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On tight cuts in matching covered graphs*

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Barrier cuts and 2-separation cuts are two familiar types of tight cuts in matching covered graphs. See Lovász ([7], 1987). We refer to these two types of tight cuts as *ELP-cuts*. A fundamental result of matching theory, due to Edmonds, Lovász, and Pulleyblank ([6], 1982) states that if a matching covered graph has a nontrivial tight cut, then it also has a nontrivial ELP-cut. Their proof of this result was based on linear programming techniques. An easier and purely graph theoretical proof was given by Szigeti ([9], 2002). This note is inspired by Szigeti's paper. Using properties of barriers in matchable graphs, which we call Dulmage-Mendelsohn barriers, we give an alternative proof of the Edmonds-Lovász-Pulleyblank (ELP) Theorem.

We conjecture that, given any nontrivial tight cut C in a matching covered graph that is not an ELP-cut, there exists a nontrivial ELP-cut D in that graph which does not cross C. Here we give a short proof of the validity of this conjecture for bicritical graphs and also for matching covered graphs with at most two bricks.

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1. Introduction

All the graphs considered in this paper are loopless. Graph theoretical terminology and notation we use, not surprisingly, is essentially that of Bondy and Murty [1]. For the terminology that is specific to matching covered graphs, we follow Lovász [7].

We denote the number of odd components of a graph G by $o(G)$. Tutte established the following characterization of graphs which have a perfect matching ([10], 1947):

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1.1 (Tutte's Theorem)**.** *A graph G has a perfect matching if and only if*

$$o(G - S) \leq |S|,$$

for every subset S of $V(G)$. □

An edge of a graph is *admissible* if there is a perfect matching of the graph which contains it. A nontrivial graph is *matchable* if it has at least one perfect matching, and is *matching covered* if it is connected and each of its edges is admissible. The following assertion is easy to prove:

1.2. *If v is a cut vertex of a graph G, then some edge incident with v is not admissible in G. (Thus, every matching covered graph on four or more vertices is 2-connected.)* □

1.1. Tight cuts

For any subset X of the set of vertices of a graph G, the set of all edges of G with exactly one end in X is denoted by $\partial(X)$, and is referred to as a *cut* of G. If G is connected and $C := \partial(X) = \partial(Y)$, then either $Y = X$ or $Y = \overline{X} := V(G) \setminus X$; and we refer to X and $\overline{X}$ as the *shores* of C. A cut is *trivial* if either of its shores is a singleton. For any cut $C := \partial(X)$ of a graph G, we denote the graph obtained from G by shrinking the shore X to a single vertex x by $G/(X \to x)$, or simply by G/X if the name of the vertex to which X is shrunk is irrelevant. The two graphs G/X and $G/\overline{X}$ are referred to as the two C*-contractions* of G.

Let G be a matching covered graph. A cut $C := \partial(X)$ of G is *tight* if $|C \cap M| = 1$, for every perfect matching M of G. The significance of this notion is that if C is a tight cut of G, then both the C-contractions G/X and $G/\overline{X}$ are also matching covered. If C is nontrivial, then both the C-contractions are strictly smaller than G.

A matching covered graph which is free of nontrivial tight cuts is a *brace* if it is bipartite, and a *brick* if it is nonbipartite. An important result due to Lovász [7] states that, given any matching covered graph, by means of tight-cut-contractions with respect to nontrivial tight cuts, one may obtain a list of bricks and braces and, more significantly, any two applications of this decomposition procedure yield the same list of bricks and braces (up to multiple edges). In particular, any two decompositions of a matching covered graph G yield the same number of bricks. We denote by $b(G)$ the number of bricks in every tight cut decomposition of a matching covered graph G. Informally, we refer to a matching covered graph G such that $b(G) \leq 2$ as "a matching covered graph having at most two bricks" and as "a matching covered graphs with two bricks", if $b(G) = 2$.

We shall make use of the following known facts about tight cuts.

1.3 ([7]). *Let G be a matching covered graph, and let C be a tight cut of G. Then both C-contractions are matching covered. Moreover, if G' is a C-contraction of G then a tight cut of G' is also a tight cut of G. Conversely, if a tight cut of G is a cut of G' then it is also tight in G'.* □

Corollary 1.4. *Let G be a matching covered graph, and let $C = \partial(X)$ be a tight cut of G. Then, both shores X and $\overline{X}$ of C induce connected graphs.*

Proof. The C-contraction $G' := G/(X \to x)$ of G is matching covered, hence it is 2-connected, by (1.2). Thus, $G' - x$ is connected. In other words, $\overline{X}$ induces a connected subgraph of G. Likewise, X also induces a connected subgraph of G. □

1.5 ([6]). *Let G be a matching covered graph and let $\partial(X)$ and $\partial(Y)$ be two tight cuts such that $|X \cap Y|$ is odd. Then $\partial(X \cap Y)$ and $\partial(X \cup Y)$ are also tight in G. Furthermore, no edge connects $X \cap \overline{Y}$ to $\overline{X} \cap Y$.* □

1.2. ELP-cuts

A *barrier* in a matchable graph G is a nonempty subset B of $V(G)$ for which $o(G - B) = |B|$. For any vertex v of G, it is easy to see that $\{v\}$ is a barrier of G; such a barrier is a *trivial barrier.* A barrier is *nontrivial* if it has two or more vertices. The following assertion elucidates the connection between barriers and perfect matchings in a matchable graph.

Lemma 1.6. *Let G be a matchable graph, and let B denote a barrier of G. Any perfect matching M of G has precisely one edge in the cut $\partial(V(K))$ for each odd component K of $G - B$. Consequently, if G is matching covered, then no edge of G has both its ends in B, and $G - B$ has no even components.*

Proof. Let M be a perfect matching of G. As B is a barrier, by definition, $G - B$ has $|B|$ odd components. For any such component of $G - B$, clearly $|M \cap \partial(V(K))| \geq 1$. A simple counting argument shows that, in fact, $|M \cap \partial(V(K))| = 1$ for each odd component K of $G - B$, and also that any edge of M that has one end in B has its other end in an odd component of $G - B$. These conclusions hold for any perfect matching of G. It follows that (i) no edge with both its ends in B is admissible and that (ii) no edge with one end in B and one end in an even component of $G - B$ is admissible.

Now suppose that G is matching covered. Since, by definition, every edge of G is admissible, it follows from the first observation made above that no edge has both its ends in B. Also, by definition, G is connected. Thus, if $G - B$ had any even component K, there would at least be one edge between B and $V(K)$ and such an edge would be inadmissible by the second observation. □

For any barrier B of a matching covered graph G, and any (odd) component K of $G - B$, the cut $\partial(V(K))$ is tight in G; tight cuts in matching covered graphs which arise this way are known as *barrier cuts* (Figure 1(a)).

