then, $ha = kb$. It follows that $ha_1 = kb_1$, where $a_1 = \frac{a}{\gcd(a,b)}$ and $b_1 = \frac{b}{\gcd(a,b)}$. Clearly, since $\gcd(a_1, b_1) = 1$, it follows that there exists $n \in \mathbb{N}^*$ such that $h = nb_1$ and $k = na_1$. Thus, the path contains $n \cdot \ell(a, b)$ steps, so that it ends at the point $(n \cdot \ell(a, b), 0)$. □

The lattice of all Dyck-like paths of type (a, b) having length $n \cdot \ell(a, b)$ will be denoted by $\mathcal{D}_n^{(a,b)}$.

Set $h = \frac{b}{\gcd(a,b)} \cdot n$, $k = \frac{a}{\gcd(a,b)} \cdot n$ and $r_1 + r_2 + \cdots r_i = \lfloor \frac{ia}{b} \rfloor$, for every $i \in [h]$. $\mathcal{D}_n^{(a,b)}$ is a distributive lattice of height $\binom{h}{2}$, with minimum $\widehat{0} = UD^{r_1}UD^{r_2} \cdots UD^{r_h}$ (see Figure 1 for some examples) and maximum $\widehat{1} = U^h D^k$, with $h-1$ atoms of the form $UD^{r_1}UD^{r_2} \cdots UD^{r_i-1}UD^{r_{i+1}+1} \cdots UD^{r_h}$, and just 1 coatom $U^{h-1}DUD^{k-1}$. It is easy to see that the socle of $\mathcal{D}_n^{(a,b)}$ is the path $s = UD^{r_1-1}UD^{r_2} \cdots UD^{r_{h-1}}UD^{r_h+1}$ and that the principal filter $\uparrow s$ is isomorphic to $\mathcal{D}_{h-1}^{(a,b)}$, whenever $n \geq 1$. The special (and important) case of ordinary Dyck lattices $\mathcal{D}_n$ corresponds to $(a, b) = (1, 1)$, and for them we can also observe that the principal ideal $\downarrow s$ is isomorphic to the Boolean algebra B_{n-1} having $n - 1$ atoms. For Dyck lattices, see also Figure 3.

Our next goal is to find some suitable representation results for the lattices $\mathcal{D}_n^{(a,b)}$. We begin by giving an alternative description of $\mathcal{D}_n^{(a,b)}$. For each path in $\mathcal{D}_n^{(a,b)}$, consider the path obtained by replacing each step $(1, a)$ with a step $(1, 1)$ and each step $(1, -b)$ with a step $(1, -1)$. If $a \neq b$, the paths of the resulting set $\overline{\mathcal{D}}_n^{(a,b)}$ terminates below the x-axis.

Lemma 3.2. *The lattice* $\overline{\mathcal{D}}_n^{(a,b)}$ *consists of all the paths that start from the origin, end at the point* $(n \cdot \frac{a+b}{\gcd(a,b)}, n \cdot \frac{b-a}{\gcd(a,b)})$ *and never go below the line* $y = \frac{b-a}{a+b}x$.
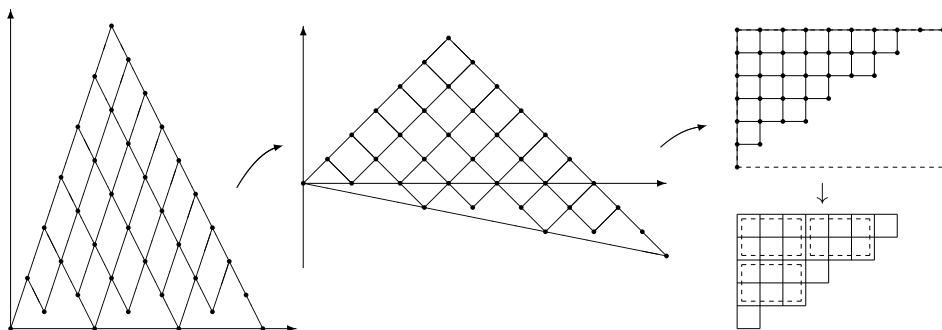
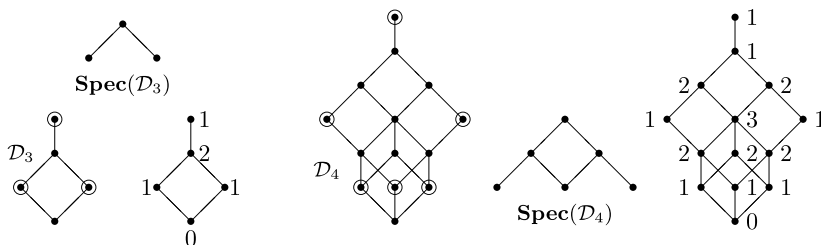Figure 2: A path in $\mathcal{D}_n^{(a,b)}$ and the associated partition for $a = 3$, $b = 2$ and $n = 3$.

Figure 3: The Dyck lattices $\mathcal{D}_3$ and $\mathcal{D}_4$, their spectra and the distribution of the characteristic.

Proof. If (x_0, y_0) is the endpoint of the paths in $\overline{\mathcal{D}}_n^{(a,b)}$ then x_0 is equal to the length of the paths in $\mathcal{D}_n^{(a,b)}$ obtained in Proposition 3.1, and y_0 can be obtained by subtracting from the number h of steps $(1,1)$ the number k of steps $(1,-1)$. Since h and k are defined as in Proposition 3.1, it follows that $y_0 = n \cdot \frac{b-a}{\gcd(a,b)}$. The last part of the thesis is obtained by determining the line passing through the origin and the point (x_0, y_0). $\square$

A careful inspection of some examples (as the one in Figure 2) suggests that the paths in $\overline{\mathcal{D}}_n^{(a,b)}$ (or, equivalently, in $\mathcal{D}_n^{(a,b)}$) are in bijection with the integer partitions whose Ferrers diagram is included in the Ferrers diagram obtained by taking simultaneously the minimum and the maximum paths in $\overline{\mathcal{D}}_n^{(a,b)}$, then rotating by 45° anticlockwise and finally considering the squares obtained by drawing the part of the lattice grid included between the two paths. We will denote such a partition $\lambda_n^{(a,b)}$, and call it the *partition associated with* $\overline{\mathcal{D}}_n^{(a,b)}$ (or $\mathcal{D}_n^{(a,b)}$).

All this implies

Proposition 3.3. *The distributive lattice $\mathcal{D}_n^{(a,b)}$ is isomorphic to the dual of the Young lattice $\mathcal{Y}_{\lambda_n^{(a,b)}}$.*

Next proposition describes the partition $\lambda_n^{(a,b)}$ in terms of the positive integers a, b and n, when $\gcd(a,b) = 1$ (the general case follows immediately).

Proposition 3.4. *Let $\gcd(a,b) = 1$. For any $n \in \mathbb{N}$,*

$$\lambda_n^{(a,b)} = (\lambda_{n,b}, \ldots, \lambda_{n,1}, \lambda_{n-1,b}, \ldots, \lambda_{n-1,1}, \ldots, \lambda_{2,b}, \ldots, \lambda_{2,1}, \lambda_{1,b}, \ldots, \lambda_{1,2})$$

is a partition with $nb - 1$ parts, where $\lambda_{h,k} = (h-1)a + \lfloor (k-1) \cdot a/b \rfloor$. In particular,

$$|\lambda_n^{(a,b)}| = ab\frac{n(n-1)}{2} + n|\lambda_1^{(a,b)}|.$$

Proof. From the expression of $\widehat{0}$ it follows that the i-th row of $\lambda_n^{(a,b)}$ (counting upwards) contains $r_1 + r_2 + \cdots + r_i = \lfloor \frac{ia}{b} \rfloor$ squares, for every $i \in [nb-1]$. Since $\lambda_{h,k}$ is the number of squares which are contained in the $((h-1)b+(k-1))$-th row of $\lambda_n^{(a,b)}$, where $(h,k) \in [\lfloor a \rfloor \times \lfloor b \rfloor] \setminus \{(1,1)\}$, we obtain that

$$\lambda_{h,k} = \left\lfloor \frac{(h-1)b + (k-1)}{b} a \right\rfloor = h - 1 + \left\lfloor \frac{(k-1)a}{b} \right\rfloor. \qquad \square$$

Since Dyck paths are Dyck-like with $(a,b) = (1,1)$, the following corollary holds.

