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The effect of vertex or edge deletion on the metric dimension of graphs

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The *metric dimension* $\dim(G)$ of a graph G is the minimum cardinality of a set of vertices such that every vertex of G is uniquely determined by its vector of distances to the chosen vertices. Let v and e respectively denote a vertex and an edge of a graph G. We show that, for any integer k, there exists a graph G such that $\dim(G-v)-\dim(G)=k$. For an arbitrary edge e of any graph G, we prove that $\dim(G-e) \leq \dim(G)+2$. We also prove that $\dim(G-e) \geq \dim(G)-1$ for G belonging to a rather general class of graphs. Moreover, we give an example showing that $\dim(G)-\dim(G-e)$ can be arbitrarily large.

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1. Introduction

Let $G=(V(G),E(G))$ be a simple, undirected, connected, and nontrivial graph with order $|V(G)|$. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, is the number of edges that are incident to v in G; an *end-vertex* is a vertex of degree one. We denote by K_n, C_n, and P_n the complete graph, the cycle, and the path on n vertices, respectively. The *distance* between two vertices $v,w \in V(G)$ is denoted by $d_G(v,w)$; we drop G if it is clear in the context. For other terminologies in graph theory, we refer to [4].

A vertex $x \in V(G)$ *resolves* a pair of vertices $u,v \in V(G)$ if $d(u,x) \neq d(v,x)$. A set of vertices $S \subseteq V(G)$ *resolves* G if every pair of distinct vertices of G is resolved by a vertex in S; then S is called a *resolving set* of G. For an ordered set $S=\{u_1,u_2,\ldots,u_k\} \subseteq V(G)$ of distinct vertices, the *metric code* (or *code*, for short) of $v \in V(G)$ with respect to S is the k-vector $code_S(v)=(d(v,u_1),d(v,u_2),\ldots,d(v,u_k))$. The *metric dimension* of G, denoted by $\dim(G)$, is the minimum of $|S|$ as S varies over all resolving sets of G.

Slater [14, 15] introduced the concept of a resolving set for a connected graph under the term *locating set*; he referred to a minimum resolving set as a *reference set*, and the cardinality of a minimum resolving set as the *location number* of a graph. Independently, Harary and Melter [8] studied these concepts under the term *metric dimension*. Metric dimension as a graph parameter has numerous applications, among them are robot navigation [10], sonar [14], combinatorial optimization [12], and pharmaceutical chemistry [3]. It was noted in [7] that determining the metric dimension of a graph is an NP-hard problem. Metric dimension has been heavily studied. For a survey on metric dimension and some variations, see [5] by Chartrand and Zhang. For a comparative study of metric dimension and graph parameters of more algebraic flavor, see [1] by Bailey and Cameron.

The question as to the effect of the deletion of a vertex or of an edge on the metric dimension of a graph was raised as a fundamental question in graph theory by Chartrand and Zhang in [5]. We address the question as follows: We show graphs G such that $\dim(G-v)$ is arbitrarily large (or small) relative to $\dim(G)$. For $e \in E(G)$, we prove that $\dim(G-e) \le \dim(G)+2$ for any graph G, and we prove that $\dim(G-e) \ge \dim(G)-1$ for G belonging to a rather general class of graphs. In general, we show that $\dim(G)-\dim(G-e)$ can be arbitrarily large.

2. The effect of vertex deletion on metric dimension of graphs

We first recall some basic facts on metric dimension for background.

Theorem 2.1. *[3] For a connected graph G of order $n \ge 2$ and diameter d,*

$$f(n,d) \le \dim(G) \le n-d,$$

where $f(n,d)$ is the least positive integer k for which $k+d^k \ge n$.

A generalization of Theorem 2.1 has been given in [9] by Hernando et al.

