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Characterization of $\mathcal{B}(\infty)$ using marginally large tableaux and rigged configurations in the A_n case via integer sequences

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Marginally large tableaux are semi-standard Young tableaux of special form that give a combinatorial realization of the crystals $\mathcal{B}(\infty)$. Rigged configurations are combinatorial objects prominent in the study of solvable lattice models, and give combinatorial realizations of the crystals $\mathcal{B}(\lambda)$ and $\mathcal{B}(\infty)$ in simply-laced and affine Kacs-Moody types. However, $\mathcal{B}(\infty)$ rigged configurations have not yet been characterized explicitly at the time of this writing.

We introduce certain nice integer sequences, called cascading sequences, to characterize marginally large tableaux. Then we use cascading sequences and a known non-explicit crystal isomorphism between marginally large tableaux and rigged configurations to give an explicit characterization of the latter set in the A_n case, revealing interesting structural properties of rigged configurations along the way, and then to give an explicit bijection between the two sets.

Keywords and phrases: Crystal, marginally large tableau, rigged configuration, Kashiwara operator.

1. Introduction

Kashiwara introduced the crystal $\mathcal{B}(\infty)$, which is the crystal base of the negative part $U_q^-(\mathfrak{g})$ of a quantum group, in [4], and used it to study the Demazure crystals that were conjectured by Littelmann [5]. As $\mathcal{B}(\infty)$ reveals much about the structure of the quantum group $U_q(\mathfrak{g})$ itself, it is an active topic of research. By the work of Hong and Lee [3], $\mathcal{B}(\infty)$ can be realized as crystals consisting of combinatorial objects called marginally large tableaux, which are a special class of semi-standard Young tableaux.

Schilling [11] gave an explicit $U_q(\mathfrak{g})$-crystal structure to combinatorial objects called rigged configurations, which naturally serve as indexes for the eigenvalues and eigenvectors of the Hamiltonian in the Bethe Ansatz. A crystal model for $\mathcal{B}(\infty)$ in terms of rigged configurations was given [9] by Salisbury and Scrimshaw for affine simply-laced types, who also established [10] an isomorphism between rigged configurations and marginally large tableaux as crystals. However, this isomorphism is not explicit, and the $\mathcal{B}(\infty)$ rigged configurations have not yet been explicitly characterized at the writing of this paper.

The purpose of this paper is to characterize the rigged configurations of the A_n type in $\mathcal{B}(\infty)$ and to give an explicit bijection between marginally large tableaux and $\mathcal{B}(\infty)$ rigged configurations of the A_n type. We will achieve this by introducing special integer sequences that will be called cascading sequences. Any element of a highest weight crystal is obtained by acting on the highest weight vector via a sequence of Kashiwara operators, though this sequence is not necessarily unique. A cascading sequence can be viewed as the "canonical" sequence of Kashiwara operators leading to any crystal from the highest weight crystal. The desired bijection will be obtained by first establishing a bijection between the marginally large tableaux and the cascading sequences, and then establishing a bijection between the cascading sequences and the rigged configurations.

This paper is organized as follows. In Subsection 2.1, we recall the definition of marginally large tableaux. In Subsection 2.2, we introduce cascading sequences and use them to characterize marginally large tableaux. In Subsection 2.3, we introduce an aspect of cascading sequences called lanes that will later be used in the characterization of rigged configurations. In Subsection 3.1, we recall the definition of $\mathcal{B}(\infty)$ rigged configurations in the A_n case. In Subsection 3.2, we show in Lemma 3.2.4 that Kashiwara operators for rigged configurations act nicely when arranged in a cascading sequence, which allows us to obtain an interesting structural property Theorem 3.2.7 of rigged configurations. In Subsection 3.3, we show that lanes of a cascading sequence correspond to columns of rigged partitions in the corresponding rigged configuration, and we obtain the first half of the characterization of rigged configurations Theorem 3.3.11. In Subsection 3.4, we give the rough idea of our growth algorithm for characterizing rigged configurations. In Subsection 3.5, we introduce special cascading sequences called p-plateaus that will be used in the growth algorithm. In Subsection 3.6, we show how to modify a cascading sequence to achieve the effect of adding boxes to a rigged partition in the corresponding rigged configuration. In Subsection 3.8, we give two formulations (Theorem 3.8.16 and Theorem 3.8.18) of our growth algorithm for characterizing rigged configurations. In Subsection 3.9, we give an algorithm for obtaining the cascading sequence of any rigged configuration in Theorem 3.9.1.

2. Marginally large tableaux and cascading sequences

We give a bijection between the set of A_n marginally large tableaux and a special set of integer sequences that we call cascading n-sequences.

2.1. Marginally large tableaux

In this subsection, we recall the definition of marginally large tableaux for type A_n, as given in [3].

Definition 2.1.1. *We call a semi-standard tableau T* ***large*** *if it has exactly n nonempty rows, and the ith row has strictly more i-boxes than the total number of boxes in the $(i+1)$st row, for each $1 \leq i < n$.*

Definition 2.1.2. *By a* ***marginally large tableau*** *in the A_n case we will mean a Young tableau with exactly n rows whose entries come from the alphabet $J = \{1 < 2 < \cdots < n < n+1\}$ that satisfies the following conditions:*

1. *The ith row of the leftmost column is a single i-box, for each $1 \leq i \leq n$.*
2. *Entries increase weakly as we go from left to right along each row.*
3. *The number of i-boxes in the ith row exceeds by exactly one the total number of boxes in the $(i+1)$st row, for each $1 \leq i < n$.*

Let $T(\infty)$ denote the set of A_n marginally large tableaux. As shown in [3], $T(\infty)$ has a *crystal structure*, given as follows.

Procedure 2.1.3. We describe how to apply the Kashiwara operator f_i to any marginally large tableau:

1. Apply f_i to this tableau in the usual way, by writing the tableau as a tensor product, applying the tensor product rule, and assembling the result back into tableau form.
2. We are done if the result we obtain is a large tableau, as it will be marginally large automatically.
3. If the result we obtain is not a large tableau, then f_i must have acted on the rightmost i-box of the ith row. Insert a single column of height i to the left of this box that f_i acted upon. For $1 \leq k \leq i$, the kth row of the added column must be a k-box.

Example 2.1.4. Given the marginally large tableau

$$S = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \cline{1-3} 3 & 4 & & \\ \cline{1-2} \end{array} \quad ,$$

we have

$$f_2(S) = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2&2&2&3& \\ \cline{1-4} 3&4& & & \\ \cline{1-2}\end{array} \quad .$$

Procedure 2.1.5. We describe how to apply the Kashiwara operator e_i to any marginally large tableau:

1. Apply e_i to this tableau in the usual way.
2. We are done if the result we obtain is zero or a marginally large tableau.
3. Otherwise, the result is a large tableau that is not marginally large. e_i must have acted on the box to the right of the rightmost i-box of the ith row. Remove the column that contains this changed box. This column will have height i, and its kth row consists of a single k-box, for $1 \leq k \leq i$.

Example 2.1.6. Given the marginally large tableau

$$S = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2&2&2& \\ \cline{1-3} 3&4& & \\ \cline{1-2}\end{array} \quad ,$$

we have

$$e_3(S) = \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2&2& \\ \cline{1-2} 3& & \\ \cline{1-1}\end{array} \quad .$$

2.2. Cascading sequences and a bijection

For any $m \in [n] = \{1, 2, \ldots, n\}$, we will call any subinterval $[a, m] = \{a, a+1, \ldots, m\}$ of $[n]$ an m**-lower subinterval**. For example, $[3, 5]$ is a 5-lower subinterval of $[6]$. By an m**-component**, we will mean a sequence of finitely many (allowed to be zero) m-lower subintervals of $[n]$ ordered by nonincreasing length.

Definition 2.2.1. *By a* ***cascading*** n***-sequence*** *we will mean an integer sequence formed by concatenating an* n*-component, an* $(n-1)$*-component, an* $(n-2)$*-component,* $\ldots$*, in that order. Let* $\bar{A}_n$ *denote the set of cascading* n*-sequences.*

Example 2.2.2. $(1, 2, 3, 4, 5, 3, 4, 5, 3, 4, 5, 5, 2, 3, 4, 3, 4, 2, 3, 3, 2, 2, 1)$ is an element of $\bar{A}_5$ where the lower subintervals (written as tuples) are $(1, 2, 3, 4, 5)$, $(3, 4, 5)$, $(3, 4, 5)$, (5), $(2, 3, 4)$, $(3, 4)$, $(2, 3)$, (3), (2), (2), (1).

We will follow the English notation for the Young tableau, with weakly increasing row length as we move up the tableau. Let M^{A_n} denote the set of marginally large tableaux (MLT) in the A_n case. We now define a map $\Phi : M^{A_n} \to \bar{A}_n$ which will be shown to be a bijection. Given a marginally large tableau T, we will give an f-string (sequence of Kashiwara operators $f_1, f_2, \ldots, f_n$; also called Lusztig data [6]) with nice properties that gives rise to T upon acting on the highest weight MLT. We will write this f-string as its corresponding sequence of indices, and we will see that this sequence is an element of $\bar{A}_n$. Let $T(i)$ denote the portion of the ith row of T without the i boxes.

Define $\Phi(T)$ as follows. The f-string that we give will add the $(n+1)$-boxes, the n-boxes, the $(n-1)$-boxes, and so on in that order. Let $x_{i,j}$ denote the number of $(j+1)$-boxes in the ith row of T. The n-component of $\Phi(T)$ consists of $x_{i,n}$ copies of $(i, i+1, \ldots, n)$ for $i = 1, 2, \ldots$. In general, the m-component of $\Phi(T)$ consists of $x_{i,m}$ copies of $(i, i+1, \ldots, m)$ for $i = 1, 2, \ldots$; each copy of $(i, i+1, \ldots, m)$ adds an $(m+1)$-box to the ith row.

Remark 2.2.3. The cascading sequences can actually be rewritten as BZL data (in [1], [2]) for the reduced word

$$(s_1)(s_2 s_1) \ldots (s_n \ldots s_2 s_1).$$

However, to the best of our knowledge, the use of such sequences for the purpose of characterization given in this paper is new.

Example 2.2.4. Given

$T =$

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	3	3	3	6
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2								
3	3	3	3	3	3	3	4	4	5	5	6	6	6									
4	4	4	4	4	5																	
5	6	6	6																			

in M^{A_5}, we have

$$\Phi(T) = (1,2,3,4,5,3,4,5,3,4,5,3,4,5,5,5,5,3,4,3,4,4,3,3,1,2,1,2,1,\\ 2,1,1,1)$$

where the lower subintervals $(1,2,3,4,5)$, $(3,4,5)$, $(3,4,5)$, $(3,4,5)$, (5), (5), (5) add all the 6-boxes of T, the lower subintervals $(3,4), (3,4), (4)$ add all the 5-boxes of T, the lower subintervals $(3), (3)$ add all the 4-boxes of T, the lower subintervals $(1,2), (1,2), (1,2)$ add all the 3-boxes of T, and the lower subintervals $(1), (1), (1)$ add all the 2-boxes of T.

Proposition 2.2.5. *The map $\Phi : M^{A_n} \to \bar{A}_n$ defined above is a bijection.*

Remark 2.2.6. Thus, we can take the cascading n-sequences to be "canonical" f-strings for M^{A_n}.

Proof. The inverse map Φ^{-1} can be described as follows. Given an f-string $\alpha \in \bar{A}_n$, we can read off all its lower subintervals in left-right order. Each such lower subinterval $[i, m]$ gives an $(m+1)$-box in the ith row of the MLT resulting from α acting on the highest weight element. Thus, each such lower subinterval $[i, m]$ specifies that there must be an $(m+1)$-box in the ith row of $\Phi^{-1}(\alpha)$. In this way, the MLT $\Phi^{-1}(\alpha)$ is completely determined, since $\Phi^{-1}(\alpha)(i)$ is completely determined for each row i. □

Remark 2.2.7. Notice that the elements α of $\overline{A}_n$ are particularly convenient as f-strings for MLT's, as we can obtain the corresponding MLT $\Phi^{-1}(\alpha)$ (which is the same MLT obtained by having α act on the highest weight element) by simply reading off the lower subintervals of α, *without* having to apply the Kashiwara operators on the highest weight element. For instance, we see in Example 2.2.4 that we can immediately obtain T from the f-string by noting that T has exactly one 6-box in the first row specified by the lower subinterval $(1, 2, 3, 4, 5)$, exactly two 5-boxes in the third row specified by the lower subintervals $(3, 4), (3, 4)$, and so on.

Finally, we mention that the cascading sequence characterization in this section can also be applied to regular Young tableaux, with slight modification.

2.3. Lanes of cascading sequences

As already shown in [9], the marginally large tableaux are isomorphic to the rigged configurations as crystals, so we can use cascading sequences to characterize the latter objects (which are in bijection with cascading sequences), which have not yet been characterized explicitly.