Corollary 3.5. *The Dyck lattice $\mathcal{D}_n$ is isomorphic to the dual of the Young lattice $\mathcal{Y}_{\lambda_n}$, where $\lambda_n = (n-1, n-2, \ldots, 2, 1)$.*

Before proceeding further, it will be useful to characterize the join-irreducible elements of $\mathcal{D}_n^{(a,b)}$.

Proposition 3.6. *A path in $\mathcal{D}_n^{(a,b)}$ (resp. $\overline{\mathcal{D}}_n^{(a,b)}$) is join-irreducible if and only if it contains exactly one peak (weakly) above the line $y = a+b$ (resp. $y = \frac{b-a}{a+b}x + 2$). Furthermore, this peak determines the path uniquely.*

Proof. In $\mathcal{D}_n^{(a,b)}$ a path x is covered by a path y if it can be obtained from y by changing a peak UD into a valley DU. The paths with a unique peak (weakly) above the line $y = a+b$ are the only paths for which this operation can be performed just in one way. This argument can be trivially adapted to obtain the corresponding statement for $\overline{\mathcal{D}}_n^{(a,b)}$. $\qquad \square$

An easy corollary concerning Dyck paths is the following.

Corollary 3.7. *The join-irreducibles of the lattice $\mathcal{D}_n$ are the paths with exactly one peak (weakly) above the line $y = 2$, that is the paths of the form*
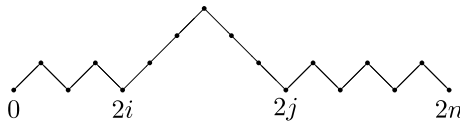
In particular, the spectrum of $\mathcal{D}_n$ is isomorphic to the poset of the intervals of a chain $\mathcal{C}_{n-2}$ with $n - 1$ elements, i.e. $\mathbf{Spec}(\mathcal{D}_n) \simeq \{(i,j) \in \mathcal{C}_n^2 : i \leq j - 2\} \simeq \mathbf{Int}(\mathcal{C}_{n-2})$.

Proof. The last statement can be proved by simply observing that every join-irreducible in $\mathcal{D}_n$ is uniquely determined by the interval corresponding to its unique peak (weakly) above the line $y = 2$. □

Now, we can state our main result concerning the representation of Dyck-like paths.

Theorem 3.8. *The distributive lattice $\mathcal{D}_n^{(a,b)}$ is isomorphic to $\mathcal{J}(P_n^{(a,b)})$, where $P_n^{(a,b)}$ is the set of points $(x,y) \in [0, n \cdot \ell(a,b)] \times \mathbb{Z}$ such that x and y have the same parity, $y \leq x$, $y \leq -x + n\frac{2b}{\gcd(a,b)}$ and $y \geq \frac{b-a}{a+b} + 2$, ordered coordinatewise.*

Proof. This follows from Proposition 3.3, by noticing that $P_n^{(a,b)}$ coincides with the set of the peaks weakly above the line $y \geq \frac{b-a}{a+b} + 2$ of the join-irreducibles in $\overline{\mathcal{D}}_n^{(a,b)}$. □

The rank of a path $x \in \mathcal{D}_n^{(a,b)}$ can be easily expressed in terms of its *area* $\mathcal{A}(x)$, i.e. the area of the region determined by the path and the x-axis.

Proposition 3.9. *The rank of any element $x \in \mathcal{D}_n^{(a,b)}$ is*

$$(2) \qquad r(x) = \frac{\mathcal{A}(x) - \mathcal{A}(\widehat{0})}{a + b}.$$

Proof. We will prove that the function defined in (2) satisfies the properties of a rank function. First we have $r(\widehat{0}) = \frac{\mathcal{A}(\widehat{0}) - \mathcal{A}(\widehat{0})}{a+b} = 0$. Suppose now that x is covered by y. Then the path y is obtained from x by replacing a valley with a peak. This implies that the area $A(y)$ is obtained from $\mathcal{A}(x)$ by adding the area of a parallelogram of area $a + b$. Hence $r(y) = \frac{\mathcal{A}(x) + a + b - \mathcal{A}(\widehat{0})}{a+b} = r(x) + 1$. □

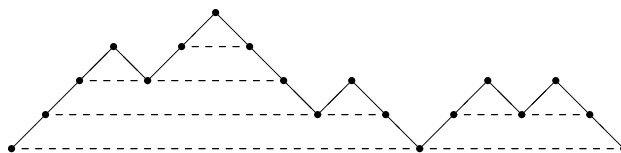In the special case of Dyck lattices, the above formula reads $r(x) = (\mathcal{A}(x) - n)/2$.

Figure 4: A Dyck path γ of length 18 with its 9 tunnels (two 0-tunnels, four 1-tunnels, two 2-tunnels and one 3-tunnel).

Before closing the present section, we mention some important facts concerning Dyck lattices which will be fundamental in what follows.

A *pyramid* in a Dyck path is a maximal sequence of consecutive steps of the form $U^h D^h$, for some $h \geq 1$, which can be replaced with $U^{h-1}DUD^{h-1}$ still remaining inside the class of Dyck paths. The positive integer h is called the *dimension* of the pyramid, whereas the height of the vertex is called the *height* of the pyramid.

A *k-tunnel* of a Dyck path x is any segment, not reducing to a point, on the horizontal line $y = k$ having in common with x only its extreme points [10, 11] (see Figure 4 for an example). Clearly every tunnel corresponds to a factor of the form UyD where the steps U and D are at the same level (see again Figure 4). In particular, the 0-tunnels correspond to the primitive factors of the paths.

Proposition 3.10. *The meet of two join-irreducibles of $\mathcal{D}_n$ is $\widehat{0}$ or a join-irreducible, i.e. $\{\widehat{0}\} \cup \mathbf{Spec}(\mathcal{D}_n)$ is a ranked sub-meet-semilattice of $\mathcal{D}_n$. Moreover, the rank of a path in the poset $\{\widehat{0}\} \cup \mathbf{Spec}(\mathcal{D}_n)$ is equal to the maximum height of its tunnels (i.e. $r(x) = k$ whenever x has a k-tunnel but not a $(k+1)$-tunnel).*

Proof. Two join-irreducible Dyck paths either meet in $\widehat{0}$ or have pyramids intersecting in a single point with integer coordinates. $\qquad\qquad\square$

We observe that an analogous result also holds for the lattices $\mathcal{D}_n^{(a,b)}$, i.e. it can be shown that $\{\widehat{0}\} \cup \mathbf{Spec}(\mathcal{D}_n^{(a,b)})$ is a sub-meet-semilattice of $\mathcal{D}_n^{(a,b)}$, even if it is in general not ranked.

3.2. Characteristic

A *Dyck-like lattice* is a distributive lattice whose spectrum is a ranked poset admitting a labelling of its elements with the following properties: all labels are positive integers and every antichain $S = \{s_1, \ldots, s_n\}$ of join-irreducibles can be linearly ordered so that the labels of the elements of S are distinct and, if s_1 and s_n are the elements having minimum and maximum labels,

respectively, then $s_1 \wedge s_n = s_1 \wedge s_2 \wedge \cdots \wedge s_{n-1} \wedge s_n$. In a Dyck-like lattice, any labelling of the join-irreducibles satisfying the above properties will be called a *Dyck-like labelling*. Similarly, if $\{s_1, \ldots, s_n\}$ is an antichain of join-irreducibles as above, the ordered n-tuple $(s_1, \ldots, s_n)$ will be called a *Dyck-like antichain*.

Proposition 3.11. *For any $a, b \in \mathbb{N}$ and for every $n \in \mathbb{N}$, the lattice $\mathcal{D}_n^{(a,b)}$ is a Dyck-like lattice.*

Proof. Consider the lattice $\overline{\mathcal{D}}_n^{(a,b)}$ isomorphic to $\mathcal{D}_n^{(a,b)}$, as defined in Section 3.1. Label each join-irreducible with the abscissa of its highest peak. Such a labelling is a Dyck-like labelling. The fact that the spectrum of $\overline{\mathcal{D}}_n^{(a,b)}$ is ranked is a consequence of Theorem 3.8. $\square$

We conjecture that a sort of converse of the previous proposition holds. More precisely, the following assertion seems plausible:

Conjecture 3.12. *Every finite Dyck-like lattice can be represented as a sublattice of a lattice of Dyck-like paths of suitable length.*

We will say that an element x of a distributive lattice $\mathcal{D}$ is *quasi-join-irreducible* when there exists an ordered k-tuple $(s_1, \ldots, s_k)$ forming an antichain of join-irreducibles such that $x = s_1 \vee \cdots \vee s_k$ and $s_i \wedge s_{i+1} \neq \hat{0}$, for every $i = 1, 2, \ldots, k - 1$.