Theorem 2.2. *[9] Let G be a graph of order n, diameter $d \ge 2$, and metric dimension k. Then*

$$n \le \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1 \right)^k + k \sum_{i=1}^{\lceil \frac{d}{3} \rceil} (2i-1)^{k-1}.$$

Theorem 2.3. *[3] Let G be a connected graph of order $n \ge 2$. Then*

(a) $\dim(G)=1$ *if and only if* $G=P_n$,
(b) $\dim(G)=n-1$ *if and only if* $G=K_n$,
(c) for $n \ge 4$, $\dim(G)=n-2$ *if and only if* $G=K_{s,t}$ $(s,t \ge 1)$, $G=K_s+\overline{K}_t$ $(s \ge 1, t \ge 2)$, *or* $G=K_s+(K_1 \cup K_t)$ $(s,t \ge 1)$*; here,* $A+B$

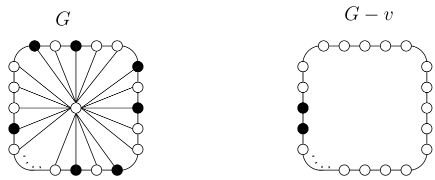

Figure 1: A graph G such that $\dim(G) - \dim(G - v)$ can be arbitrarily large.

denotes the graph obtained from the disjoint union of graphs A and B by joining every vertex of A with every vertex of B, and $\overline{C}$ denotes the complement of a graph C.

The following definitions are stated in [3]. Fix a graph G. A vertex of degree at least three is called a *major vertex*. An end-vertex u is called *a terminal vertex of a major vertex* v if $d(u, v) < d(u, w)$ for every other major vertex w. The *terminal degree* of a major vertex v is the number of terminal vertices of v. A major vertex v is an *exterior major vertex* if it has positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of all major vertices of G, and let $ex(G)$ denote the number of exterior major vertices of G. Two vertices $u, v \in V(G)$ are called *twins* if $N(u) - \{v\} = N(v) - \{u\}$, where $N(u)$ is the set of all vertices adjacent to u in G. Notice that $S \cap \{u, v\} \neq \emptyset$ if S is a resolving set and u, v are twins for any graph. We now recall two theorems useful in the two examples which follow.

Theorem 2.4. *[3, 10, 11] If T is a tree that is not a path, then* $\dim(T) = \sigma(T) - ex(T)$.

Theorem 2.5. *[2, 13] For $n \geq 3$, let $W_{1,n} = C_n + K_1$ be the wheel graph on $n + 1$ vertices. Then*

$$\dim(W_{1,n}) = \begin{cases} 3 & \text{if } n = 3 \text{ or } n = 6, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

The following example appeared in [2].

Example 2.6. There exists a graph G such that $\dim(G) - \dim(G - v)$ can be arbitrarily large; take $G = W_{1,n}$ for $n \geq 7$ and let v be the central vertex of degree n in G (see Figure 1). Notice that $\dim(G - v) = 2$ since $G - v \cong C_n$, whereas $\dim(G) = \lfloor \frac{2n+2}{5} \rfloor$ by Theorem 2.5.

Example 2.7. There exists a graph G such that $\dim(G - v) - \dim(G)$ can be arbitrarily large. For $k \geq 6$, let $G - v$ be a tree with k exterior major vertices, $u_1, u_2, \ldots, u_k$, and three terminal vertices $\ell_{i,1}, \ell_{i,2}, \ell_{i,3}$ for each u_i,

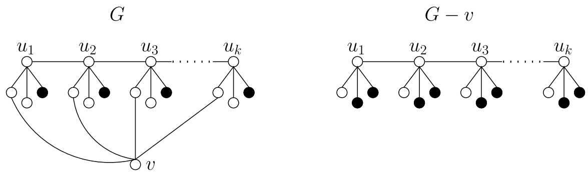

Figure 2: A graph G such that $\dim(G-v)-\dim(G)$ can be arbitrarily large.