Given two tuples $u = (u_1, \dots, u_i), v = (v_1, \dots, v_j)$ we define

$$u \oplus v = (u_1, \dots, u_i, v_1, \dots, v_j).$$

If u, v are lower subintervals, we define their intersection $u \cap v$ in the natural way. For example, we have $(3, 5, 2) \oplus (5) = (3, 5, 2, 5)$ and we have $(7, 8, 9) \cap (6, 7, 8) = (7, 8)$.

We first introduce the aspects of cascading n-sequences that will be useful in describing A_n rigged configurations. Let $\alpha \in \overline{A}_n$ be a cascading n-sequence.

For the remainder of this subsection, we partition α into subsequences that we will call **lanes**. As subsequences, lanes will be written as tuples. For any tuple, the first entry will be called the **head** of the tuple and the last entry will be called the **tail** of the tuple. Also, for any lane L of α, let $|L|$ denote the length of L. Formation of these lanes will reflect the way Kashiwara operators act on rigged configurations in Lemma 3.2.4. Furthermore, we will show that the ith l-lane corresponds to the ith column of the lth partition in the corresponding rigged configuration. Label the lower subintervals of α as $I_1, I_2, \ldots, I_P$ from left to right. Denote by $I_i(j)$ the jth entry of I_i, and by $\overline{I}_i(j)$ the integer value (in $[n]$) of $I_i(j)$; in Example 2.2.2, $I_5 = (2,3,4)$ and $\overline{I}_5(3) = 4$. Lanes will be formed, via the following iterative procedure, for each integer in α; i.e. for $m \in [n]$ there will be lanes $L_1(m), L_2(m), \ldots$ at the end of the procedure. The **lane forming procedure** builds the lanes in stages, as follows:

At the outset, we form lanes using entries of I_1, by setting $L_1(\overline{I}_1(j)) := (I_1(j))$ for each j. In general, suppose a collection of lanes $M^1, M^2, \ldots, M^a$ has been formed from the lower subintervals $I_1, I_2, \ldots, I_{b-1}$. Set $L_p(q) := \emptyset$ for any $L_p(q) \notin \{M^1, M^2, \ldots, M^a\}$. We will form new lanes using entries of I_b. First, pick the maximal d_1 such that $L_{d_1}(\overline{I}_b(1)) \in \{M^1, M^2, \ldots, M^a\}$, and set $L_{d_1+1}(\overline{I}_b(1)) := (I_b(1))$; if no such d_1 exists, take $d_1 = 0$. In general, for any entry $I_b(k)$ with $k > 1$, pick the maximal $d_k \le d_{k-1}$ such that $|L_{d_k}(\overline{I}_b(k))| > |L_{d_k+1}(\overline{I}_b(k))|$, and set $L_{d_k+1}(\overline{I}_b(k)) := L_{d_k+1}(\overline{I}_b(k)) \oplus (I_b(k))$; take $d_k = 0$ if no such d_k exists. Finally, we fix all other preexisting lanes. At the end of this iterative procedure, we obtain the lanes partitioning α.

Example 2.3.1. Consider the cascading 10-sequence

$$(8,9,10,8,9,10,7,8,9,7,8,9,7,8,9,8,9,6,7,8,7,8),$$

whose lower subintervals are $I_1 = (8,9,10), I_2 = (8,9,10), I_3 = (7,8,9), I_4 = (7,8,9)$, $I_5 = (7,8,9), I_6 = (8,9), I_7 = (6,7,8), I_8 = (7,8)$. The lanes are formed in the following processes (with exactly one entry added to the lane at each stage):

1. $L_1(8) : (I_1(1)) \to (I_1(1), I_3(2)) \to (I_1(1), I_3(2), I_7(3))$
2. $L_1(9) : (I_1(2)) \to (I_1(2), I_3(3))$
3. $L_1(10) : (I_1(3))$
4. $L_2(8) : (I_2(1)) \to (I_2(1), I_4(2))$
5. $L_2(9) : (I_2(2)) \to (I_2(2), I_4(3))$
6. $L_2(10) : (I_2(3))$

7. $L_1(7) : (I_3(1)) \to (I_3(1), I_7(2))$
8. $L_2(7) : (I_4(1))$
9. $L_3(7) : (I_5(1))$
10. $L_3(8) : (I_5(2)) \to (I_5(2), I_8(2))$
11. $L_3(9) : (I_5(3))$
12. $L_4(8) : (I_6(1))$
13. $L_4(9) : (I_6(2))$
14. $L_1(6) : (I_7(1))$
15. $L_4(7) : (I_8(1))$

Written another way, the lower subintervals and lanes are $I_1 = (8^1, 9^1, 10^1)$, $I_2 = (8^2, 9^2, 10^2)$, $I_3 = (7^1, 8^1, 9^1)$, $I_4 = (7^2, 8^2, 9^2)$, $I_5 = (7^3, 8^3, 9^3)$, $I_6 = (8^4, 9^4)$, $I_7 = (6^1, 7^1, 8^1)$, $I_8 = (7^4, 8^3)$, where lane i has been marked with a superscript i.

Example 2.3.2. Let us now look at a more complex example. The cascading 10-sequence

$$(6,7,8,9,10,7,8,9,10,7,8,9,10,8,9,10,6,7,8,9,6,7,8,9,7,8,9,5,6,7,8,\\ 5,6,7,8,5,6,7,8,6,7,8)$$

has lower subintervals with lanes $I_1 = (6^1, 7^1, 8^1, 9^1, 10^1)$, $I_2 = (7^2, 8^2, 9^2, 10^2)$, $I_3 = (7^3, 8^3, 9^3, 10^3)$, $I_4 = (8^4, 9^4, 10^4)$, $I_5 = (6^2, 7^1, 8^1, 9^1)$, $I_6 = (6^3, 7^2, 8^2, 9^2)$, $I_7 = (7^4, 8^3, 9^3)$, $I_8 = (5^1, 6^1, 7^1, 8^1)$, $I_9 = (5^2, 6^2, 7^2, 8^2)$, $I_{10} = (5^3, 6^3, 7^3, 8^3)$, $I_{11} = (6^4, 7^4, 8^4)$, where lane i has been marked with a superscript i.

We now show with more detail the formation of lanes at the stage where I_6 is acting.

The lanes formed before I_6 are:

$L_1(6) = (I_1(1))$
$L_1(7) = (I_1(2), I_5(2))$
$L_1(8) = (I_1(3), I_5(3))$
$L_1(9) = (I_1(4), I_5(4))$
$L_1(10) = (I_1(5))$
$L_2(6) = (I_5(1))$
$L_2(7) = (I_2(1))$
$L_2(8) = (I_2(2))$
$L_2(9) = (I_2(3))$
$L_2(10) = (I_2(4))$
$L_3(7) = (I_3(1))$
$L_3(8) = (I_3(2))$

$L_3(9) = (I_3(3))$
$L_3(10) = (I_3(4))$
$L_4(8) = (I_4(2))$
$L_4(9) = (I_4(3))$
$L_4(10) = (I_4(4))$

Lastly, we have $L_3(6) = L_3(7) = L_3(8) = L_3(9) = L_4(7) = L_1(5) = L_2(5) = L_3(5) = L_4(6) = ()$.

Assignment of entries of I_6:

We have $L_3(6) := () \oplus (I_6(1)) = (I_6(1))$, since $d_1 = 2$ for integer value 6.

We have $L_2(7) := (I_2(1)) \oplus (I_6(2)) = (I_2(1), I_6(2))$, since $d_2 = 1$ for integer value 7.

We have $L_2(8) := (I_2(2)) \oplus (I_6(3)) = (I_2(2), I_6(3))$, since $d_3 = 1$ for integer value 8.

We have $L_2(9) := (I_2(3)) \oplus (I_6(4)) = (I_2(3), I_6(4))$, since $d_4 = 1$ for integer value 9.

3. Cascading sequences and rigged configurations

We use cascading sequences to give an explicit characterization (with a growth algorithm) of $\mathcal{B}(\infty)$ rigged configurations in the A_n case, and we give an explicit bijection between these rigged configurations and cascading sequences. This results in an explicit bijection between the marginally large tableaux and A_n rigged configurations.

3.1. Rigged configurations

The definition of $\mathcal{B}(\infty)$ rigged configurations in the A_n case is given in [9], based on work done in [11]. We now recall this definition. Let $\mathfrak{g}$ be a simply-laced Kac-Moody algebra with index set I, and let $\mathcal{H} := I \times \mathbb{Z}_{>0}$. Fix a multiplicity array

$$L = (L_i^{(a)} \in \mathbb{Z}_{>0} : (a, i) \in \mathcal{H}).$$

We typically define a partition to be a multiset of integers sorted in decreasing order. Define a **rigged partition** to be a multiset of integer pairs (i, x) with $i > 0$, with these pairs sorted in decreasing lexicographic order. We will call each (i, x) a *string*, with i the *size* or *length* of the string and x the *quantum number*, *label*, or *rigging* of the string. By a **rigged configuration** we will mean a pair (ν, J) where $\nu = \{\nu^{(a)} : a \in I\}$ is a sequence of rigged partitions and $J = (J_i^{(a)})_{(a,i)\in\mathcal{H}}$ where each $J_i^{(a)}$ is the weakly increasing sequence of riggings of strings in $\nu^{(a)}$ whose length is i. The **vacancy number** of ν is defined as

$$p_i^{(a)} = p_i^{(a)}(\nu) = -\sum_{(b,j)\in\mathcal{H}} A_{ab}\min(i,j)m_j^{(b)},$$

where $m_j^{(b)}$ is the number of parts in the partition $\nu^{(b)}$ with length j. The **coquantum number** or **colabel** of a string (i,x) is defined to be $p_i^{(a)} - x$. The ath part of (ν, J) is often denoted by $(\nu, J)^{(a)}$ for brevity.

To give the definition of $\mathcal{B}(\infty)$ rigged configurations, denoted $RC(\infty)$, let $\nu_\emptyset$ be the multipartition with all parts empty; that is, set $\nu_\emptyset = (\nu^{(1)},\ldots,\nu^{(n)})$ where $\nu_i^{(a)} = \emptyset$ for all $(a,i)\in\mathcal{H}$. Therefore the rigging $J_\emptyset$ of $\nu_\emptyset$ must be $J_i^{(a)} = \emptyset$ for all $(a,i)\in\mathcal{H}$.

Definition 3.1.1. *The Kashiwara operators e_a and f_a act on elements $(\nu, J)\in RC(\infty)$ as follows: Fix $a\in I$, and let x denote the smallest label of $(\nu,J)^{(a)}$, assuming $(\nu,J)^{(a)}\neq\emptyset$.*

1. *Set $e_a(\nu,J) = 0$ if $x\geq 0$. Otherwise, let l denote the smallest length of all strings which have label x in (ν,J). We obtain the rigged configuration $e_a(\nu,J)$ by replacing the string (l,x) with $(l-1,x+1)$ and then changing all the other labels to ensure that all colabels are preserved.*
2. *Add the string $(1,-1)$ to $(\nu,J)^{(a)}$ if $x>0$. Otherwise, let l denote the greatest length of all strings which have label x in $(\nu,J)^{(a)}$. Replace the string (l,x) by $(l+1,x-1)$, then change all the other labels to ensure that all colabels are preserved. The result is $f_a(\nu,J)$.*
   If $(\nu,J)^{(a)}$ is empty, then f_a adds the string $(1,-1)$ to $(\nu,J)^{(a)}$.

$RC(\infty)$ is the graph generated by $(\nu_\emptyset, J_\emptyset)$ using e_a and f_a, for $a\in I$.

We now give the remaining part of the crystal structure:

$$\begin{aligned}
\epsilon_a(\nu,J) &= \max\{k\in\mathbb{Z}_{\geq 0} : e_a^k(\nu,J)\neq 0\},\\
\phi_a(\nu,J) &= \epsilon_a(\nu,J) + \langle h_\alpha, \mathrm{wt}(\nu,J)\rangle,\\
\mathrm{wt}(\nu,J) &= -\sum_{(a,i)\in\mathcal{H}} i m_i^{(a)}\alpha_a = -\sum_{a\in I}|\nu^{(a)}|\alpha_a,
\end{aligned}$$

where $\{\alpha_a\}_{a\in I}$ denotes the simple roots.

3.2. Kashiwara operators acting in a cascading sequence arrangement

We show in this section that $RC(\infty)$ Kashiwara operators act in a nice way when arranged in a cascading sequence. Let $R = (\nu_1, \nu_2, \ldots, \nu_n)$ be a $B(\infty)$ rigged configuration of A_n type where ν_i is the ith rigged partition whose jth row has rigging rig_i^j.

Notation 3.2.1. Whenever we write a rigged configuration in the form

$$R = (\nu_1, \nu_2, \ldots, \nu_n),$$

it is understood that each ν_i is a rigged partition carrying the riggings information rig_i^j.

Fix rigged partition ν_m. If $\nu_m = \emptyset$, then we regard ν_m as a single empty row r_1 whose length is zero and whose rigging is zero. Generally, if we label the rows of ν_m from top to bottom by $r_1, r_2, \ldots, r_k$, then we regard ν_m has having an "empty row" r_{k+1} beneath r_k, where r_{k+1} is understood to have zero length and a rigging of zero.