Lemma 3.13. *Let $\mathcal{D}$ be a Dyck-like lattice and $x \in \mathcal{D}$ a quasi-join-irreducible. Then x can be expressed as $x = t_1 \vee \cdots \vee t_k$ where $(t_1, \ldots, t_k)$ is a Dyck-like antichain and $t_i \wedge t_{i+1} \neq \hat{0}$, for every $i = 1, 2, \ldots, k - 1$.*

Proof. Let $x = t_1 \vee \cdots \vee t_k$, where $(t_1, \ldots, t_k)$ is a Dyck-like antichain. Suppose there exists an index $j < k$ such that $t_j \wedge t_{j+1} = \hat{0}$. This would imply $t_a \wedge t_b = \hat{0}$, for any $a \leq j$ and $b \geq j + 1$. However, in any rearrangement of the t_i's, at least a pair of adjacent elements must appear such that one has index $\leq j$ and the other has index $\geq j + 1$, and this contradicts the hypothesis that x is quasi-join-irreducible. $\square$

Lemma 3.13 asserts that in a Dyck-like lattice the antichain of join-irreducibles in the definition of a quasi-join-irreducible element can be taken to be a Dyck-like antichain.

Before proceeding further we notice that, in the finite case, the notion of a quasi-join-irreducible element can be related to an interesting property.

Proposition 3.14. *Suppose that $\mathcal{D}$ is a finite distributive lattice. If $x \in \mathcal{D}$ is quasi-join-irreducible then the following property holds:*

$$(3) \qquad \text{if } x = y \vee z \text{ and } y \wedge z = \widehat{0}, \text{ then } x = y \text{ or } x = z.$$

Proof. In a finite distributive lattice, every element has an essentially unique factorization (up to the order of the factors) as a join of join-irreducible elements. Let $x = y \vee z$ and $y \wedge z = \widehat{0}$. Let $x = s_1 \vee \cdots \vee s_k$, with $s_1, \ldots, s_k$ join-irreducibles as in the definition of a quasi-join-irreducible. Then necessarily both y and z can be expressed as the join of some of the s_i's. Moreover, there cannot exist any i such that s_i appears in the factorization of both y and z, otherwise $y \wedge z \neq \widehat{0}$. Therefore the set $\{s_1, \ldots s_k\}$ can be partitioned into two blocks (the elements appearing in the factorization of y and those appearing in the factorization of z, respectively) in such a way that, if s_i and s_j belong to different blocks, then $s_i \wedge s_j = \widehat{0}$. The fact that x is quasi-join-irreducible implies that one of the two blocks must be empty, whence $x = y$ or $x = z$. □

We can also prove a converse of the above proposition in the hypothesis that the lattice is Dyck-like.

Proposition 3.15. *Suppose that $\mathcal{D}$ is a finite Dyck-like lattice. If $x \in \mathcal{D}$ satisfies property (3) (for all y and z), then it is quasi-join-irreducible.*

Proof. Set $x = s_1 \vee \cdots \vee s_k$, where the s_i's are join-irreducibles and $(s_1, \ldots, s_k)$ is a Dyck-like antichain. If x were not quasi-join-irreducible, then there would exist i such that $s_i \wedge s_{i+1} = \widehat{0}$. Define $y = s_1 \vee \cdots \vee s_i$ and $z = s_{i+1} \vee \ldots \vee s_k$. Of course $x = y \vee z$. Moreover, for every $a \leq i$ and for every $b \geq i+1$, $s_a \wedge s_b = s_a \wedge \cdots \wedge s_i \wedge s_{i+1} \wedge \cdots \wedge s_b = \widehat{0}$, whence $y \wedge z = \widehat{0}$. Therefore $x = y$ or $x = z$, which is impossible since x has a unique factorization as the join of join-irreducibles. □

Remark. In the above proofs, the finiteness hypothesis is used in order to ensure the uniqueness of the factorization as join of join-irreducibles. Actually, such a property is guaranteed also if we relax the hypothesis, and simply ask that the distributive lattice $\mathcal{D}$ satisfies the *descending chain condition*, i.e. every nonempty subset of $\mathcal{D}$ has minimal elements (see, for instance, [7]). So the above propositions can be reformulated in this more general setting.

A *special Dyck-like lattice* is a Dyck-like lattice where the meet of any two join-irreducibles is $\widehat{0}$ or a join-irreducible. In view of Proposition 3.10 and Proposition 3.11, it follows that $\mathcal{D}_n^{(a,b)}$ is a special Dyck-like lattice.

Proposition 3.16. *In a special Dyck-like lattice $\mathcal{D}$, every quasi-join-irreducible element has Euler characteristic equal to 1.*

Proof. Let x be a quasi-join-irreducible element of $\mathcal{D}$. Then $x = s_1 \vee \cdots \vee s_k$, where $s_1, \ldots, s_k$ are incomparable join-irreducibles such that $s_i \wedge s_{i+1} \neq \widehat{0}$ for every $i < k$. If $k = 1$ then x is join-irreducible and $\chi(x) = 1$. From

formula (1) we have

$$\chi(x) = \chi(s_1 \vee \cdots \vee s_k) = \sum_{\substack{S \subseteq [k] \\ S \neq \emptyset}} (-1)^{|S|-1} \chi \left(\bigwedge_{i \in S} s_i \right). \tag{4}$$

By Lemma 3.13, $(s_1, \ldots, s_k)$ can be taken to be a Dyck-like antichain. Hence it follows at once that $\bigwedge_{i \in S} s_i = s_{i_1} \wedge s_{i_2}$, where $i_1 = \min S$ and $i_2 = \max S$. If S is a subset of $[k]$ with $\min S = \max S$, then clearly $|S| = 1$. Hence the contribution of these subsets to the sum in (4) is $\chi(s_1) + \cdots + \chi(s_k) = k$. If S is a subset of $[k]$ with $\max S = \min S + 1$, then it follows that $S = \{i, i+1\}$. Since $s_i \wedge s_{i+1}$ is a join-irreducible, the contribution of these subsets to the sum is

$$-\sum_{i=1}^{k-1} \chi(s_i \wedge s_{i+1}) = -(k-1).$$

Finally, since the subsets S of $[k]$ with $\max S - \min S = j \geq 2$, having minimum i, maximum $i+j$ and cardinality $h+2$ $(h \geq 0)$ are exactly $\binom{j-1}{h}$, the contribution of all these subsets is

$$\sum_{h=0}^{j-1} \binom{j-1}{h} (-1)^{h+1} \chi(s_i \wedge s_{i+j}) = -(1-1)^{j-1} \chi(s_i \wedge s_{i+j}) = 0.$$

In conclusion, we have $\chi(x) = k - (k-1) = 1$. $\square$

An element x of a finite distributive lattice is said to have a *quasi-join-irreducible decomposition* when it can be expressed as a join of quasi-join-irreducible elements $x_1, \ldots, x_k$ such that $x_i \wedge x_j = \widehat{0}$, for every $i \neq j$.

Proposition 3.17. *Every element $x \neq \widehat{0}$ of a finite Dyck-like lattice has a quasi-join-irreducible decomposition.*

Proof. Let $x = s_1 \vee \cdots \vee s_k$, where $(s_1, \ldots, s_k)$ is a Dyck-like antichain. If i is the first index such that $s_i \wedge s_{i+1} = \widehat{0}$ then $s_1 \wedge \cdots \wedge s_i$ is quasi-join-irreducible. If j is the second index satisfying the above condition (and so $s_j \wedge s_{j+1} = \widehat{0}$), then obviously $s_{i+1} \wedge \cdots \wedge s_j$ is quasi-join-irreducible and $(s_1 \wedge \cdots \wedge s_i) \wedge (s_{i+1} \wedge \cdots \wedge s_j) = \widehat{0}$. Repeating this argument one obtains the desired decomposition. $\square$

We observe that the uniqueness of the quasi-join-irreducible decomposition can be proved for a generic finite distributive lattice.

