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How many alphabets can a Schur function accommodate?

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Four.

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1. Introduction

Let $\mathbf{x} = \mathbf{x}_n = \{x_1, \ldots, x_n\}$ be a finite set of indeterminates, $\mathfrak{Sym}$ be the ring of symmetric polynomials in $\mathbf{x}$. *Schur functions* constitute the fundamental basis of $\mathfrak{Sym}$. The classical way to express them is either as a determinant of complete functions of $\mathbf{x}$, or as a quotient of a determinant of powers of $x_1, \ldots, x_n$ by the *Vandermonde* $\Delta(\mathbf{x}) = \prod_{1 \le i < j \le n}(x_j - x_i)$.

The first choice may be generalized by replacing complete functions of $\mathbf{x}$ by complete functions of $\mathbf{x} - \mathbf{y}_k$, that is, by the coefficients of z in the Taylor expansion of $\prod_{j=1}^{k}(1 - zy_j)\prod_{i=1}^{n}(1 - zx_i)^{-1}$, with $\mathbf{y}_k = \{y_1, \ldots, y_k\}$ a second set of indeterminates (*flag Schur functions [9, 11, 19]*).

For example, Schubert calculus on Graßmannians interprets $x_1, \ldots, x_n$ as the Chern roots of the so-called tautological quotient bundle. A second set $\mathbf{y}_\infty$ enables flag conditions defining Schubert varieties. Algebraically, the class in the cohomology ring of the Schubert variety corresponding to the partition $\lambda \in \mathbb{N}^n$ is

$$\det\big(S_{\lambda_i + j - i}(\mathbf{x} - \mathbf{y}_{k_i})\big)_{i,j=1\ldots n},$$

with $[k_1, \ldots, k_n]$ depending upon λ, and $S_j()$ being the complete function of degree j. Specializing $\mathbf{y}$ to $\mathbf{0} = \{0, 0, \ldots\}$ gives the *Schur function* $S_\lambda(\mathbf{x})$ *of index* λ.

The class of a Schubert variety in the Grothendieck ring of the Graßmannian may be expressed by a similar determinant that I give in [9].

Considerations about Weyl's dimension formula for the representations of the unitary groups led Biedenharn and Louck [1] to define *factorial Schur functions* as the quotient

$$\det\left(x_j^{\langle \lambda_i+n-i\rangle}\right)\Delta(\mathbf{x})^{-1},$$

where $x^{\langle k\rangle}$ is the modified power $x(x-1)\dots(x-k+1)$. Okounkov and Olshanski [18] use these functions to study the asymptotic character theory of the symmetric and unitary group.

A small further step is to take $x^{\langle k,q\rangle}=(x-1)(x-q)\dots(x-q^{k-1})$ instead of $x^{\langle k\rangle}$, the ensuing functions being the *q-factorial Schur functions*. Molev [17] takes as building blocks the Newton's polynomials $x^{\langle k,\mathbf{y}\rangle}=(x-y_1)\dots(x-y_k)$. Macdonald surveys in [15] these different variations about Schur functions.

As a matter of fact, all the preceding generalizations of Schur functions are special cases of Schubert polynomials, and consequently, may be easily obtained using divided differences only. Since Schubert polynomials may also be obtained by enumeration of Young tableaux, the combinatorial description of these functions also follows from the combinatorics of Schubert polynomials [10]. For the correspondence between tableaux and paths, we refer to [5], and for a description of flag Schur functions in terms of lattice paths, to [2].

We give below a deformation of Schur functions involving, apart from $\mathbf{x}$, three other alphabets $\mathbf{a},\mathbf{b},\mathbf{y}$, as promised by the abstract.

As can be expected, these last functions satisfy properties similar to those of Schur functions: Jacobi-Trudi like determinants, Giambelli like determinants, Cauchy decomposition. But before writing precise formulas, we need to recall some facts about Schubert polynomials.

2. Schubert polynomials

The symmetric group $\mathfrak{S}_n$ acts on the ring of polynomials in $\mathbf{x}$, with coefficients in $\mathbf{a}=\{a_1,a_2,\dots\}$, $\mathbf{b}=\{b_1,b_2,\dots\}$, $\mathbf{y}=\{y_1,y_2,\dots\}$. Specifically, the *simple transposition* s_i, $i=1,\dots,n-1$, acts by transposing x_i,x_{i+1}, fixing the other x_j, as well as $\mathbf{a},\mathbf{b},\mathbf{y}$. The action is denoted exponentially $f\to f^{s_i}$.

The *divided difference* ∂_i, acting on its left, is the operator

$$f(\mathbf{x}_n)\to f(\mathbf{x}_n)\,\partial_i=(f-f^{s_i})\,(x_i-x_{i+1})^{-1}. \tag{1}$$

To any permutation $\sigma \in \mathfrak{S}_n$ corresponds a divided difference ∂_σ which may be obtained as a product of ∂_i corresponding to a reduced decomposition of σ. The divided difference ∂_ω associated to the maximal permutation $\omega = [n, \dots, 1]$ can also be interpreted as the *Cauchy-Jacobi operator* [10, Prop. 7.6.2]

$$f \to \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} f^\sigma \frac{1}{\Delta}.$$

Notice that, given n functions $f_1, \dots, f_n$ of a single variable, the image of the product $f_1(x_1) \dots f_n(x_n)$ under ∂_ω is equal to

$$\det\left(f_i(x_j)\right) \Delta(\mathbf{x})^{-1}.$$

Thus the above determinants of (modified) powers are of this type.

The *Schubert polynomials* $Y_v(\mathbf{x}, \mathbf{a})$, $v \in \mathbb{N}^n$, are polynomials in $\mathbf{x} = \{x_1, \dots, x_n\}$ with coefficients in $\mathbf{a} = \{a_1, a_2, \dots, a_\infty\}$ which are defined starting from the case where v is *dominant*, i.e. v is a partition: $v = \lambda = [\lambda_1, \dots, \lambda_n]$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. In that case

$$Y_\lambda(\mathbf{x}, \mathbf{a}) := \prod_{i=1}^{n} \prod_{j=1}^{v_i} (x_i - a_j).$$

The general polynomials are then defined recursively by applying divided differences, while sorting and decreasing the indices

$$Y_v(\mathbf{x}, \mathbf{a})\, \partial_i = Y_{\dots, v_{i-1}, v_{i+1}, v_i - 1, v_{i+2}, \dots}(\mathbf{x}, \mathbf{a}),\ v_i > v_{i+1}$$

(in the case where $v_i \leq v_{i+1}$, then $Y_v(\mathbf{x}, \mathbf{a})\, \partial_i = 0$ since $\partial_i \partial_i = 0$).

