Journal of Combinatorics
Volume 8, Number 2, 305–321, 2017

# The index problem of group connectivity

Miaomiao Han, Hong-jian Lai, and Yehong Shao

Let $G$ be a connected graph and $L(G)$ be its line graph. Define $L^0(G) = G$ and for any integer $k \geq 0$, the *kth iterated line graph* of $G$, denoted by $L^k(G)$, is defined recursively as $L^{k+1}(G) = L(L^k(G))$. For a graphical property $\mathcal{P}$, the $\mathcal{P}$-index of $G$ is the smallest integer $k \geq 0$ such that $L^k(G)$ has property $\mathcal{P}$. In this paper, we investigate the indices of group connectivity, and determine some best possible upper bounds for these indices. Let $A$ be an abelian group and let $i_A(G)$ be the smallest positive integer $m$ such that $L^m(G)$ is $A$-connected. A path $P$ of $G$ is a normal divalent path if all internal vertices of $P$ are of degree 2 in $G$ and if $|E(P)| = 2$, then $P$ is not in a 3-cycle of $G$. Let

$$l(G) = \max\{m : \ G \text{ has a normal divalent path of length } m\}.$$

In particular, we prove the following.
(i) If $|A| \geq 4$, then $i_A(G) \leq l(G)$. This bound is best possible.
(ii) If $|A| \geq 4$, then $i_A(G) \leq |V(G)| - \Delta(G)$. This bound is best possible.
(iii) Suppose that $|A| \geq 4$ and $d = diam(G)$. If $d \leq |A| - 1$, then $i_A(G) \leq d$; and if $d \geq |A|$, then $i_A(G) \leq 2d - |A| + 1$.
(iv) $i_{\mathbb{Z}_3}(G) \leq l(G) + 2$. This bound is best possible.

## 1. Introduction

Throughout this paper, we use $\mathbb{Z}$ to denote the set of all integers and $\mathbb{N}$ to denote the set of all natural numbers. For an $m \in Z$ with $m > 1$, we use $\mathbb{Z}_m$ to denote the set of all integers modulo $m$ as well as the cyclic group of order $m$. We use [2] for terminology and notation not defined here. Graphs considered in this paper may have multiple edges but no loops. Following [2], for a graph $G$, $\kappa(G)$, $\kappa'(G)$, $\delta(G)$ and $\Delta(G)$ denote the connectivity, the edge-connectivity, the minimum degree and the maximum degree of $G$, respectively. The **line graph** $L(G)$ of a graph $G$ is defined as the graph whose vertices are the edges of $G$ and where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ are incident to a common vertex.

We define $L^0(G) = G$ and for integers $k \geq 0$, define recursively $L^{k+1}(G) = L(L^k(G))$. Each $L^k(G)$ is called **the $k$th iterated line graph** of $G$, or just an iterated line graph of $G$. For an integer $n > 0$, let $P_n$ and $C_n$ denote the path on $n$ vertices and the cycle of order $n$, called an $n$-path and an $n$-cycle, respectively. By the definition of line graphs, if $G \in \{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, then the iterated line graph of $G$ is either stable as a cycle, or diminishing when $k$ becomes bigger. Therefore, throughout this paper, we always assume that $G$ is a connected graph that is not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$.

The **hamiltonian index** $i_h(G)$ of $G$ is the smallest positive integer $k$ such that $L^k(G)$ is hamiltonian. The concept of hamiltonian index was first introduced by Chartrand and Wall [3], who showed that (Theorem A of [3]) if a connected graph $G$ is not a path, then $i_h(G)$ exists as a finite number. Clark and Wormald [4] considered other indices related to hamiltonicity of the iterated line graphs. More generally, the following is proposed in [16].

**Definition 1.1.** *For a graphical property $\mathcal{P}$ and a connected nonempty simple graph $G$ which is not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, define the $\mathcal{P}$-index of $G$, denoted $\mathcal{P}(G)$, as*

$$\mathcal{P}(G) = \begin{cases} \min\{k | L^k(G) \text{ has property } \mathcal{P}\} & \text{if at least one such integer } k \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

The index problem has been investigated by many, including [3], [4], [6], [13], [17]. [21], [29], among others. The purpose of this paper is to investigate the indices for group connectivity of graphs.

Throughout this paper, $A$ denotes an (additive) abelian group with identity 0, and $A^* = A - \{0\}$. Assume that $G$ has an orientation $D(G)$. If an edge $e \in E(G)$ is oriented from a vertex $u$ to a vertex $v$, then let $\textbf{tail}(e) = u$ and $\textbf{head}(e) = v$. For a vertex $v \in V(G)$, define

$$E_D^+(v) = \{e \in E(G) | v = \text{ tail}(e)\}, \text{ and } E_D^-(v) = \{e \in E(G) | v = \text{ head}(e)\}.$$

Following Jaeger et al. [11], we define $F(G, A) = \{f | f : E(G) \to A\}$ and $F^*(G, A) = \{f | f : E(G) \to A^*\}$. For a function $f : E(G) \to A$, define $\partial f : V(G) \to A$ by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in $A$.

A mapping $b : V(G) \to A$ is an **$A$-valued zero sum function** on $G$ if $\sum_{v \in V(G)} b(v) = 0$. The set of all $A$-valued zero sum functions on $G$ is denoted by $Z(G,A)$. A function $f \in F(G,A)$ is an **$A$-flow** of $G$ if $\partial f(v) = 0$ for every vertex $v \in V(G)$. An $A$-flow $f$ is a **nowhere-zero $A$-flow** (abbreviated as $A$-NZF) if $f \in F^*(G,A)$. For a mapping $b \in Z(G,A)$, a function $f \in F^*(G,A)$ is **a nowhere-zero $(A,b)$-flow** (abbreviated as $(A,b)$-NZF) if $\partial f = b$. A graph $G$ is **$A$-connected** if for any $b \in Z(G,A)$, $G$ has an $(A,b)$-NZF. Let $\langle A \rangle$ be the family of graphs that are $A$-connected. The **group connectivity number** of a graph $G$ is defined as

$$\Lambda_g(G) = \min\{k \mid G \in \langle A \rangle \text{ for every abelian group } A \text{ with } |A| \geq k\}.$$

The concept of group connectivity was first introduced by Jaeger, Linial, Payan and Tarsi in [11] as a nonhomogeneous form of the nowhere-zero flow problem. The nowhere-zero flow problem was first introduced by Tutte [27] in his way to attach the 4-color-conjecture. Tutte left with several fascinating conjectures in this area, which remain open as of today.

**Conjecture 1.2.** *(Tutte [27], [10])*
*(i) Every graph $G$ with $\kappa'(G) \geq 2$ has a nowhere-zero $\mathbb{Z}_5$-flow.*
*(ii) Every graph $G$ with $\kappa'(G) \geq 2$ without a subgraph contractible to the Peterson graph admits a nowhere-zero $\mathbb{Z}_4$-flow.*
*(iii) Every graph $G$ with $\kappa'(G) \geq 4$ admits a nowhere-zero $\mathbb{Z}_3$-flow.*

Many efforts towards these conjectures have been made, as surveyed in [10]. Seymour [22] proves that every 2-edge-connected graph has a nowhere zero 6-flow. Jaeger et al. improve this result by showing that if $G$ is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$. More recently, a break through on $\mathbb{Z}_3$-connectivity has been made by Thomassen and by Lovaze et al.