Lemma 3.18. *Let x be a quasi-join-irreducible of a finite distributive lattice. If $x \leq y \vee z$ and $y \wedge z = \widehat{0}$, then $x \leq y$ or $x \leq z$.*

Proof. From the hypothesis we get $x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Moreover, from $y \wedge z = \widehat{0}$, it follows that $(x \wedge y) \wedge (x \wedge z) = \widehat{0}$. Since x is quasi-join-irreducible, Proposition 3.14 implies that $x = x \wedge y$ or $x = x \wedge y$, that is $x \leq y$ or $x \leq y$.   □

Proposition 3.19. *In a finite distributive lattice, the quasi-join-irreducible decomposition of an element is unique (up to the order of factors).*

Proof. Let $x = x_1 \vee \cdots \vee x_s = y_1 \vee \cdots \vee y_r$ be two quasi-join-irreducible decompositions of x. For every $i \leq s$, the above lemma implies that there exists $j \leq r$ such that $x_i \leq y_j$ and, analogously, that there exists $k \leq s$ such that $y_j \leq x_k$. Therefore $x_i \leq x_k$, whence $x_i = x_k$, and so also $x_i = y_j$. We have thus shown that $\{x_1, \ldots, x_s\} \subseteq \{y_1, \ldots, y_t\}$, and the opposite inequality holds for a similar reason.   □

Theorem 3.20. *Let $\mathcal{D}$ be a finite special Dyck-like lattice. Then, for every $x \in \mathcal{D} \setminus \{\widehat{0}\}$, $\chi(x)$ is the number of quasi-join-irreducibles in a decomposition of x.*

Proof. If $x = x_1 \vee \cdots \vee x_k$ is a quasi-join-irreducible decomposition of x, then it follows at once that $\chi(x) = \chi(x_1) + \cdots + \chi(x_k) = k$.   □

In the case of Dyck paths, the characteristic can be interpreted combinatorially as follows.

Proposition 3.21. *A Dyck path $x \in \mathcal{D}_n$ is quasi-join-irreducible if and only if it has precisely one 1-tunnel.*

Proof. If x has precisely one 1-tunnel, then it is of the form $(UD)^s UyD(UD)^t$, for suitable $s, t \in \mathbb{N}$, where y is an elevated Dyck path. Then x can be expressed as $x = x_1 \vee \cdots \vee x_k$, where $x_1, \ldots, x_k$ are the join-irreducibles uniquely determined by the peaks of y. The fact that y is elevated implies that $x_i \wedge x_{i+1} \neq \widehat{0}$, for every $i < k$, and consequently that x is quasi-join-irreducible.

On the other hand, suppose that $x \in \mathcal{D}_n$ is quasi-join-irreducible. If x had no 1-tunnels, then $x = \widehat{0}$, which is impossible (since $\widehat{0}$ is not quasi-join-irreducible). If x had more than one 1-tunnel, then x would have at least two elevated factors (i.e. two Dyck factors starting at level 1). In this situation, any expression of x as a join of join-irreducibles would contain join-irreducible elements determined by the peaks of all the elevated factors of x. Thus, if $x = x_1 \vee \cdots \vee x_k$ is any join-irreducible decomposition of x, then there exists at least one $i < k$ such that x_i and x_{i+1} are join-irreducibles determined by the peaks of two different elevated factors, and so $x_i \wedge x_{i+1} = \widehat{0}$. In conclusion, x has exactly one 1-tunnel.   □

Finally, as an immediate consequence of Theorem 3.20 and Proposition 3.21, we have

Theorem 3.22. *The characteristic of a Dyck path is the number of its 1-tunnels.*

3.3. Generalized characteristics

Suppose that $\mathcal{D}$ is a finite special Dyck-like lattice with spectrum P such that $\widehat{P} = \{\widehat{0}\} \cup P$ is ranked with rank function $r_{\widehat{P}}$. For every $k \in \mathbb{N}^*$, we define the *generalized characteristic* χ_k as the valuation on $\mathcal{D}$ such that

$$\chi_k(x) = \begin{cases} 1 & \text{if } r_{\widehat{P}}(x) \geq k \\ 0 & \text{if } r_{\widehat{P}}(x) < k \end{cases}$$

for every join-irreducible x. Clearly χ_1 is equal to the ordinary characteristic χ. Our aim is to evaluate $\chi_k(x)$ for every $x \in \mathcal{D}$.

Proposition 3.23. *Let $x = x_1 \vee \cdots \vee x_m$ where $m \geq 1$, each x_i is a join-irreducible and $r_{\widehat{P}}(x_j \wedge x_{j+1}) \geq k$ for every $j < m$. Then $\chi_k(x) = 1$.*

Proof. The proof follows the same lines of that of Proposition 3.16.

$\square$

Proposition 3.24. *Let $x = x_1 \vee \cdots \vee x_m$, where $m \geq 1$, and each $x_i = x_{i,1} \vee \cdots \vee x_{i,\ell_i}$ is a join of join-irreducibles such that $r_{\widehat{P}}(x_{i,t} \wedge x_{i,t+1}) \geq k$, for every $t = 1, \ldots, \ell_i - 1$, and $r_{\widehat{P}}(x_{i,t} \wedge x_{j,s}) < k$ for $t = 1, \ldots, \ell_i$, $s = 1, \ldots, \ell_j$, $i \neq j$. Then $\chi_k(x) = m$.*

Proof. Using formula (1), we have

$$\chi_k(x) = \chi_k(x_1 \vee \cdots \vee x_m) = \sum_{\substack{S \subseteq [m] \\ S \neq \emptyset}} (-1)^{|S|-1} \chi_k \left(\bigwedge_{i \in S} x_i \right).$$

Observe that, for every $S \subseteq [m]$, $|S| \geq 2$, $\bigwedge_{i \in S} x_i$ can be expressed as the join of terms of the form $x_{i,t} \wedge x_{j,s}$, whose generalized characteristic χ_k is equal to 0. So, applying again formula (1), it is immediately seen that, in these cases, $\chi_k(\bigwedge_{i \in S} x_i) = 0$ and then $\chi_k(x) = \chi_k(x_1) + \cdots + \chi_k(x_m)$. Finally, the claim follows applying Proposition 3.23. $\square$

Proposition 3.25. *Every element x of $\mathcal{D}$ can be written as $x = (x_1 \vee \cdots \vee x_h) \vee (x_{h+1} \vee \cdots \vee x_m)$ where*

1. $x_i = x_{i,1} \vee \cdots \vee x_{i,\ell_i}$ *is a join of join-irreducibles with* $r_{\widehat{P}}(x_{i,t} \wedge x_{i,t+1}) \geq k$
   for every $t = 1, \ldots, \ell_i - 1$, *and* $r_{\widehat{P}}(x_{i,t} \wedge x_{j,s}) < k$ *for* $t = 1, \ldots, \ell_i$,
   $s = 1, \ldots, \ell_j$, *whenever* $i \neq j$, *for every* $i, j = 1, \ldots, h$;
2. x_i *is a join-irreducible with* $r_{\widehat{P}}(x_i) < k$, *for every* $i = h + 1, \ldots, m$.

Proof. Let $x = p_1 \vee \cdots \vee p_s$, where $(p_1, \ldots, p_s)$ is a Dyck-like antichain. For
the first element p_1 there are two possible cases. If $r_{\widehat{P}}(p_1) < k$, then p_1 is
one of the x_i's. Otherwise, if $r_{\widehat{P}}(p_1) \geq k$, then consider the first index i such
that $r_{\widehat{P}}(p_i \wedge p_{i+1}) < k$: then $p_1 \vee \cdots \vee p_i$ is a join of join-irreducibles such
that $r_{\widehat{P}}(p_j \wedge p_{j+1}) \geq k$, for $j < i$, and so it is one of the x_i's. Repeating this
argument and rearranging the x_i in the correct order, we obtain the desired
decomposition. □

Any decomposition of the kind described in Proposition 3.25 will be
said a *k-quasi-join-irreducible decomposition* of the element $x \in \mathcal{D}$, and
the elements $x_1, \ldots, x_h$ appearing in such a decomposition will be called
k-quasi-join-irreducibles.