where $1 \le i \le k$; let G be the graph obtained by joining $\ell_{1,1}, \ell_{2,1}, \dots, \ell_{k,1}$ to a new vertex v (see Figure 2). By Theorem 2.4, $\dim(G-v) = 2k$. We will show that $\dim(G) = k$. Since $\ell_{i,2}$ and $\ell_{i,3}$ are twins for each i $(1 \le i \le k)$ in G, $\dim(G) \ge k$. On the other hand, $d_G(\ell_{i,3}, v) = 3$ implies that $d_G(\ell_{i,3}, \ell_{i+3,1}) = 4$ and $d_G(\ell_{i,3}, \ell_{i+3,2}) = 5$. So, if $k \ge 6$, then $\{\ell_{i,3} \mid 1 \le i \le k\}$ forms a resolving set for G; thus $\dim(G) \le k$.

3. The effect of edge deletion on metric dimension of graphs

Next, we consider how the metric dimension of a graph changes upon deletion of an edge. The following theorem is stated in [3], with a correct proof given in [6].

Theorem 3.1. *[3, 6] Let T be a tree of order at least three. If $e \in E(\overline{T})$, then*

$$\dim(T) - 2 \le \dim(T+e) \le \dim(T) + 1.$$

It turns out that the lower bound in the preceding theorem holds for all graphs.

Theorem 3.2. *For any graph G and any edge $e \in E(G)$, we have*

$$\dim(G-e) \le \dim(G) + 2.$$

Proof. Let S be a minimum resolving set for G, and let u and v be the endpoints of the edge e. We will show that $S' = S \cup \{u, v\}$ is a resolving set for $G-e$. Let x and y be distinct vertices in $V(G-e) = V(G)$ which, in the graph G, are resolved by $z \in S$. Suppose x and y, in the graph $G-e$, are not resolved by z; then $d_{G-e}(x,z) = d_{G-e}(y,z)$. We consider two cases.

Case I. For one of x and y, say y, the distance to z is not changed by removing edge e; so $d_{G-e}(y,z) = d_G(y,z)$. In this case, $d_G(y,z) = d_{G-e}(y,z) = d_{G-e}(x,z) > d_G(x,z)$ and the edge e must lie on every $x-z$ geodesic in G. Thus, up to transposing the labels u and v, we have $d_G(x,u) + d_G(u,v) +$

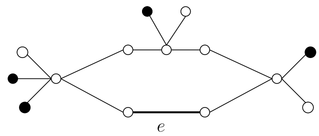

Figure 3: A graph G with $\dim(G-e) = \dim(G) + 2$.

$d_G(v,z) = d_G(x,z)$. Notice that $d_G(x,u) = d_{G-e}(x,u)$, since there is an $x-u$ geodesic in G that does not use edge e. Since $d_G(x,u) + d_G(u,z) = d_G(x,z) < d_G(y,z) \le d_G(y,u) + d_G(u,z)$, we must have $d_G(x,u) < d_G(y,u)$. But then $d_{G-e}(x,u) = d_G(x,u) < d_G(y,u) \le d_{G-e}(y,u)$, so $u \in S'$ resolves x and y.

Case II. For both x and y, the distance to z is increased by removing the edge e. In this case, the edge e must lie on every $x-z$ geodesic and on every $y-z$ geodesic in G. Notice that if a geodesic from some vertex a to another vertex c traverses the edge e in the order u, v (as opposed to v, u), then a geodesic containing e from any vertex b to c must also traverse e in the order u, v: For the sake of contradiction, let an $a-c$ geodesic have the form $a, \dots, u, v, \dots, c$ and let some $b-c$ geodesic have the form $b, \dots, v, u, \dots, c$. The presence of the $a-c$ geodesic implies that $d(u,v) + d(v,c) = d(u,c)$, and the presence of the $b-c$ geodesic implies that $d(v,u) + d(u,c) = d(v,c)$. The sum of the two equations simplifies to $d(u,v) = 0$, a contradiction. Suppose that u is traversed before v by a $x-z$ geodesic and a $y-z$ geodesic (directed towards z) in G, then a $x-u$ geodesic and a $y-u$ geodesic, neither containing the edge e, are obtained by truncating a common $u-z$ geodesic in G; thus, u resolves x and y in $G-e$. To complete the proof, simply swap the letters u and v in the preceding sentence. □