Since ∂_i sends polynomials to polynomials symmetrical in x_i, x_{i+1}, the Schubert polynomial $Y_v(\mathbf{x}, \mathbf{a})$ has this symmetry whenever $v_i \leq v_{i+1}$. In particular, if v is *anti-dominant* (i.e. $v_1 \leq \dots \leq v_n$), then $Y_v(\mathbf{x}, \mathbf{a})$ is symmetrical in $x_1, \dots, x_n$ and is called a *Graßmannian Schubert polynomial.* It specializes to the Schur function in $\mathbf{x}$ of index $v_n, \dots, v_1$ for $\mathbf{a} = \mathbf{0}$, to the factorial Schur function for $\mathbf{a} = \{0, 1, 2, \dots\}$, to the q-factorial Schur function for $\mathbf{a} = \{1, q, q^2, \dots\}$.

When there exists $k : v_{k+1} = 0 = \dots = v_n$, then $Y_v(\mathbf{x}, \mathbf{a})$ is of degree 0 in $x_{k+1}, \dots, x_n$. In that case, one can change the notations and replace v by $[v_1, \dots, v_k]$, or $\mathbf{x}$ by $x_1, \dots, x_k$, or make both changes. From now on, we shall suppose $\mathbf{x}$ to be infinite, $Y_v(\mathbf{x}, \mathbf{y})$ being of degree 0 in $x_{n+1}, x_{n+2}, \dots$ if $v \in \mathbb{N}^n$.

The Graßmannian Schubert polynomial $Y_v(\mathbf{x},\mathbf{a})$ is the image of $Y_{v_n+n-1}(x_1,\mathbf{a})\cdots Y_{v_1+0}(x_n,\mathbf{a})$ under ∂_ω, and therefore can be written as

$$Y_{v_n+n-1}(x_1,\mathbf{a})\cdots Y_{v_1+0}(x_n,\mathbf{a}) = \det\Big(Y_{v_j+j-1}(x_i,\mathbf{a})\Big)\,\Delta^{-1}.$$

Thanks to [10, Prop. 9.3.1], the above determinant can be transformed, giving still another expression of a Graßmannian Schubert polynomial:
(2)

$$Y_v(\mathbf{x},\mathbf{a}) = \begin{vmatrix} Y_{v_1}(\mathbf{x},\mathbf{a}) & Y_{v_2+1}(\mathbf{x},\mathbf{a}) & \cdots & Y_{v_n+n-1}(\mathbf{x},\mathbf{a}) \\ Y_{0,v_1-1}(\mathbf{x},\mathbf{a}) & Y_{0,v_2}(\mathbf{x},\mathbf{a}) & \cdots & Y_{0,v_n+n-2}(\mathbf{x},\mathbf{a}) \\ \vdots & \vdots & & \vdots \\ Y_{0^{n-1},v_1-n+1}(\mathbf{x},\mathbf{a}) & Y_{0^{n-1},v_2-n+2}(\mathbf{x},\mathbf{a}) & \cdots & Y_{0^{n-1},v_n}(\mathbf{x},\mathbf{a}) \end{vmatrix}.$$

The non-zero images of a Graßmannian polynomial can be computed from this expression. Given $u,v\in\mathbb{N}^n$ anti-dominant, such that $u_1\le v_1,\ldots,$ $u_n\le v_n$, let the *skew Graßmannian Schubert polynomial* $Y_{v/u}(\mathbf{x},\mathbf{a})$ be

$$\begin{aligned} Y_{v/u}(\mathbf{x},\mathbf{a}) &= Y_{0^{u_1},v_1-u_1,0^{u_2-u_1},v_2-u_2,\ldots,0^{u_n-u_{n-1}},v_n-u_n}(\mathbf{x},\mathbf{a}) \\ &= Y_v(\mathbf{x},\mathbf{a})(\partial_n\cdots\partial_{n+u_n-1})\,(\partial_{n-1}\cdots\partial_{n+u_{n-1}-2})\cdots(\partial_1\cdots\partial_{u_1}). \qquad (3) \end{aligned}$$

The determinant appearing in (2) is such that its first row is independent of $x_2,\ldots,x_n$, the second row, is independent of $x_3,\ldots,x_n,\ldots$. Therefore the product $(\partial_n\cdots\partial_{n+u_n-1})$ acts only on the last row, the product $(\partial_{n-1}\cdots\partial_{n+u_{n-1}-2})$ acts only on the row just above, and so on. One recognizes in the final outcome the minor taken on rows $u_1+1,\ u_2+2,\ldots,u_n+n$ of the matrix

$$\Big[Y_{0^i,v_1-i}(\mathbf{x},\mathbf{a}),\ Y_{0^i,v_2+1-i}(\mathbf{x},\mathbf{a}),\ \ldots,\ Y_{0^i,v_n+n-1-i}(\mathbf{x},\mathbf{a})\Big]_{i\ge 0}.$$

For example,

$$Y_{457/134}(\mathbf{x},\mathbf{a}) = Y_{457}(\partial_3\partial_4\partial_5\partial_6)\,(\partial_2\partial_3\partial_4)\,(\partial_1) = Y_{0300203}(\mathbf{x},\mathbf{a})$$

is equal to the determinant

$$\begin{vmatrix} Y_{0,3}(\mathbf{x},\mathbf{a}) & Y_{0,5}(\mathbf{x},\mathbf{a}) & Y_{0,8}(\mathbf{x},\mathbf{a}) \\ Y_{0^4,0}(\mathbf{x},\mathbf{a}) & Y_{0^4,2}(\mathbf{x},\mathbf{a}) & Y_{0^4,5}(\mathbf{x},\mathbf{a}) \\ 0 & Y_{0^6,0}(\mathbf{x},\mathbf{a}) & Y_{0^6,3}(\mathbf{x},\mathbf{a}) \end{vmatrix}.$$

The top row is symmetrical in x_1,x_2, the middle row is symmetrical in $x_1,\ldots,x_5$, the last row symmetrical in $x_1,\ldots,x_7$, the Schubert polynomial

$Y_{0300203}(\mathbf{x}, \mathbf{a})$ itself being symmetrical in x_1, x_2, in $x_3, \ldots, x_5$ and in x_6, x_7, as told by decomposing its index 03 002 03 into three increasing blocks.