Theorem 3.26. *The generalized characteristic* $\chi_k(x)$ *of an element* $x \in$
$\mathcal{D}$ *is equal to the number of k-quasi-join-irreducibles in any k-quasi-join-irreducible decomposition of* x.

Proof. By Proposition 3.25, every element x of $\mathcal{D}$ admits a k-quasi-join-irreducible decomposition $x = x_1 \vee \cdots \vee x_h \vee x_{h+1} \vee \cdots \vee x_m$. Applying
formula (1) we have

$$\chi_k(x) = \sum_{\substack{S \subseteq [m] \\ S \neq \emptyset}} (-1)^{|S|-1} \chi_k \left(\bigwedge_{i \in S} x_i \right).$$

If S contains an i such that $r_{\widehat{P}}(x_i) < k$ then $\chi_k \left(\bigwedge_{i \in S} x_i \right) = 0$. Therefore
in the computation of $\chi_k(x)$ all the join-irreducibles with rank strictly less
than k in $\widehat{P}$ can be discarded, i.e.

$$\chi_k(x) = \sum_{\substack{S \subseteq [h] \\ S \neq \emptyset}} (-1)^{|S|-1} \chi_k \left(\bigwedge_{i \in S} x_i \right) = \chi_k(x_1 \vee \cdots \vee x_h).$$

Finally, the claim follows from Proposition 3.24. □

As a consequence of Theorem 3.26 it follows that the number of k-
quasi-join-irreducibles in any k-quasi-join-irreducible decomposition of an
element x is constant. Moreover, from Theorem 3.26 and Proposition 3.10,
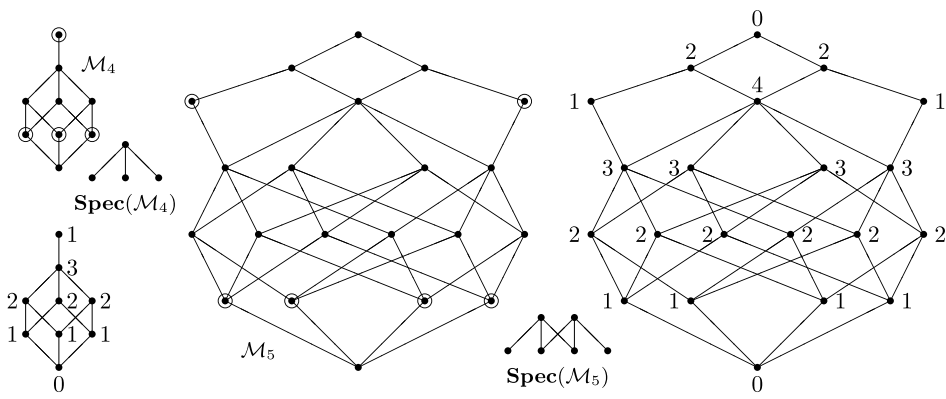
Figure 5: The Motzkin lattices $\mathcal{M}_4$ and $\mathcal{M}_5$, their spectra and the distribution of the characteristic.

we have the following interpretation of the generalized characteristics of Dyck lattices.

Theorem 3.27. *The generalized characteristic $\chi_k(x)$ of an element x of the Dyck lattice $\mathcal{D}_n$ is equal to the number of k-tunnels of x.*

4. Motzkin lattices

4.1. Representation

The *Motzkin lattice* $\mathcal{M}_n$ is a distributive lattice of height $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, with minimum H^n, and maximum $U^k D^k$ when $n = 2k$ or $U^k H D^k$ when $n = 2k + 1$, with $n - 1$ atoms of the form $H^k U D H^{n-k-2}$ and just one coatom $U^{k-1} H D^{k-1}$ when $n = 2k$ and two coatoms $U^{k-1} H U D^k$ and $U^k D H D^{k-1}$ when $n = 2k+1$, It is easy to see that the socle is the path $s = U H^{n-2} D$ and that the principal ideal $\downarrow s$ is isomorphic to a Boolean algebra B_{n-1} while the principal filter $\uparrow s$ is isomorphic to a Motzkin lattice $\mathcal{M}_{n-2}$, whenever $n \geq 1$ (see Figure 5). The rank function is given by the area determined by the path, i.e. $r(x) = \mathcal{A}(x)$.

Proposition 4.1. *The join-irreducibles of the Motzkin lattice $\mathcal{M}_n$ are the paths having the form*
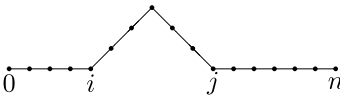
The spectrum of $\mathcal{M}_n$ is isomorphic to the poset of the intervals of even length of a chain $\mathcal{C}_n$ having $n + 1$ elements, i.e. $\mathbf{Spec}(\mathcal{M}_n) \simeq \{(i, j) \in \mathcal{C}_n^2 : \exists k \in \mathbb{N} \text{ such that } j - i = 2k + 2\}$.

Proof. If $x \in \mathcal{M}_n$ has an horizontal step at height > 0, then it is easy to see that x can be obtained as the join of two smaller paths. Hence a join-irreducible cannot have horizontal steps at height > 0. If x had more than one peak, then it could be expressed as the join of all paths obtained from x by replacing each peak with a couple of horizontal steps, one peak at a time. So x can have only one peak. □

Remark. Motzkin paths are not Dyck-like paths, nevertheless the lattice $\mathcal{M}_n$ of Motzkin paths of length n is a Dyck-like lattice. Notice that $\mathcal{M}_n$ is isomorphic to the lattice of Dyck paths of length $2n$ without three consecutive down steps [4]. This agrees with our previous conjecture on the representation of Dyck-like lattices.

Motzkin lattices are not special Dyck-like lattices. For instance, in $\mathcal{M}_5$ the meet of the paths U^2D^2H and HU^2D^2 is $HUHDH$, which is different from the minimum $\widehat{0}$ and not join-irreducible. However, we can prove a result similar to Proposition 3.10 for the meet of two join-irreducibles in $\mathcal{M}_n$, which will allow to compute the Euler characteristic also in this case.

4.2. Characteristic

To give a combinatorial description of the characteristic for Motzkin lattices we cannot use the theory developed in the previous sections, since in a Motzkin lattice it could happen that the meet of two join-irreducibles is neither $\widehat{0}$ nor a join-irreducible, as we have seen in Section 4.1. However, the arguments developed for Dyck paths can be adapted to the Motzkin case and lead to analogous results.

A *truncated pyramid* of a Motzkin path x is a sequence of $k \geq 1$ up steps followed by a sequence of $m \geq 1$ horizontal steps followed by a sequence of k down steps, i.e. $U^kH^mD^k$. The positive integer k is called the *dimension* of the truncated pyramid, whereas m is its *length*. Moreover, we say that a truncated pyramid has *height h* when the sequence of horizontal steps lies on the line $y = h$. In the sequel, we will denote by $T_{n,m,k}$ the set of Motzkin paths of length n having only horizontal steps at height 0, except for a unique truncated pyramid of dimension k and length m. An element of $T_{n,m,k}$ will be called a *Motzkin path with a unique truncated pyramid* of length m and dimension k (see Figure 6).
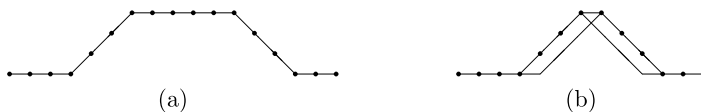
Figure 6: (a) A Motzkin path in $T_{16,5,3}$. (b) A Motzkin path in $T_{12,1,3}$ and the two join-irreducibles it covers.

Now we are ready to state our result on the meet of two join-irreducible Motzkin paths.