**Theorem 1.3.** *(Thomassen [25]) If $\kappa'(G) \geq 8$, then $G$ is strongly $\mathbb{Z}_3$-connected.*

This lower bound in Theorem 1.3 has recently been improved.

**Theorem 1.4.** *(Lovasz, Thomassen, Wu and Zhang [19], Wu [28]) If $\kappa'(G) \geq 6$, then $G$ is $\mathbb{Z}_3$-connected.*

The goal of this research is to show that if $G \notin \{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, then for any $A$, there exists a finite integer $m \in \mathbb{N}$ such that $L^m(G) \in \langle A \rangle$. The smallest such $m$ is denoted by $i_A(G)$, called **the $A$-connected index** of $G$. We shall to determine best possible upper bounds for the indices of $A$-connectedness of graphs, for all abelian groups $A$. In Section 2, we display the tools we will use in the arguments. Best possible upper bounds of group connectivity are studied in the last section.

## 2. Triangular and triangulated connected indices

Throughout this section, $G$ denotes a connected graph that is not in $\{K_{1,3}\}\cup\{P_n, C_n | n \in \mathbb{N}\}$. For each $i \in \mathbb{N}$, let $D_i(G)$ denote the set of all vertices of degree $i$ in $G$, for an integer subset $I$, let $D_I(G) = \cup_{i\in I} D_i(G)$, and so $D_{\leq i}(G) = \cup_{1\leq j\leq i} D_j(G)$. A graph $G$ is **triangular** if every edge of $G$ lies in a 3-cycle of $G$.

As our arguments will be back and forth between $G$ and $L(G)$, for each edge $e \in E(G)$, we will often use, in the proof arguments throughout the rest of this paper, $v_e$ to denote the vertex in $L(G)$ corresponding to $e \in E(G)$. Likewise, if $u \in V(L(G))$, then we often use $e(u)$ to denote the edge in $G$ corresponding to $u$ in $L(G)$.

**Proposition 2.1.** *The following are equivalent.*
*(i) $L(G)$ is triangular;*
*(ii) For any $v \in D_1(G)$, $N_G(v) \subseteq D_{\geq 3}(G)$; for any $v \in D_2(G)$, there exists an $K_3 \subseteq G$ such that $v \in V(K_3)$.*

**Proof.** Suppose that (i) holds, or $L(G)$ is triangular. We argue by contradiction to prove (ii). Assume first that for some $v_1 \in D_1(G)$, the only vertex $w$ in $N_G(v_1)$ has degree at most 2. Since $G$ is not a path, we have $w \in D_2(G)$. Thus the vertex in $L(G)$ corresponding to the edge $v_1w \in E(G)$ is a vertex of degree 1, contrary to the assumption that $L(G)$ is triangular. Thus every vertex in $D_1(G)$ must be adjacent to a vertex in $D_{\geq 3}(G)$. Next, we assume that $G$ has a vertex $v_2 \in D_2(G)$ with $N_G(v_2) = \{w_1, w_2\}$. If $w_1w_2 \notin E(G)$, then by the definition of line graphs, the edge in $L(G)$ joining the vertices $w_1v_2$ and $v_2w_2$ in $L(G)$ is not in a 3-cycle, contrary to the assumption that $L(G)$ is triangular. This proves (ii).

Conversely, assume that $G$ satisfies (ii). Let $e_1, e_2 \in E(G)$ be an arbitrary pair of adjacent vertices in $L(G)$. Then $L(G)$ has an edge $f$ linking $e_1$ and $e_2$. Then for some $v \in V(G)$, both $e_1$ and $e_2$ are incident with $v$. If $d_G(v) = k \geq 3$, then by the definition of line graphs, edges incident with $v$ are vertices in $L(G)$ which induce a complete subgraph on $k \geq 3$ vertices. As $k \geq 3$, $f$ lies in a 3-cycle of $L(G)$. Therefore, we assume that $d_G(v) = 2$. By (ii), $v$ lies in a 3-cycle of $G$. Since $e_1$ and $e_2$ are the only edges incident with $v$, the 3-cycle in $G$ containing $v$ must also contain $e_1$ and $e_2$. By the definition of line graphs, the edges of this 3-cycle is also a 3-cycle in $L(G)$, and so $f$ lies in a 3-cycle in this case also. This proves that $L(G)$ must be triangular, and so (i) holds. □

For any graph $\Gamma$, and for distinct edges $e, e' \in E(\Gamma)$, an $(e, e')$**-path** of $\Gamma$ is a path $P$ whose initial edge is $e$ and whose terminal edge is $e'$. The edges in $E(P) - \{e, e'\}$ are called the internal edges of $P$. By the definition of connectedness, a graph $\Gamma$ is connected if and only if for any pair of distinct edges $e, e' \in E(\Gamma)$, $\Gamma$ has an $(e, e')$-path.

For any $e, e'$ in a graph $G$, define $e \sim e'$ if and only if $e = e'$ or there exists a sequence $C_1, C_2, \cdots, C_k$ of cycles of length at most 3, such that $e \in E(C_1)$ and $e' \in E(C_k)$ and for any $1 \leq i \leq k-1$, $E(C_i) \cap E(C_{i+1}) \neq \emptyset$. Such a sequence of 3-cycles is called an **triangular sequence** connecting $e$ and $e'$. It is routine to verify that $\sim$ is an equivalence relation on $E(G)$. Each equivalence class induces a subgraph which is called a **triangularly connected component** of $G$. If $E(G)$ is a triangularly connected component, then $G$ is **triangularly connected**.

**Proposition 2.2.** *Let $G$ be a connected graph not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$ with $|E(G)| \geq 3$. The following are equivalent.*
*(i) $L(G)$ is triangularly connected.*
*(ii) For any pair of distinct edges $e, e' \in E(G)$, $G$ has an $(e, e')$-path $P$ such that every internal edge of $P$ lies in a 3-cycle of $G$.*

**Proof.** Assume that (ii) holds. Let $H_1, H_2, \cdots, H_c$ be the triangularly connected components of $L(G)$. Since $G$ is connected, $L(G)$ is also connected. We may assume that $V(H_1) \cap V(H_2)$ contains a vertex $v_e$, corresponding to an edge $e \in E(G)$. By definition of $v_e$, there exists a vertex $v_{e_1} \in V(H_1)$ and a vertex $v_{e_2} \in V(H_2)$ such that $e$ is incident with $e_1$ and $e_2$ in $G$. Therefore, we assume that for $i \in \{1, 2\}$, $G$ has vertices $v_1, v_2$ such that $e_i, e$ are incident with $v_i$. Since $v_{e_1}$ and $v_{e_2}$ are not in the same triangularly connected component of $L(G)$, $v_1 \neq v_2$. Thus $e_1$ and $e_2$ are distinct edges in $G$. By (ii), $G$ has an $(e_1, e_2)$-path $P$ such that every internal edge of $P$ lies in a 3-cycle of $G$. Thus by the definition of $L(G)$, for the two edges $ee_1$ and $ee_2$, $L(G)$ has a triangular sequence connecting $ee_1$ and $ee_2$. It follows that $ee_1$ and $ee_2$ are in the same triangularly connected component, whence $H_1 = H_2$, contrary to the fact that $H_1 \neq H_2$. This contradiction justifies that (ii) implies (i) of Lemma 2.2.