3. Four alphabets

We have now the tools to define new functions, using Schubert polynomials in different alphabets.

Definition 1. Given $r \in \mathbb{N}$, $\lambda \in \mathbb{N}^n$ a partition, $\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y}$ four infinite alphabets, let

$$\varphi_r(x_j, \mathbf{a}, \mathbf{b}, \mathbf{y}) = \sum_{i=0}^{r} Y_i(x_j, \mathbf{a}) Y_{0^i, r-i}(\mathbf{b}, \mathbf{y}),$$
$$\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y}) = \varphi_{\lambda_1+n-1}(x_1, \mathbf{a}, \mathbf{b}, \mathbf{y}) \varphi_{\lambda_2+n-2}(x_2, \mathbf{a}, \mathbf{b}, \mathbf{y}) \cdots \varphi_{\lambda_n+0}(x_n, \mathbf{a}, \mathbf{b}, \mathbf{y})\, \partial_\omega .$$

This definition by divided differences is easily implemented. However, the number of terms in the expansion of such functions grows very fast. For $n = 3$, $\lambda = [2, 1, 0]$, the Schur function $S_{210}(\mathbf{x}_3)$ is a sum of 7 monomials, while the function $\varphi_{210}(\mathbf{x}_3, \mathbf{a}, \mathbf{b}, \mathbf{y})$ is a sum of 416 terms that we shall avoid displaying for ecological reasons. Nevertheless, we are going to illustrate that, at the pure algebraic level, the theory of these functions is not more complicated than the theory of factorial Schur functions.

Thanks to the expression of ∂_ω as a summation over the symmetric group, one can also write $\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y})$ determinantaly:

$$\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y})\, \Delta = \left|\varphi_{\lambda_j+n-j}(x_i, \mathbf{a}, \mathbf{b}, \mathbf{y})\right|_{i,j=1\ldots n} . \tag{4}$$

The expression of $\varphi_r(x_j, \mathbf{a}, \mathbf{b}, \mathbf{y})$ as a sum of products entails that the matrix (4) factorizes into the product of the $n \times \infty$ and $\infty \times n$ matrices

$$\Big[1,\, Y_1(x_i, \mathbf{a}),\, Y_2(x_i, \mathbf{a}),\, Y_3(x_i, \mathbf{a}) \ldots\Big]_{i=1,\ldots,n}$$

and

$$\Big[Y_{0^j, \lambda_n - j}(\mathbf{b}, \mathbf{y}),\, Y_{0^j, \lambda_{n-1}+1-j}(\mathbf{b}, \mathbf{y}), \ldots, Y_{0^j, \lambda_1+n-1-j}(\mathbf{b}, \mathbf{y})\Big]_{j=0,1,2,\ldots} .$$

Minors of the second matrix are skew-Graßmannian Schubert polynomials, according to (3), while the minors of the other matrix are Graßmannian polynomials times the Vandermonde. The Binet-Cauchy theorem for minors of a product of two matrices gives the following expansion of $\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y})$.

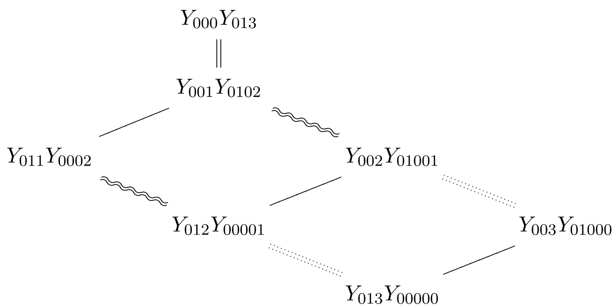
Theorem 2. *Let $\lambda \in \mathbb{N}^n$ be a partition, $v = [\lambda_n, \dots, \lambda_1]$. Then*

$$\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y}) = \sum_u Y_u(\mathbf{x}, \mathbf{a}) Y_{v/u}(\mathbf{b}, \mathbf{y}),$$

sum over all $u \in \mathbb{N}^n$, u antidominant, $u_1 \le v_1, \dots, u_n \le v_n$.

Thus the function $\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y})$ decomposes into a sum of products of a deformation of a Schur function times a deformation of a skew Schur function.

For example, writing $Y_u Y_w$ for $Y_u(\mathbf{x}, \mathbf{a}) Y_w(\mathbf{b}, \mathbf{y})$, here is the expansion of $\varphi_{310}(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y})$:



One can interpret such a graph as obtained using divided differences in $\mathbf{b}$ and $\mathbf{a}$. Indeed the different polynomials $Y_{v/u}(\mathbf{b}, \mathbf{y})$ are obtained by applying divided differences in $\mathbf{b}$ to $Y_{013}(\mathbf{b}, \mathbf{y})$. Thanks to a symmetry [10, 10.2.6] of Schubert polynomials exchanging the two alphabets $\mathbf{x}, \mathbf{a}$, the polynomials $Y_u(\mathbf{x}, \mathbf{a})$ are obtained using $-\partial_i^a$ for the edge corresponding to ∂_i^b (but this time reading upwards), starting from $Y_{013}(\mathbf{x}, \mathbf{a})$. For example, the edge connecting $Y_{011}Y_{0002}$ and $Y_{012}Y_{00001}$ corresponds to $Y_{0002}(\mathbf{b}, \mathbf{y})\partial_4^b = Y_{00001}(\mathbf{b}, \mathbf{y})$, as well as $Y_{012}(\mathbf{x}, \mathbf{a})\partial_4^a = -Y_{011}(\mathbf{x}, \mathbf{a})$.