Consider a pair $\{u, v\}$ of two distinct vertices of a matching covered graph G. Suppose that $G - u - v$ is not connected. From the fact that $|V(G)|$ is even, we have that the number of odd components of $G - u - v$ is even. As G is matchable, the number of odd components of $G - u - v$ is two or less. That is, $o(G - u - v) \in \{0, 2\}$. If $o(G - u - v) = 2$ then $\{u, v\}$ is a barrier. If $o(G - u - v) = 0$ then the pair $\{u, v\}$ is referred to as a *2-separation*.

In other words, a pair $\{u, v\}$ of two distinct vertices of a matching covered graph G is a *2-separation* if $\{u, v\}$ is not a barrier but $G - u - v$ is not connected. If G happens to have a 2-separation $\{u, v\}$ as described above, then every component of $G - u - v$ is even, and for any partition of $G - u - v$ in two disjoint subgraphs G_1 and G_2, the cuts $\partial(V(G_1) \cup \{u\})$ and $\partial(V(G_2) \cup \{u\})$ are tight cuts of G; tight cuts which arise in this manner are known as 2-*separation cuts* (Figure 1(b)). We shall refer to barrier cuts and 2-separation cuts, collectively, as *ELP-cuts*.

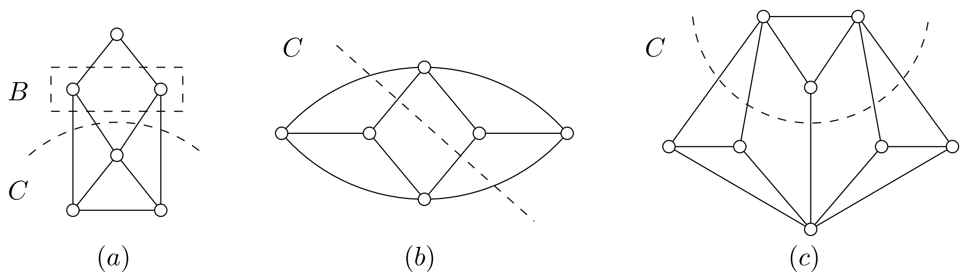


Figure 1: (a) Barrier cut; (b) 2-separation cut; (c) a tight cut which is not an ELP-cut.

When the set X of vertices of a connected bipartite graph G is odd, the two parts of the bipartition of G have distinct cardinalities; the larger part is called *the majority part*, the other *the minority part*; we denote the majority part of X by X_+, and the minority part by X_-. The following result is easily proved:

1.7. *Suppose that G is a matching covered graph and that v is a vertex of G. If the subgraph $G - v$ is bipartite, then G is also bipartite. Furthermore, the majority part of $G - v$ has just one more vertex than its minority part.* □

As a consequence we have:

Theorem 1.8. *Let $C := \partial(X)$ be a tight cut in a matching covered graph G. If the subgraph $G[X]$ of G induced by X is bipartite, then the majority part X_+ of $G[X]$ is a barrier of G and C is a barrier cut of G.*

Proof. Consider the C-contraction $G' := G/(\overline{X} \to \overline{x})$. Then $G' - \overline{x} = G[X]$ is bipartite by the hypothesis. Thus, by (1.7), G' is also bipartite, and the vertex $\overline{x}$ is adjacent in G' only to the vertices in X_+. Furthermore, $|X_+| = |X_-| + 1$. Thus, $G - X_+$ has precisely $|X_+|$ odd components, one of them being $G[\overline{X}]$, and the remaining being the trivial components corresponding to the vertices in the minority part X_-. □

As an immediate consequence, we have:

Corollary 1.9. *Every tight cut in a bipartite matching covered graph is a barrier cut.* □

In a bipartite matching covered graph every tight cut is a barrier cut, hence an ELP-cut. But a nonbipartite matching covered graph may have a tight cut which is not an ELP-cut. For example, the cut shown in Figure 1(c) is such a tight cut.

1.3. Properties of barriers

We record here the properties of barriers in matchable graphs which will be used in the proof of the main theorem. The following result may be established by a straightforward application of Tutte's Theorem 1.1.

1.10. *Let u and v be any two vertices of a matchable graph G. Then the graph $G - u - v$ is matchable if and only if there is no barrier of G which contains both u and v.* □

A nontrivial graph G is *bicritical* if $G - u - v$ has a perfect matching for any two distinct vertices u and v of G. The following two assertions are simple consequences of the above result.

1.11. *An edge $e = uv$ of a matchable graph G is admissible if and only if no barrier of G contains both u and v.* □

1.12. *A matchable graph is bicritical if and only if it is free of nontrivial barriers.* □

A graph G is *critical* (*factor-critical*, *hypomatchable*) if $G-v$ is matchable for any $v \in V(G)$. The result stated below may also be derived from Tutte's Theorem 1.1.

1.13. *Let B denote a maximal barrier of a matchable graph G. Then, every component of $G - B$ is odd and critical.* □

1.4. Cores of matchable graphs

Let G be a matchable graph, and let B denote a nonempty barrier of G. The bipartite graph obtained from G by deleting the vertices in the even components of $G - B$, contracting every odd component to a single vertex, and deleting the edges with both ends in B, is denoted by $\mathbb{H}(B)$. We refer to this bipartite graph as the *core* of G with respect to the barrier B. The following property is a direct consequence of this definition.

1.14. *Let B be any barrier of a matchable graph G. Then $o(G - B)$ is equal to $o(\mathbb{H}(B) - B)$, and for any perfect matching M of G, the set $M \cap E(\mathbb{H}(B))$ is a perfect matching of $\mathbb{H}(B)$.* □

In our paper on Pfaffian orientations ([3], 2012), we were able to obtain several useful results by exploiting the relationship between a matchable graph and its core with respect to a chosen maximal barrier of the graph.

2. The Dulmage-Mendelsohn barriers

One of the important tools we use in our proof of the ELP Theorem is a property of bipartite matchable graphs which is due to Dulmage and Mendelsohn [5] (also see [8]).

2.1. *Given a bipartite matchable graph $G[U, W]$, there exists a subset S of U such that the subgraph of G induced by $S \cup N(S)$ is matching covered. Consequently, $N(S)$ is a barrier of G, and the odd components of $G - N(S)$ are the trivial graphs induced by the vertices of S (Figure 2).* □

A *Dulmage-Mendelsohn decomposition* of a matchable bipartite graph $G[U, W]$ consists of a partition of U into subsets $A_1, A_2, \ldots, A_k$ and a partition of W into subsets $B_1, B_2, \ldots, B_k$ such that, for $1 \leq i \leq k$, (i) the subgraph of G induced by $A_i \cup B_i$ is matching covered, and (ii) $N_G(A_i) \subseteq B_1 \cup B_2 \cup \cdots \cup B_i$. Given any subset S of U with the property described in (2.1), a Dulmage-Mendelsohn decomposition of G maybe obtained with $A_1 = S$ and $B_1 = N(S)$, and we shall say that B_1 is the *principal barrier* of G associated with that decomposition.