Conversely, assume that (i) holds. Let $e, e'$ be distinct edges in $G$. If $e$ and $e'$ are adjacent in $G$, then the path in $G[\{e, e'\}]$ is a path satisfying (ii). Thus we assume that $e$ and $e'$ are not adjacent in $G$. Since $G$ is connected, there exist edges $e_1, e_2 \in E(G)$ such that $e, e_1$ are adjacent in $G$, and $e'$ and $e_2$ are adjacent in $G$. Thus $ee_1$ and $e'e_2$ are edges in $L(G)$. Since $L(G)$ is triangularly connected, there exists a triangular sequence $C_1, C_2, \cdots, C_k$ connecting the two edges $ee_1$ and $e'e_2$ in $L(G)$. Among all such sequences, choose one such that $k$ is minimized. Let $v_e$, $v_{e'}$ denote the vertices in $L(G)$ corresponding to the edges $e$ and $e'$ in $G$, respectively. Let $P'$ be a $(v_e, v_{e'})$-path in $L(G)$ with $V(P') \subset \cup_{i=0}^{k} V(C_i)$. As $V(P') \subseteq E(G)$, we define $P = G[V(P')]$. Since

$k$ is minimized, there is no 3-cycle in $P$, and so $P$ is a path. Let $xy$ be any internal edge of $P$. By the definition of $P'$, we have $v_{xy} \in V(C_i)$, for some $1 \le i \le k$. Let $uv_{xy}$ be the common edge of $C_{i-1}$ and $C_i$ in $L(G)$. Then we may assume that $V(C_i) = \{u, v_{xy}, v\}$ and, $V(C_{i-1}) = \{u, v_{xy}, w\}$, for some vertices $v, w \in V(G)$. If $G[e(u) \cup e(v) \cup xy] = C_3$, then $xy$ lies in this 3-cycle in $G$. Thus we may assume that $G[e(u) \cup e(v) \cup xy] = K_{1,3}$.

Since $w$ is adjacent to $u$ and $v_{xy}$, and $G[\{e(u), e(w), xy\}] = C_3$, if $xy$ is not in any 3-cycle in $G$, then we may assume that $x$ is a common vertex in $e(u)$, $e(v)$ and $e(w)$, and so $G[\{e(u), e(v), xy, e(w)\}] = K_{1,4}$, contrary to the assumption that $k$ is minimized. This proves (ii). □

**Corollary 2.3.** *Each of the following holds.*
*(i) If $G$ is triangular, then $L(G)$ is triangularly connected.*
*(ii) If a graph $G$ is triangularly connected, then $L(G)$ is also triangularly connected.*

**Proof.** (i) Let $e, e'$ be any pair of distinct edges in $G$, and $e = u_1u_2$, $e' = v_1v_2$. Since $G$ is connected, there is $(u_1, v_1)$-path $P$ in $G$. Since $G$ is triangular, every edge of $P$ is in a 3-cycle $C_3$. By Proposition 2.2, $L(G)$ is triangularly connected.

(ii) Since triangularly connected graph $G$ is also a triangular graph, by (i), it follows that $L(G)$ is also triangularly connected. □

Given a connected graph $G$, a path $P$ of $G$ is a **divalent path** of $G$ if every internal vertex of $P$ has degree 2 in $G$. By this definition, if an edge is incident with two vertices neither of which is of degree 2, then this edge $e$ induces a divalent path of $G$. We call $P$ a normal divalent path of $G$, if all internal vertices of $P$ are of degree 2 in $G$ and if $|E(P)| = 2$, then $P$ is not in a 3-cycle of $G$. Let $\mathcal{P}(G)$ denote the set of all normal divalent path of $G$, and define,

$$l(G) = \max\{m | \ G \text{ has a normal divalent path of length } m\}.$$

As in the literature, many studies have used $l(G)$ as an invariant to investigate the hamiltonian index as well as other hamiltonian related indices, see [3], [4], [6], [13], [29], among others. We present the following.

**Proposition 2.4.** *Let $G$ be a connected graph with at least 3 edges not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$, and let $l = l(G)$. Each of the following holds.*
*(i) (Lemma 3.2 [26]) $L^l(G)$ is triangular.*
*(ii) $L^{l+1}(G)$ is triangularly connected.*

**Proof.** It suffices to prove (ii). By (i), $L^l(G)$ is triangular. Then, by Corollary 2.3 (i), $L^{l+1}(G)$ is triangularly connected. □

## 3. Group connectivity indices

Throughout this section, we always assume that $A$ is a finite abelian group with at least 3 elements and $G$ is a connected graph not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$. Define the $A$-connected index of $G$ as

$$i_A(G) = \min\{m \in \mathbb{N} \cup \{\infty\} \mid \ L^m(G) \text{ is } A\text{-connected}\}.$$

We shall show that for any abelian group $A$, if $G$ is not in $\{K_{1,3}\} \cup \{P_n, C_n | n \in \mathbb{N}\}$ then $i_A(G)$ exists as a finite number. We will determine best possible upper bounds for these indices. The following will be used in our arguments.

**Lemma 3.1.** *Let $A$ be an abelian group with $|A| \geq 3$ and let $T$ be a connected spanning subgraph of a graph $G$. Each of the following holds.*
*(i) (Lemma 3.3 of [14]) If $G$ is a cycle of length $n \geq 2$, then $G$ is $A$-connected if and only if $|A| \geq n + 1$.*
*(ii) (Lemma 2.1 of [15]) If for each edge $e \in E(T)$, $G$ has an $A$-connected subgraph $H_e$ with $e \in E(H_e)$, then $G$ is $A$-connected.*

For a subset $X \subset E(G)$, the contraction $G/X$ is the graph obtained from $G$ by identifying the two ends of each edge in $X$ and then deleting all loops generated by this process. Note that even if $G$ is simple, $G/X$ may have multiple edges. For simplicity, we write $G/e$ for $G/\{e\}$, where $e \in E(G)$. If $H$ is a subgraph of $G$, then $G/H$ denotes $G/E(H)$. If $v \in V(G/H)$ is obtained by contracting a connected subgraph $H$ of $G$, then $H$ is called **the preimage** of $v$, and $v$ is called the **image** of $H$.