Choosing $\mathbf{a} = \{0, 1, 2, \dots\}$ or $\mathbf{a} = \{1, q, q^2, \dots\} = (1-q)^{-1}$ gives a function $\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y})$ which is a linear combination of factorial Schur functions (resp. q-factorial Schur functions) with coefficients which are skew Schubert polynomials in $\mathbf{b}, \mathbf{y}$. Choosing $\mathbf{a} = \{0, 1, 2, \dots\}$, $\mathbf{b} = \{1, q, q^2, \dots\}$ while keeping $\mathbf{y}$ generic gives a mixture of factorial, q-factorial Schur functions and Graßmannian Schubert polynomials.

When $\mathbf{a} = \mathbf{b}$, then the Cauchy formula for Schubert polynomials [10, Th.10.2.6] shows that

$$\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{a}, \mathbf{y}) = Y_{\lambda_n, \dots, \lambda_1}(\mathbf{x}, \mathbf{y}),$$

i.e. is equal to the Graßmannian Schubert polynomial in the alphabets $\mathbf{x}, \mathbf{y}$ of index $[\lambda_n, \dots, \lambda_1]$. When $\mathbf{b} = \mathbf{y}$, all the Schubert polynomials $Y_{v/u}(\mathbf{y}, \mathbf{y})$

vanish, except for $u = v$, and therefore $\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{y}, \mathbf{y}) = Y_{\lambda_n,\dots,\lambda_1}(\mathbf{x}, \mathbf{a})$, which is the same polynomial but for the exchange of $\mathbf{y}$ and $\mathbf{a}$.

Other specializations may be of interest. For example, when $\mathbf{a} = \mathbf{0} = \mathbf{y}$, and $\mathbf{b} = \{1, q, q^2, \dots\}$, then $\varphi_\lambda(\mathbf{x}, \mathbf{0}, (1-q)^{-1}, \mathbf{0})$ is a sum of products of Schur functions times a determinant of q-binomials. *Rogers-Szegő polynomials*, since $Y_k(\mathbf{x}, \mathbf{0}, (1-q)^{-1}, \mathbf{0})$ reduces to the usual Rogers-Szegő polynomial $\sum_{i=0}^n x^i \begin{bmatrix} n \\ i \end{bmatrix}$ when $n = 1$, writing $\begin{bmatrix} n \\ i \end{bmatrix}$ for the q-binomial $(1-q^n)\cdots(1-q^{n-i+1})$ $(1-q)^{-1}\cdots(1-q^i)^{-1}$.

We have already said that $\varphi_\lambda(\mathbf{x}, (1-q)^{-1}, \mathbf{0}, \mathbf{0})$ is a q-factorial Schur function. If we take now $\mathbf{b} = \{q, q^2, \dots\} = q(1-q)^{-1}$ instead of $\mathbf{0}$, then

$$\varphi_r(x, (1-q)^{-1}, q(1-q)^{-1}, \mathbf{0}) = x^r + (q^r - 1)x^{r-1}.$$

Consequently

$$\varphi_\lambda\big(\mathbf{x}, (1-q)^{-1}, q(1-q)^{-1}, \mathbf{0}\big) = \sum_\mu \frac{c_\lambda}{c_\mu} S_\mu, \tag{5}$$

where the sum runs over all partitions μ such that λ/μ be a vertical strip, with $c_\lambda = \prod_{i=1}^n \prod_{j=1}^{\lambda_i+n-i} (q^j - 1)$.

For example,

$$\begin{aligned}\varphi_{522}\left(\mathbf{x}, \frac{1}{1-q}, \frac{q}{1-q}, \mathbf{0}\right) &= S_{522} + (q^7-1)S_{422} + (q^2-1)S_{521} \\ &\quad + (q^2-1)(q^3-1)S_{521} + (q^2-1)(q^7-1)S_{421} \\ &\quad + (q^2-1)(q^3-1)(q^7-1)S_{411}.\end{aligned}$$

The case $\mathbf{a} = 1/(1-q)$, $\mathbf{b} = b/(1-q) = \{b, bq, bq^2, \dots\}$ is a little more elaborate and is worth some comments which are given in the next section.

4. Still another q-Schur function

For any partition λ, let

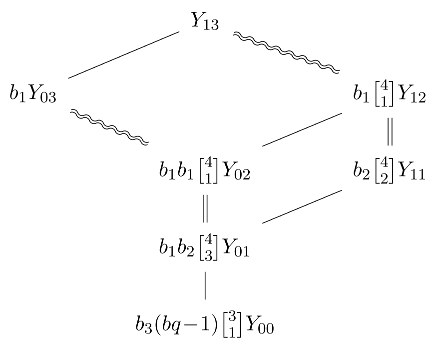
$$\psi_\lambda(\mathbf{x}) = \varphi_\lambda\left(\mathbf{x}, \frac{1}{1-q}, \frac{b}{1-q}, \mathbf{0}\right).$$

For $b = 0$, $\psi_\lambda(\mathbf{x})$ is a q-factorial Schur function, for $b = q$, it was given in the preceding subsection, while it reduces to the usual Schur function for $b = 1$.

As for Schur functions, one extends the indexing to all $v \in \mathbb{Z}^n$, by the convention that $\psi_i(x) = 0$, $i < 0$, when $n = 1$, and

$$\psi_v(\mathbf{x}) = \psi_{v_1+n-1}(x_1) \cdots \psi_{v_n}(x_n)\, \partial_\omega .$$

Writing Y_{ij} for $Y_{ij}(\mathbf{x}, \mathbf{0})$, b_i for $(b-1)(b-q)\cdots(b-q^{i-1})$, here is for example the expansion of $\psi_{31}(\mathbf{x})$:



Recall that $S_r(b/(1-q)) = b^r/(q:q)_r$, with $(a:q)_r := (1-a)(1-aq)\cdots(1-aq^{r-1})$. In the case where $\lambda = r \in \mathbb{N}$, then

$$\begin{aligned}\psi_r(\mathbf{x}) &= \sum_i S_i\left(\frac{x_1-1}{1-q}\right)(q:q)_i b^{r-i}\begin{bmatrix} r \\ r-i\end{bmatrix}\\ &= (q:q)_r \sum_i S_i\left(\frac{x_1-1}{1-q}\right) S_{r-i}\left(\frac{b}{1-q}\right)\\ &= (q:q)_r\, S_r\left(\frac{x_1-1+b}{1-q}\right).\end{aligned}$$

Hence, with $c_\lambda = (q:q)_{\lambda_1+n-1}\cdots(q:q)_{\lambda_n+0}$,

$$\psi_\lambda(\mathbf{x}) = c_\lambda\, S_{\lambda_1+n-1}\left(\frac{x_1+b-1}{1-q}\right)\cdots S_{\lambda_n+0}\left(\frac{x_n+b-1}{1-q}\right)\partial_\omega . \tag{6}$$

To compute the expansion of $\psi_\lambda(\mathbf{x})$ in the basis of Schur functions in $\mathbf{x}$, one can proceed as follows.