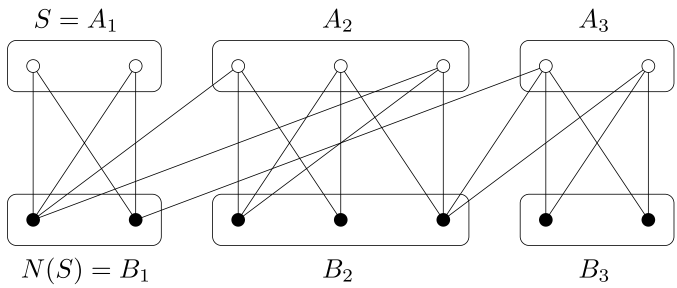


Figure 2: A Dulmage-Mendelsohn decomposition of a matchable bipartite graph.

Now we use (2.1) to establish the existence of barriers in matchable graphs which satisfy two special properties. These properties were exploited by Szigeti [9] in his proof of the ELP theorem.

Let G be a matchable graph, and let $B^\star$ be a maximal barrier of G. Let $\mathbb{H}(B^\star) = \mathbb{H}[A^\star, B^\star]$ be the core of G with respect to $B^\star$, where $A^\star$ is the part of the bipartition of $\mathbb{H}(B^\star)$ different from $B^\star$. By (2.1) and (1.13), $\mathbb{H}(B^\star)$ has a barrier B that satisfies the following two properties:

DMB-1: each odd component of $G - B$ is critical, and
DMB-2: the core $\mathbb{H}(B)$ of G with respect to B is matching covered.

It can be shown that any barrier B of G satisfying properties DMB-1 and DMB-2 is a principal barrier corresponding to a Dulmage-Mendelsohn decomposition of the core with respect to some maximal barrier of G. For this reason we shall refer to a barrier B of G satisfying these two properties as a *Dulmage-Mendelsohn barrier*, or briefly as a *DM-barrier* of G. (DM-barriers are equivalent to Strong barriers used by Szigeti [9].) The following results concerning DM-barriers summarize the above arguments.

2.2. *Every maximal barrier of a matchable graph G contains a subset which is a DM-barrier of G.* □

2.3. *Let B denote a DM-barrier of a matchable graph G. Then, every edge e of the core $\mathbb{H}(B)$ with respect to B is admissible in G.* □

2.1. A key lemma

The following assertion plays a crucial role in our proof of the ELP Theorem, where it is applied to derive properties of suitable subgraphs of a matching covered graph which are matchable but are not matching covered.

2.4 (Key Lemma)**.** *Let G be a matchable graph, and let X be a nonempty proper subset of $V(G)$ such that:*

- *both the subgraphs $G[X]$ and $G[\overline{X}]$ are connected, and*
- *no edge in the cut $\partial(X)$ is admissible in G.*

Then G has a DM-barrier B which is a subset of X or of $\overline{X}$. Furthermore, the vertex sets of all the odd components of $G - B$ are also subsets of that same shore.

Proof. By hypothesis, G has perfect matchings but no edge of $\partial(X)$ is admissible. It follows that the graphs $G[X]$ and $G[\overline{X}]$ both have perfect matchings.

Consider first the case in which $\partial(X)$ is empty. Let $B^\star$ be any maximal barrier of $G[X]$. By (2.2), $G[X]$ has a DM-barrier B that is a subset of $B^\star$. This barrier B of $G[X]$ is also a DM-barrier of G. The assertion holds in this case.

We may thus assume that $\partial(X)$ is nonempty. Let $e := uv$ denote an edge of $\partial(X)$, where $u \in X$ and $v \in \overline{X}$. By hypothesis, e is not admissible. By (1.11), G has a barrier that contains both u and v. Let $B^\star$ be a maximal barrier of G that contains both u and v. By (2.2), some subset B of $B^\star$ is a DM-barrier of G.

Let us first show that if K is any odd component of $G - B$, then $V(K)$ is a subset of one of the shores of the cut $\partial(X)$. Suppose that this is not the case. Then, both $V(K) \cap X$ and $V(K) \cap \overline{X}$ are nonempty. One of these sets has to be even and the other odd because $V(K)$ is an odd set. Without loss of generality, assume that $|V(K) \cap X|$ is even. Since, by hypothesis, $G[X]$ is connected, there is some edge, say e_1, which joins a vertex in $V(K) \cap X$ to a vertex in $B \cap X$. That edge e_1, being an edge of the graph $\mathbb{H}(B)$, is admissible in G, by (2.3). Let M_1 be a perfect matching of G containing e_1. Since K is an odd component of $G - B$, the edge e_1 is the only edge of M_1 in $\partial(V(K))$. However, since $V(K) \cap X$ is an even set, $|M_1 \cap \partial(V(K) \cap X)|$ is even. This implies that some edge with one end in $V(K) \cap X$ and one end in $V(K) \cap \overline{X}$ is in M_1. This is impossible because, by hypothesis, no edge in $\partial(X)$ is admissible. We conclude that $V(K)$ is a subset of one of the shores of $\partial(X)$.

Now observe that since B is a DM-barrier, graph $\mathbb{H}(B)$ is matching covered. Thus $\mathbb{H}(B)$ is connected and each of its edges is admissible in G. But by hypothesis, no edge of $\partial(X)$ is admissible in G. It follows that B and the vertex sets of all the odd components of $G - B$ are all subsets of one and the same shore of the cut $\partial(X)$. □

3. Tight cuts with minimal shores

Our main objective is to show that any matching covered graph G which has a nontrivial tight cut also has a nontrivial ELP-cut. As a first step towards the proof of this statement we show that any nontrivial tight cut of G with a minimal shore has certain special properties. We exploit these properties in the proof of the assertion stated above.

Lemma 3.1 ([9, Claims 25 and 26]). *Let G be a matching covered graph which has nontrivial tight cuts, and let X be a minimal subset of $V(G)$ such that the cut $C := \partial(X)$ is nontrivial and tight. Then there exists an edge $e := uv \in C$, with $u \in X$ and $v \in \overline{X}$, such that both the subgraphs $G[X - u]$ and $G[\overline{X} - v]$ are connected.*

Proof. Consider the two C-contractions $G/\overline{X} := G/(\overline{X} \to \overline{x})$ and $G/X := G/(X \to x)$ of G. Note that $G[X] = (G/\overline{X}) - \overline{x}$, and $G[\overline{X}] = (G/X) - x$.

As C is a tight cut of G, both $G/\overline{X}$ and G/X are matching covered. Furthermore, by the minimality of X, the graph $G/\overline{X}$ is free of nontrivial tight cuts.