**Proposition 3.2.** *(Propostion 3.2 of [14])*
*(i) If $H \in \langle A \rangle$ and if $e \in E(H)$, then $H/e \in \langle A \rangle$.*
*(ii) If $H \in \langle A \rangle$, then $G/H \in \langle A \rangle$ if and only if $G \in \langle A \rangle$.*

Let $H$ be an induced subgraph of $G$. We define $I_1(H)$ to be $L(G)[E(H)]$, the subgraph of $L(G)$ induced by $E(H)$. Let $I_1 : H \to L(G)[E(H)]$ be a mapping from the set of all induced subgraph $H$ of $G$ to the set of all induced subgraphs of $L(G)$. We define $I_1^-(I_1(H)) = H$. Inductively, if $I_k$ and $I_k^-$ are defined, then $I_k(H)$ is an induced subgraph of $L^k(G)$, and so $I_{k+1}(H) = I_1(I_k(H))$ is an induced subgraph of $L^{k+1}(G)$, and define $I_{k+1}^-(H) = I_1^-(I_k^-(H))$. We adopt the notation $I_{k+1}^-(e)$ if $I_{k+1}(H)$ is a path induced by an edge $e$. Let $G$ be a graph. Define $E' = E'(G) = \{e \in E(G) | e \text{ is in a cycle of } G \text{ of length at most } 3\}$ and $E''(G) = E(G) - E'(G)$. Also define

$$P(G) = \{P | P \text{ is a divalent path in } G \text{ with } |E(P)| = l\}.$$

**Lemma 3.3.** *(Lemma 12 of [17]) Let $d > 0$ be an integer and let $e \in E''(L^{d-1}(G))$. Then $I^{-}_{d-1}(e)$ is a divalent path in $G$ with length at least $d$.*

Let $G^* = G - E(P(G))$, and let $G^*_1, G^*_2, \cdots, G^*_t$ be the components of $G^*$, where $t \geq 1$. Let $G'$ be the graph obtained from $G$ by contracting every $G^*_i \in G$ into a vertex, for any $1 \leq i \leq t$ and replace every $P \in P(G)$ with one edge. By definition, if $G \in \langle A \rangle$, then $\kappa'(G) \geq 2$.

**Theorem 3.4.** *Let $G$ be a connected graph with $l = l(G)$, and $A$ be an abelian group with $|A| \geq 4$. Each of the following holds:*
*(i) If $l = 1$, then $L(G) \in \langle A \rangle$.*
*(ii) If $l > 1$, then $i_A(G) \leq l$, and the equality holds if and only if $G' \notin \langle A \rangle$.*

**Proof.** (i). Assume that $l = 1$. By the definition of divalent paths, $l = 1$ if and only if one of the following holds:
(A) $\delta(G) \geq 3$, or
(B) $\delta(G) \leq 2$ and every vertex of degree 2 is contained in a triangle.
For every edge $e_1e_2 \in E(L(G))$, there exists a vertex $v \in V(G)$ such that $e_1, e_2$ are both incident with $v$ in $G$. If (A) holds, then the edge $e_1e_2$ in $L(G)$ lies in a complete subgraph of order $d_G(v) \geq \delta(G) \geq 3$. It follows by Lemma 3.1 that $L(G) \in \langle A \rangle$. If (B) holds, then $G$ has a 3-cycle containing both $e_1$ and $e_2$, hence $L(G)$ has a 3-cycle containing the edge $e_1e_2$. Again by Lemma 3.1 $L(G) \in \langle A \rangle$. This proves (i).
(ii). Suppose that $l \geq 2$. By Proposition 2.4, every edge $e$ of $L^l(G)$ is in a 3-cycle. By Lemma 3.1(i), $K_3 \in \langle A \rangle$, and so by Lemma 3.1(ii), $L^l(G) \in \langle A \rangle$. This implies that $i_A(G) \leq l$; and that $i_A(G) = l$ if and only if $L^{l-1}(G) \notin \langle A \rangle$.

By the definition of $P(G)$, we have, for any $1 \leq i \leq t$, $i_A(G^*_i) \leq l(G^*_i) \leq l - 1$. Thus $I_{l-1}(G^*_i) \in \langle A \rangle$. By the definition of line graphs, every divalent path of length $l$ in a graph $G$ will become a divalent path of length $l - 1$ in $L(G)$. It follows that if $P \in P(G)$, then $I_{l-1}(P) \cong K_2$. By Proposition 3.2 (ii), $L^{l-1}(G) \notin \langle A \rangle$ if and only if $G' \notin \langle A \rangle$. □

Let $\Delta \geq 3$ be an integer and $G(\Delta)$ be the graph obtained from $K_{1,\Delta}$ and $P_{n-\Delta}$ by identifying a vertex in $D_1(K_{1,\Delta})$ and a vertex in $D_1(P_{n-\Delta})$. We observe that $\Delta$ is the maximum degree of $G(\Delta)$.

**Theorem 3.5.** *Let $G$ be a connected simple graph on $n > 3$ vertices, $\Delta = \Delta(G)$ and $A$ be an abelian group with $|A| \geq 4$. Each of the following holds.*
*(i) $i_A(G) \leq n - \Delta$.*
*(ii) Equality in (i) holds if and only if $G = G(\Delta)$.*

**Proof.** (i) Note that since $G$ is not a cycle nor a path, we have $\Delta \geq 3$. By the definition of line graphs, $L(G)$ contains a $K_\Delta$ as a subgraph. Since $\Delta \geq 3$, by Lemma 3.1, $K_\Delta \in \langle A \rangle$. By Proposition 3.2 (ii), $L(G) \in \langle A \rangle$ if and only if $L(G)/K_\Delta \in \langle A\rangle$. Let $w \in V(L(G)/K_\Delta)$ be the vertex in $L(G)/K_\Delta)$ onto which $K_\Delta$ is contracted. By Theorem 3.4, $i_A(L(G)) \le l(L(G)/K_\Delta)$.

If $l(G) = 1$, then by Theorem 3.4(i), we have $i_A(G) \le 1 \le n - \Delta$. hence we may assume that $l(G) \ge 2$. As every divalent path of length $l$ in $G$ will become a divalent path of length $l-1$ in $L(G)$. To prove $i_A(G) \le n - \Delta$, it suffices to prove $l(L(G)/K_\Delta) \le n - \Delta - 1$. Let $P$ be any divalent path in $L(G)/K_\Delta$ with $|E(P)| = l(L(G)/K_\Delta)$.

**Case 1.** $w \in V(P)$.

Suppose that $d_{L(G)/K_\Delta}(w) = 1$, or that $d_{L(G)/K_\Delta}(w) \ge 3$. Let $P' = P - \{w\}$. Then $I_1^-(P')$ is also a divalent path in $G$, and so

$$l(L(G)/K_\Delta) = |E(P)| \le |E(I_1^-(P'))| \le l \le n - \Delta - 1.$$

Thus we assume that $d_{L(G)/K_\Delta}(v) = 2$. Let $P^1$ and $P^2$ be the two component of $P - \{w\}$. Then $I_1^-(P^1)$ and $I_1^-(P_2)$ are divalent paths in $G$. It follows that

$$\begin{aligned} l(L(G)/K_\Delta) &= |E(P)| \le |E(P^1)| + |E(P^2)| + 2 \\ &\le |E(I_1^-(P^1))| + |E(I_1^-(P^2))| \le n - \Delta - 1. \end{aligned}$$

**Case 2.** $w \notin V(P)$.

Fix a vertex $v_0 \in D_\Delta(G)$. Then $I_1^-(P)$ is also a divalent path in $G$ with $V(I_1^-(P)) \cap N_G(v_0) = \emptyset$. Hence $l(L(G)/K_\Delta) = |E(P)| \le |E(I_1^-(P))| - 1 \le n - \Delta - 1$.