Let α denote the cascading sequence of R. Recall how the vacancy number changes when a Kashiwara operator acts on R:

If the Kashiwara operator f_a adds a box to a row of length l in ν_a, then the vacancy numbers of R are changed using the formula

$$p_i^{(b)} = \begin{cases} p_i^{(b)} & \text{if } i \leq l \\ p_i^{(b)} - A_{ab} & \text{if } i > l \end{cases}$$

where $p_i^{(b)}$ denotes the vacancy number of a row of length i in ν_b, and

$$A_{ab} = \begin{cases} -1 & \text{if } b = a \pm 1 \\ 2 & \text{if } b = a \\ 0 & \text{otherwise} \end{cases}$$

For each partition λ, we will denote by λ^b the bth part (row) of λ and by $\tilde{\lambda}^b$ the portion of λ^b that has no boxes beneath it; we call $\tilde{\lambda}^b$ the **stretch** of λ^b. For instance, T in Example 2.2.4 has $\tilde{T}^4 = \boxed{4}\boxed{5}$. If λ is a rigged partition, by the *rigging of the stretch* $\tilde{\lambda}^b$ we will always mean the rigging of the row λ^b. Also, letting λ^t denote the transpose of λ, $l(\lambda) := \max(\lambda^t)$ is then the number of rows λ has.

By an integer sequence $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ **acting** on a rigged configuration R' we will always mean the corresponding sequence of Kashiwara operators $\{f_i | i \in \gamma\}$ acting on R'. More precisely, γ acts on R' by $f_\gamma R' = f_{\gamma_p} f_{\gamma_{p-1}} \cdots f_{\gamma_1} R'$.

When working with cascading sequences, we can rely on the following useful lemmas:

Let $I = (a, a+1, \ldots, m)$ be an m-lower subinterval of the cascading sequence α. Denote by α_I the subsequence of α before I. Let $R_I = (\mu_1, \mu_2, \ldots, \mu_n)$ denote the preexisting rigged configuration corresponding to α_I. Whenever I acts on R_I, it adds one box to each of the partitions $\mu_a, \mu_{a+1}, \ldots, \mu_{m-1}, \mu_m$. As will be proven below, the box added to any μ_j is of two forms: contributing and noncontributing. A **contributing box** is a single box

$$\square\, -1$$

which contributes -1 to the rigging of the row to which this box is added, contributes -1 to the rigging of any row of μ_j longer than the row to which it is added, and contributes $+1$ to the rigging of any row of μ_{j+1} longer than the row to which it is added, but does not change the riggings of μ_b for $b \neq j, j+1$. A **noncontributing box** is a single box

$$\square\, 0$$

with rigging 0, which does not change the riggings of any rigged partition.

Remark 3.2.2. The contributing and noncontributing boxes describe the *cumulative* effect of the action of f_I, demonstrated in Lemma 3.2.4.

Let us analyze in more detail how I acts on the preexisting rigged configuration R_I corresponding to α_I. For any partition λ let $\overline{\lambda}$ denote the portion of λ beneath the top row.

Lemma 3.2.3. *Let λ_1, λ_2 be two partitions satisfying $\overline{\lambda_2} \subset \lambda_1 \subset \lambda_2$. Fix a positive integer p. Let u_1 be the uppermost row of λ_1 with length p and u_2 be the uppermost row of λ_2 with $|u_2| \leq p$. Then every row of λ_1 below u_1 is no longer than u_2.*

Proof. Suppose $u_1 = \lambda_1^b$ and $u_2 = \lambda_2^c$. Since $\lambda_1 \subset \lambda_2$, we must have $c \geq b$. Since $\overline{\lambda_2} \subset \lambda_1$, we must have $c \leq b+1$. If $c = b$, then we have $|u_1| = |u_2|$, and the claim follows immediately. Suppose $c = b+1$. Then we have $|\lambda_1^d| \leq |\lambda_2^{b+1}| = |u_2|$ for any $d \geq b+1$, since $\lambda_1 \subset \lambda_2$. ∎

Lemma 3.2.4 (Main Lemma)**.** *For $I = (a, a+1, \ldots, m)$, $R_I = (\mu_1, \mu_2, \ldots, \mu_n)$ satisfies the following properties. Let r_a be the top row of μ_a, and r_i be the uppermost row of μ_i with $|r_i| \leq |r_{i-1}|$.*

1. *The partitions $\mu_1, \mu_2, \ldots, \mu_{m-1}$ have all riggings equal to zero. For $l \in [m-1]$ we have $\overline{\mu_{l+1}} \subset \mu_l \subset \mu_{l+1}$. By Lemma 3.2.3 it follows in particular that:*
   Fix a positive integer p. For $l \in [m-1]$ let u_l be the uppermost row of μ_l with length p, and let u_{l+1} be the uppermost row of μ_{l+1} such that $|u_{l+1}| \leq p$. Then every row of μ_l below u_l must be no longer than u_{l+1}.
2. *The rows of the rigged partition μ_m above r_m have the same rigging as r_m, and this rigging is non-positive and minimal in μ_m.*
3. *I acts on R_I by adding a noncontributing box to the rows $r_a, r_{a+1}, \ldots, r_{m-1}$ and a contributing box to the row r_m.*

Proof. Induction. R_I clearly satisfies these properties if I is the first or second lower subinterval of α. Now consider the general case, assuming that R_I satisfies these properties.

Let $R'_I = (\mu'_1, \mu'_2, \ldots, \mu'_n)$ denote the rigged configuration corresponding to $\alpha'_I := \alpha_I \oplus I$. We apply I to R_I to obtain R'_I and prove that it satisfies these properties as well.

We first check Property 3 and the first statement of Property 1 for R'_I. By Property 1 for R_I, Kashiwara operator f_a adds a box to the first row r_a of μ_a, adding -1 to its rigging, adding $+1$ to the vacancy number as well as the rigging of rows of μ_{a+1} longer than r_a, and not changing the riggings of $\mu_1, \ldots, \mu_{a-1}$. f_{a+1} then adds a box to the uppermost row r_{a+1} of μ_{a+1} with $|r_{a+1}| \leq |r_a|$, adding -1 to its rigging, adding -2 to the vacancy number as well as the rigging of rows of μ_{a+1} longer than r_{a+1} (so these rows end up with a rigging of $1-2=-1$), adding $+1$ to the vacancy number and the rigging of rows of the ath partition longer than r_{a+1} (which by Property 1 gives μ'_a with zero riggings), and adding $+1$ to the vacancy number as well as the rigging of rows of μ_{a+2} longer than r_{a+1}. f_{a+2} then adds a box to the uppermost row r_{a+2} of μ_{a+2} with $|r_{a+2}| \leq |r_{a+1}|$, adding -1 to its rigging, adding -2 to the vacancy number and the rigging of rows of μ_{a+2} longer than r_{a+2} (so these rows end up with a rigging of $1-2=-1$), adding $+1$ to the vacancy number and the rigging of rows of the $(a+1)$st partition longer than r_{a+2} (which by Property 1 gives μ'_{a+1} with zero riggings), and adding $+1$ to the vacancy number as well as the rigging of rows of μ_{a+3} longer than r_{a+2}. Iterating this process, for $j = 0, 1, \ldots, m-a-2$ we obtain μ'_{a+j} by adding a noncontributing box to row r_{a+j} of μ_{a+j} so μ'_{a+j} has zero riggings.

Now, after f_{m-1} added a box to row r_{m-1} of μ_{m-1}, all rows of the resulting $(m-1)$st partition with length at least $|r_{m-1}|+1$ have rigging -1. By Property 2 for R_I, this action of f_{m-1} must have contributed $+1$ to the vacancy number and the rigging of all rows of μ_m longer than r_{m-1}, and consequently these rows of μ_m now have greater rigging than r_m does, so r_m is now the longest row of μ_m with the smallest rigging. Finally, f_m adds a box to r_m, adding -1 to its rigging, adding $+1$ to the vacancy number and the rigging of rows of the $(m-1)$st partition longer than $|r_m|$ (which by Property 1 gives μ'_{m-1} with zero riggings), adding -2 to the vacancy number and the rigging of rows of the mth partition longer than r_m (so these rows now have the same rigging as rows of length $|r_m|+1$), and adding $+1$ to the vacancy number as well as the rigging of rows of μ_{m+1} longer than $|r_m|$. This shows that μ'_{m-1} is obtained from μ_{m-1} by adding a noncontributing box to r_{m-1}, and that μ'_m is obtained from μ_m by adding a contributing box to r_m.

Now we verify Property 2 for $R'_I = (\mu'_1, \mu'_2, \ldots, \mu'_n)$. By above, we conclude that rows of μ'_m with length at least $|r_m|+1$ have identical rigging, and this rigging is minimal and non-positive. Let $a' \geq a$. Let $r'_{a'}$ be the top row of $\mu'_{a'}$, and let r'_k be the uppermost row of μ'_k with $|r'_k| \leq |r'_{k-1}|$, for $k = a'+1, a'+2, \ldots, m$. Notice that we have $|r'_{a'}| = |r_a|+1$ if $a' = a$ and we have $|r'_{a'}| \geq |r_a|+1$ if $a' > a$. Since μ'_k contains a row with length $|r_k|+1$, we have $|r'_k| \geq |r_k|+1$, for $k = a', a'+1, \ldots, m$. It follows that the rows of μ'_m above r'_m have the same rigging as r'_m, and this rigging is non-positive and minimal in μ'_m.

Finally, we verify the second statement of Property 1 for $R'_I = (\mu'_1, \mu'_2, \ldots, \mu'_n)$. If $l, l+1 < a$, then the claim follows by hypothesis. If $l = a-1$, then the claim follows since $\mu'_{l+1} = \mu'_a$ is obtained from μ_a by adding a single box to the first row. Now suppose $l \in [a, m-1]$. μ'_l and μ'_{l+1} are obtained from μ_l and μ_{l+1}, respectively, by adding a box via Property 3. Let $r_l = \mu_l^c$ and $r_{l+1} = \mu_{l+1}^d$ be the rows of μ_l and μ_{l+1}, respectively, to which the box was added. Then r_{l+1} is the uppermost row of μ_{l+1} no longer than r_l. Since $\mu_l \subset \mu_{l+1}$, we must have $d \geq c$. Since $\overline{\mu_{l+1}} \subset \mu_l$, we must have $d \leq c+1$. Thus, either $d = c$ or $d = c+1$. Suppose $d = c$. Then $|r_l| = |r_{l+1}|$, and it follows immediately that $\mu'_l \subset \mu'_{l+1}$. If r_{l+1} is the first row, then $\overline{\mu'_{l+1}} \subset \mu'_l$ by hypothesis. If r_{l+1} is not the first row, we still have $\overline{\mu'_{l+1}} \subset \mu'_l$ since $|(\mu'_l)^{c-1}| = |\mu_l^{c-1}| \geq |r_{l+1}|+1 = |(\overline{\mu'_{l+1}})^{c-1}|$. Suppose $d = c+1$. Then we must have $|r_l| < |\mu_{l+1}^c|$, so $|(\mu'_l)^c| = |r_l|+1 \leq |\mu_{l+1}^c| = |(\mu'_{l+1})^c|$ and thus $\mu'_l \subset \mu'_{l+1}$. Since $|r_{l+1}| = |\mu_{l+1}^{c+1}| \leq |\mu_l^c| = |r_l|$, we have $|(\mu'_{l+1})^{c+1}| = |r_{l+1}|+1 \leq |\mu_l^c|+1 = |(\mu'_l)^c|$, and thus we have $\overline{\mu'_{l+1}} \subset \mu'_l$. This completes the induction. □

Remark 3.2.5. It is easy to see that the first containment $\overline{\mu_{l+1}} \subset \mu_l$ holds for all $l \in [n-1]$, since no more boxes will be added to the $(l+1)$st partition once all the $(l+1)$-lower subintervals have acted.

It follows immediately from Lemma 3.2.4 that

Lemma 3.2.6. *The following are true.*

1. *If α ends in a p-lower subinterval, then ν_q of R has zero riggings for all $q \leq p-1$.*
2. *All contributing boxes (and hence negative riggings) to the ν_m of R are added by m-lower subintervals of α.*
3. *All positive contributions to the riggings of ν_m are added by $(m-1)$-lower subintervals of α, which add no boxes to ν_m.*
4. *Suppose I_1, I_2 are m-lower subintervals of α with I_1 preceding I_2. If I_1 adds a contributing box to column i_1 and I_2 adds a contributing box to column i_2 of the mth partition, then we have $i_1 < i_2$.*

Proof. The first three items follow immediately from the lemma. For the fourth item, note that $I_1 = (a_1, \ldots, m)$ and $I_2 = (a_2, \ldots, m)$, where $a_1 \leq a_2$. Let r_{a_1} be the top row of μ_{a_1}, and r_i be the uppermost row of μ_i with $|r_i| \leq |r_{i-1}|$. Let r'_{a_2} be the top row of μ_{a_2}, and r'_i be the uppermost row of μ_i with $|r'_i| \leq |r'_{i-1}|$. Notice that $|r'_k| > |r_k|$ for $k = a_2, a_2+1, \ldots, m$. Applying Property 3 of Lemma 3.2.4, we deduce that the contributing box added by I_2 must be strictly to the right of the contributing box added by I_1. □

We thus obtain the following interesting result.