Proposition 4.2. *In $\mathcal{M}_n$ the meet of two join-irreducibles is either $\widehat{0}$ or a join-irreducible or an element of $T_{n,1,k}$, $k \leq n-2$ (i.e., a Motzkin path with a unique truncated pyramid of length 1).*

Proof. Two join-irreducible Motzkin paths that do not meet at $\widehat{0}$ must intersect at a point (x_0, y_0) with $y_0 > 0$. Then, their meet is a join-irreducible path if $y_0 \in \mathbb{N}$, whereas it belongs to $T_{n,1,k}$ if $y_0 \notin \mathbb{N}$. $\square$

Remark. The above proof can be reformulated in a bijective fashion as follows. Let E_n be the set obtained by taking all join-irreducibles of $\mathcal{M}_n$ and all elements of $T_{n,1,h}$, for $h = 1, 2, \ldots, n-2$. E_n with the induced order is isomorphic to $\mathbf{Spec}(\mathcal{D}_n)$. Indeed, in E_n each element of $T_{n,1,h}$ covers precisely two join-irreducibles (see Figure 6(b)) and each join-irreducible covers precisely two elements of $T_{n,1,h}$, for a suitable h. Alternatively, map each element of $\mathbf{Spec}(\mathcal{D}_n)$ to the element of E_n obtained by performing the following substitutions, when reading the path from left to right: $UU \to U$, $UD \to H$, $DD \to D$. The resulting map is an isomorphism, which maps $(UD)^i U^k D^k (UD)^j$ into $H^i U^{\lfloor \frac{k}{2} \rfloor} H^{k-2\lfloor \frac{k}{2} \rfloor} D^{\lfloor \frac{k}{2} \rfloor} H^j$.

First of all, we compute the characteristic of some particular Motzkin paths.

Lemma 4.3. *If $x \in T_{n,m,h}$, then $\chi(x) = (-1)^{h+1} m + 1$.*

Proof. Since a Motzkin path $x \in T_{n,1,1}$ is the join of two join-irreducibles whose meet is $\widehat{0}$, we have $\chi(x) = 2$. Similarly, since a Motzkin path $x \in T_{n,1,2}$ is the join of two join-irreducibles whose meet is a Motzkin path belonging to $T_{n,1,1}$, we have $\chi(x) = 1 + 1 - 2 = 0$. Iterating this argument it follows that for every $x \in T_{n,1,h}$ the characteristic is $\chi(x) = 2$ when h is odd and $\chi(x) = 0$ when h is even, i.e. $\chi(x) = (-1)^{h+1} + 1$. We now proceed by induction on the length m. If $x \in T_{n,m+1,h}$, then $x = H^a U^h H^{m+1} D^h H^b$ and hence $x = x_1 \vee x_2$ where $x_1 = H^a U^h H^m D^h H^{b+1}$, $x_2 = H^{a+1} U^h H^m D^h H^b$.

Since $x_1 \wedge x_2 = H^{a+1}U^h H^{m-1}D^h H^{b+1} \in T_{n,m-1,h}$, we have

$$\chi(x) = \chi(x_1) + \chi(x_2) - \chi(x_1 \wedge x_2)$$
$$= (-1)^{h+1}m + 1 + (-1)^{h+1}m + 1 - (-1)^{h+1}(m-1) - 1$$

that is $\chi(x) = (-1)^{h+1}(m+1) + 1$. So, the lemma is proved. □

Remark. The maximum $\widehat{1}$ of $\mathcal{M}_n$ is join-irreducible when $n = 2k$ and belongs to $T_{2k+1,1,k}$ when $n = 2k + 1$. Hence, from Lemma 4.3, it follows that $\chi(\mathcal{M}_n) = 1$ when n is even, $\chi(\mathcal{M}_n) = 0$ when $n = 4k+1$ and $\chi(\mathcal{M}_n) = 2$ when $n = 4k + 3$.

In any Motzkin lattice each quasi-join-irreducible has a particular join-irreducible decomposition, coming directly from the definition of quasi-join-irreducible element. However, for our purposes, another kind of decomposition will be useful in representing quasi-join-irreducibles. In fact, every quasi-join-irreducible element $x \in \mathcal{M}_n$ can be expressed, *in a unique way*, as $x = s_1 \vee \cdots \vee s_k$, where each s_i is either join-irreducible or it belongs to $T_{n,m,h}$, and $s_i \wedge s_{i+1} \neq \widehat{0}$, for every $i < k$. To see this, just write $x = t_1 \vee \cdots \vee t_r$ as a join of join-irreducibles and group together all the consecutive join-irreducibles whose join gives rise to a Motzkin path belonging to some $T_{n,m,h}$. Such a decomposition will be called the *Motzkin decomposition* of the quasi-join-irreducible x.

Now, we are ready to state and proof the fundamental step in the determination (and combinatorial interpretation) of the characteristic of $\mathcal{M}_n$. Our main proposition will be preceded by a technical lemma.

Lemma 4.4. *Let $x \in \mathcal{M}_n$ be a quasi-join-irreducible element and $x = s_1 \vee \cdots \vee s_k$ its Motzkin decomposition. Then, for every $i < k$, $s_i \wedge s_{i+1}$ is join-irreducible.*

Proof. If s_i and s_{i+1} are both join-irreducibles, the conclusion follows from the definition of Motzkin decomposition (since otherwise s_i and s_{i+1} would "contribute" to a truncated pyramid in the Motzkin decomposition of x). If at least one of the two is an element of some $T_{n,m,h}$, then $s_i \wedge s_{i+1}$ is equal to the meet of two join-irreducibles. Indeed, if s_i is join-irreducible and $s_{i+1} \in T_{n,m,h}$, then $s_i \wedge s_{i+1} = s_i \wedge x$, where $s_{i+1} = x \vee x_1 \vee \cdots \vee x_r$ is a join-irreducible decomposition of s_{i+1} and x is the join-irreducible having minimum abscissa; then an argument similar to the above one shows that $s \wedge x$ is join-irreducible. The remaining cases can be dealt with in a similar way. □

Let $x \in \mathcal{M}_n$. We will write $o(x)$ for the number of horizontal steps at odd height and $e(x)$ for the number of horizontal steps at even nonzero height (i.e. at even height and not lying on the x-axis).

Proposition 4.5. *Let $x \in \mathcal{M}_n$ be a quasi-join-irreducible. Then $\chi(x) = o(x) - e(x) + 1$.*

Proof. Let $x = s_1 \vee \cdots \vee s_k$ be the Motzkin decomposition of x. From formula (1) we have

$$(5) \qquad \chi(x) = \chi(s_1 \vee \cdots \vee s_k) = \sum_{\substack{S \subseteq [k] \\ S \neq \emptyset}} (-1)^{|S|-1} \chi \left(\bigwedge_{i \in S} s_i \right).$$

Since $\mathcal{M}_n$ is a Dyck-like lattice, we can proceed as in Proposition 3.16, noticing that a result analogous to Lemma 3.13 holds for the Motzkin decomposition. We first observe that, if $\min S = \max S$, then the contribution to the sum is $\chi(s_1) + \cdots + \chi(s_k)$. Some of the s_i's are join-irreducibles (and so their contribution is 1), but some of them could be paths in $T_{n,m,h}$. If $s_i \in T_{n,m,h}$, then from Lemma 4.3 it follows that $\chi(s_i) = (-1)^{h+1} m + 1$. Starting from this remark, it is not difficult to show that $\chi(s_1) + \cdots + \chi(s_k) = k + o(x) - e(x)$. Now, if $\max S = \min S + 1$, the contribution to the sum is $-\sum_{i<k} \chi(s_i \wedge s_{i+1}) = -(k-1)$, since all meets $s_i \wedge s_{i+1}$ are necessarily join-irreducible by Lemma 4.4. Finally, if $\max S = \min S + r$, with $r > 1$, using an argument completely analogous to the one used in the Dyck case, we find that the contribution of these subsets to the sum in (5) is zero. So, in conclusion, we have $\chi(x) = k + o(x) - e(x) - k + 1 = o(x) - e(x) + 1$. $\qquad\square$

Since Motzkin lattices are Dyck-like, from Proposition 3.17 it follows that every element of a Motzkin lattice has a quasi-join-irreducible decomposition. Let $\|x\|$ be the number of all quasi-join-irreducibles in a decomposition of $x \in \mathcal{M}_n$, and let $o'(x)$ be the number of horizontal steps at odd height different from 1.

Theorem 4.6. *The characteristic of a Motzkin path $x \neq \widehat{0}$ is $\chi(x) = \|x\| + o'(x) - e(x)$.*

Proof. If $x = x_1 \vee \cdots \vee x_k$ is a quasi-join-irreducible decomposition of x, then $\chi(x) = \chi(x_1) + \cdots + \chi(x_k) = o(x_1) - e(x_1) + 1 + \cdots + o(x_k) - e(x_k) + 1 = o'(x) - e(x) + k$. $\qquad\square$

To properly state our last result of the present section, we characterize the quasi-join-irreducible elements of Motzkin lattices, thus obtaining an analog of Proposition 3.21. The easy proof is left to the reader.