Since

$$\psi_r(\mathbf{x})\frac{1}{(q:q)_r} = S_r\left(\frac{x_1}{1-q} + \frac{b-1}{1-q}\right) = \sum_i \frac{x_1^{r-i}}{(q:q)_{r-i}} S_i\left(\frac{b-1}{1-q}\right)$$

one can replace, with $v = [\lambda_1+n-1, \ldots, \lambda_n]$, the product $\psi_{v_1}(x_1)\cdots\psi_{v_n}(x_n)$ by $S_{v_1}(x_1+y_1)\cdots S_{v_n}(x_n+y_n)$, at the cost of introducing extra indeterminates $y_1, \ldots, y_n$. The monomial $x_1^{u_1}\cdots x_n^{u_n}$, $u \in \mathbb{N}^n$, which is sent to $S_{u_1+n-1,\ldots,u_n}(\mathbf{x})$ under ∂_ω, has coefficient

$$\frac{1}{(q:q)_{u_1}\cdots(q:q)_{u_n}} S_{v_1-u_1}\left(\frac{b-1}{1-q}\right)\cdots S_{v_n-u_n}\left(\frac{b-1}{1-q}\right)$$

in the first function, and

$$y_1^{v_1-u_1}\cdots y_n^{v_n-u_n}$$

in the second.

However, the image of $S_{v_1}(x_1+y_1)\cdots S_{v_n}(x_n+y_n)$ under ∂_ω is the multi-Schur function

$$S_\lambda(\mathbf{x}+y_1,\ldots,\mathbf{x}+y_n) := \begin{vmatrix} S_{\lambda_1}(\mathbf{x}+y_1) & \cdots & S_{\lambda_1+n-1}(\mathbf{x}+y_1) \\ \vdots & & \vdots \\ S_{\lambda_n-n+1}(\mathbf{x}+y_n) & \cdots & S_{\lambda_n}(\mathbf{x}+y_n) \end{vmatrix}. \tag{7}$$

Putting all this together, one obtains the following statement.

Proposition 3. *Let $\lambda \in \mathbb{N}^n$ be a partition. The function $\psi_\lambda(\mathbf{x})$ is the image of*

$$c_\lambda S_\lambda(\mathbf{x}+y_1,\ldots,\mathbf{x}+y_n)$$

under the linear morphism $y_j^i \to (b-1)\cdots(b-q^{i-1})(q:q)_i^{-1}, j = 1\ldots n$, $S_\mu(\mathbf{x}) \to c_\mu^{-1} S_\mu(\mathbf{x})$.

For example, let $\lambda = [5,3,2]$. The coefficient of $S_{111}(\mathbf{x})$ in the expansion of $\psi_{532}(\mathbf{x})$ corresponds to the terms

$$y^{421}x^{321},\ y^{511}x^{231},\ y^{430}x^{312},\ y^{610}x^{132}$$

in the expansion of $S_7(x_1+y_1)S_4(x_2+y_2)S_2(x_3+y_3)$. The image of $y^{421} - y^{511} - y^{430} + y^{610}$ is

$$\frac{q(bq^2-1)(bq-1)(b-1)^2(b-q)(b-q^2)(b-q^3)}{(1-q)^3(1-q^2)(1-q^3)(1-q^4)(1-q^6)}.$$

Correcting by $c_{532}/c_{111} = (q:q)_7(q:q)_4(q:q)_2/((q:q)_3(q:q)_2(q:q)_1)$, one finds that the coefficient of $S_{111}(\mathbf{x})$ is

$$\frac{q(bq^2-1)(bq-1)(b-1)^2(b-q)(b-q^2)(b-q^3)(1-q^7)(1-q^5)(1-q^4)}{(1-q)^3}.$$

In accordance with the case of Schur functions, let us call *Pieri formula* the expansion of the product

$$(x_1+z)\cdots(x_n+z)\,\psi_\lambda(\mathbf{x})$$

for all partitions $\lambda \in \mathbb{N}^n$, z being an extra indeterminate.

Since $(x_1+z)\cdots(x_n+z)$ is symmetrical in $\mathbf{x}$, it commutes with ∂_ω, so that

$$(8)\qquad \begin{aligned}&(x_1+z)\cdots(x_n+z)\,\psi_\lambda(\mathbf{x})\\ &\quad=\Big((x_1+z)\psi_{\lambda_1+n-1}(x_1)\ \Big)\cdots\Big((x_n+z)\psi_{\lambda_n}(x_n)\ \Big)\,\partial_\omega.\end{aligned}$$

One checks that, for any $r \in \mathbb{N}^n$,

$$\begin{aligned}x\psi_r(x) = \psi_{r+1}(x) + (1-b)\Big(&q^r\psi_r(x) + q^{r-1}b(q^r:q^{-1})_1\psi_{r-1}(x)\\ &+ q^{r-2}b(q^r:q^{-1})_2\psi_{r-2}(x)+\cdots\Big).\end{aligned}$$

Using these equalities in (8), one obtains the following proposition.

Proposition 4. *Let $\lambda \in \mathbb{N}^n$ be a partition. Let $d(r,j)$ be the function on $\mathbb{N}\times\mathbb{Z}$ such that*

$$d(r,1)=0\ ,\ d(r,0)=z+(1-b)q^r\ ,\ d(r,j)=(1-b)q^{r+j}b^{-j}(q^r:q^{-1})_{-j},\ j<0.$$

Then, with $v=[\lambda_1+n-1,\ldots,\lambda_n]$,

$$(9)\qquad (x_1+z)\cdots(x_n+z)\,\psi_\lambda(\mathbf{x}) = \sum_{u:u\le[1,\ldots,1]}\left(\prod_i d(v_i,u_i)\right)\,\psi_{\lambda+u}(\mathbf{x}),$$

sum over all $u\in\mathbb{Z}^n$, $u_1\le 1,\ldots,u_n\le 1$.