Let us first show that C contains an edge $e := uv$, where v lies in $\overline{X}$, such that $G[\overline{X} - v]$ is connected. The graph G/X is matching covered, hence 2-connected, by (1.2). Therefore the graph $G[\overline{X}] = (G/X) - x$ is connected. If $G[\overline{X}]$ happens to be 2-connected then $G[\overline{X} - v]$ would be connected, for every vertex v of $\overline{X}$. We may thus assume that $G[\overline{X}]$ has two or more blocks. Then it has a block F that contains precisely one cut vertex, say w, of $G[\overline{X}]$. Now, as G/X is 2-connected, it follows that F contains a vertex v, different from w, which is incident with an edge $e = uv$ of C. As $v \neq w$, vertex v is not a cut vertex of $G[\overline{X}]$, and hence $G[\overline{X} - v]$ is connected. (See Figure 3.)

To complete the proof, we now proceed to show that the graph $G[X - u]$ is connected. For this, assume the contrary that $G[X - u]$ is disconnected. As the graph $G/\overline{X}$ is matching covered, and $G[X - u] = (G/\overline{X}) - \overline{x} - u$, it follows that $\{u, \overline{x}\}$ is either a barrier or a 2-separation of $G/\overline{X}$. But the ends of e in $G/\overline{X}$ are u and $\overline{x}$. Moreover, as $G/\overline{X}$ is matching covered, the edge e is admissible in $G/\overline{X}$. Consequently, $\{u, \overline{x}\}$ is not a barrier of $G/\overline{X}$. It follows that $\{u, \overline{x}\}$ is a 2-separation of $G/\overline{X}$. Thus, $G/\overline{X}$ has nontrivial tight cuts, contradicting the minimality of X. This proves the assertion. □

4. The ELP theorem

4.1 (ELP Theorem). *If a matching covered graph G has a nontrivial tight cut then it has a nontrivial ELP-cut.*

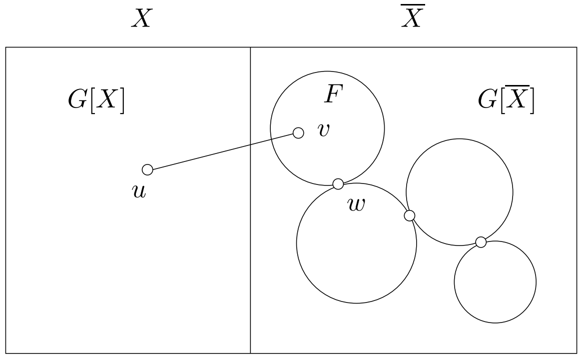


Figure 3: An edge uv in C such that $G[\overline{X} - v]$ is connected.

Proof. Assume that G has nontrivial tight cuts. By Corollary 1.9, every tight cut in a bipartite graph is a barrier cut. We may thus assume that G is nonbipartite. Suppose that B is a nontrivial barrier of G. By Lemma 1.6, every component K of $G - B$ is odd and the cut $\partial(V(K))$ is tight. Moreover, as G is not bipartite, at least one component of $G - B$ is nontrivial. In sum, $G - B$ has a nontrivial component K and the cut $\partial(K)$ is a nontrivial barrier cut of G. Thus, in order to prove that G has a nontrivial barrier cut, it is enough to prove that it has a nontrivial barrier.

By Lemma 3.1, G has a nontrivial tight cut $C := \partial(X)$ and an edge $e := uv \in C$ such that $u \in X$, $v \in \overline{X}$ and the graphs $G[X - u]$ and $G[\overline{X} - v]$ are both connected.

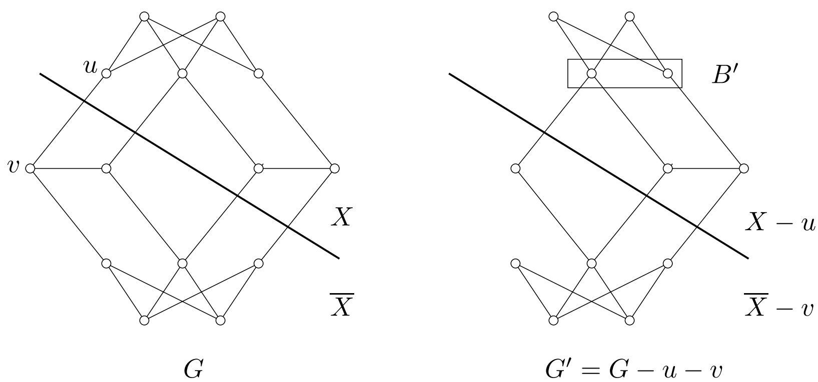


Figure 4: Case 4.1.1: $B := B' \cup \{u\}$ is a nontrivial barrier of G.

4.1.1. *If v is the only neighbor of u in $\overline{X}$ and u is the only neighbor of v in X then G has a nontrivial barrier.*

Proof. Let $G' := G - u - v$. Graph G is matching covered. Thus, G has a perfect matching, say M, that contains edge e. Then $M - e$ is a perfect matching of G', and we deduce that G' has perfect matchings. Moreover, $C - e = \partial_{G'}(X - u) = \partial_{G'}(\overline{X} - v)$ is a cut of G'. For every perfect matching M' of G', the set $M' \cup \{e\}$ is a perfect matching of G. As C is tight in G, it follows that no edge of $C - e$ is admissible in G'. By (2.4), G' has a DM-barrier B' such that B', as well as the vertex sets of all the odd components of $G' - B'$, are subsets of one of $X - u$ and $\overline{X} - v$. Adjust notation so that $B' \subseteq X - u$. Let $B := B' \cup \{u\}$. By hypothesis, u is the only vertex of X adjacent to v. Thus, all the $|B| - 1$ odd components of $G' - B'$ are also odd components of $G - B$. Consequently, B is a nontrivial barrier of G (see Figure 4 for an illustration). □

We may thus assume that either u has two or more neighbours in $\overline{X}$, or that v has two or more neighbours in X. Adjust notation so that u has two or more neighbours in $\overline{X}$. Let $R := \partial(u) \setminus C$. Now consider the graph $G'' := G - R$, together with the cut $D := \partial(X - u)$ (see Figure 5).

4.1.2. *The graphs $G''[X - u]$ and $G''[\overline{X} + u]$ are both connected.*

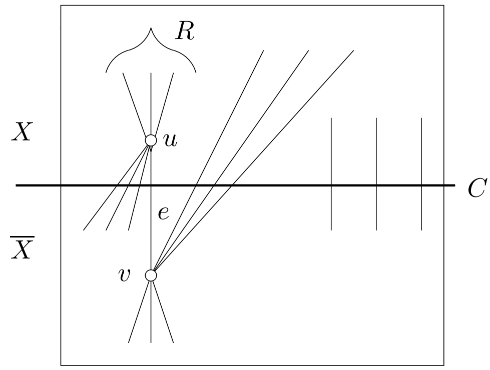
Proof. Note that the graph $G''[X - u]$ is the same as the graph $G[X - u]$, which is connected.