Since $i_A(G) - 1 = i_A(L(G))$, Combining Cases 1 and 2, we have proved that $i_A(G) \le n - \Delta$, and so (i) must hold.

(ii) If $G = G(\Delta)$, then $L^{n-\Delta-1}(G)$ has one cut edge, and so $L^{n-\Delta-1}(G) \notin \langle A\rangle$. Thus $i_A(G) = n - \Delta$. Conversely, assume that $i_A(G) = n - \Delta$. By Theorem 3.4, $l(G) \ge i_A(G) = n - \Delta$. Thus $G$ must have a divalent path of length at least $n - \Delta$. Since $\Delta \ge 3$, we conclude that $G = G(\Delta)$. □

The distance of two vertices $u, v \in V(G)$, denoted $dist_G(u, v)$, is the length of a shortest path from $u$ to $v$ of $G$. The diameter of $G$, denoted by $diam(G)$, is defined as

$$diam(G) = \max\{dist_G(u, v) \mid u, v \in V(G)\}.$$

Let $G_0$ be a graph obtained from a cycle $C_{2d}$ by identifying a pendant edge, and for any finite abelian group $A$ with $|A| \ge 4$, define

$$\mathcal{F}_A = \{G : G \text{ has a subgraph } H \text{ such that } G/H \text{ is a cycle of length at least } d + |A|\}.$$

**Theorem 3.6.** *Let $G$ be a connected graph with $d = diam(G) \geq 2$ and $A$ be an abelian group with $|A| \geq 4$.*
*(i) If $d \leq 3$, then $i_A(G) \leq d$.*
*(ii) If $d \geq 4$, then $i_A(G) \leq d$ if and only if $G \notin \mathcal{F}_A$.*
*(iii) If $d \leq |A| - 1$, then $i_A(G) \leq d - 1$.*
*(iv) If $d \geq |A|$, then $i_A(G) \leq 2d - |A| + 1$.*

**Proof.** Let $l = l(G)$. If $d \geq l$, by Theorem 3.4, then $i_A(G) \leq l \leq d$. Thus we may assume that $l \geq d + 1$. Fix a divalent path $P_0 \in P(G)$. Let $u$ and $v$ denote the two end vertices of $P_0$. If $u \neq v$, then there exists a $(u, v)$-path $P'$ in $G$ with $|E(P')| = d' \leq d$. Since $l > d$, and since $P$ is a divalent path, we have $V(P_0) \cap V(P') = \{u, v\}$ and $l \leq 2d$. If $u = v$, then $P_0$ is a cycle. Since $G \neq C_n$, we also have $l \leq 2d$.
(i). $d \leq 3$. Then $l \leq 2d \leq d+3$. For any divalent path $Q \in P(G)$, we observe that $I_d(Q)$ is a divalent path with length at most 3 in $L^d(G)$. We claim that $L^d(G)$ is triangular. If not, there exists one edge $e \in E(L^d(G))$ such that $e \in E''(L^d(G))$. By Lemma 3.3, $I^-_{d-1}(e)$ is a divalent path $Q'$ in $G$ with length at least $d$. Take a midpoint $w$ of $P_0$ and a midpoint $z$ of $Q'$. Then $dist_G(w, z) \geq l/2 + d/2 \geq (2d + 1)/2 > d$, contrary to the assumption that $d = diam(G)$. Hence $L^d(G)$ must be triangular. By Lemma 3.1, we conclude that $i_A(G) \leq d$. Thus (i) must hold.
(ii). $d \geq 4$. Suppose that $G$ has no subgraph $H$ such that $G/H$ is a cycle of length at least $d + |A|$. We claim that $l \leq d + |A| - 1$. If not, then there exists a divalent path $P \in P(G)$ with $|E(P)| \geq d + |A|$. Let $P^o$ denote the set of all internal vertices of $P$. If $G - P^o$ is connected, then $G/(G - P^o)$ is a cycle of length $|E(P)| \geq d + |A|$, contrary to the assumption. Hence every edge in $E(P)$ is a cut edge of $G$. Since $G$ is not a path, at least one end of $P$ has degree at least 3 in $G$. It follows that $d \geq l$, contrary to the assumption that $l \geq d + 1$. Thus we must have $l \leq d + |A| - 1$. It follows that $l(L^d(G)) \leq |A| - 1$. If there exists an edge $e \in E(L^d(G))$ which is not in a cycle of length at most $|A| - 1$ in $L^d(G)$, then as $|A| \geq 4$, we note that $e \in E''(L^d(G))$. By Lemma 3.3, $I^-_{d-1}(e)$ is a divalent path $Q$ in $G$ with length at least $d$. Take a midpoint $w$ of $P_0$ and a midpoint $z$ of $Q$. Then $dist_G(w, z) \geq l/2 + d/2 \geq (2d + 1)/2 > d$, contrary to the fact that $d = diam(G)$. Hence every edge of $L^d(G)$ lies in a cycle of length at most $|A| - 1$. By Lemma 3.1, $i_A(G) \leq d$.

Conversely, assume that $d \geq 4$ and $i_A(G) \leq d$. By contradiction, suppose that there exist $H$ such that $G/H$ is a cycle of length at least $d+|A|$. Thus $E(G/H)$ induces a divalent path $Q$ in $G$, and $Q' = I_d(Q)$ is a divalent path with length at least $|A|$ in $L^d(G)$. Let $(Q')^o$ denote the set of all internal