4.1. More determinants

Newton used divided differences in $x_1, x_2, \ldots$ to interpolate a function $f(z)$ at points $x_1, x_2, \ldots$. With a change of notations, his formula reads

$$\begin{aligned}f(z) &= f(x_1) + f(x_1)\partial_1\,(z-x_1) + f(x_1)\partial_1\partial_2\,(z-x_1)(z-x_2)+\cdots\\ &= f(x_1) + f(x_1)\partial_1\,Y_1(z,\mathbf{x}) + f(x_1)\partial_1\partial_2\,Y_2(z,\mathbf{x})+\cdots\end{aligned}$$

In particular,

$$\begin{aligned} f(x_2) &= f(x_1) + f(x_1)\partial_1\,(x_2 - x_1) \\ f(x_3) &= f(x_1) + f(x_1)\partial_1\,(x_3 - x_1) + f(x_1)\partial_1\partial_2\,(x_3 - x_1)(x_3 - x_2) \\ &\cdots \end{aligned}$$

Therefore, Eq. (4) can be transformed, factoring out the Vandermonde, and permuting the columns:

$$\begin{aligned} \varphi_\lambda(\mathbf{x},\mathbf{a},\mathbf{b},\mathbf{y}) &= \Big|\varphi_{v_j}(x_1,\mathbf{a},\mathbf{b},\mathbf{y})\,\partial_1\cdots\partial_{i-1}\Big| \\ &= \Big|\varphi_{v_j-i+1,0^{i-1}}(\mathbf{x},\mathbf{a},\mathbf{b},\mathbf{y})\Big|, \end{aligned} \tag{10}$$

with $v = [\lambda_n + 0, \ldots, \lambda_1 + n - 1]$.

This last determinant does not specialize to the usual Jacobi-Trudi determinant expressing a Schur function as a determinant of complete functions, because its entries are not symmetrical $x_1, \ldots, x_n$, apart from the ones in the last row, though $\varphi_\lambda(\mathbf{x},\mathbf{a},\mathbf{b},\mathbf{y})$ is. But one can recover symmetry by using the operator

$$\pi_\omega : f \to f\, x^{n-1,\ldots,0}\,\partial_\omega,$$

which sends dominant monomials onto Schur functions, but without decreasing degrees contrary to ∂_ω. This operator has the property that, given $\mu^1, \ldots, \mu^n \in \mathbb{N}^n$, then

$$S_{\mu^1}(x_1)S_{\mu^2}(x_1 + x_2)\cdots S_{\mu^n}(x_1 + \cdots + x_n)\,\pi_\omega = S_{\mu^1}(\mathbf{x})S_{\mu^2}(\mathbf{x})\cdots S_{\mu^n}(\mathbf{x}).$$

Therefore, one can replace the entries of the determinant (10) by their images under π_ω without altering its value. Since

$$Y_{0^{i-1},r}(\mathbf{x},\mathbf{a}) = S_r(x_1 + \cdots + x_i - a_1 - \cdots - a_{r+i-1}),$$

this last function is sent under π_ω onto

$$S_r(\mathbf{x} - a_{i+1-n} - \cdots - a_0 - \cdots - a_{r+i-1})\Big|_{a_j=0, j<0} = Y_{0^{n-1},r}(\mathbf{x},\mathbf{a}^{[n-i]}),$$

writing $\mathbf{a}^{[n-i]}$ for the shifted alphabet $\{\underbrace{0,\ldots,0}_{n-i}, a_1, a_2, \ldots\}$.

By linearity,

$$\varphi_{r,0^{i-1}}(\mathbf{x},\mathbf{a},\mathbf{b},\mathbf{y})\,\pi_\omega = \varphi_{r,0^{n-1}}(\mathbf{x},\mathbf{a}^{[n-i]},\mathbf{b}^{[n-i]},\mathbf{y}^{[n-i]}).$$

In the end, one obtains the following Jacobi-Trudi like expression of $\varphi_\lambda(\mathbf{x},\mathbf{a},\mathbf{b},\mathbf{y})$.

Proposition 5. *Let $\lambda \in \mathbb{N}^n$ be a partition, $v = [\lambda_n + 0, \dots, \lambda_1 + n - 1]$. Then one has*

$$\varphi_\lambda(\mathbf{x},\mathbf{a},\mathbf{b},\mathbf{y}) = \left|\varphi_{v_j-i+1,0^{n-1}}(\mathbf{x},\mathbf{a}^{[n-i]},\mathbf{b}^{[n-i]},\mathbf{y}^{[n-i]})\right|_{i,j=1\dots n}. \tag{11}$$

For example, for $\lambda = [7,4,2]$, one has $v = [2,5,9]$ and $\varphi_{742} =$

$$\begin{vmatrix} \varphi_{200}(\mathbf{x},\mathbf{a}^{[2]},\mathbf{b}^{[2]},\mathbf{y}^{[2]}) & \varphi_{500}(\mathbf{x},\mathbf{a}^{[2]},\mathbf{b}^{[2]},\mathbf{y}^{[2]}) & \varphi_{900}(\mathbf{x},\mathbf{a}^{[2]},\mathbf{b}^{[2]},\mathbf{y}^{[2]}) \\ \varphi_{100}(\mathbf{x},\mathbf{a}^{[1]},\mathbf{b}^{[1]},\mathbf{y}^{[1]}) & \varphi_{400}(\mathbf{x},\mathbf{a}^{[1]},\mathbf{b}^{[1]},\mathbf{y}^{[1]}) & \varphi_{800}(\mathbf{x},\mathbf{a}^{[1]},\mathbf{b}^{[1]},\mathbf{y}^{[1]}) \\ \varphi_{000}(\mathbf{x},\mathbf{a}^{[0]},\mathbf{b}^{[0]},\mathbf{y}^{[0]}) & \varphi_{300}(\mathbf{x},\mathbf{a}^{[0]},\mathbf{b}^{[0]},\mathbf{y}^{[0]}) & \varphi_{700}(\mathbf{x},\mathbf{a}^{[0]},\mathbf{b}^{[0]},\mathbf{y}^{[0]}) \end{vmatrix}.$$

This expression looks very similar to the one given in [15, 9th Variation]. However, Macdonald takes an automorphism where we take a projection when shifting alphabets (indeterminates of index smaller than 1 are sent to 0 in our case), and the functions $\varphi_\lambda(\mathbf{x},\mathbf{a},\mathbf{b},\mathbf{y})$ are not a special case of those considered by Macdonald.