Proposition 4.7. *A Motzkin path is quasi-join-irreducible if and only if either it has a unique elevated factor having no horizontal steps at height 1 or it has a unique peak at height 1.*

A *reverse truncated pyramid* of height h is any factor of a Motzkin path of the form DH^kU, where $k \geq 1$ and the sequence H^k of horizontal steps lies on the line $y = h$. The height of a peak is given by its ordinate. If $f_h(x)$, $p_h(x)$, $t_h(x)$ and $r_h(x)$ are respectively the number of truncated pyramids, peaks, maximal sequences of consecutive 1-tunnels and reverse truncated pyramid of height h in x, then Theorem 4.6 can be interpreted combinatorially as stated in

Theorem 4.8. *The characteristic of a Motzkin path $x \in \mathcal{M}_n$ is*

$$\chi(x) = o(x) - e(x) + t_1(x) + f_1(x) + p_1(x) - r_1(x).$$

Proof. We have only to give a combinatorial interpretation of the term $\|x\|$. By virtue of Proposition 4.7, the quasi-join-irreducibles in a decomposition of a Motzkin path x can be of two types only. They can be Motzkin paths with a unique elevated factor having no horizontal steps at height 1, whose number is equal to the number of maximal sequences of consecutive 1-tunnels of x. Otherwise they can be Motzkin paths having a unique peak at height 1. In this case, we have an isolated peak in x, or a sequence of consecutive peaks whose abscissas differ by 1. In this last case, what we see in x is a sequence of horizontal steps at height 1. We have three possible configurations:

 a) a truncated pyramid at height 1, UH^mD, obtained as the join of $m+1$ peaks at height 1,
 b) either UH^mU or DH^mD: in both cases, the configuration is obtained as the join of m peaks at height 1,
 c) a reverse truncated pyramid DH^mU at height 1 obtained as the join of $m - 1$ peaks at height 1.

Hence $\|x\| = t_1(x) + p_1(x) + h_1(x) + f_1(x) - r_1(x)$, where $h_1(x)$ is the number of horizontal steps of x at height 1. □

5. Schröder lattices

5.1. Representation

The Schröder lattice $\mathcal{S}_n$ is a distributive lattice of height n^2, with minimum H^{2n} and maximum U^nD^n, with $n - 1$ atoms of the form $H^{2k}UDH^{2n-2k-2}$ and one coatom $U^{n-1}H^2D^{n-1}$. The socle is the path $s = (UD)^n$ and the principal ideal $\downarrow s$ is isomorphic to a Boolean algebra B_n. This time, however, the principal filter $\uparrow s$ just contains an isomorphic copy of the Schröder lattice $\mathcal{S}_{n-1}$ consisting of the principal filter generated by the path $s' = UH^{2n-2}D$. Moreover, the interval $[s, s']$ is a Boolean algebra B_{n-2} (see Figure 7). The rank function is given by the area under the path (as for Motzkin lattices).
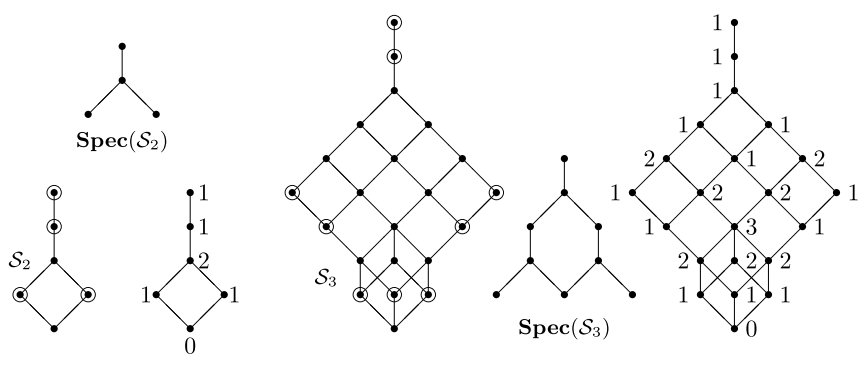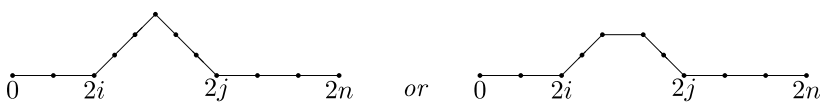
Figure 7: The Schröder lattices $\mathcal{S}_2$ and $\mathcal{S}_3$, their spectra and the distribution of the characteristic.

Proposition 5.1. *The join-irreducibles of the Schröder lattice $\mathcal{S}_n$ are the paths of the form*

In particular, $\mathbf{Spec}(\mathcal{S}_n) \simeq \{(i, j, k) \in \mathcal{C}_n^2 \times \mathcal{C}_1 : j - i \geq k + 1\}$.

Proof. Given a Schröder path, there are only two possible ways of getting a path which is covered by the starting one: either replace an occurrence of UD with a double horizontal step or replace a double horizontal step with DU. □

There are at least two further ways of describing the poset $\mathbf{Spec}(\mathcal{S}_n)$. They are essentially equivalent, but the first one is expressed in purely algebraic language whereas the second one can be considered a sort of combinatorial interpretation.

1. The *lexicographic product* $P \circ Q$ of two posets P and Q is the set $P \times Q$ endowed with the order defined by setting $(x_1, y_1) \leq (x_2, y_2)$ when $x_1 < x_2$ or $x_1 = x_2$ and $y_1 \leq y_2$. Then $\mathbf{Spec}(\mathcal{S}_n)$ is isomorphic to the poset obtained by $\mathbf{Spec}(\mathcal{D}_{n+1}) \circ \mathcal{C}_1$ when all the minimal elements are removed (see Figure 7).

2. Denote by $\overrightarrow{\mathbf{Int}}(\mathcal{C}_n)$ the set of oriented intervals of $\mathcal{C}_n$. An interval I of a poset P is said to be *oriented upward* (*downward*) when its elements are listed in such a way that, if $x < y$ in P, then x precedes (follows) y in the above listing (in this way I is not just a set but, more precisely, an ordered t-uple). If P is a chain, $P = \mathcal{C}_n$, then its intervals can

have only two orientations, say *negative* (from top to bottom) and *positive* (from bottom to top). We can introduce a partial order on $\overrightarrow{\mathbf{Int}}(\mathcal{C}_n)$ by simply declaring that, for any $I, J \in \overrightarrow{\mathbf{Int}}(\mathcal{C}_n)$, $I \leq J$ when either I is contained in J as ordinary intervals or $I = J$ as ordinary intervals but I is negative and J is positive. It is not difficult to see that, endowed with this partial order, $\overrightarrow{\mathbf{Int}}(\mathcal{C}_{n-1}) \simeq \mathbf{Spec}(\mathcal{S}_n)$ (observe that, in $\overrightarrow{\mathbf{Int}}(\mathcal{C}_n)$, as far as singleton intervals are concerned, there is no distinction between positive and negative intervals).

Proposition 5.2. *The Schröder lattices $\mathcal{S}_n$ are special Dyck-like lattices where, in particular, the meet of any two join-irreducibles is $\widehat{0}$ or a join-irreducible having a unique peak.*

Proof. As we did for Dyck paths, label each join-irreducible of $\mathcal{S}_n$ with the abscissa of its unique (truncated) pyramid: this labelling satisfies the condition in the definition of a Dyck-like lattice. Since two Schröder paths of the same length can cross only at points with integer coordinates and cannot cross in the middle point of a double horizontal step of one of them, if the intersection of two join-irreducibles is not the minimum $\widehat{0}$, then it is necessarily a join-irreducible with a unique peak. $\square$

5.2. Characteristic

The characteristic of Schröder lattices admits a combinatorial interpretation analogous to the one given for Dyck lattices. Indeed, following the same lines of the proofs of Proposition 3.21 and Corollary 3.22 and using Proposition 5.1, we have

Theorem 5.3. *A Schröder path is quasi-join-irreducible if and only if it has exactly one 0-tunnel.*

Theorem 5.4. *The characteristic of a Schröder path equals the number of its 0-tunnels.*

6. Final remarks on rank unimodality

An interesting property common to several sequences arising in combinatorics is unimodality. Specifically, a *sequence* $\{a_0, a_1, \ldots, a_n\}$ of positive integers is *unimodal* when there exists an index k such that $a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$, and a *polynomial* is *unimodal* when the sequence of its coefficients is unimodal. In the case of ranked posets this is often a property of the distribution of the elements of given rank. More precisely, a (finite) *poset P* is *rank unimodal* when it is ranked and its rank poly-

nomial (or equivalently the sequence of its Whitney numbers) is unimodal. The *Whitney number* $W_k(P)$ is the number of all elements of P with rank k whereas the *rank polynomial* is $R(P; q) = W_0(P) + W_1(P)q + \cdots + W_h(P)q^h$, where $h = r(P)$ is the height of P.