vertices of $Q'$. It follows that $C' = L^d(G)/(L^d(G) - (Q')^o)$ is a cycle of length at least $|A|$. By Lemma 3.1 (i), $C' \notin \langle A \rangle$. Since $L^d(G) \in \langle A \rangle$, by Proposition 3.2 (ii), $C' = L^d(G)/(L^d(G) - (Q')^o) \in \langle A \rangle$. This contradiction justifies that $G \notin \mathcal{F}_A$.
(iii) We claim that $\kappa'(L^{d-1}(G)) \geq 2$. If $e$ is a cut edge of $L^{d-1}(G)$, then by Lemma 3.3, $I^-_{d-1}(e)$ is a divalent path $P$ of length at least $d$ in $G$ such that every edge of $P$ is a cut edge of $G$. Let $P$ be a $(u, v)$-path of $G$. Since $G$ is not a path, we may assume that $d_G(u) \geq 3$, and so $N_G(u) - V(P)$ has vertex $w$. It follows that $d \geq dist_G(w, v) \geq |E(P)| + 1 \geq d + 1$, a contradiction. This proves our claim. Now suppose that $L^{d-1}(G)$ has an induced cycle $C$ of length $|E(C)| \geq |A| \geq 4$. For each edge $e \in E(C)$, by Lemma 3.3, $I^-_{d-1}(e)$ is a divalent path of length at least $d \geq 2$ in $G$. Hence $G$ has a pair of vertices whose distance in $G$ is least $d + 1$, contrary to the fact that $d = diam(G)$. Hence we conclude that every induced cycle of $G$ must have length at most $|A| - 1$. Since $\kappa'(L^{d-1}(G)) \geq 2$, it follows that every edge of $L^{d-1}(G)$ lies in a cycle of length at most $|A| - 1$. By Lemma 3.1, $L^{d-1}(G)$ is $A$-connected.
(iv) Now assume that $d \geq |A|$. By Theorem 3.4, if $l(G) \leq d$, then $i_A(G) \leq d < 2d - |A| + 1$. Hence we may assume that $l(G) \geq d + 1$. Note that for any divalent path $P \in P(G)$, $I_{2d-|A|+1}(P)$ is a divalent path with length at most $|A| - 1$ in $L^{2d-|A|+1}(G)$. If there exists an edge $e \in E(L^{2d-|A|+1}(G))$ which is not in a cycle of length at most $|A| - 1$ in $L^{2d-|A|+1}(G)$, then as $|A| \geq 4$, we have $e \in E''(L^{2d-|A|+1}(G))$. By Lemma 3.3, $I^-_{2d-|A|+1}(e)$ is a divalent path $Q$ in $G$ with length at least $2d - |A| + 2$. Take the midpoint $w$ of $P$ and a midpoint $z$ of $Q$. We observe that $d \geq dist_G(w, z) \geq l/2 + (2d - |A| + 2)/2$ $\geq d + 1 + (l - |A|)/2 \geq d + 1$, a contradiction. Thus every edge in $L^{2d-|A|+1}(G)$ is in a cycle of length at most $|A| - 1$. By Lemma 3.1, $i_A(G) \leq 2d - |A| + 1$. □

A **wheel** $W_n$ is the graph obtained from $C_n$ by adding one vertex and joining it to each vertex of $C_n$. A **fan** $F_n$ is the graph obtained from $P_n$ by adding one vertex and joining it to each vertex of $P_n$. As examples, $K_4 \cong W_3$ and $K_3 \cong F_2$. Let $G_1, G_2$ be two disjoint graphs. As in [7], $G_1 \oplus_2 G_2$, called the **parallel connection** of $G_1$ and $G_2$, is defined to be the graph obtained from $G_1 \cup G_2$ by identifying exactly one edge. Let $\mathcal{WF}$ be the family of graphs that satisfy the following conditions:
(i) $K_3, W_{2n+1} \in \mathcal{WF}$;
(ii) If $G_1, G_2 \in \mathcal{WF}$, then $G_1 \oplus_2 G_2 \in \mathcal{WF}$.

**Theorem 3.7.** *(Theorem 1.4 of [7]) Let $G$ be a triangularly connected graph with $|V(G)| \geq 3$. Then $G$ is not $\mathbb{Z}_3$-connected if and only if $G \in \mathcal{WF}$.*

Beineke [1] and Robertson [20] showed that any line graph cannot have an induced subgraph isomorphic to $W_5$ or $K_{1,3}$. As for $n \geq 3$, any induced

$W_{2n+1}$ contains an induced $K_{1,3}$, Beineke and Robertson in fact proved the following.

**Theorem 3.8.** *(Beineke [1] and Robertson [20], see also page 74 of [9]) If a connected graph $G$ is a line graph, then $G$ has no induced subgraph isomorphic to $W_{2n+1}$ for $n \geq 2$.*

**Lemma 3.9.** *If $G$ is triangularly connected, then $L(G) \in \langle \mathbb{Z}_3 \rangle$.*

**Proof.** By Corollary 2.3(ii), $L(G)$ is also triangularly connected. By Theorem 3.7, to prove $L(G) \in \langle \mathbb{Z}_3 \rangle$, it suffices to prove that $L(G) \notin \mathcal{WF}$. By contradiction, we assume that $L(G) \in \mathcal{WF}$. By the definition of $\mathcal{WF}$, either $L(G) = G_1 \oplus_2 K_3$, or $L(G) = G_1 \oplus_2 W_{2n+1}$. By Theorem 3.8, we must have $n = 1$.

**Case 1.** $L(G) = G_1 \oplus_2 K_3$.

Let $V(K_3) = \{v_1, v_2, v_3\}$ in $L(G)$, where $d_{L(G)}(v_2) = 2$. Then $G[\{e(v_1), e(v_2), e(v_3)\}] \in \{K_3, K_{1,3}\}$ in $G$. Since $G$ is triangularly connected, we must have $G[\{e(v_1), e(v_2), e(v_3)\}] = K_3$. Let $u_1, u_2, u_3$ denote the vertices of this $K_3$ in $G$ such that $e(v_1) = u_1u_2$, $e(v_2) = u_2u_3$ and $e(v_3) = u_3u_1$. Since $G \neq K_3$, we may assume that $G - \{u_1, u_2, u_3\}$ has a vertex $u_4$ such that $u_1u_4 \in E(G)$. Since $G$ is triangularly connected, there must be a 3-cycle sequence connecting $u_1u_4$ and $u_2u_3$. It follows that there must be a vertex $u_5 \in V(G) - \{u_1, u_2, u_3\}$ such that $u_5u_2$ or $u_5u_3 \in E(G)$. It follows that $d_{L(G)}(v_2) \geq 3$, contrary to the fact that $d_{L(G)}(v_2) = 2$. This contradiction indicates that Case 1 cannot occur.

**Case 2.** $L(G) = G_1 \oplus_2 W_{2n+1}$, where $n = 1$.

If $n = 1$, then $W_3 = K_4$ is a subgraph of $L(G)$. Let $V(W_3) = \{e_1, e_2, e_3, e_4\} \subset E(G)$, by the definition of line graphs, $G[\{e_1, e_2, e_3, e_4\}] \cong K_{1,4}$. Since $G$ is triangular, we may assume that for some $e \in E(G)$, $G[e_1, e_2, e\}]$ is a 3-cycle. It follows that in $L(G)$, $e$ as a vertex is adjacent to both vertices $e_1$ and $e_2$, contrary to the fact that $L(G) = G_1 \oplus_2 W_3$. This contradiction indicates that Case 2 cannot occur as well.