Example 3.3. For the sharpness of the upper bound of Theorem 3.2, see Figure 3. Notice that $\dim(G) = 4$ (the solid vertices in Figure 3 form a minimum resolving set of G). By Theorem 2.4, $\dim(G-e) = 6$, and hence $\dim(G-e) = \dim(G) + 2$.

Next, we consider how small the metric dimension of G could become upon deleting an edge of G. The following theorem is really an example; we are calling it a theorem in deference to its importance and the effort expended in its discovery!

Theorem 3.4. *There exists a graph G such that* $\dim(G) - \dim(G-e)$ *can be arbitrarily large. Let G be the graph in Figure 4 for $k \ge 2$, and let $e = AB \in E(G)$. Then* $\dim(G) = 2k$ *and* $\dim(G-e) = k+1$.

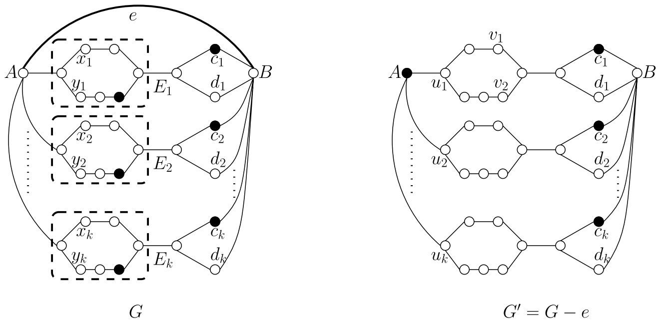

Figure 4: A graph G such that $\dim(G) - \dim(G-e)$ can be arbitrarily large.

Proof. Let S be a minimum resolving set for G, and let S' be a minimum resolving set for $G' = G - e$. Notice that, for each i ($1 \leq i \leq k$), $|S \cap \{c_i, d_i\}| \geq 1$ since c_i and d_i are twin vertices in G; similarly, $|S' \cap \{c_i, d_i\}| \geq 1$. Without loss of generality, we may assume $S_0 = \{c_i \mid 1 \leq i \leq k\} \subseteq S \cap S'$. For the sake of complete clarity, let $code_S(x, G)$ denote the code vector of x with respect to the set of vertices S in the graph G.

First, we show that $\dim(G) = 2k$. Notice that, for each i ($1 \leq i \leq k$), $code_{S_0}(x_i, G) = code_{S_0}(y_i, G)$. Further, if $S \cap E_i = \emptyset$ for some i, then $code_S(x_i, G) = code_S(y_i, G)$, contradicting the assumption that S is a resolving set for G, and thus $|S \cap E_i| \geq 1$ for each i ($1 \leq i \leq k$). So, $\dim(G) \geq 2k$. Since the solid vertices of G in Figure 4 form a resolving set for G, $\dim(G) = 2k$.

Next, we show that $\dim(G') = k+1$. Since, for instance, $code_{S_0}(v_1, G') = code_{S_0}(v_2, G')$, we have $|S' - S_0| \geq 1$, implying that $\dim(G') \geq k+1$. Since $\{A\} \cup S_0$ forms a resolving set for G', $\dim(G') = k+1$. □

In [6], it's proved that $\dim(G+e) \leq \dim(G)+1$ when G is a tree; a key idea used there is the notion of "strong resolution", identified but not named in the paper [11] by Poisson and Zhang: we say vertices u and v are strongly resolved by a set of vertices W if $code_W(u) - code_W(v) \neq (a, \ldots, a)$ for any $a \in \mathbb{Z}$. In fact, the proof in [6] shows that $\dim(G+e) \leq \dim(G)+1$ holds for a more general class of graphs than just trees.