By (1.4), the shore $\overline{X}$ of C induces a connected subgraph of G. Vertex u is adjacent to vertices of $\overline{X}$ (v is one such vertex). Thus, it follows that the second graph $G''[\overline{X} + u]$ is connected as well. □

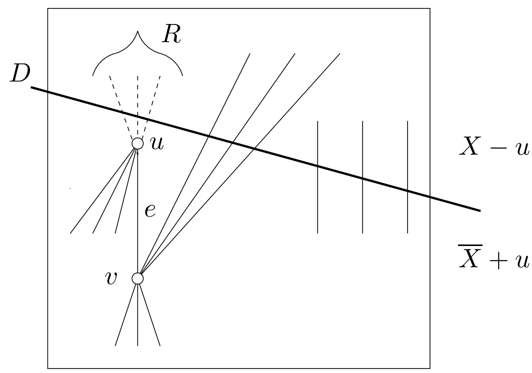
Every perfect matching of G that contains edge e is also a perfect matching of G''. Thus, G'' has perfect matchings. Both shores in G'' of cut $D := \partial(X - u)$ are even. Every perfect matching M of G'' is also a perfect matching of G. Moreover, $|M \cap C| = |M \cap D| + 1$. As C is tight in G, it follows that no edge of D is admissible in G''. By (2.4), G'' has a DM-barrier B'' such that B'' and the vertex sets of all the odd components of $G'' - B''$ are subsets of one of $X - u$ and $\overline{X} + u$.

The rest of the analysis depends on where the vertex u is in relation to the barrier B''.

4.1.3. *If vertex u does not lie in either B'' or in the vertex set of one of the odd components of $G'' - B''$, then G has a nontrivial barrier.*



(a) G



(b) G''

Figure 5: The graphs G and G''; dashed lines indicate removed edges.

Proof. In this case, u is a vertex of some even component of $G'' - B''$, whence $B := B'' \cup \{u\}$ is a nontrivial barrier of G''. But $G'' - B = G - B$, and therefore B is a nontrivial barrier of G. □

Suppose that u is either in B'' or in the vertex set of one of the odd components of $G'' - B''$. Since u is in $\overline{X} + u$, it follows that B'' and the vertex sets of all the odd components of $G'' - B''$ are subsets of $\overline{X} + u$. We conclude that $X - u$ is a subset of the vertex set of an even component, say L, of the graph $G'' - B''$.

4.1.4. *If vertex u lies in B'' then G has a nontrivial barrier.*

Proof. In that case, $G'' - B'' = G - B''$. Consequently, L is an even component of $G - B''$. Let w be any vertex of L. Then, $B'' \cup \{w\}$ is a nontrivial barrier of G. □

We may thus assume that u lies in some (odd) component K of $G'' - B''$. Note that B'' is a barrier of G'', and not only B'', but also the set of vertices of each odd component of $G'' - B''$ is a subset of $\overline{X} + u$. However, in the graph G, the edges of R join u to vertices of $X - u$. Thus u is the only vertex in an odd component of $G'' - B''$ that is adjacent to vertices of X in the graph G.

4.1.5. *Barrier B'' of G'' is also a barrier of G.*

Proof. By (4.1.2), $G''[X - u]$ is connected. Thus, $K \cup L$ is an odd component of $G - B''$. Therefore, B'' is a barrier of G (see Figure 6.) □

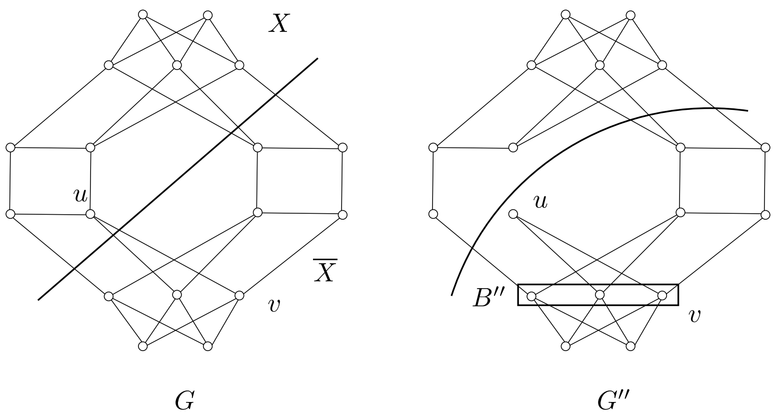


Figure 6: Case 4.1.5: B'' is a barrier of G.

If B'' is nontrivial, then we are done. Assume thus that B'' is trivial. Vertex u lies in $V(K)$ and has at least two neighbours in G''. (This is the only reason for requiring u to have degree two or more in G''.) Thus, at least one neighbour of u lies also in $V(K)$, whence K is nontrivial. Let w denote the only vertex of B''. The graph $G - w - u$ has at least two components; one is a subgraph of $K - u$ and another includes L. Thus, $\{u, w\}$ is a nontrivial barrier or a 2-separation of G. The assertion holds. □

Bipartite graphs of order four or more are not bicritical. The ELP Theorem implies that if a matching covered graph of order four or more is not a brick, then it is either not bicritical, or is not 3-connected. In other words, bricks are precisely the matching covered graphs of order four or more which are 3-connected and bicritical.

5. A conjecture

Two cuts $C := \partial(X)$ and $D := \partial(Y)$ of a matching covered graph *cross* if each of the four sets $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$, and $\overline{X} \cap \overline{Y}$ is nonempty. Thus, C and D do not cross if and only if one of the two shores of C is a subset of one of the two shores of D.

The ELP Theorem (4.1) says that any matching covered graph which has a nontrivial tight cut also has a nontrivial ELP-cut. In ([2], 2002) we were able to prove a stronger statement in a special case: if G is a brick and e is an edge of G such that $G-e$ is matching covered with two bricks, then for every nontrivial tight cut C of $G-e$, the graph $G-e$ has a nontrivial ELP-cut that does not cross C. (This was an important ingredient in our proof of a conjecture due to Lovász, presented in [2].) We venture to conjecture that this is true for all matching covered graphs:

Conjecture 5.1. *Let C be a nontrivial tight cut of a matching covered graph G. Then, G has an ELP-cut that does not cross C.*

The above conjecture may be rephrased in terms of the following notion: a tight cut C of a matching covered graph G is *essentially an ELP-cut* of G if there is a sequence $G_1 = G, G_2, \ldots, G_r$, $r \geq 1$ of matching covered graphs, such that (i) for $i = 1, 2, \ldots, r-1$, G_i has an ELP-cut, C_i, and G_{i+1} is a C_i-contraction of G_i, and (ii) cut C is an ELP-cut of G_r. (Trivially, every ELP-cut is an essentially ELP-cut.) It can be seen that Conjecture 5.1 is equivalent to the statement that every nontrivial tight cut of a matching covered graph is essentially an ELP-cut.