Theorem 3.2.7. *Identical rows of ν_m of R have equal riggings, for any $m \in [n]$.*

Proof. Before any m-lower subinterval has acted, the mth partition has zero riggings. After the first m-lower subinterval adds a contributing box to row r, every row with length at least $|r|+1$ has rigging -1, while the rigging of every row with length at most $|r|$ remains unchanged. In general, assume that the jth m-lower subinterval has added a box to row r' of the mth partition, so that rows with length at least $|r'|+1$ have equal rigging, and that identical rows with length at most $|r'|$ have equal rigging. By the fourth item of Lemma 3.2.6, the $(j+1)$st m-lower subinterval adds a contributing box to row r'' with $|r''| \geq |r'|+1$. In the resulting mth partition, rows of length at most $|r''|$ have unchanged riggings, which is the same for identical rows, while the new row with length $|r''|+1$ and other rows with length at least $|r''|+1$ receive a -1 contribution to their identical riggings. This shows that the riggings of identical rows remain equal after all the m-lower subintervals have acted.

Similarly, each time an $(m-1)$-lower subinterval acts, all the rows of ν_m no longer than a certain length k experience no change in rigging, while all the rows of ν_m longer than k receive $+1$ contribution to the rigging.

Therefore, the riggings of identical rows remain equal after all the $(m-1)$-lower subintervals have acted. □

3.3. Obtaining the rigged configuration from the cascading sequence

Now we relate the concepts in Subsection 2.3 to the $\mathcal{B}(\infty)$ rigged configurations in the A_n case. Let $R = (\nu_1, \nu_2, \ldots, \nu_n)$ be an A_n rigged configuration. Let α denote the cascading sequence of R. We can obtain any partition in the corresponding rigged configuration without doing explicit calculation via the Kashiwara operators involved. This is done by partitioning α into lanes and then analyzing the relevant lanes.

Each column of ν_l ends in exactly one of the stretches of ν_l. We denote by $\mathrm{col}(\tilde{\nu_l}^b)$ the set of columns of ν_l ending in the stretch $\tilde{\nu_l}^b$, and let $W_b^l := \{\text{lanes } L_i(l) \,|\, |L_i(l)| = b\}$.

By Lemma 3.2.4, the l-lanes correspond precisely to the columns of ν_l, and we have the following useful facts:

Lemma 3.3.1. *The set* $\mathrm{col}(\tilde{\nu_l}^b)$ *corresponds to the set* W_b^l. *In fact, under the definition of lanes,* $L_i(l)$ *corresponds exactly to the* i*th column of* ν_l, *with the height of the column given by* $|L_i(l)|$.

Proof. Adding a box to the longest row r with $|r| \le c$ is the same as adding a box to the dth column for maximal $d \le c+1$ whose height is strictly less than that of the $(d-1)$st column. □

Remark 3.3.2. In particular, the number of columns of height b in ν_l is given by the number of l-lanes $L_i(l)$ with $|L_i(l)| = b$ in the corresponding cascading n-sequence.

Roughly speaking, the riggings of ν_l are determined by the number of l-lanes that contain the right endpoint of some lower subinterval and by the number of $(l-1)$-lanes that contain the right endpoint of some lower subinterval. In Example 2.3.2, if we fix $l = 9$, then the 9-lane $L_2(9)$ is an example of the former because it contains the right endpoint of I_5, and the 8-lane $L_3(8)$ is an example of the latter because it contains the right endpoint of I_9.

Lemma 3.3.3. *Suppose* r *is a row of* ν_l. *Let* V_r^l *denote the set of* l*-lanes* $L_i(l)$ *ending at a right endpoint, where* $i \le |r|$. *Let* V_r^{l-1} *denote the set of* $(l-1)$*-lanes* $L_i(l-1)$ *ending at a right endpoint, where* $i \le |r|$. *Then the rigging of* r *is given by* $-|V_r^l| + |V_r^{l-1}|$.

Proof. Lemma 3.2.6 gives us that at most one contributing box can be added to a column. The rigging of r is determined by the number of contributing boxes (negative contribution) that were added to the columns of the lth partition occupied by r and the number of contributing boxes (positive contribution) that will have been added to the same columns of the $(l-1)$st partition; the former number corresponds to the term $-|V_r^l|$, while the latter number corresponds to the term $|V_r^{l-1}|$. $\square$

For any entry χ of α with value $|\chi|$, if χ is the vth entry of the lane $L_u(|\chi|)$, we say that χ has **lane depth** v, χ has **lane number** u, and we also refer to u as the **lane number** of $L_u(|\chi|)$. In Example 2.3.2, the entry 8 of I_9 has lane number 2 and depth 3.

Lemma 3.3.4. *ν_l has at most* $\mathrm{maxr}_l := \min(n-l, l-1)+1 = \min(n-l+1, l)$ *rows.*

Proof. It suffices to consider $L_1(l)$, which corresponds to the first column of ν_l. Each entry of $L_1(l)$ is an entry of some lower subinterval I with $\min I \in [l]$ and $\max I \in [l, n]$. Suppose that $L_1(l)(j)$ and $L_1(l)(j+1)$ are contained in an m_1-lower subinterval and an m_2-lower subinterval, respectively. By Lemma 3.2.6 Property 4, we must have $m_1 \neq m_2$. Since the m_1-lower subintervals must precede the m_2-lower subintervals, it follows that $m_2 < m_1$. Hence we have $|L_1(l)| \leq |[l, n]| = n-l+1$. On the other hand, notice that ν_1 can only be a single row, by Lemma 3.2.4. By Lemma 3.2.4, for any lower subinterval I', the lane depth of entry $b+1$ of I' exceeds the lane depth of entry b of I' by at most one. Thus, inductively we have $|L_1(l)| \leq |[l]| = l$ as well. $\square$

Lemma 3.3.5. *If $l > \frac{n+1}{2}$, then* $\mathrm{maxr}_l = n-l+1$. *If $l \leq \frac{n+1}{2}$, then* $\mathrm{maxr}_l = l$.

Proof. If $l > \frac{n+1}{2}$, then $n-l+1 < n - \frac{n+1}{2} + 1 = \frac{n+1}{2} < l$. If $l \leq \frac{n+1}{2}$, then $n-l+1 \geq n - \frac{n+1}{2} + 1 = \frac{n+1}{2} \geq l$. The claims then follow by Lemma 3.3.4. $\square$

Lemma 3.3.6. *We have* $\mathrm{maxr}_l \leq \mathrm{maxr}_{l-1}$ *if and only if $l-1 > n-l$ (equivalently $l > \frac{n+1}{2}$).*

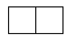
Example 3.3.7. Consider the element

$$(7,8,9,10,7,8,9,10,8,9,10,6,7,8,9,6,7,8,9,7,8,9,5,6,7,8,5,6,7,8,7,8)$$

of $\overline{A}_{10}$, whose lower subintervals are $(7^1, 8^1, 9^1, 10^1)$, $(7^2, 8^2, 9^2, 10^2)$, $(8^3, 9^3, 10^3)$, $(6^1, 7^1, 8^1, 9^1)$, $(6^2, 7^2, 8^2, 9^2)$, $(7^3, 8^3, 9^3)$, $(5^1, 6^1, 7^1, 8^1)$, $(5^2, 6^2, 7^2, 8^2)$, $(7^4, 8^4)$, where the lanes have been marked with superscripts. From this information, we can tell, for example, that the 9th partition of this rigged configuration has exactly three columns of height 2, the 8th partition has exactly one column of height 2, and the 7th partition has height 1 for both its third and fourth columns.

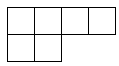
We can apply Lemmas 3.3.1 and 3.3.3 to obtain the lth partition in the rigged configuration as well as its riggings, given the corresponding cascading n-sequence. We illustrate this in the following

Example 3.3.8. In the corresponding rigged configuration in Example 2.3.1, the 10th partition is



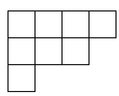
with rigging $-2+2=0$, since $L_1(10), L_2(10)$ end at right endpoints (contributing $-1-1$ to the rigging) and since $L_1(9), L_2(9)$ also end at right endpoints (contributing $+1+1$ to the rigging).

The 9th partition is



with rigging -1 for the second row and rigging -2 for the first row, since $L_1(9)$, $L_2(9)$, $L_3(9)$, $L_4(9)$ all end at right endpoints (contributing $-1-1$ to the rigging of the second row and $-1-1-1-1$ to the rigging of the first row) with $|L_1(9)| = |L_2(9)| = 2$ and $L_3(9) = L_4(9) = 1$, and since $L_1(8), L_4(8)$ end at right endpoints (contributing $+1$ to the rigging of the second row and $+1+1$ to the rigging of the first row).

Similarly, the 8th partition is



with riggings -1 for the third row, -2 for the second row, and -2 for the first row.

Example 3.3.9. In Example 2.3.2, the cascading 10-sequence

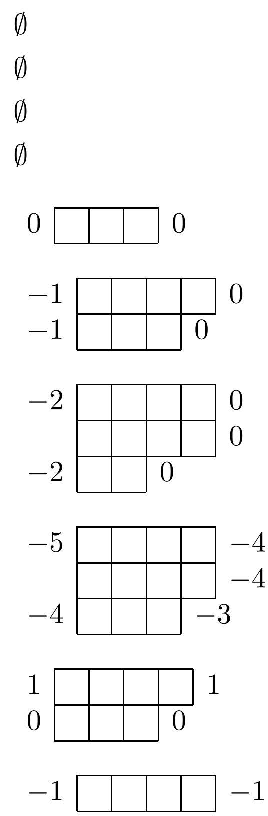
$$(6,7,8,9,10,7,8,9,10,7,8,9,10,8,9,10,6,7,8,9,6,7,8,9,7,8,9,5,6,7,8,\\ 5,6,7,8,5,6,7,8,6,7,8)$$

has lower subintervals with lanes $I_1 = (6^1, 7^1, 8^1, 9^1, 10^1)$, $I_2 = (7^2, 8^2, 9^2, 10^2)$, $I_3 = (7^3, 8^3, 9^3, 10^3)$, $I_4 = (8^4, 9^4, 10^4)$, $I_5 = (6^2, 7^1, 8^1, 9^1)$, $I_6 = (6^3, 7^2, 8^2, 9^2)$, $I_7 = (7^4, 8^3, 9^3)$, $I_8 = (5^1, 6^1, 7^1, 8^1)$, $I_9 = (5^2, 6^2, 7^2, 8^2)$, $I_{10} = (5^3, 6^3, 7^3, 8^3)$, $I_{11} = (6^4, 7^4, 8^4)$, where lane i has been marked with a superscript i.

Looking at these lanes, we can tell that

1. ν_{10} has four columns of length one, with $\text{rig}_{10}^1 = -4+3 = -1$, since $L_1(10), \ldots, L_4(10)$ end at right endpoints, and since $L_1(9), L_2(9), L_3(9)$ end at right endpoints
2. ν_9 has three columns of length two, and one column of length one, with $\text{rig}_9^2 = -3+3 = 0$ and $\text{rig}_9^1 = -3+3+1 = 1$, since $L_1(9), L_2(9), L_3(9)$ end at right endpoints but $L_4(9)$ does not, and since $L_1(8), \ldots, L_4(8)$ end at right endpoints
3. ν_8 has three columns of length three and one column of length two, with $\text{rig}_8^3 = -3$ and $\text{rig}_8^2 = \text{rig}_8^1 = -3-1 = -4$, since $L_1(8), \ldots, L_4(8)$ end at right endpoints

The rigged configuration (with ν_i in top-bottom order) in its entirety is



Lastly, the following theorem imposing constraints (in a recursive manner, starting from the last partition) on the range of possible legitimate $\mathcal{B}(\infty)$ rigged configurations of type A also follows from Lemma 3.2.4 (which states that at most one noncontributing box can be added to each column), Lemma 3.2.6 Property 4 (which states that at most one contributing box can be added to each column), and Lemma 3.3.4. This result is the first half of our classification of rigged configurations. For convenience of description, we will regard $\overline{\nu_m}$ as having $|\nu_m^1| - |\nu_m^2|$ columns of height zero to the right of its first row (i.e. $\overline{\nu_m}$ has an empty row of length $|\nu_m^1| - |\nu_m^2|$ to the right of its first row).

Theorem 3.3.10. *ν_{m-1} is obtained from ν_m in three stages:*

1. *Add at most one noncontributing box to each column of $\overline{\nu_m}$, resulting in a partition $\widehat{\nu_{m-1}}$.*
2. *Add at most one contributing box to each column of $\widehat{\nu_{m-1}}$, resulting in a partition $\widehat{\nu_{m-1}}'$.*
3. *Finally add a number of contributing boxes to the first row of $\widehat{\nu_{m-1}}'$.*

In this process, any column of $\overline{\nu_m}$ with height $\max((\overline{\nu_m})^t)$ must receive at most $\min(\text{maxr}_{m-1} - \max((\overline{\nu_m})^t), 2)$ boxes.