Theorem 3.5. *Suppose there exists an induced cycle C in $G+e$ which contains the edge e, with the vertices of C cyclically labeled as $c_0, \ldots, c_{n-1}$. Let G_i be the subgraph of $G+e$ rooted at c_i; i.e., G_i is the maximal subgraph of $G+e$ such that $c_i \in V(G_i)$ and $E(G_i) \cap \{c_{i-1}c_i, c_ic_{i+1}\} = \emptyset$ (the indices*

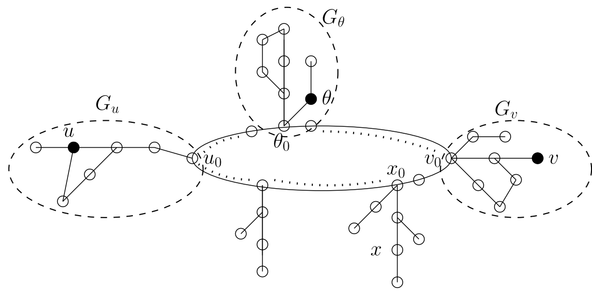

Figure 5: The set $\{u, v, \theta\}$ resolves the subgraphs G_i's, $i \in \{u, v, \theta\}$, from each other, where two of $\{u_0, v_0, \theta_0\} \subseteq \{c_0, c_1, \ldots, c_{n-1}\}$ attain the diameter of the cycle C.

of vertices being taken modulo n). Suppose further that $V(G_i) \cap V(G_j) = \emptyset$ *for* $i \neq j$. *Then* $\dim(G + e) \leq \dim(G) + 1$ *(see Figure 5).*

Proof. Exactly as in [6]; see Appendix A. □

Definition 3.6. We say a "graph G has no even cycles" if, whenever there exists a (not necessarily induced) subgraph of G isomorphic to a cycle C_n, n must be an odd integer.

Lemma 3.7. *Suppose G has no even cycles; then any two (odd) cycles of G intersect in at most one vertex.*

Proof. Suppose two cycles A' and B' share two distinct vertices u and v. Then there exist two cycles A and B and a fixed $u - v$ path P^2 such that A is the concatenation of a path P^1 with P^2 and B is the concatenation of a path P^3 with P^2. Since the length of A is odd, the length of P^1 and the length of P^2 must have opposite parity. Thus, either the concatenation of P^3 with P^1 or the concatenation of P^3 with P^2 forms an even cycle, and we have a contradiction. □

Thus, we have the following

Corollary 3.8. *Suppose that a connected graph G has no even cycles; then the following results hold: (1) Every cycle occurring as a subgraph of G occurs as an induced subgraph of G; (2) There is a unique geodesic between any pair of vertices of G; (3)* $\dim(G - e) \geq \dim(G) - 1$.

Proof. Parts (1) and (2) readily follow from Lemma 3.7. To obtain part (3), apply part (1) of the present corollary, Lemma 3.7, and Theorem 3.5 to G. □

Appendix A. Proof of Theorem 3.5

The following is an excerpt from reference [6] (by Eroh, Kang, and Yi; arXiv:1408.5943); we post it herewith so that the present paper is self-contained.

The *cycle rank* of a graph G, denoted by $r(G)$, is defined as $|E(G)| - |V(G)| + 1$. For a tree T, $r(T) = 0$. If a graph G has $r(G) = 1$, we call it a *unicyclic* graph. By $T + e$, we shall mean a unicyclic graph obtained from a tree T by attaching a new edge $e \in E(\overline{T})$. In [11], the notion of a resolving set W with the property $code_W(u) - code_W(v) \neq (a, \ldots, a)$ for any $a \in \mathbb{Z}$ was identified and shown to be very useful. We will say that "G is *strongly resolved* by W" if $code_W(u) - code_W(v) \neq (a, \ldots, a)$ for any $a \in \mathbb{Z}$ and any $u, v \in V(G)$. Still following [11], observe that $u \sim_W v$ if and only if $code_W(u) - code_W(v) = (a, \ldots, a)$ for some $a \in \mathbb{Z}$ defines an equivalence relation $\sim_W$ on $V(G)$; let $[u]_W$ denote the equivalence class of u under this relation.