In support of Conjecture 5.1, we establish its validity for bicritical graphs and also for graphs with only two bricks. We shall make use of the following known fact about tight cuts.

5.2 ([8])**.** *Let C be a 2-separation cut of a matching covered graph G. If G is bicritical then both C-contractions of G are bicritical.* □

5.1. Validity of the conjecture for bicritical graphs

Theorem 5.3. *Let G be a bicritical graph, and let $C := \partial(X)$ be a nontrivial tight cut of G. Then, G has a 2-separation cut that does not cross C.*

Proof. As the graph G has nontrivial tight cuts, by the ELP Theorem 4.1, it has nontrivial ELP-cuts. Let $D := \partial(Y)$ be a nontrivial ELP-cut of G such that Y is minimal. Since G is bicritical, by the hypothesis, it cannot have nontrivial barrier cuts, and hence D is a 2-separation cut. Suppose that

$\{u, v\}$ is the 2-separation of G which gives rise to D, and adjust notation so that $u \in Y$, and $v \in \overline{Y}$.

If D does not cross C then we are done. We may thus assume that C and D cross. Adjust notation so that $|X \cap Y|$ is odd.

5.3.1. *One of u and v lies in X, the other lies in $\overline{X}$.*

Proof. By (1.4), the subgraph $G[X]$ of G induced by the shore X of C is connected. Thus, there is at least one edge joining a vertex in $X \cap Y$ to a vertex in $X \cap \overline{Y}$. Similarly, $G[\overline{X}]$ is connected and thus, there is at least one edge joining a vertex in $\overline{X} \cap Y$ to a vertex in $\overline{X} \cap \overline{Y}$. As the cut $\partial(Y)$ is a 2-separation cut associated with the 2-separation $\{u, v\}$, each edge of this cut is incident either with u or v. The desired conclusion follows. □

5.3.2. *Graph $G' := G/(\overline{Y} \to \overline{y})$ is a brick.*

Proof. By (5.2), G' is bicritical. Let us now show that G' is 3-connected, and conclude that G' is a brick. For this, assume the contrary, and let v_1, v_2 denote two vertices of G' such that $G' - v_1 - v_2$ is not connected. As G' is bicritical, it follows that $\{v_1, v_2\}$ is a 2-separation of G'.

Consider first the case in which $\overline{y}$ does not lie in $\{v_1, v_2\}$. In that case, $\{v_1, v_2\}$ is a 2-separation of G as well. Let K be a component of $G' - v_1 - v_2$ that does not contain $\overline{y}$. Then, $V(K) + v_1$ is the shore of a 2-separation cut of G and a proper subset of Y, in contradiction to the minimality of Y.

Consider next the case in which $\overline{y}$ lies in $\{v_1, v_2\}$. Adjust notation so that $\overline{y} = v_2$. Let L denote a connected component of $G' - v_1 - \overline{y}$ that does not contain vertex u. Let e be an edge of $\partial(V(L))$ that is not incident with v_1. Then, e is incident with $\overline{y}$, whence it is an edge of D. Every edge of D is incident with a vertex in $\{u, v\}$. Vertex u lies in $Y \setminus V(L)$. Thus, e is incident with v. This conclusion holds for each edge e of $\partial(V(L))$ that is not incident with v_1. It follows that $\{v_1, v\}$ is a 2-separation of G. Thus, $V(L) + v_1$ is the shore of a 2-separation cut of G and a proper subset of Y, in contradiction to the minimality of Y. Thus, G' is a brick. □

The cut $\partial(X \cap Y)$ is tight in the brick G'. Thus, $X \cap Y$ is a singleton, say $\{w\}$. Suppose that $u \in \overline{X}$ and $v \in X$. In this case, $u \in \overline{X} \cap Y$, $v \in X \cap \overline{Y}$. By (1.5), $G[\overline{Y}]$ is an odd component of $G - \{u, w\}$. It follows that $\{u, w\}$ is a barrier of G. This is absurd because G is bicritical. Thus, we may suppose that $u \in X$ and $v \in \overline{X}$. In this case, $u \in X \cap Y$, $v \in \overline{X} \cap \overline{Y}$ and $w = u$, hence $\{v, w\}$ is a 2-separation of G, and the corresponding cut $\partial((\overline{X} \cap Y) \cup \{v\})$ is a 2-separation cut that does not cross C. □

5.2. Validity of the conjecture for graphs with at most two bricks

Theorem 5.4. *Let G be a matching covered graph such that $b(G) \leq 2$, and let $C = \partial(X)$ be a nontrivial tight cut of G. Then, G has a nontrivial ELP-cut that does not cross C.*

Proof. By induction on $|V(G)|$. Let us first consider the base case. The smallest matching covered graph with two bricks is the graph shown in Figure 7. It has only two nontrivial tight cuts, each of which is a 2-separation cut; therefore, every nontrivial tight cut of that graph is itself an ELP-cut. We may thus assume that G has order eight or more.

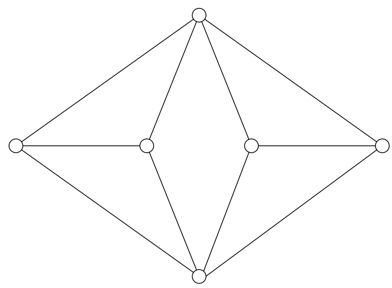


Figure 7: The smallest matching covered graph with two bricks.

Let $G_1 := G/\overline{X}$ and let $G_2 := G/X$. Suppose that one of the C-contractions, say G_1, is bipartite. Every C-contraction is matching covered. Thus, G_1 is a matching covered bipartite graph. As C is nontrivial, G_1 has four or more vertices, hence the part of G_1 that does not contain the contraction vertex is a nontrivial barrier of G. In that case, C is a barrier cut of G and we are done. We may thus assume that both C-contractions of G are nonbipartite. By hypothesis, $b(G) \leq 2$. Thus, $b(G) = 2$ and $b(G_1) = 1 = b(G_2)$. If G is bicritical then the assertion holds by Theorem 5.3. We may also assume that G is not bicritical. Therefore G has nontrivial barriers.