We will use the following restatement extensively.

Theorem 3.3.11 (Restatement of Theorem 3.3.10)**.** *ν_{m-1} is obtained from ν_m by adding at most two boxes to each column of $\overline{\nu_m}$, the first box being noncontributing and the second box being contributing. Moreover, we have the following constraints:*

1. *At most $\min(\text{maxr}_{m-1} - \max((\overline{\nu_m})^t), 2)$ boxes can be added to any column of height $\max((\overline{\nu_m})^t)$.*
2. *At most one box can be added to the dth column for $d > |\nu_m^1|$, and this box must be contributing.*
3. *In any row of ν_{m-1}, no contributing box precedes a noncontributing box.*

Remark 3.3.12*.* Item 1 simply states that the resulting $(m-1)$st partition cannot have more rows than maxr_{m-1}. Any box added to a column of $\overline{\nu_m}$ with height zero will be in the first row of the resulting partition. Needless to say, there cannot be any gaps between the boxes added to any row, as the result would not be a valid partition.

Even though we have not specified the rigging of ν_m here, this theorem gives us the “at most two boxes to each column” constraint. Precisely how the rigging of ν_m constrains ν_{m-1} will be handled in later sections.

3.4. Rough idea of the algorithm

Given an m-lower subinterval $I = (a, a+1, \ldots, m)$, we say that the lower subinterval $I_+ = (a-1, a, a+1, \ldots, m)$ is the **lengthening** of I, and we say that we **lengthen** I to obtain I_+.

Our characterization for the A_n rigged configurations will be an algorithm for growing rigged configurations starting from the last (nth) rigged partition; this growth algorithm can determine whether any given n-tuple of rigged partitions is a legitimate A_n rigged configuration. In other words, given the last partition (which consists of a row with any number of boxes), we can give the range of all possible $(n-1)$st partitions and its riggings. In general, given the nth, $(n-1)$st, ..., $(n-i)$th partitions, we can give the range of all possible $(n-i-1)$st partitions and its riggings.

Growing the rigged configuration in our algorithm corresponds to growing its corresponding cascading n-sequence. Note that any cascading n-sequence can be constructed by first adding copies of n to the (initially empty) string, then copies of $n-1$ to the string, then copies of $n-2$ to the string, and so on, such that we have a cascading n-sequence at each stage. It follows *a fortiori* that any cascading n-sequence can be constructed by first adding copies of $i \leq n$ to the (initially empty) string, then copies of $i \leq n-1$ to the string, then copies of $i \leq n-2$ to the string, and so on, such that we have a cascading n-sequence at each stage. Hence any A_n rigged configuration can be constructed (by applying the Kashiwara operators in the order of the cascading n-sequence at each stage) via this type of iterative process, which constructs the nth partition, $(n-1)$st partition, $(n-2)$nd partition, and so on, in that order. What we need to do is to fine tune this process so that the already constructed nth, $(n-1)$st, ..., $(n-i)$th partitions and their riggings do not change when we construct the $(n-i-1)$st partition. More precisely, at the ith stage, we will add all the copies of $n-i+1$ along with minimal copies of $j < n-i+1$ necessary to preserve the previously constructed rigged partitions; we will elaborate on this in the next few subsections.

Remark 3.4.1. We mention that, by Lemma 3.3.1, if ν is the lth partition in the rigged configuration, then the stretch $\tilde{\nu}^b$ corresponds to the set $\{L_i(l) || L_i(l)| = b\}$. In other words, the stretch $\tilde{\nu}^b$ corresponds to the set of l-lanes of length b, or equivalently the set of columns of ν with height b.

Let $R = (\nu_1, \nu_2, \ldots, \nu_n)$ be a rigged configuration we want to construct by our growth algorithm. To construct the compatible rigged partition ν_{i-1} given that we have already constructed $\nu_i, \nu_{i+1}, \ldots, \nu_n$, where the riggings of

$\nu_i, \nu_{i+1}, \ldots, \nu_n$ are fixed, we will add noncontributing boxes and contributing boxes beneath the stretches of $\overline{\nu_i}$ (which has zero riggings by default). Roughly speaking, at most two rows of boxes will be added beneath each stretch of $\overline{\nu_i}$, with the first row consisting of noncontributing boxes and the second row consisting of contributing boxes. This is justified by Theorem 3.3.11. Of course, the contributing boxes added beneath each stretch of $\overline{\nu_i}$ must account for the riggings of ν_i, by Lemma 3.3.3. Before describing how to add boxes to $\overline{\nu_i}$, we need the notion of *plateaus* to delineate the stretches of a rigged partition to which boxes can be added.

3.5. Plateaus as base for construction

Definition 3.5.1. *We say that a cascading sequence β (as well as its corresponding rigged configuration) is a (p, q, r)-**plateau** if it satisfies the following property:*

1. *For every i for which $L_i(p)$ exists, we have $|L_i(p-1)| = |L_i(p)| - 1$ whenever $|L_i(p-1)| < q$.*
2. *For every $(p-1)$-lane $L_i(p-1)$ with $|L_i(p-1)| \leq r$, $L_i(p-1)$ does not end at a right endpoint.*
3. *For any $k < p-1$ no k-lane ends at a right endpoint.*

*If the above property holds for $q = \infty$ and $r = \infty$, then we call β a p-**plateau**. If β is an m-plateau for every $m \in [p]$, then we call β a p^*-**plateau**; here (and in what follows) we will exclude the case $m = 1$ from consideration, as the letter 0 does not occur in β.*

Lemma 3.5.2. *If β is a p^*-plateau corresponding to the rigged configuration $(\mu_1, \mu_2, \ldots, \mu_n)$, then $\mu_{l-1} = \overline{\mu_l}$ with zero riggings for $l \in [p]$.*

Proof. First we show that a right endpoint can only exist at the end of a lane. Suppose a right endpoint occurs in an l-lane $L_j(l)$, and let I denote the lower subinterval containing this right endpoint. By the definition of cascading sequences, the only lower subintervals after I containing l as an entry must have l as a right endpoint. However, any lower subinterval after I that contains l as a right endpoint must add its right endpoint to a lane $L_k(l)$ where $k > j$, by Lemma 3.2.6.

Since right endpoints can only occur at the end of a lane, we have $\mu_{l-1} = \overline{\mu_l}$ by the definition of p^*-plateau, and we have that μ_l has zero riggings by Lemma 3.3.3, for $l \in [p]$ □

Example 3.5.3. The cascading sequence consisting of lower subintervals $(7^1, 8^1, 9^1)$, $(8^2, 9^2)$, $(8^3, 9^3)$, (9^4), $(6^1, 7^1, 8^1)$, $(7^2, 8^2)$, (8^4) is a 7^*-plateau and a $(8, 2, \infty)$-plateau. The cascading sequence consisting of the lower subintervals $(7^1, 8^1, 9^1, 10^1)$, $(7^2, 8^2, 9^2, 10^2)$, $(8^3, 9^3, 10^3)$, $(8^4, 9^4, 10^4)$, $(8^5, 9^5, 10^5)$, $(6^1, 7^1, 8^1, 9^1)$, $(7^3, 8^2, 9^2)$, $(5^1, 6^1, 7^1, 8^1)$, $(6^2, 7^2, 8^2)$, $(6^3, 7^3, 8^3)$, $(5^2, 6^2, 7^2)$ is a 7^*-plateau and an $(8, 2, 2)$-plateau.

Remark 3.5.4. As a start, notice that the cascading sequence consisting of the (singleton) lower subintervals $(n), (n), \ldots, (n)$ is an n^*-plateau.

We now present procedures for adding at most two boxes to each column of the $(p-1)$st partition of a rigged configuration that is both a (p, q, r)-plateau and a $(p-1)^*$-plateau, generating all possible $(p-1)$st partitions compatible with the predetermined pth, $(p+1)$st, $\ldots$, nth partitions. Rigged configurations that are both a (p, q, r)-plateau and a $(p-1)^*$-plateau will serve as the "skeletons" upon which boxes are added in our growth algorithm.

3.6. Adding boxes to a stretch

Since any stretch s of a partition λ corresponds to all columns of some fixed height $\mathrm{ht}(s)$, we will also refer to $\mathrm{ht}(s)$ as the **height** of the stretch s.

Convention 3.6.1. For any A_n rigged configuration $R' = (\nu_1, \nu_2, \ldots, \nu_n)$, by Remark 3.2.5 we already know that $\nu_{i-1} \supset \overline{\nu_i}$. If we label the stretches of ν_i from bottom to top by $g_1, g_2, \ldots, g_k$, and the stretches of $\overline{\nu_i}$ from bottom to top by $g'_1, g'_2, \ldots, g'_k$, then clearly g'_i is identical to g_i for $i \in [k-1]$. Here g'_k is identical to g_k if $|\nu_i^1| = |\nu_i^2|$, and is empty if $|\nu_i^1| > |\nu_i^2|$. In the case $|\nu_i^1| > |\nu_i^2|$, we will refer to g'_k as an "invisible" stretch above the first row of $\overline{\nu_i}$, with $|g'_k| = |g_k|$. Thus, in either case, we will regard $\overline{\nu_i}$ as having identical copies of all the stretches of ν_i; this will be convenient for when we talk about adding boxes to $\overline{\nu_i}$ to form ν_{i-1}, where a box added beneath g'_k will be in the first row of the resulting partition in the case $|\nu_i^1| > |\nu_i^2|$.

Now, fix cascading sequence β that is both a $(p-1)^*$-plateau and a (p, q, r)-plateau, with its corresponding rigged configuration $R = (\mu_1, \mu_2, \ldots, \mu_n)$. We give two procedures for adding respectively noncontributing boxes and contributing boxes beneath the stretches $z_1, z_2, \ldots, z_a$ of μ_{p-1} (ordered from bottom to top, following Convention 3.6.1), which fixes μ_x for all $x > p$ and also fixes the shape of μ_p (though not necessarily the rigging, which will depend on the resulting $(p-1)$st partition). Assume that $\mathrm{ht}(z_1) < \mathrm{maxr}_{p-1}$; otherwise no boxes can be added beneath z_1. Both procedures output both the desired cascading sequence and its corresponding rigged configuration, and can be applied repeatedly to add boxes to multiple stretches sequentially.

Procedure 3.6.2 (Adding Noncontributing Boxes to a Given Stretch). Suppose $\mathrm{ht}(z_i) < \min(q,r)$. The following algorithm adds n_i noncontributing boxes beneath z_i, where $0 \leq n_i \leq |z_i|$:

We will add n_i copies of $p-1$, n_i copies of $p-2$, …, n_i copies of $p-(\mathrm{ht}(z_i)+1)$ to β as follows. Let B_v denote the set of lower subintervals with head v. We will delete a number of elements from each B_v before lengthening the first n_i of the remaining elements.

Label from left to right by $\breve{I}_1^{p-\mathrm{ht}(z_i)-1}$, $\breve{I}_2^{p-\mathrm{ht}(z_i)-1}$, …, $\breve{I}_w^{p-\mathrm{ht}(z_i)-1}$ the elements of $B_{p-\mathrm{ht}(z_i)-1}$. Then delete the element $\breve{I}_1^{p-\mathrm{ht}(z_i)} \in B_{p-\mathrm{ht}(z_i)}$ nearest $\breve{I}_1^{p-\mathrm{ht}(z_i)-1}$ left of $\breve{I}_1^{p-\mathrm{ht}(z_i)-1}$, delete the element $\breve{I}_2^{p-\mathrm{ht}(z_i)} \in B_{p-\mathrm{ht}(z_i)}$ nearest $\breve{I}_2^{p-\mathrm{ht}(z_i)-1}$ left of $\breve{I}_2^{p-\mathrm{ht}(z_i)-1}$, …, delete the element $\breve{I}_w^{p-\mathrm{ht}(z_i)} \in B_{p-\mathrm{ht}(z_i)}$ nearest $\breve{I}_w^{p-\mathrm{ht}(z_i)-1}$ left of $\breve{I}_w^{p-\mathrm{ht}(z_i)-1}$. Let $\overline{B}_{p-\mathrm{ht}(z_i)}$ denote the subset obtained from $B_{p-\mathrm{ht}(z_i)}$ after performing this sequence of deletions. Next, delete the element $\breve{I}_1^{p-\mathrm{ht}(z_i)+1} \in B_{p-\mathrm{ht}(z_i)+1}$ nearest $\breve{I}_1^{p-\mathrm{ht}(z_i)}$ left of $\breve{I}_1^{p-\mathrm{ht}(z_i)}$, delete the element $\breve{I}_2^{p-\mathrm{ht}(z_i)+1} \in B_{p-\mathrm{ht}(z_i)+1}$ nearest $\breve{I}_2^{p-\mathrm{ht}(z_i)}$ left of $\breve{I}_2^{p-\mathrm{ht}(z_i)}$, …, delete the element $\breve{I}_w^{p-\mathrm{ht}(z_i)+1} \in B_{p-\mathrm{ht}(z_i)+1}$ nearest $\breve{I}_w^{p-\mathrm{ht}(z_i)}$ left of $\breve{I}_w^{p-\mathrm{ht}(z_i)}$. Let $\overline{B}_{p-\mathrm{ht}(z_i)+1}$ denote the subset obtained from $B_{p-\mathrm{ht}(z_i)+1}$ after performing this sequence of deletions. In general, delete the element $\breve{I}_1^{p-\mathrm{ht}(z_i)+c} \in B_{p-\mathrm{ht}(z_i)+c}$ nearest $\breve{I}_1^{p-\mathrm{ht}(z_i)+c-1}$ left of $\breve{I}_1^{p-\mathrm{ht}(z_i)+c-1}$, delete the element $\breve{I}_2^{p-\mathrm{ht}(z_i)+c} \in B_{p-\mathrm{ht}(z_i)+c}$ nearest $\breve{I}_2^{p-\mathrm{ht}(z_i)+c-1}$ left of $\breve{I}_2^{p-\mathrm{ht}(z_i)+c-1}$, …, delete the element $\breve{I}_w^{p-\mathrm{ht}(z_i)+c} \in B_{p-\mathrm{ht}(z_i)+c}$ nearest $\breve{I}_w^{p-\mathrm{ht}(z_i)+c-1}$ left of $\breve{I}_w^{p-\mathrm{ht}(z_i)+c-1}$. Let $\overline{B}_{p-\mathrm{ht}(z_i)+c}$ denote the subset obtained from $B_{p-\mathrm{ht}(z_i)+c}$ after performing this sequence of deletions.