Theorem A.1. *[3] If T is a tree of order at least three and e is an edge of $\overline{T}$, then*

$$\dim(T + e) \leq \dim(T) + 1.$$

Proof (as in [6]). The claim holds when T is a path P_n, as the two end-vertices of P_n form a basis (minimum resolving set) for $P_n + e$: If $e = v_i v_j$ where $i < j$, then v_i and v_j, being adjacent vertices, resolve vertices on the unique cycle C of $P_n + e$ among themselves (whence we say "v_i and v_j resolve C"). But then $W = \{v_1, v_n\}$ resolves C since for any $v \in V(C)$, $code_{W'}(v) = code_W(v) + (a_1, a_2)$, where $W' = \{v_i, v_j\}$ and (a_1, a_2) is a fixed vector. Further, v_1 and v_n obviously resolve vertices in $V(P_n + e) - V(C)$ among themselves and from $V(C)$.

So, let T be a tree which is not a path, and thus $\dim(T) \geq 2$. Cyclically label the vertices lying on the unique cycle C of $T + e$ ($e \in E(\overline{T})$) by $u_1, \ldots u_k$ ($k \geq 3$). Denote by T_i the subtree rooted at u_i (in other words, the component of $(T + e) - E(C)$ which contains u_i). Given any basis B of T, partition B into the disjoint union of sub-bases B_i, where $B_i \subseteq V(T_i)$, $1 \leq i \leq k$; assume, without loss of generality, that $B_1 \neq \emptyset$. If $B_i = \emptyset$ for each $i \neq 1$, then $T - T_1$ must be a path (for B to be a basis of T); in this case, either $B \cup \{u_2\}$ or $B \cup \{u_k\}$ is a resolving set for $T + e$.

So, assume there exists $1 < i \leq k$ such that $B_i \neq \emptyset$. If there exist two non-empty sub-bases B_i and B_j such that $d_{T+e}(u_i, u_j) = m = \lfloor \frac{k}{2} \rfloor$, then let $b_0 \in V(C) - \{u_i, u_j\}$ and put $B_0 = \{b_i, b_j, b_0\}$ (also put $B'_0 = \{u_i, u_j, b_0\}$)

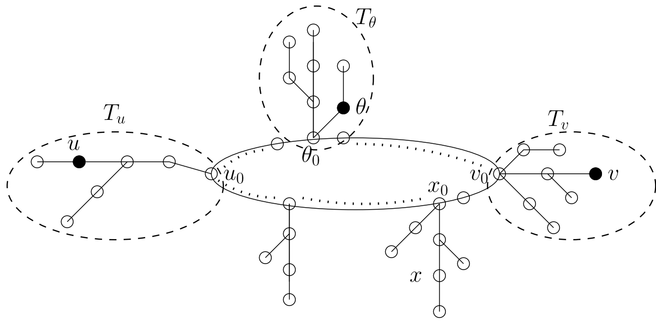

Figure 6: The set $\{u, v, \theta\}$ resolves the subtrees T_i's from each other.

where $b_i \in B_i$ and $b_j \in B_j$; otherwise, let $b_0 = u_{m+1}$ and put $B_0 = \{b_1, b_0, b_s\}$ (also put $B'_0 = \{u_1, b_0, u_s\}$), where $b_1 \in B_1$ and $b_s \in B_s \neq \emptyset$ for some $s \neq 1, m+1$. (The point here is to arrange a resolving set for $T + e$ that contains elements in three subtrees (the T_i's), two of which having roots (the u_i's) attaining the diameter of the cycle C.) We will show that the set $\widetilde{B} = B \cup \{b_0\}$ is a resolving set for $T + e$. Notice that $B_0 \subseteq \widetilde{B}$.