Let B denote a nontrivial maximal barrier of G. As $b(G) = 2$, the graph G is not bipartite. By Lemma 1.6, $G-B$ has a nontrivial (odd) component K. Let $Y := V(K)$ and let $D := \partial(Y)$. Clearly, D is a nontrivial barrier cut of G. If C does not cross D then the assertion holds. We may thus assume that C crosses D. Let X be the shore of C such that $|X \cap Y|$ is odd. Let $I := X \cap Y$ and let $U := \overline{X} \cap \overline{Y}$. By (1.5), the cuts $C_1 := \partial(I)$ and $C_2 := \partial(U)$ are both tight. Let $H_1 := G/(\overline{Y} \to \overline{y})$ and let $H_2 := G/(Y \to y)$. Let

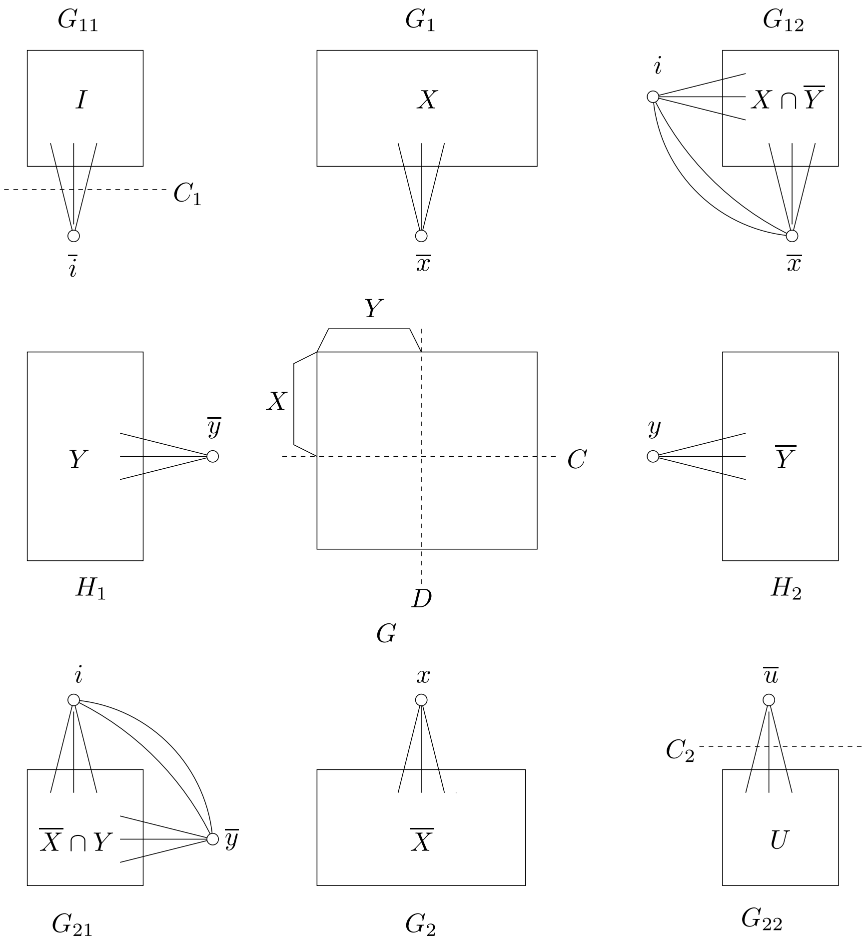


Figure 8: The graph G and its cut contractions.

$G_{11} := G/(\overline{I} \to \overline{i})$, let $G_{22} := G/(\overline{U} \to \overline{u})$, let $G_{12} := G_1/(I \to i)$ and let $G_{21} := H_1/(I \to i)$. (See Figure 8.)

5.4.1. *The vertex $\overline{y}$ does not lie in any nontrivial barrier of H_1. Consequently, the graph G_{21} is not bipartite.*

Proof. Let B' denote any barrier of H_1 that contains vertex $\overline{y}$. Then, the set $(B' - \overline{y}) \cup B$ is a barrier of G. By the maximality of B, it follows that $B' = \{\overline{y}\}$. That is, $\overline{y}$ does not lie in any nontrivial barrier of H_1.

Assume, to the contrary, that G_{21} is bipartite. Let $[A'', B'']$ denote the bipartition of G_{21}. Adjust notation so that i lies in A''. By (1.4), the subgraph

$G[X]$ induced by X is connected. This implies that $\overline{y}$ and i are adjacent. Thus, $\overline{y}$ lies in B''. Then, B'' is a nontrivial barrier of H_1 that contains vertex $\overline{y}$. This is a contradiction. □

Cut C_2 is tight in G_2, having G_{21} and G_{22} as its contractions. By (5.4.1), G_{21} is not bipartite. As $b(G_2) = 1$, it follows that G_{22} is bipartite. If U is not a singleton then $\partial(U)$ is a nontrivial barrier cut of G that does not cross C, and the assertion holds. We may thus assume that U is a singleton, say, $U = \{u\}$.

Suppose that G_{11} is bipartite. If I is not a singleton then $\partial(I)$ is a barrier cut of G that does not cross C, and the assertion holds. If I is a singleton then, by (1.5), C is a 2-separation cut of G and, again, the assertion holds. We may thus assume that G_{11} is not bipartite.

Cut C_1 is tight in H_1; one of its contractions is G_{11} and the other is G_{21}. As $b(G) = 2$, it follows that $b(G_{11}) = 1 = b(G_{21})$ and $b(H_1) = 2$.

We now apply the induction hypothesis, with the graph H_1 playing the role of G and C_1 playing the role of C. Incidentally, this is the only point in the proof of the Theorem in which induction is applied. All other cases, one might say, are the “trivial” cases, from the point of view of mathematical induction.

As C_1 is not a trivial tight cut in H_1, then, by induction, H_1 has a nontrivial ELP-cut D_1 that does not cross C_1. Let Y_1 be the shore of D_1 in H_1 that does not contain vertex $\overline{y}$. Then, Y_1 is a shore of D_1 in G itself.

5.4.2. *Either $I \subseteq Y_1$ or $Y_1 \subset I$ or $Y_1 \subset \overline{X} \cap Y$.*

Proof. As C_1 and D_1 do not cross, it follows that, in G, D_1 has a shore that is disjoint with a shore of C_1. In H_1, the vertex $\overline{y}$ does not lie in Y_1, whence, in G, $\overline{Y}$ is a subset of $\overline{Y_1} \cap \overline{I}$. Thus, at least one of the sets $\overline{Y_1} \cap I$, $Y_1 \cap \overline{I}$ and $Y_1 \cap I$ is empty. These imply, respectively, that $I \subseteq Y_1$, $Y_1 \subseteq I$ and $Y_1 \subseteq \overline{I}$. If $Y_1 \subseteq \overline{I}$, as Y_1 is odd and $\overline{X} \cap Y$ is even, it follows that $Y_1 \subset \overline{X} \cap Y$. □

Case 1. *Graph $G[Y_1]$ is bipartite.*

The cut C_1 is tight and the graph G_{11} is a C_1-contraction of G. Thus, G_{11} is matching covered. We have seen that G_{11} is not bipartite. By (1.7), neither is $G[I]$. It follows that I is not a subset of Y_1. By (5.4.2), either $Y_1 \subset I$ or $Y_1 \subset \overline{X} \cap Y$. In both alternatives, D_1 and C do not cross. Moreover, D_1 is a (nontrivial) ELP-cut of G. The assertion holds. We may thus assume that $G[Y_1]$ is not bipartite.