We have recovered symmetry in $\mathbf{x}$ at the cost of shifting the other alphabets. However, it is known that a Graßmannian Schubert polynomial can be written as a determinant without shifts, generalizing the expression of a Schur function as a determinant of hook-Schur functions[1]. In fact, Hou and Mu [6] show that, instead of hook-Schur functions, one can use more generally the Taylor coefficients (in z) of a rational function $P(z)\left(1 - ze_1(\mathbf{x}) + z^2e_2(\mathbf{x}) + \cdots + (-z)^n e_n(\mathbf{x})\right)^{-1}$.

To state the corresponding expression for the functions $\varphi_\lambda(\mathbf{x},\mathbf{a},\mathbf{b},\mathbf{y})$, we must allow some flexibility in $\mathbf{x}$. We shall write

$$\varphi_\lambda(x_i + x_j + \cdots + x_k)$$

when using $\{x_i, x_j, \dots, x_k\}$ instead of $\mathbf{x}$, keeping $\mathbf{a},\mathbf{b},\mathbf{y}$ fixed and erasing them from the notation.

Lemma 6. *Let i, r be positive integers, X be a subset of $\mathbf{x}$ not containing x_i, x_{i+1}. Then*

$$\varphi_{1^r}(X + x_i)\,\partial_i = \varphi_{1^{r-1}}(X),$$
$$\varphi_{1^r}(X + x_{i+1})\,\partial_i = -\varphi_{1^{r-1}}(X).$$

[1] which was the way that Schubert expressed the classes of Schubert varieties in the cohomology ring of Graßmannians. Giambelli later shown that one could also use determinants of complete or elementary symmetric functions [16, Ex. 9, p. 49].

Proof. Due to the expansion of φ_{1^r}, the lemma reduces to the property that, for any k, j,

$$S_{1^k}(X + x_i - a_1 - \cdots - a_j)\,\partial_i = S_{1^{k-1}}(X - a_1 - \cdots - a_j) \\ = -S_{1^k}(X + x_{i+1} - a_1 - \cdots - a_j)\partial_i.$$

This lemma allows to generalize the expansion of hook-Schur functions as alternating sums of products of a complete function times an elementary symmetric function. □

Lemma 7. *For any non-negative integers r, n, k: $k \le n-1$, one has*

(12)
$$\begin{aligned}\varphi_{r,1^k,0^{n-1-k}}&(x_1+\cdots+x_n)\\ &= \varphi_{r,0^{n-1}}(x_1+\cdots+x_n)\varphi_{1^k,0^{n-1-k}}(x_1+\cdots+x_{n-1})\\ &\quad - \varphi_{r,0^{n-2}}(x_1+\cdots+x_{n-1}))\varphi_{1^{k-1},0^{n-1-k}}(x_1+\cdots+x_{n-2}) + \cdots\\ &\quad + (-1)^k\varphi_{r+k,0^{n-1-k}}(x_1+\cdots+x_{n-k}))\varphi_{1^0,0^{n-1-k}}(x_1+\cdots+x_{n-1-k}).\end{aligned}$$

Proof. This is a direct consequence of Leibniz's formula: $fg\partial_i = f\,(g\partial_i) + f\partial_i\, g^{s_i}$ for divided differences. Let us illustrate/prove the statement choosing $n=5, r=5, k=2$. Then $\varphi_{51100}(x_1+\cdots+x_5)$ is the image of $\varphi_9(x_1)\varphi_{1100}(x_2+\cdots+x_5)$ under $\partial_1\cdots\partial_4$. The chain of divided differences produces

$$\varphi_{80}(x_1+x_2)\varphi_{1100}(x_1+x_3+x_4+x_5) - \varphi_9(x_1)\varphi_{100}(x_3+x_4+x_5),$$

$$\varphi_{700}(x_1+x_2+x_3)\varphi_{1100}(x_1+x_2+x_4+x_5) - \varphi_{80}(x_1+x_2)\varphi_{100}(x_1+x_4+x_5) + \varphi_9(x_1),$$

$$\begin{aligned}&\varphi_{6000}(x_1+\cdots+x_4)\varphi_{1100}(x_1+x_2+x_3+x_5)\\ &\quad - \varphi_{700}(x_1+x_2+x_3)\varphi_{100}(x_1+x_2+x_5) + \varphi_{80}(x_1+x_2),\end{aligned}$$

$$\begin{aligned}&\varphi_{50000}(x_1+\cdots+x_5)\varphi_{1100}(x_1+\cdots+x_4)\\ &\quad - \varphi_{6000}(x_1+\cdots+x_4)\varphi_{100}(x_1+x_2+x_3) + \varphi_{700}(x_1+x_2+x_3).\end{aligned}$$

This last expression is nothing else but the RHS of (12).

Relations (12) allow to transform the determinant (4) by linear combination of rows. For example,

$$\varphi_{742}(x_1+x_2+x_3) = \begin{vmatrix} \varphi_9(x_1) & \varphi_5(x_1) & \varphi_2(x_1) \\ \varphi_{80}(x_1+x_2) & \varphi_{40}(x_1+x_2) & \varphi_{10}(x_1+x_2) \\ \varphi_{700}(x_1+x_2+x_3) & \varphi_{300}(x_1+x_2+x_3) & \varphi_{000}(x_1+x_2+x_3) \end{vmatrix}$$
$$= \begin{vmatrix} \varphi_{711}(x_1+x_2+x_3) & \varphi_{311}(x_1+x_2+x_3) & 0 \\ \varphi_{710}(x_1+x_2+x_3) & \varphi_{310}(x_1+x_2+x_3) & 0 \\ \varphi_{700}(x_1+x_2+x_3) & \varphi_{300}(x_1+x_2+x_3) & 1 \end{vmatrix}$$

using that $\varphi_{k11}(x_1+x_2+x_3) = \varphi_{k00}(x_1+x_2+x_3)\varphi_{11}(x_1+x_2) - \varphi_{k+1,0}(x_1+x_2)\varphi_1(x_1)+\varphi_{k+2}(x_1)$ and that $\varphi_{k10}(x_1+x_2+x_3) = \varphi_{k00}(x_1+x_2+x_3)\varphi_{10}(x_1+x_2) - \varphi_{k+1,0}(x_1+x_2)$.