By Lemma A.2, we have $code_{B_0}(x_i) \neq code_{B_0}(x_j)$ and, a fortiori, $code_{\widetilde{B}}(x_i) \neq code_{\widetilde{B}}(x_j)$ for $x_i \in V(T_i)$ and $x_j \in V(T_j)$, when $i \neq j$. It thus suffices to show that $\forall x, y \in V(T_i)$ where $1 \leq i \leq k$, $code_{\widetilde{B}}(x) \neq code_{\widetilde{B}}(y)$. Accordingly, let $x, y \in V(T_i)$ be given for a fixed i. It's clear that if $d_T(x, b) \neq d_T(y, b)$ for some $b \in B_i$, then $d_{T+e}(x, b) \neq d_{T+e}(y, b)$; so, let $b \in B_j$ for some $j \neq i$. Notice that there exists a fixed $a \in \mathbb{N}$ such that $\forall x \in V(T_i)$, $d_{T+e}(x, b) = d_T(x, b) - a$. Thus, $d_T(x, b) \neq d_T(y, b)$ implies $d_{T+e}(x, b) \neq d_{T+e}(y, b)$ for $b \notin B_i$ as well.

We have thus proved the theorem. □

The following lemma shows that subtrees are distinguished by the B_0 chosen above; see Figure 6 for an illustration of the situation under consideration.

Lemma A.2. *Let B_0 and B'_0 be chosen as in the Proof of Theorem A.1; explicitly, let $B_0 = \{u, v, \theta\}$ and $B'_0 = \{u_0, v_0, \theta_0\} \subseteq V(C)$, where $d(u_0, v_0) = diam(C)$ and u (v, θ, respectively) is a vertex on the subtree rooted at u_0 (v_0, θ_0, respectively). Then, we have $code_{B_0}(x) \neq code_{B_0}(y)$ for vertices x and y belonging to distinct subtrees rooted at vertices of the unique cycle C of $T + e$.*

Proof. Observe that B'_0 *strongly resolves* the unique cycle C of $T + e$, because no vertex of C can have shorter distance, *by the same value*, to all vertices of B_0' than another vertex of C. Thus, B_0 *strongly resolves* C, because there exists a fixed vector (a_1, a_2, a_3) such that $\forall x \in V(C)$, $code_{B_0}(x) = code_{B_0'}(x) + (a_1, a_2, a_3)$. If $x \in V(T_i)$ where $V(T_i) \cap B_0 = \emptyset$, then $[x]_{B_0} = [x_0]_{B_0}$, where x_0 is the root of T_i: this is because any path from x *of such a subtree* T_i to a vertex in B_0 must go through x_0. Thus $[x]_{B_0} \neq [y]_{B_0}$ and, a fortiori, $code_{B_0}(x) \neq code_{B_0}(y)$ for x and y belonging to distinct subtrees which have empty intersection with B_0. If $B_0 = B_0'$, then the same reasoning applies to the subtrees containing elements of B_0. Otherwise, if suffices to check $code_{B_0}(x) \neq code_{B_0}(y)$ (1) for $x \in V(T_i)$ and $y \in V(T_u)$, (2) for $x \in V(T_i)$ and $y \in V(T_\theta)$, (3) for $x \in V(T_u)$ and $y \in V(T_v)$, and (4) for $x \in V(T_u)$ and $y \in V(T_\theta)$; here T_u, T_v, T_θ, and T_i are the subtrees containing u, v, θ, and none of B_0, respectively. Since the same argument works for all four inequalities, we will only explicitly verify (1).