Finally, lengthen the first n_i elements (in left-right order as usual) of $\overline{B}_{p-\mathrm{ht}(z_i)+d}$ in β, for $d = 0, 1, \ldots, \mathrm{ht}(z_i)$.

Notation 3.6.3. Let $C_{\mathrm{ht}(z_i)}$ denote the set of lower subintervals $\breve{I}_j^{p-\mathrm{ht}(z_i)+c}$ of β deliberately fixed (i.e. not lengthened) in Procedure 3.6.2. We will call $C_{\mathrm{ht}(z_i)}$ the set of **deleted elements** of β, or the set of **fixed elements** of β.

Remark 3.6.4. In Procedure 3.6.2, we say that the element $\breve{I}_u^{p-\mathrm{ht}(z_i)+c}$ is **paired** with the element $\breve{I}_u^{p-\mathrm{ht}(z_i)+c-1}$, for each $u \in [w]$. This pairing process used to obtain $C_{\mathrm{ht}(z_i)}$ is in fact the same pairing/bracketing process in the definition of the Kashiwara operator.

A rough illustration of the pairing/bracketing in Procedure 3.6.2: If we let a denote any lower subinterval with head j and b denote any lower subinterval with head $j-1$ (a and b will be used as shorthand here; the a's (resp. b's) are not necessarily identical), and if *aaabbaabbabaa* is the cascading subsequence of β whose lower subintervals have only j or $j-1$ as head, then the pairing and deletion process in Procedure 3.6.2 works as follows.

$$aaabbaabbabaa \to aa(ab)ba(ab)b(ab)aa \to a(ab)(ab)aa \to aaa$$

(this means that only the remaining lower subintervals aaa can be lengthened in β)

That Procedure 3.6.2 works as described will be proven in Section 3.7. Meanwhile, let us look at some examples of how this procedure works.

Example 3.6.5. Consider the cascading sequence α (with lanes marked by superscripts as usual) consisting of the lower subintervals $(8^1, 9^1, 10^1, 11^1)$, $(8^2, 9^2, 10^2, 11^2)$, $(8^3, 9^3, 10^3, 11^3)$, $(7^1, 8^1, 9^1, 10^1)$, $(7^2, 8^2, 9^2, 10^2)$, $(8^4, 9^4, 10^4)$, $(6^1, 7^1, 8^1, 9^1)$, $(7^3, 8^3, 9^3)$, which is an 8^*-plateau. We have

$$\mu_7 = \begin{array}{rll} 0 & \square\square\square & 0 \\ 0 & \square & 0 \end{array}.$$

To add two noncontributing boxes to the second stretch of μ_7, we add two copies of 7 and two copies of 6 to α; C_1 in this case consists of the lower subintervals $(6^1, 7^1, 8^1, 9^1)$, $(7^2, 8^2, 9^2, 10^2)$, $(8^3, 9^3, 10^3, 11^3)$. The resulting cascading sequence α' (where the added copies are in bold) consisting of the lower subintervals $(\mathbf{7}^1, 8^1, 9^1, 10^1, 11^1)$, $(\mathbf{7}^2, 8^2, 9^2, 10^2, 11^2)$, $(8^3, 9^3, 10^3, 11^3)$, $(\mathbf{6}^1, 7^1, 8^1, 9^1, 10^1)$, $(7^3, 8^2, 9^2, 10^2)$, $(8^4, 9^4, 10^4)$, $(6^2, 7^2, 8^1, 9^1)$, $(\mathbf{6}^3, 7^3, 8^3, 9^3)$ corresponds to the resulting rigged configuration whose seventh partition is

$$\begin{array}{rll} -2 & \square\square\square & 0 \\ & \square\square\square & 0 \end{array}.$$

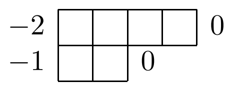
Example 3.6.6. Consider the cascading sequence α (with lanes marked by superscripts as usual) consisting of the lower subintervals $(7^1, 8^1, 9^1, 10^1, 11^1)$, $(8^2, 9^2, 10^2, 11^2)$, $(8^3, 9^3, 10^3, 11^3)$, $(6^1, 7^1, 8^1, 9^1, 10^1)$, $(7^2, 8^2, 9^2, 10^2)$, $(8^4, 9^4, 10^4)$, $(6^2, 7^2, 8^1, 9^1)$, $(7^3, 8^3, 9^3)$, which is a 7^*-plateau and an $(8, 2, 3)$-plateau. We have

$$\mu_7 = \begin{array}{rll} -1 & \square\square\square & 0 \\ -1 & \square\square & 0 \end{array}$$

and

$$\mu_8 = \begin{array}{rll} -2 & \square\square\square\square & 0 \\ -2 & \square\square\square & 0 \\ -1 & \square & 0 \end{array}.$$

To add one noncontributing box to the third stretch (which has length 1, since $|\mu_8^1| - |\mu_7^1| = 1$) of μ_7, we add one copy of 7 to α; C_0 in this case consists of the lower subintervals $(7^2, 8^2, 9^2, 10^2)$, $(7^3, 8^3, 9^3)$, $(8^3, 9^3, 10^3, 11^3)$, $(8^4, 9^4, 10^4)$. The resulting cascading sequence α' (where the added copies are in bold) consisting of the lower subintervals $(7^1, 8^1, 9^1, 10^1, 11^1)$, $(\mathbf{7}^2, 8^2, 9^2, 10^2, 11^2)$, $(8^3, 9^3, 10^3, 11^3)$, $(6^1, 7^1, 8^1, 9^1, 10^1)$, $(7^3, 8^2, 9^2, 10^2)$, $(8^4, 9^4, 10^4)$, $(6^2, 7^2, 8^1, 9^1)$, $(7^4, 8^3, 9^3)$ corresponds to the resulting rigged configuration whose seventh partition is



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Example 3.6.7. Consider the cascading 11-sequence consisting of the lower subintervals $(8^1, 9^1, 10^1, 11^1)$, $(8^2, 9^2, 10^2, 11^2)$, $(8^3, 9^3, 10^3, 11^3)$, $(8^4, 9^4, 10^4, 11^4)$, $(7^1, 8^1, 9^1, 10^1)$, $(7^2, 8^2, 9^2, 10^2)$, $(7^3, 8^3, 9^3, 10^3)$, $(8^5, 9^5, 10^5)$, $(6^1, 7^1, 8^1, 9^1)$, $(6^2, 7^2, 8^2, 9^2)$, $(6^3, 7^3, 8^3, 9^3)$, $(7^4, 8^4, 9^4)$, $(7^5, 8^5, 9^5)$, $(5^1, 6^1, 7^1, 8^1)$, $(6^4, 7^4, 8^4)$.

To add boxes to the stretch of height two, notice that C_2 consists of the lower subintervals $(5^1, 6^1, 7^1, 8^1)$, $(6^3, 7^3, 8^3, 9^3)$, $(7^3, 8^3, 9^3, 10^3)$, $(8^4, 9^4, 10^4, 11^4)$. If we add three boxes to the stretch of height two, we obtain $(\mathbf{7}^1, 8^1, 9^1, 10^1, 11^1)$, $(\mathbf{7}^2, 8^2, 9^2, 10^2, 11^2)$, $(\mathbf{7}^3, 8^3, 9^3, 10^3, 11^3)$, $(8^4, 9^4, 10^4, 11^4)$, $(\mathbf{6}^1, 7^1, 8^1, 9^1, 10^1)$, $(\mathbf{6}^2, 7^2, 8^2, 9^2, 10^2)$, $(7^4, 8^3, 9^3, 10^3)$, $(8^5, 9^5, 10^5)$, $(\mathbf{5}^1, 6^1, 7^1, 8^1, 9^1)$, $(\mathbf{5}^2, 6^2, 7^2, 8^2, 9^2)$, $(6^3, 7^3, 8^3, 9^3)$, $(\mathbf{6}^4, 7^4, 8^4, 9^4)$, $(7^5, 8^5, 9^5)$, $(5^3, 6^3, 7^3, 8^1)$, $(\mathbf{5}^4, 6^4, 7^4, 8^4)$.

To add boxes to the stretch of height one, notice that C_1 consists of the lower subintervals $(6^3, 7^3, 8^3, 9^3)$, $(6^4, 7^4, 8^4, 9^4)$, $(6^1, 7^1, 8^1, 9^1, 10^1)$, $(6^2, 7^2, 8^2, 9^2, 10^2)$, $(7^4, 8^3, 9^3, 10^3)$, $(7^1, 8^1, 9^1, 10^1, 11^1)$, $(7^2, 8^2, 9^2, 10^2, 11^2)$, $(7^3, 8^3, 9^3, 10^3, 11^3)$, $(8^4, 9^4, 10^4, 11^4)$. If we add one box to the stretch of height one, we get $(7^1, 8^1, 9^1, 10^1, 11^1)$, $(7^2, 8^2, 9^2, 10^2, 11^2)$, $(7^3, 8^3, 9^3, 10^3, 11^3)$, $(8^4, 9^4, 10^4, 11^4)$, $(6^1, 7^1, 8^1, 9^1, 10^1)$, $(6^2, 7^2, 8^2, 9^2, 10^2)$, $(7^4, 8^3, 9^3, 10^3)$, $(\mathbf{7}^5, 8^5, 9^5, 10^5)$, $(5^1, 6^1, 7^1, 8^1, 9^1)$, $(5^2, 6^2, 7^2, 8^2, 9^2)$, $(6^3, 7^3, 8^3, 9^3)$, $(6^4, 7^4, 8^4, 9^4)$, $(\mathbf{6}^5, 7^5, 8^5, 9^5)$, $(5^3, 6^3, 7^3, 8^1)$, $(5^4, 6^4, 7^4, 8^4)$.

Procedure 3.6.8 (Adding Contributing Boxes to a Given Stretch). Suppose $\mathrm{ht}(z_i) < r$.

1. To add n_i contributing boxes beneath z_i, where $0 \le n_i \le |z_i|$: Add n_i copies of the lower subinterval $(p - \mathrm{ht}(z_i) - 1, p - \mathrm{ht}(z_i), \ldots, p - 1)$ to the right of β.
2. Exceptional Case: We can add any number $N \in \mathbb{Z}_{\ge 0}$ of contributing boxes to the top row of μ_{p-1} by adding N singleton lower subintervals $(p - 1)$ to the right of β.

That Procedure 3.6.8 works as described will be proven in Section 3.7.

3.7. Proof of the procedures for adding boxes

We now show that the two procedures for adding boxes to a stretch works as described.

Lemma 3.7.1. *Suppose α is a (p,q,r)-plateau and a $(p-1)^*$-plateau, with corresponding rigged configurations $R' = (\nu_1, \ldots, \nu_{p-1}, \nu_p, \ldots, \nu_n)$. Then R' (and hence α) is completely determined once we know the rigged partitions $\nu_{p-1}, \nu_p, \ldots, \nu_n$.*

Proof. Follows from Lemma 3.5.2. □

Remark 3.7.2. Hence, if we keep $\nu_p, \ldots, \nu_n$ fixed, the range of all possible such α is completely determined by the range of all possible ν_{p-1} (obtained by adding boxes to the allowed stretches of $\overline{\nu}_p$).