Such a transformation produces more generally a determinant with a diagonal block filled with hook functions, the complementary block being unitriangular, as in the case of Schur functions. It is convenient to use the Frobenius coordinates for Schur functions [16, I.1]. The preceding computations can be rephrased in the following theorem.

Theorem 8. *Let $\lambda \in \mathbb{N}^n$ be a partition, $(\alpha_1, \ldots, \alpha_r \,|\, \beta_1, \ldots, \beta_r)$ its Frobenius decomposition. Then*

$$\varphi_\lambda(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y}) = \det\Big(\varphi_{\alpha_i+1,1^{\beta_j},0^{n-j-1}}(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{y})\Big).$$

5. Extensions

One can forget symmetry, while keeping determinants, and use vexillary Schubert polynomials [10] instead of Graßmannian ones, or use *multi-Schur functions* as in [9] (under another terminology) or [10, 1.4.7]. One can furthermore forget determinants and use general Schubert polynomials. This amounts using general products of divided differences instead of only ∂_ω. The "maximal Schubert polynomials" $Y_\rho(\mathbf{x}_n, \mathbf{y})$, indexed by $\rho_n = [n-1, \ldots, 0]$ for variable n, are associated to the flag $\{y_1\} \subset \{y_1, y_2\} \subset \{y_1, y_2, y_3\} \subset \cdots$:

$$Y_{\rho_n} = \prod_{i=1}^{n-1} (x_i - y_1) \cdots (x_i - y_{n-i}),$$

the general polynomials being all their non-zero images under products of divided differences.

Instead of the functions $f_{n-i}(x_i) = S_{n-i}(x_i - \mathbf{y}_{n-i})$, one can take any family of monic polynomials $f_k(x) = x^k + \cdots$, and define, following Fulton [4], the *universal Schubert polynomials* as the non-zero images of

$$\widetilde{Y}_{\rho_n} = f_{n-1}(x_1) \cdots f_1(x_{n-1})$$

under products of divided differences.

Discarding symmetry, determinants and commutativity is easy when starting with another linear basis of the ring of polynomials $\mathfrak{Pol}(\mathbf{x}_\infty)$. Indeed, products of elementary symmetric functions in $\ldots, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1$:

$$P_v(\mathbf{x}) = e_{v_1}(\mathbf{x}_{n-1}) \cdots e_{v_{n-1}}(\mathbf{x}_1) e_{v_n}(\emptyset), \;\; v \leq \rho_n, \quad n \text{ arbitrary},$$

constitute a linear basis of $\mathfrak{Pol}(\mathbf{x}_\infty)$ [11] (identifying $P_v(\mathbf{x})$ and $P_{0,v}(\mathbf{x})$). One can replace all the $e_{v_i}(\mathbf{x}_j)$, $1 \le v_i \le j$, by arbitrary elements e_{ji} of a ring $\mathcal{R}$ (putting $e_{j0} = 1$), obtaining elements

$$P_v^e = e_{n-1,v_1} \dots e_{1,v_{n-1}}$$

for all $v \le \rho_n$, all n. The transformation $P_v(\mathbf{x}) \to P_v^e$ induces by linearity a transformation of polynomials. In particular, Schubert polynomials $Y_w(\mathbf{x}, \mathbf{y}) = \sum c_w^v P_v(\mathbf{x})$ are transformed into $\mathcal{R}$-Schubert polynomials $\sum c_w^v P_v^e$.

When $\mathcal{R}$ is commutative, one obtains the universal Schubert polynomials of Fulton [4, Lemma 2.1] as stated above. These polynomials specialize to the *quantum Schubert polynomials* of Fomin, Gelfand, Postnikov [3].

Taking e_{ji} to be the sum of all decreasing words of degree i in the letters $1, \dots, j$, one obtains Schubert elements in the free algebra generated by $1, 2, \dots$. Their images in the plactic algebra (quotient of the free algebra by the relations $ikj \equiv kij$, $jik \equiv jki$, $jii \equiv iji$, $jij \equiv jji$, $i < j < k$) can be expressed in terms of tableaux satisfying flag conditions [13], [10, 11.6.1].

Having discarded determinants, symmetry, commutativity does not suffice to cover all the generalizations of Schur functions, for example the *Macdonald polynomials* or the *k-Schur functions* [14].

However, Lapointe showed me that Lam and Shimozono [7] obtain the specialization $t = 1$ of k-Schur functions from quantum Schubert calculus. One can in fact avoid having recourse to quantum Schubert polynomials, but directly deform the usual Schubert polynomials as follows.

For given k, $n = k + 1$, $v \le \rho_n$, define

$$P_v^k = \prod_{i=1}^{n-1} S_{(n-i)^{i-1}, n-i-v_i}$$

using Schur functions in an unspecified infinite alphabet. This family $\{P_v^k\}$ induces a family of Schubert polynomials $\{Y_v^k, v \le \rho_n\}$. Lam and Shimozono's theorem implies that

Theorem 9. *Given $k = n-1$, the image of a Schubert polynomial $Y_v(\mathbf{x}, \mathbf{0})$, $v \le \rho_n$, under the transformation $P_v(\mathbf{x}) \to P_v^k$ is equal to the specialization $t = 1$ of some k-Schur function.*

For example, for $k=4$, $Y_{20100}(\mathbf{x},\mathbf{0}) = P_{02010}(\mathbf{x}) - P_{03000}(\mathbf{x})$, and therefore

$$\begin{aligned} Y^4_{20100} = P^4_{02010} - P^4_{03000} &= S_4 S_{31} S_{222} S_{111} - S_4 S_3 S_{222} S_{1111} \\ &= S_4 S_{222}(S_{31}S_{111} - S_3 S_{1111}) = S_4 S_{222}(S_{421} + S_{3211}) \end{aligned}$$

is equal to some 4-Schur function at $t=1$. To be more explicit would require introducing the combinatorics of k-Schur functions, we prefer to refer to [8].

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