Suppose, for the sake of contradiction, $code_{B_0}(y) = code_{B_0}(x)$; i.e., $(d(y,u), d(y,v), d(y,\theta)) = (d(x,u), d(x,v), d(x,\theta))$ for vertices $y \in V(T_u)$ and $x \in V(T_i)$. Equating the first two coordinates and expanding, we get $d(y,u) = d(x,x_0) + d(x_0,u_0) + d(u_0,u)$ and $d(y,u_0) + d(u_0,v_0) + d(v_0,v) = d(x,x_0) + d(x_0,v_0) + d(v_0,v)$, where x_0 is the root of the subtree containing x. Subtracting the two equations and rearranging terms, we get $d(y,u) = d(y,u_0) + d(x_0,u_0) + d(u_0,u) + d(u_0,v_0) - d(x_0,v_0)$. Now, since $d(u_0,v_0) = diam(C)$, we have $d(u_0,v_0) - d(x_0,v_0) = d(u_0,x_0)$. And we have $d(y,u) = d(y,u_0) + d(u_0,u) + 2d(u_0,x_0)$. Since $x \in V(T_i)$ and $T_i \neq T_u$, $d(u_0,x_0) > 0$, and we have $d(y,u) > d(y,u_0) + d(u_0,u)$, violating the triangle inequality which $d(\cdot,\cdot)$ must satisfy as a metric. □

Remark A.3. Notice that Lemma A.2 still holds if each "subtree T_i rooted at u_i" is replaced by "subgraph G_i rooted at u_i" with G_i and G_j disjoint for $i \neq j$.

References

[1] R. F. Bailey and P. J. Cameron (2011). Base size, metric dimension and other invariants of groups and graphs. *Bull. London Math. Soc.* **43(2)** 209–242. MR2781204

[2] P. S. Buczkowski, G. Chartrand, C. Poisson and P. Zhang (2003). On k-dimensional graphs and their bases. *Period. Math. Hungar.* **46(1)** 9–15. MR1975342

[3] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann (2000). Resolvability in graphs and the metric dimension of a graph. *Discrete Appl. Math.* **105** 99–113. MR1780464

[4] G. Chartrand and P. Zhang (2004). *Introduction to graph theory.* McGraw-Hill, Kalamazoo, MI.

[5] G. Chartrand and P. Zhang (2003). The theory and applications of resolvability in graphs. A Survey. *Congr. Numer.* **160** 47–68. MR2049102

[6] L. Eroh, C. X. Kang and E. Yi. A Comparison between the Metric Dimension and Zero Forcing Number of Trees and Unicyclic Graphs. arXiv:1408.5943.

[7] M. R. Garey and D. S. Johnson (1979). *Computers and intractability: A guide to the theory of NP-completeness.* Freeman, New York. MR0519066

[8] F. Harary and R. A. Melter (1976). On the metric dimension of a graph. *Ars Combin.* **2** 191–195. MR0457289

[9] C. Hernando, M. Mora, I. M. Pelayo, C. Seara and D. R. Wood (2010). Extremal graph theory for metric dimension and diameter. *Electron. J. Combin.* **17(1)** #R30. MR2595490

[10] S. Khuller, B. Raghavachari and A. Rosenfeld (1996). Landmarks in graphs. *Discrete Appl. Math.* **70** 217–229. MR1410574

[11] C. Poisson and P. Zhang (2002). The metric dimension of unicyclic graphs. *J. Combin. Math. Combin. Comput.* **40** 17–32. MR1887964

[12] A. Sebö and E. Tannier (2004). On metric generators of graphs. *Math. Oper. Res.* **29** 383–393. MR2065985

[13] B. Shanmukha, B. Sooryanarayana and K. S. Harinath (2002). Metric dimension of wheels. *Far East J. Appl. Math.* **8(3)** 217–229. MR1944130

[14] P. J. Slater (1975). Leaves of trees. *Congress. Numer.* **14** 549–559. MR0422062

[15] P. J. Slater (1998). Dominating and reference sets in a graph. *J. Math. Phys. Sci.* **22** 445–455. MR0966610

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