It follows that $L(G) \notin \mathcal{WF}$, and so by Theorem 3.7, $L(G) \in \langle \mathbb{Z}_3 \rangle$. □

**Example 3.10.** *We consider two examples, which are useful in our discussions below.*

*(i) A tree $T$ is a (3,1)-tree if every vertex in $T$ has degree equaling 3 or 1. Let $T_n$ denote a $(3,1)$-tree on $n \geq 4$ vertices. Then $l(T_n) = 1$. Direct computation indicates that $L^2(T_n)$ can be obtained from $K_3$ and $K_4$ via parallel connections. Hence $L^2(T_n) \in \mathcal{WF}$. It follows by the theorem below that $L^3(T_n) \in \mathbb{Z}_3$. This shows that $i_{\mathbb{Z}_3}(T_n) = l(T_n) + 2$.*

*(ii) Let $d \geq 3$ and $l \geq 1$ be integers. Define $J(d, l)$ to be the graph obtained from $K_{1,d}$ by replacing one edge of $K_{1,d}$ by a path of length $l$. Thus*

*$J(d,1)=K_{1,d}$. Since any $J(d,l)$ has $n=d+l$ vertices with $d=\Delta(J(d,l))$ being the maximum degree, if $G(\Delta)$ has $n$ vertices, then $G(\Delta)=J(\Delta,n-\Delta)$. Direct computation yields that $L^2(J(3,2))=K_4-e$ and $L^3(J(3,2))=W_4$. Therefore, $i_{\mathbb{Z}_3}(J(3,2))=3$.*

**Lemma 3.11.** *Each of the following holds.*
*(i) Let $k>0$ be an integer. If $H$ is a subgraph of $G$ such that $H\notin\{K_{1,3}\}\cup\{P_n,C_n|n\in\mathbb{N}\}$, then $L^k(H)$ is a subgraph of $L^k(G)$.*
*(ii) (Lemma 2.4 of [7]) Let $G$ be a triangularly-connected graph. Then $G$ is $\mathbb{Z}_3$-connected if and only if $G$ has a nontrivial $\mathbb{Z}_3$-connected subgraph.*
*(iii) Let $G$ be a connected graph with a vertex $v$ of $d_G(v)=1$. If $G-v$ is triangular-connected, then $L(G)$ is $\mathbb{Z}_3$-connected.*
*(iv) If $l(G)\geq 2$, then $i_{\mathbb{Z}_3}(G)\leq l(G)+1$.*

**Proof.** (i). By the definition of a line graph, $L(H)$ is a subgraph of $L(G)$. As $H\notin\{K_{1,3}\}\cup\{P_n,C_n|n\in\mathbb{N}\}$, we note that $L(H)\notin\{K_{1,3}\}\cup\{P_n,C_n|n\in\mathbb{N}\}$, and so Lemma 3.11 (i) follows from induction.
(iii). Let $H=G-v$. Since $H$ is triangular-connected, both $\delta(H)\geq 2$ and, by Lemma 3.9, $L(H)$ is $\mathbb{Z}_3$-connected. Let $e$ denote the only edge incident with $v$ in $G$. Then by the definition of line graphs, $L(G)-e=L(G-v)=L(H)$. Since $\delta(H)\geq 2$, the vertex $e$ is adjacent to at least 2 vertices in $L(H)$. It follows that $L(G)/L(H)$ is spanned by a 2-cycle, which, by Lemma 3.1(i), is $\mathbb{Z}_3$-connected. Since $L(H)$ is $\mathbb{Z}_3$-connected, it follows by Proposition 3.2(ii) that $L(G)$ is $\mathbb{Z}_3$-connected. This justifies Lemma 3.11 (iii).
(iv). By Proposition 2.4, $L^{l+1}(G)$ is triangularly-connected. By (ii), it suffices to show that $L^{l+1}(G)$ contains a nontrivial subgraph $H$ such that $H$ is $\mathbb{Z}_3$-connected. Since $l(G)=l$, there exists a maximal divalent path $P$ of $G$ with $|E(P)|=l(G)\geq 2$. Since $G$ is not a path, we may assume that $P$ has an end vertex $u$ with $d_G(u)=d\geq 3$. Thus $G$ contains a subgraph $J(3,l)$ with $l\geq 2$. We shall show that Let $H=L^{l+1}(J(3,l))$. By Lemma 3.11 (i), $H$ is a subgraph of $L^{l+1}(G)$.

To show that $H$ is $\mathbb{Z}_3$-connected, we argue by induction on $k\geq 2$ to show that $L^{k+1}(J(3,k))$ is triangularly-connected and $\mathbb{Z}_3$-connected. If $k=2$, then by Example 3.10(ii), $L^3(J(3,2))$ is triangularly-connected and $\mathbb{Z}_3$-connected. Assume that $k\geq 3$, and that $L^k(J(3,k-1))$ is triangularly-connected and $\mathbb{Z}_3$-connected. By direct computation, $L^k(J(3,k))$ has a unique vertex $v$ of degree 1 such that $L^k(J(3,k))-v=L^k(J(3,k-1))$. By Lemma 3.11 (iii), we conclude that $L^{k+1}(J(3,k))$ is triangularly-connected and $\mathbb{Z}_3$-connected. Hence $H$ is $\mathbb{Z}_3$-connected. As $H$ is a subgraph of $L^{l+1}(G)$, and as $L^{l+1}(G)$ is triangularly-connected, it follows by Lemma 3.11 (ii) that $L^{l+1}(G)$ is $\mathbb{Z}_3$-connected. □

**Theorem 3.12.** *Let $A = \mathbb{Z}_3$ denote the cyclic group of order 3. For an integer $d > 0$, define*

$$\mathcal{F}_d = \{G : G \text{ has a subgraph } H \text{ such that } G/H \text{ is a cycle of length at least } d+5\}.$$

*If $G$ is a connected graph with $diam(G) = d$ and $l = l(G)$, then each of the following holds.*
*(i) $i_A(G) \le l+2$, and the equality holds if and only if $G$ is a (3,1)-tree.*
*(ii) If $d \le 3$, then $i_A(G) \le d+2$.*
*(iii) If $d \ge 4$, then $i_A(G) \le d+2$ if and only if $G \notin \mathcal{F}_d$.*

**Proof.** (i) By Proposition 2.4, $L^{l+1}(G)$ is triangularly connected. By Lemma 3.9, we have $i_A(G) \le l+2$. By Lemma 3.11 (iv), $i_A(G) = l+2$ if and only if $l(G) = 1$ and $L^{l+1}(G) \notin \langle Z_3 \rangle$. This happens, by Theorem 3.7, if and only if $L^2(G) \in \mathcal{WF}$. By Theorem 3.8, $L^2(G) \in \mathcal{WF}$ if and only if $L^2(G)$ can be built via parallel-connected from $K_3$ and $K_4$. By Example 3.10(i), if $G$ is a $(3,1)$-tree, then $i_{\mathbb{Z}_3}(G) = 3$. Conversely, since $L^(G)$ is triangular, if $L^2(G)$ can be built via parallel-connected from $K_3$ and $K_4$, then direct computation indicates that $G$ must be a $(3,1)$-tree. This proves (i).

If $d \ge l$, then by Proposition 2.4 and Lemma 3.9, $i_A(G) \le d+2$. Hence we assume that $d < l$. Pick any divalent path $P \in P(G)$. Then $|E(P)| = l \ge d+1$. Let $u$ and $v$ denote the two end vertices of $P$. Since $l \ge d+1$, there exists a $(u,v)$-path $P'$ in $G$ with $|E(P')| = d' \le d$ such that $V(P) \cap V(P') = \{u, v\}$. Note that $u = v$ is possible. Since $G$ is not a cycle, we always have $d+1 \le l \le 2d$.
(ii). Assume that $d \le 3$. Then $l \le 2d \le d+3$. For any divalent path $L \in P(G)$, $I_d(L)$ is a divalent path with length at most 3, and so $L^d(G)$ is triangular. By Corollary 2.3 and Lemma 3.9, $i_A(G) \le d+2$. This proves (ii).
(iii). Assume that $d \ge 4$. Fix a divalent path $P \in P(G)$, and let $P^o$ denote the internal vertices of $P$. Since $d < l$, edges in $P$ cannot be cut edges of $G$, and so $H_P = G - P^o$ is connected. Hence $G/H_P$ is a cycle of length $l$. It follows that if $G$ has no subgraph $H$ such that $G/H$ is a cycle of length at least $d+5$. then $l \le d+4$. We claim that $L^d(G)$ is triangular. If not, then there exists an edge $e \in E''(L^d(G))$. By Lemma 3.3, $I^-_{d-1}(e)$ is a divalent path $Q$ in $G$ with length at least $d$. Take the midpoint $w$ of $P$ and the midpoint $z$ of $Q$. Direct computation yields then $dist_G(w,z) \ge l/2 + d/2 \ge (2d+1)/2 > d$, a contradiction. By Corollary 2.3 (i), and Lemma 3.9, $i_A(G) \le d+2$.