Dyck lattices. Let $D_n(q)$ be the rank polynomial of $\mathcal{D}_n$. Since every non-empty Dyck path decomposes as $U\gamma'D\gamma''$ (where γ' and γ'' are arbitrary Dyck paths), we have the recurrence

$$D_{n+1}(q) = \sum_{k=0}^{n} q^k D_k(q) D_{n-k}(q)$$

with the initial condition $D_0(q) = 1$. It is easy to see that the generating series for these polynomials satisfies the identity $D(q, t) = (1 - tD(q, qt))^{-1}$ from which it is possible to obtain an expansion as a continued fraction [23]. The polynomials $D_n(q)$ define a q-analog of Catalan numbers, namely $q^{\binom{n}{2}} D_n(1/q) = C_n(q)$, where the $C_n(q)$'s are the *q-Catalan numbers* defined as the sum $\sum_x q^{\mathcal{A}(x)}$ over all lattice paths from $(0, 0)$ to (n, n), with steps $(1, 0)$ and $(0, 1)$, never rising above the line $y = x$, where $\mathcal{A}(x)$ is the area of the region determined by the path and the x-axis [14] [23, p. 235]. The Whitney numbers of $\mathcal{D}_n$ appears in [21] (essentially) as sequence A129182. It is still an open problem [6] to prove or disprove that the rank polynomials $D_n(q)$ are unimodal for every $n \in \mathbb{N}$. This problem is also mentioned in [24], where it is conjectured that the Young's lattices associated with the staircase partitions $(n, n-1, \ldots, 2, 1)$ are rank unimodal.

Motzkin lattices. Let $M_n(q)$ be the rank polynomial of $\mathcal{M}_n$. Since every non-empty Motzkin path decomposes as $H\gamma$ or $U\gamma'D\gamma''$ (where γ, γ' and γ'' are arbitrary Motzkin paths), we have the recurrence

$$M_{n+2}(q) = M_{n+1}(q) + \sum_{k=0}^{n} q^{k+1} M_k(q) M_{n-k}(q)$$

with the initial conditions $M_0(q) = M_1(q) = 1$. Their generating series $M(q, t)$ satisfies the identity $M(q, t) = (1 - t - qt^2 M(q, qt))^{-1}$ and hence admits an expansion as a continued fraction. The Whitney numbers of $\mathcal{M}_n$ appear in [21] as sequence A129181 and also in this case we can conjecture that the lattices $\mathcal{M}_n$ are rank-unimodal.

Schröder lattices. Let $S_n(q)$ be the rank polynomial of $\mathcal{S}_n$. Since every non-empty Schröder path decomposes as $HH\gamma$ or $U\gamma'D\gamma''$ (where γ, γ' and γ''

are arbitrary Schröder paths), we have the recurrence

$$S_{n+1}(q) = S_n(q) + \sum_{k=0}^{n} q^{2k+1} S_k(q) S_{n-k}(q)$$

with the initial condition $S_0(q) = 1$. Their generating series $S(q, x)$ satisfies the identity $S(q, x) = (1 - x - qxS(q, q^2x))^{-1}$ and hence also this time it has an expansion as a continued fraction. The Whitney numbers of $\mathcal{S}_n$ appear in [21] as sequence A129179. Also in this case it is still an open problem [6] to prove the rank-unimodality of the lattices $\mathcal{S}_n$.

Acknowledgment

References

[1] M. Aigner, *Combinatorial Theory*, Springer-Verlag, New York, 1979. MR0542445.

[2] C. Banderier, P. Flajolet, *Basic analytic combinatorics of directed lattice paths*, Theoret. Comput. Sci. **281** (2002), 37–80. MR1909568.

[3] E. Barcucci, A. Bernini, L. Ferrari, M. Poneti, *A distributive lattice structure connecting Dyck paths, noncrossing partitions and* 312-*avoiding permutations*, Order **22** (2005), 311–328. MR2220342.

[4] A. Bernini, L. Ferrari, *Order properties of the Motzkin and Schröder families*, Australasian J. Combin. **39** (2007), 259–272. MR2351206.

[5] G. Birkhoff, *Lattice Theory*, third edition, Amer. Math. Soc. Colloquium Publications, Providence, 1966.

[6] J. Bonin, L. Shapiro, R. Simion, *Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths*, J. Statist. Plann. Inference **34** (1993), 35–55. MR1209988.

[7] P. Crawley, R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, N.J., 1973.

[8] B. A. Davey, H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, New York, 2002. MR1902334.

[9] P. Duchon, *On the enumeration and generation of generalized Dyck words*, Discrete Math. **255** (2000), 121–135. MR1798327.

[10] S. Elizalde, *Fixed points and excedances in restricted permutations*, Proceedings of FPSAC, 2003.

[11] S. Elizalde, *Multiple pattern-avoidance with respect to fixed points and excedances*, Electron. J. Combin. **11** (2004), #R51 (40pp.). MR2097317.

[12] L. Ferrari, *Some combinatorics related to central binomial coefficients: Grand-Dyck paths, coloured noncrossing partitions and signed pattern avoiding permutations*, Graphs and Combinatorics **26** (2010), 51–70. MR2606618.

[13] L. Ferrari, R. Pinzani, *Lattices of lattice paths*, J. Statist. Plann. Inference **135** (2005), 77–92. MR2202340.

[14] J. Fürlinger, J. Hofbauer, *q-Catalan numbers*, J. Combin. Theory Ser. A **40** (1985), 248–264. MR0814413.

[15] D. A. Klain, G.-C. Rota, *Introduction to Geometric Probability*, Cambridge University Press, New York, 1997. MR1608265.

[16] V. Klee, *The Euler characteristic in combinatorial geometry*, Amer. Math. Monthly **79** (1963), 119–127. MR0146101.

[17] E. Munarini, *On the Euler characteristic of finite distributive lattices*, pp. 165–186, in From Combinatorics to Philosophy. The Legacy of G.-C. Rota, E. Damiani, O. D'Antona, V. Marra, F. Palombi, editors, Springer, 2009.

[18] G.-C. Rota, *On the Combinatorics of the Euler Characteristic*, Studies in Pure Mathematics, Academic Press, London, 1971, pp. 221–233. MR0282892.

[19] L. Santocanale, *On the join dependency relation in multinomial lattices*, Order **24** (2007), 155–179. MR2358079.

[20] A. Sapounakis, I. Tasoulas, P. Tsikouras, *On the dominance partial ordering of Dyck paths*, J. Integer Seq. **9** (2006), #06.2.5 (17 pp.). MR2217231.

[21] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, published electronically at http://oeis.org.

[22] R. P. Stanley, *The Fibonacci lattice*, Fibonacci Quart. **13** (1975), 215–232. MR0387143.

[23] R. P. Stanley, *Enumerative Combinatorics*, Volume 2, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, Cambridge, 1999. MR1676282.

[24] D. Stanton, *Unimodality and Young's lattice*, J. Combin. Theory Ser. A **54** (1990), 41–53. MR1051777.

Luca Ferrari
Dipartimento di Sistemi e Informatica
Universitá di Firenze
viale Morgagni 65, 50134 Firenze
Italy
E-mail address: ferrari@dsi.unifi.it

Emanuele Munarini
Dipartimento di Matematica
Politecnico di Milano
Piazza Leonardo da Vinci 32, 20133 Milano
Italy
E-mail address: emanuele.munarini@polimi.it