For any cascading sequence α with corresponding rigged configuration R, let $\alpha[i]$ denote the sub-cascading sequence formed by the first i lower subintervals of α, and let $R[i]$ denote the rigged configuration corresponding to $\alpha[i]$; we call $\alpha[i]$ the **initial i-segment of** α. For any lower subinterval I of α, let α_I denote the portion of α preceding I, and let R_I denote the rigged configuration corresponding to α_I. The following lemmas show that, if α is an l-plateau, then it has certain nice properties, which we will use in the proof of the main lemma of this section.

Lemma 3.7.3. *Suppose that α is a cascading sequence with corresponding rigged configuration R such that $R[i] = (\mu_1, \mu_2, \ldots, \mu_n)$ has the property that the jth column of $\overline{\mu_k}$ is shorter than the jth column of μ_{k-1} for all $j \geq b$ for some b. Then $R[i+1] = (\mu'_1, \mu'_2, \ldots, \mu'_n)$ has the property that the jth column of $\overline{\mu'_k}$ is shorter than the jth column of μ'_{k-1} for all $j \geq c$ for some $c \geq b$.*

Remark 3.7.4. This lemma is used to prove Lemma 3.7.5.

Proof. Let I denote the lower subinterval after $\alpha[i]$. We may assume that I has head at most k. If I has head k, then it adds a box to the first row of μ_k, and the conclusion is still true for $c = b$. Suppose I has head less than k. Suppose I adds a box to the pth column of μ_{k-1}. Then it must add a box to the qth column of μ_k, where $p \geq q$. If $q < b$, then the conclusion is still true for $c = b$. Suppose that $q \geq b$. Then the conclusion is true for $c = p$. □

Lemma 3.7.5. *Suppose that α is an l-plateau with corresponding rigged configuration $R = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then no lower subinterval of α with head less than l can contain the head of some l-lane.*

Proof. Suppose for a contradiction that α has a lower subinterval J with head less than l containing the head of some l-lane. Without loss of generality assume that J is the leftmost such lower subinterval. Let $\alpha[i]$ denote the portion of α preceding J, and let $R[i] = (\mu_1, \mu_2, \ldots, \mu_n)$ denote the corresponding rigged configuration. By definition, J adds a box to the top row of μ_l. By Lemma [18], we have $\mu_{l-1} \subset \mu_l$, so J must add a box to the top row of μ_{l-1} as well; in fact, the top rows of μ_{l-1} and μ_l must be identical. Then, again by Lemma [18], $R[i+1] = (\nu_1, \nu_2, \ldots, \nu_n)$ has the property that the jth column of $\overline{\nu_l}$ is shorter than the jth column of ν_{l-1} for all $j \geq m$, where m is the length of the top row of ν_{l-1}. Repeatedly applying Lemma 3.7.3, we conclude that the dth column of $\overline{\lambda_l}$ is shorter than the dth column of λ_{l-1}, for some d. This contradicts the l-plateau assumption on α. □

Lemma 3.7.6. *Suppose that α is an l-plateau. Then the following hold:*

1. *$\alpha[i]$ is also an l-plateau for any i.*
2. *For any lower subinterval I of $\alpha[i]$, the entries $l-1$ and l of I have the same lane number.*

Proof. Notice that, by definition, I must contain l whenever it contains an entry less than l. The two items are vacuously true if every lower subinterval of $\alpha[i]$ has head greater than l. In the base case that $\alpha[i]$ ends in J, where J is the first lower subinterval of α with head at most l, J must contain the head of some l-lane, so J must have head l by Lemma 3.7.5, and hence the two items hold for $\alpha[i]$. Now we suppose that the two items hold for $\alpha[i]$, and prove them for $\alpha[i+1]$. Let I denote the lower subinterval following $\alpha[i]$. If I has head at least l, clearly the two items hold true (vacuously true for the second item). Suppose I has head less than l. By Lemma 3.7.5, I must add a box beneath the top row of the lth partition. Since the first item holds for $\alpha[i]$, the $(l-1)$st partition is exactly the portion of the lth partition beneath the first row, in the corresponding rigged configuration. If I adds a box to row r^{l-1} of the $(l-1)$st partition (where r^{l-1} is the uppermost row of its length), then it must add a box to row r^l of the lth partition, where r^l is the uppermost row of the lth partition with length $|r^{l-1}|$. This shows that $\alpha[i+1]$ is still an l-plateau, and that the entries $l-1$ and l of I have the same lane number $|r^{l-1}|+1$, completing the induction. □

Lemma 3.7.7. *Suppose that α is an l-plateau. For any i and any $j < l$, $\alpha[i]$ has no fewer lower subintervals with head l than lower subintervals with head j. We say that α satisfies the **Lyndon property** for letter l.*

Remark 3.7.8. In particular, α can be considered as a left Lyndon word in the letters l and j.

Proof. Fix i and $j < l$. By Lemma 3.7.6, $\alpha[i]$ is also an l-plateau. Let $R[i] = (\nu_1, \nu_2, \ldots, \nu_n)$ denote the rigged configuration corresponding to $\alpha[i]$. By Lemma 3.2.4 and by the definition of an l-plateau, we have $\nu_j \subset \nu_{j+1} \subset \ldots \subset \nu_l$. By Lemma 3.7.5, any lower subinterval that adds a box to the first row of the lth partition must have head l. Since any lower subinterval of $\alpha[i]$ with head j adds a box to the top row of the jth partition and any lower subinterval of $\alpha[i]$ with head l adds a box to the top row of the lth partition, it follows that $\alpha[i]$ has no fewer lower subintervals with head l than lower subintervals with head j. □

Lemma 3.7.9. *Suppose that α satisfies the Lyndon property for all letters at most l. If I is a lower subinterval of α with head $j \leq l$, then all entries of I at most l have the same lane number, and the depth of entry l' is $l' - j + 1$ for any $j \leq l' \leq l$.*

Proof. The claim is obvious for the base case of the initial segment $\alpha[i_1]$ ending in the first lower subinterval with head l. Now suppose that I is a lower subinterval of α with head $j \leq l$, and suppose that the claim holds for α_I. We show that I satisfies the desired properties. Let I' denote the lower subinterval of α_I with head j nearest I; if I' does not exist, then α_I has no lower subinterval with head smaller than $j + 1$, so all entries of I at most l have lane number one and the claim follows immediately (by the inductive hypothesis, the depth of entry $c \leq l$ of I is one more than the depth of entry c of a lower subinterval with head $j + 1$). By the inductive hypothesis, all entries of I' at most l have the same lane number k, and the depth of entry l' is $l' - j + 1$ for any $j \leq l' \leq l$. Fix $j \leq l' \leq l$. We show that entry l' of I has lane number $k + 1$ and depth $l' - j + 1$. Let I'' denote any lower subinterval of α_I after I'. By definition, I'' has head $j'' \neq j$. Let $j \leq d \leq l'$. If $j'' < j$, then entry d of I'' has depth greater than that of entry d of I' by inductive hypothesis. If $j'' > j$, then entry d of I'' has depth less than that of entry d of I' by inductive hypothesis. It follows that the number of d-lanes of length at least $d - j + 1$ in α_I is k; equivalently, the dth partition of the rigged configuration corresponding to α_I has exactly k columns of height at least $d - j + 1$. Recall that $d - j + 1$ is the depth of entry d of I'. Moreover, since α_I has more lower subintervals with head $j + 1$ than those with head j by the Lyndon property, α_I has some d-lanes of length $d - (j + 1) + 1 = d - j$. Since the head j of I clearly has lane number $k + 1$, we deduce that the entry d of I must also have lane number $k + 1$ and depth $d - j + 1$, for $d = j, j + 1, \ldots, l'$ in that order, by Lemma 3.2.4. This completes the induction. □

We now complete the proof of the two procedures for adding boxes. If a cascading sequence γ has entry g and lanes L, L' such that $L' = L \oplus (g)$, we say that L' is the **lengthening** of L, and that we **lengthen** L to obtain L'.

Lemma 3.7.10 (Main Lemma)**.** *Let β be both a $(p-1)^*$-plateau and a (p,q,r)-plateau, whose corresponding rigged configuration is $R = (\mu_1, \mu_2, \ldots, \mu_n)$. Label the stretches of μ_{p-1} from bottom to top as $z_1, z_2, \ldots, z_a$. Then the following is true.*

1. *Suppose $\mathrm{ht}(z_i) < \min(q,r)$. Let $\beta^{@}$ denote the cascading sequence obtained from β via Procedure 3.6.2. $\beta^{@}$ thus obtained is a $(p-1)^*$-plateau and a $(p, \mathrm{ht}(z_i), r)$-plateau, and $\beta^{@}$ corresponds to the rigged configuration obtained after adding n_i noncontributing boxes beneath z_i, where $0 \le n_i \le |z_i|$, and fixing μ_x for all $x \ge p$.*
2. *Suppose $\mathrm{ht}(z_i) < \min(q,r)$, and let $0 < a' \le \mathrm{ht}(z_i)$ be an integer.*
   - *(a) There exists j_1 such that $\beta[j_1]$ contains all the lower subintervals with head $p-1$ containing the head of some p-lane, and that $\beta[j_1]$ contains no more lower subintervals with head $p-1$ than lower subintervals with head $p-2$. Let such j_1 be minimal. Then there exists $j_2 \ge j_1$ such that $\beta[j_2]$ contains no more lower subintervals with head $p-2$ than lower subintervals with head $p-3$. Let such j_2 be minimal. Then there exists $j_3 \ge j_2$ such that $\beta[j_3]$ contains no more lower subintervals with head $p-3$ than lower subintervals with head $p-4$. This continues until, for minimal $j_{a'-1}$ there exists $j_{a'} \ge j_{a'-1}$ such that $\beta[j_{a'}]$ contains no more lower subintervals with head $p-a'$ than lower subintervals with head $p-a'-1$.*
   - *(b) If I is an element of B_{p-h} outside $\beta[j_{p-(p-h)}] = \beta[j_h]$ where $1 \le h \le \mathrm{ht}(z_i)$, then I has entry p of depth $p-(p-h)+1 = h+1$, entry $p-1$ of depth $p-1-(p-h)+1 = h$, and one common lane number for all entries not exceeding p.*
   - *(c) If $\widehat{I}$ is an element of B_{p-1} containing the head of some p-lane, then all elements of B_p in $\beta_{\widehat{I}}$ are elements of $C_{\mathrm{ht}(z_i)}$.*
3. *Suppose $\mathrm{ht}(z_i) < r$.*
   - *(a) Suppose $\beta^!$ is the cascading sequence obtained from β via the Procedure 3.6.8(1). $\beta^!$ thus obtained is a $(p-1)^*$-plateau and a $(p, q, \mathrm{ht}(z_i))$-plateau, and $\beta^!$ corresponds to the rigged configuration obtained after adding n_i contributing boxes beneath z_i in μ_{p-1}, where $0 \le n_i \le |z_i|$, and fixing μ_x for all $x > p$ as well as fixing the shape of μ_p.*
   - *(b) Suppose $\beta^!$ is the cascading sequence obtained from β via the Procedure 3.6.8(2). Then $\beta^!$ thus obtained is a $(p-1)^*$-plateau and a $(p,q,0)$-plateau, and corresponds to the rigged configuration obtained after adding any number $N \in \mathbb{Z}_{\ge 0}$ of contributing boxes to*

the top row of μ_{p-1}, and fixing μ_x for all $x > p$ as well as fixing the shape of μ_p.

Remark 3.7.11. Keep the following in mind for the proof that follows.

1. Item 2 is a technical fact about β that will be used in the proof of Procedure 3.6.2 (i.e. Item 1) in an induction argument.
2. Although adding boxes beneath z_i changes the partition, it does not change the stretches z_x for any $x > i$, so we will continue referring to the stretches z_x even though they may well belong to a partition different from μ_{p-1}.
3. Convention: Let α and α' be cascading sequences with corresponding rigged configurations S and S', respectively. We say that $\alpha[i]$ and $\alpha'[i]$ *have the same l-lanes* or *have identical l-lanes* if the following holds: l appears as an entry in $\alpha[i]$ the same number of times as l appears as an entry in $\alpha'[i]$, and the jth occurrence of l in $\alpha[i]$ has the same lane number as the jth occurrence of l in $\alpha'[i]$. In particular, this implies that the lth partitions of $S[i]$ and $S'[i]$ are identical.

Proof. We prove these items by induction on i (i.e. one stretch at a time by decreasing height); observe that conditions on β become less restrictive with smaller q and r, while the number of stretches beneath which boxes can be added decreases. In the base case where β is a p^*-plateau, Item 2 holds vacuously with $j_1 = 0$, by Lemma 3.7.9.

In the general case, suppose β is both a $(p-1)^*$-plateau and a (p, q, r)-plateau. We will prove Item 3, executability of Procedure 3.6.2, Item 1, and finally Item 2, in that order.