Case 2. *The cut D_1 is a barrier cut of H_1.*

Let B_1 denote a nontrivial barrier of H_1 with which D_1 is associated. Every edge of D_1 is incident with a vertex of B_1. As B_1 is nontrivial, it follows by (5.4.1) that $\overline{y}$ does not lie in B_1. Thus, no edge of D_1 is incident with $\overline{y}$. Graph G has an edge e that joins a vertex of I to a vertex of $X \cap \overline{Y}$. In H_1, that edge is incident with $\overline{y}$. It follows that e does not lie in D_1. Consequently, I is not a subset of Y_1. By (5.4.2), it follows that D_1 and C do not cross. As $\overline{y}$ does not lie in B_1, we conclude that D_1 is a (nontrivial) barrier cut of G. The assertion holds in this case.

We may thus assume that D_1 is a 2-separation cut of H_1. Let $\{v_1, v_2\}$ denote a 2-separation of H_1 with which D_1 is associated. Adjust notation so that v_1 lies in Y_1, whereupon v_2 does not lie in Y_1 and $v_1 \neq \overline{y}$.

Case 3. $Y_1 \subset \overline{X} \cap Y$.

In that case, D_1 and C do not cross. If v_2 is not $\overline{y}$ then $\{v_1, v_2\}$ is a 2-separation of G and the assertion holds. We may thus assume that $v_2 = \overline{y}$. Every edge of D_1 is incident with v_1 or $\overline{y}$. As $Y_1 \subset \overline{X} \cap Y$, it follows by (1.5) that every edge of D_1 not incident with v_1 is incident in G with u, the only vertex of U. We conclude that D_1 is a 2-separation cut of G associated with the 2-separation $\{u, v_1\}$. The assertion holds in this case.

Case 4. $I \subseteq Y_1$.

Consider first the case in which $v_2 = \overline{y}$. Let $Z' := Y \setminus Y_1$, let $Y' := Z' + u$, let $D' := \partial(Y')$. Clearly, Y' is a subset of $\overline{X}$. Thus, the cuts C and D' do not cross. In H_1, every edge of $\partial(Z')$ not incident with v_1 is incident with $\overline{y}$. Thus, in G, every edge of $\partial(Z')$ not incident with v_1 is incident with u, the only vertex of U. Thus, D' is a 2-separation cut of G that does not cross C. The assertion holds.

Suppose now that v_2 and $\overline{y}$ are distinct. Let R denote the set of edges of D_1 that, in G, join a vertex of I to a vertex of $\overline{Y}$. In H_1, the edges of R are incident with $\overline{y}$. The vertices v_2 and $\overline{y}$ are distinct. Moreover, v_2 does not lie in Y_1, in turn a superset of I. Every edge of R must be incident with a vertex in $\{v_1, v_2\}$. It follows that every edge of R is incident with v_1. We conclude that all edges of D are incident in G with a vertex in $\{u, v_1\}$. Consequently, the cut $\partial((X \cap \overline{Y}) + v_1)$ is a 2-separation cut of G that does not cross C. The assertion holds in this case.

Case 5. $Y_1 \subset I$.

In this case, D_1 and C do not cross. If v_2 and $\overline{y}$ are distinct then $\{v_1, v_2\}$ constitutes a 2-separation of G and D_1 is a 2-separation cut of G that does

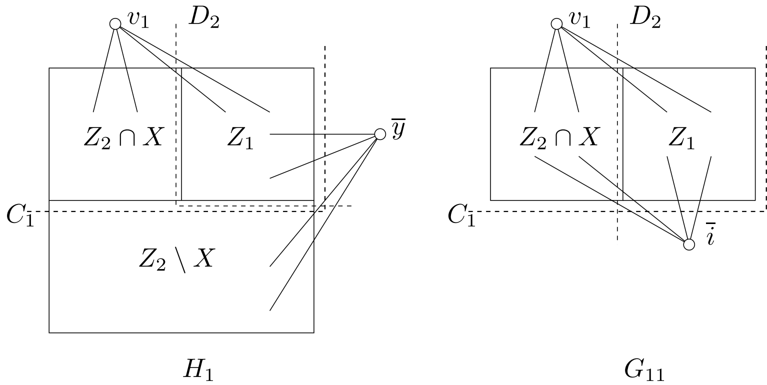


Figure 9: The Case $Y_1 \subset I$.

not cross C, whence the assertion holds. We may thus assume that $v_2 = \overline{y}$. Let $Z_1 := Y_1 - v_1$, let $Z_2 := Y \setminus Y_1$ (Figure 9).

In the graph H_1, the cut D_2 that has $Z_1 + \overline{y}$ as a shore is a tight cut associated with the 2-separation $\{v_1, \overline{y}\}$. Thus, in G_{11}, D_2 is a tight cut associated with the 2-separation $\{v_1, \overline{i}\}$. We have seen that $b(G_{11}) = 1$, and therefore, one of the two D_2-contractions of G_{11} is bipartite. The D_2-contraction of G_{11} that contains $\overline{i}$ is isomorphic to the D_2-contraction of H_1 that contains $\overline{y}$. We have seen that no nontrivial barrier of H_1 contains $\overline{y}$. Thus, the D_2-contraction of H_1 that contains $\overline{y}$ is not bipartite, whence the D_2-contraction of G_{11} that contains $\overline{i}$ is not bipartite. We conclude that $G_{11}[Y_2]$ is bipartite, where $Y_2 := (Z_2 \cap X) + v_1$. But $G_{11}[Y_2] = G[Y_2]$. Thus, $\partial(Y_2)$ is a (nontrivial) barrier cut of G that does not cross C. The assertion holds. □

The above result is a simpler version of a more complete result which characterizes tight cuts in a matching covered graph with two bricks, as explained below.

A tight cut $C := \partial(X)$ of a matching covered graph G is an *essentially 2-separation cut* if there is a *companion* tight cut $D := \partial(Y)$ which crosses C such that $|X \cap Y|$ is odd and both $G[X \cap Y]$ and $G[\overline{X} \cap \overline{Y}]$ are bipartite subgraphs of G. In the example depicted in Figure 10, C is an essentially 2-separation cut with D as its companion.

Theorem 5.5 ([4]). *Every tight cut of a matching covered graph with at most two bricks is either a barrier cut or is an essentially a 2-separation cut.* □

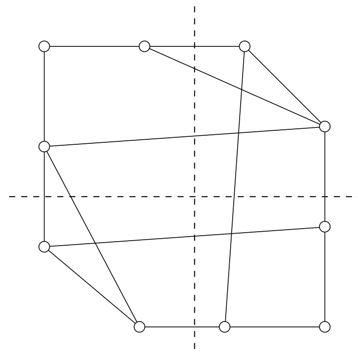


Figure 10: An example of essentially 2-separation cuts.

We dedicate this paper to our friend Adrian Bondy for his many valuable contributions to graph theory. We submitted it in 2014 with the intention of having it published in a special issue to mark Adrian's seventieth birthday.

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