Conversely, assume that $i_A(G) \le d+2$. By contradiction, we assume further that $G$ contains a subgraph $H$ such that $G/H$ is a cycle of length at least $d+5$. Thus $P_0 = G[E(G/H)]$ is a divalent path in $G$; and $C' = I_{d+2}(P_0)$

is a divalent path with length at least 4 in $L^{d+2}(G)$. By Lemma 3.1 (i), $C' \notin \langle A\rangle$. On the other hand, since $L^{d+2}(G) \in \langle A\rangle$, by Proposition 3.2 (ii), $C' = L^{d+2}(G)/L^{d+2}(H) \in \langle A\rangle$. Thus a contradiction is obtained. This completes the proof of (iii). □

## References


[1] L. Beineke, Derived graphs and digraphs, Beiträge zur Graphentheorie, Teubner, Leipzig, 1968.

[2] J. A. Bondy, U. S. R. Murty, Graph theory with applications, Macmillan, London and Elsevier, New York, 1976. MR0411988

[3] G. Chartrand and C. E. Wall, On the hamiltonian index of a graph, Studia Sci. Math. Hungar., 8 (1973), 43–48. MR0335349

[4] L. H. Clark and N. C. Wormald, Hamiltonian like indices of graphs, Ars Combinatoria, 15 (1983), 131–148. MR0706294

[5] Z. H. Chen, H.-J. Lai and H. Y. Lai, Nowhere zero flows in line graphs, Discrete Math., 230 (2001), 133–141. MR1812347

[6] E. Eschen, H.-J. Lai, Y. Shao, and L. Zhang, The $s$-Hamiltonian index, Discrete Math., 308 (2008), 4779–4785. MR2438182

[7] G. Fan, H.-J. Lai, R. Xu, C.-Q. Zhang and C. Zhou, Nowhere-zero 3-flows in triangularly connected graphs, J. Combin. Theory Ser. B, 98 (2008), 1325–1336. MR2462322

[8] R. Gould, On line graphs and the Hamiltonian index, Discrete Math., 34 (1981), 111–117. MR0611423

[9] F. Harary, Graph theory, Edison-Wesley Publishing Company, Reading, (1969). MR0256911

[10] F. Jaeger, Nowhere-zero flow problems, in "Selected Topics in Graph Theory" (L. Beineke and R. Wilson, Eds), Academic Press, London/New York, 3 (1988), 91–95. MR1205397

[11] F. Jaeger, N. Linial, C. Payan and M. Tarsi, Group connectivity of graphs – a nonhomogeneous analogue of nowhere-zero flow properties, J. Combin. Theory Ser. B, 56 (1992), 165–182. MR1186753

[12] M. Kochol, An equivalent version of the 3-flow conjecture, J. Combin. Theory Ser. B, 83 (2001), 258–261. MR1866721

[13] H.-J. Lai, On the Hamiltonian index, Discrete Math., 69 (1988), 43–53. MR0935026

[14] H.-J. Lai, Group connectivity in 3-edge-connected chordal graph, Graphs and Combinatorics, 16 (2000), 165–176. MR1771220

[15] H.-J. Lai, Nowhere-zero 3-flows in locally connected graphs, J. Graph Theory, 42 (2003), 211–219. MR1958145

[16] H.-J. Lai and Y. Shao, Problems related to Hamiltonian line graphs, AMS/IP Studies in Advanced Mathematics, 39 (2007), 149–159. MR2338624

[17] H.-J. Lai, L. Xiong and H. Yan, Hamilton-connected indices of graphs, Discrete Math., 309 (2009), 4819–4827. MR2533144

[18] H.-J. Lai, X. Li, Y. H. Shao and M. Zhan Group connectivity and group colorings of graphs – a survey, Acta Mathematica Sinica, English Series, 27 (2011), 405–434. MR2770816

[19] L. M. Lovasz, C. Thomassen, Y. Wu and C. Q. Zhang, Nowhere-zero 3-flows and $\mathbb{Z}_3$-connectivity for 6-edge-connected graphs, J. Combin. Theory Ser. B, 103 (2013), 587–598. MR3096333

[20] N. Robertson, unpublished notes, privated communication.

[21] M. L. Saražin, A simple upper bound for the hamiltonian index of a graph, Discrete Math., 134 (1994), 85–91. MR1303397

[22] P. D. Seymour, Nowhere-zero 6-flows, J. Combin. Theory Ser. B, 30 (1981), 130–135. MR0615308

[23] Y. Shao, Claw-Free Graphs and Line Graphs, PhD Dissertation, West Virginia University, (2005). MR2707853

[24] Y. Shao, Connectivity of interated line graphs, Discrete Appl. Math., 158 (2010), 2081–2087. MR2726477

[25] C. Thomassen, The weak 3-flow conjecture and the weak circular flow conjecture, J. Combin. Theory Ser. B, 102 (2012), 521–529. MR2885433

[26] L. Zhang, E. Elaine, H.-J. Lai, Y. Shao, Note the $s$-Hamiltonian index, Discrete Math., 308 (2008), 4779–4785. MR2438182

[27] W. T. Tutte, A contribution on the theory of chromatic polynomial, Canad. J. Math., 6 (1954), 80–91. MR0061366

[28] Y. Wu, Integer Flows and Modulo Orientations, PhD Dissertation, West Virginia University, 2012. MR3093976

[29] L. Zhang, S. Shao, G. Chen, X. Xu, J. Zhou, s-Vertex Pancyclic index, Graphs and Combinatorics, 28 (2012), 393–C406.

MIAOMIAO HAN
DEPARTMENT OF MATHEMATICS
WEST VIRGINIA UNIVERSITY
MORGANTOWN, WV 26506
USA
*E-mail address:* mahan@mix.wvu.edu

HONG-JIAN LAI
DEPARTMENT OF MATHEMATICS
WEST VIRGINIA UNIVERSITY
MORGANTOWN, WV 26506
USA
*E-mail address:* hjlai@math.wvu.edu
URL: www.math.wvu.edu/~hjlai

YEHONG SHAO
ARTS AND SCIENCE
OHIO UNIVERSITY SOUTHERN
IRONTON, OH 45638
USA
*E-mail address:* shaoy@ohio.edu

RECEIVED 12 JANUARY 2015