Proof of Item 3

We first prove Item 3, that Procedure 3.6.8 works as stated. Suppose $\mathrm{ht}(z_i) < r$. By Lemma 3.7.9, if I is a lower subinterval whose entry $p-1$ has depth $\mathrm{ht}(z_{i-1})$, then $I \in B_{p-\mathrm{ht}(z_{i-1})}$. Since $p - \mathrm{ht}(z_{i-1}) \leq p - \mathrm{ht}(z_i) - 1$, and since β is a $(p-1)^*$-plateau, the entry $p-1$ of any of the added $(p - \mathrm{ht}(z_i) - 1, p - \mathrm{ht}(z_i), \ldots, p-1)$ has depth not exceeding $p - 1 - (p - \mathrm{ht}(z_{i-1})) + 1 = \mathrm{ht}(z_{i-1})$. Notice that β has $|B_{p-\mathrm{ht}(z_i)}| = |B_{p-\mathrm{ht}(z_{i-1})}| + |z_i|$ and $|B_{p-\mathrm{ht}(z_i)-1}| = |B_{p-\mathrm{ht}(z_{i-1})}|$ by definition. Since $|B_{p-\mathrm{ht}(z_i)}| = |B_{p-\mathrm{ht}(z_i)-1}| + |z_i|$, and since β is a $(p-1)^*$-plateau, the entry $p-1$ of any of the added $(p - \mathrm{ht}(z_i) - 1, p - \mathrm{ht}(z_i), \ldots, p-1)$ has depth exceeding $p - 1 - (p - \mathrm{ht}(z_i)) + 1 = \mathrm{ht}(z_i)$. By Lemma 3.2.6, the entries $p-1$ of the n_i added copies of $(p - \mathrm{ht}(z_i) - 1, p - \mathrm{ht}(z_i), \ldots, p-1)$ must occupy n_i distinct columns, so it follows that these entries $p-1$ must have depth $\mathrm{ht}(z_i) + 1$, as desired. Clearly, $\beta^!$ is a $(p-1)^*$-plateau as well.

That Part 2 of this procedure works for adding boxes to the top row is obvious from Lemma 3.2.4. Finally, the property of Item 2 is clearly preserved by Procedure 3.6.8. This concludes our proof of Item 3.

Now assume that $\mathrm{ht}(z_i) < \min(q, r)$, for which Item 2 holds for β. We show that Items 1 and 2 hold for $\beta^{@}$. Let $\widehat{B}_{p-1}$ denote the set of elements of B_{p-1} containing the head of some p-lane. Let $B_v^{@}$ denote the number of lower subintervals of $\beta^{@}$ with head v. Let $\widehat{B}_{p-1}^{@}$ denote the set of elements of $B_{p-1}^{@}$ containing the head of some p-lane.

Proof of Executability

We first prove that Procedure 3.6.2 is executable. By Lemma 3.7.7, it immediately follows that the deletions (recall that a deleted lower subinterval is ultimately fixed by the procedure) and lengthening specified in Procedure 3.6.2 are executable for all pairs B_{p-j}, B_{p-j-1} for all $j \geq 1$. More precisely, by Lemma 3.7.9, after deleting all the elements of $C_{\mathrm{ht}(z_i)}$ from $B_{p-\mathrm{ht}(z_i)}$, there will be exactly $|z_i|$ elements of $B_{p-\mathrm{ht}(z_i)}$ remaining; after deleting all the elements of $C_{\mathrm{ht}(z_i)}$ from $B_{p-\mathrm{ht}(z_i)+1}$, there will be at least $|z_i|$ elements of $B_{p-\mathrm{ht}(z_i)+1}$ remaining; after deleting all the elements of $C_{\mathrm{ht}(z_i)}$ from $B_{p-\mathrm{ht}(z_i)+2}$, there will be at least $|z_i|$ elements of $B_{p-\mathrm{ht}(z_i)+2}$ remaining; and so on.

We now verify that the deletion and lengthening are executable for the pair B_p, B_{p-1}. By Item 2 and Lemma 3.7.7, all elements of $\widehat{B}_{p-1}$ are elements of $C_{\mathrm{ht}(z_i)}$. By Lemma 3.7.9, no element of B_l contains the head of some p-lane or the head of some $(p-1)$-lane, for any $l < p-1$. Hence only the elements of $\widehat{B}_{p-1}$ and B_p contain the head of some p-lane. It follows that any initial segment of β contains no fewer elements of B_p than elements of $B_{p-1} - \widehat{B}_{p-1}$; if an initial segment contained fewer elements of B_p than elements of $B_{p-1} - \widehat{B}_{p-1}$, then some of the latter elements would have to belong to $\widehat{B}_{p-1}$, a contradiction.

Let $\tilde{I} \in \widehat{B}_{p-1}$. By Lemma 3.2.4, $\beta_{\tilde{I}}$ has the same number of p-lanes and $(p-1)$-lanes. By Item 2, every $I_1 \in B_p$ in $\beta_{\tilde{I}}$ belongs to $C_{\mathrm{ht}(z_i)}$. If $\tilde{I}$ is the leftmost element of $\widehat{B}_{p-1}$, then clearly every $I_1 \in B_p$ in $\beta_{\tilde{I}}$ must be paired with an $I_1' \in B_{p-1} - \widehat{B}_{p-1}$ in $\beta_{\tilde{I}}$. In general, suppose that $\bar{I} \in \widehat{B}_{p-1}$ precedes $\tilde{I}$ in β, where every $I_1 \in B_p$ in $\beta_{\bar{I}}$ is paired with an $I_1' \in B_{p-1} - \widehat{B}_{p-1}$ in $\beta_{\bar{I}}$. Since $\beta_{\tilde{I}}$ has the same number of p-lanes and $(p-1)$-lanes, and since no elements of $\widehat{B}_{p-1}$ exist after $\bar{I}$ in $\beta_{\tilde{I}}$, B_p has the same number of elements after $\bar{I}$ in $\beta_{\tilde{I}}$ as those of B_{p-1} after $\bar{I}$ in $\beta_{\tilde{I}}$. Hence every $I_1 \in B_p$ after $\bar{I}$ in $\beta_{\tilde{I}}$ must be paired with an $I_1' \in B_{p-1} - \widehat{B}_{p-1}$ after $\bar{I}$ in $\beta_{\tilde{I}}$. Therefore, inductively we conclude that every element of $B_p \cap C_{\mathrm{ht}(z_i)}$ must be paired with exactly one element of $B_{p-1} - \widehat{B}_{p-1}$. This shows that, after deleting all the elements of $C_{\mathrm{ht}(z_i)}$ from B_p, there are at least $|z_i|$ elements remaining in B_p that can be lengthened, because $B_{p-1} - \widehat{B}_{p-1} \supset B_{p-1} - C_{\mathrm{ht}(z_i)}$. This shows that Procedure 3.6.2 is executable.

$\beta^{@}$ satisfies the Lyndon property

We first show that $\beta^{@}$ satisfies the Lyndon property for all letters $l \leq p-1$, so that we can apply Lemma 3.7.9 to $\beta^{@}$ later on. Since $\beta^{@}$ clearly satisfies the Lyndon property for all letters $l \leq p - \mathrm{ht}(z_i) - 1$, we only need to consider $l > p - \mathrm{ht}(z_i) - 1$. Let m be any positive integer. We analyze $\beta^{@}[m]$ using $\beta[m]$.

Suppose $p - \mathrm{ht}(z_i) - 1 < l' \leq p - 1$. By Lemma 3.7.7, $\beta[m]$ has no fewer elements of $B_{l'}$ than elements of $B_{l'-1}$.

Claim 3.7.12. $\beta[m]$ has no fewer elements of $B_{l'} - C_{\mathrm{ht}(z_i)}$ than elements of $B_{l'-1} - C_{\mathrm{ht}(z_i)}$.

Proof. Suppose not. Then we must have $|B_{l'} \cap C_{\mathrm{ht}(z_i)}| > |B_{l'-1} \cap C_{\mathrm{ht}(z_i)}|$ in $\beta[m]$. Pick the minimal m' such that $|B_{l'} \cap C_{\mathrm{ht}(z_i)}| = |B_{l'-1} \cap C_{\mathrm{ht}(z_i)}|$ in $\beta[m+m']$. By definition, any element of $B_{l'}$ in $\beta[m+m']$ outside $\beta[m]$ must be an element of $C_{\mathrm{ht}(z_i)}$ and must be paired with an element of $B_{l'-1} \cap C_{\mathrm{ht}(z_i)}$ to its right in $\beta[m+m']$, and $\beta[m+m']$ must end in an element of $B_{l'-1} \cap C_{\mathrm{ht}(z_i)}$. Notice that $|B_{l'} \cap C_{\mathrm{ht}(z_i)}| = |B_{l'-1} \cap C_{\mathrm{ht}(z_i)}|$ in $\beta[m+m']$ but $|B_{l'} - C_{\mathrm{ht}(z_i)}| < |B_{l'-1} - C_{\mathrm{ht}(z_i)}|$ in $\beta[m+m']$ due to the assumption that $|B_{l'} - C_{\mathrm{ht}(z_i)}| < |B_{l'-1} - C_{\mathrm{ht}(z_i)}|$ in $\beta[m]$. This implies that $\beta[m+m']$ has fewer elements of $B_{l'}$ than elements of $B_{l'-1}$, which contradicts the Lyndon property for β, and the claim is proved. $\square$

Suppose $l' = p-1$. By Item 2, we have $\widehat{B}_{p-1} \subset C_{\mathrm{ht}(z_i)}$, so by definition only elements of $B_{p-1} - C_{\mathrm{ht}(z_i)} \subset B_{p-1} - \widehat{B}_{p-1}$ can be lengthened by Procedure 3.6.2. By Lemma 3.2.4 and Lemma 3.7.5, any lower subinterval of β whose entry p has depth one must have either p or $p-1$ as head. It follows that $\beta[j]$ has no fewer elements of B_p than elements of $B_{p-1} - \widehat{B}_{p-1}$ for any j.

Claim 3.7.13. $\beta[m]$ has no fewer elements of $B_p - C_{\mathrm{ht}(z_i)}$ than elements of $B_{p-1} - C_{\mathrm{ht}(z_i)}$.

Proof. Suppose not. Then we must have $|B_p \cap C_{\mathrm{ht}(z_i)}| > |(B_{p-1} - \widehat{B}_{p-1}) \cap C_{\mathrm{ht}(z_i)}|$ in $\beta[m]$. Pick the minimal b' such that $|B_p \cap C_{\mathrm{ht}(z_i)}| = |(B_{p-1} - \widehat{B}_{p-1}) \cap C_{\mathrm{ht}(z_i)}|$ in $\beta[m+b']$. By Item 2, all of $\widehat{B}_{p-1}$ is contained inside $\beta[j_1]$, and $\beta[j_1]$ contains no elements of $B_{p-1} - C_{\mathrm{ht}(z_i)}$. Thus, we must have $j_1 < m$, and there are no elements of $\widehat{B}_{p-1}$ outside $\beta[m]$. By definition, any element of B_p in $\beta[m+b']$ outside $\beta[m]$ must be an element of

$C_{\mathrm{ht}(z_i)}$ and must be paired with an element of $B_{p-1} \cap C_{\mathrm{ht}(z_i)}$ to its right in $\beta[m+b']$. Notice that $|B_p \cap C_{\mathrm{ht}(z_i)}| = |(B_{p-1} - \widehat{B}_{p-1}) \cap C_{\mathrm{ht}(z_i)}|$ in $\beta[m+b']$ but $|B_p - C_{\mathrm{ht}(z_i)}| < |(B_{p-1} - \widehat{B}_{p-1}) - C_{\mathrm{ht}(z_i)}|$ in $\beta[m+b']$ due to the assumption that $|B_p - C_{\mathrm{ht}(z_i)}| < |B_{p-1} - C_{\mathrm{ht}(z_i)}|$ in $\beta[m]$. This implies that $|B_p| < |B_{p-1} - \widehat{B}_{p-1}|$ in $\beta[m+b']$, which is a contradiction, and the claim is proved. □

Now let $p - \mathrm{ht}(z_i) - 1 < l'' \leq p-1$. By Lemma 3.7.7 and definition of pairing, $|B_{l''} \cap C_{\mathrm{ht}(z_i)}| \geq |B_{l''-1} \cap C_{\mathrm{ht}(z_i)}|$ in $\beta[m]$. By Claim 3.7.12 for $l'' < p-1$ or Claim 3.7.13 for $l'' = p-1$, we have $|B_{l''+1} - C_{\mathrm{ht}(z_i)}| \geq |B_{l''} - C_{\mathrm{ht}(z_i)}|$ and $|B_{l''} - C_{\mathrm{ht}(z_i)}| \geq |B_{l''-1} - C_{\mathrm{ht}(z_i)}|$ in $\beta[m]$. After running Procedure 3.6.2, we compare $|B_{l''}|$ in $\beta[m]$ with $|B^{@}_{l''}|$ in $\beta^{@}[m]$ and $|B_{l''-1}|$ in $\beta[m]$ with $|B^{@}_{l''-1}|$ in $\beta^{@}[m]$. If $|B_{l''-1} - C_{\mathrm{ht}(z_i)}| \geq n_i$ in $\beta[m]$, then Procedure 3.6.2 forms $\beta^{@}[m]$ by lengthening n_i elements of $B_{l''+1} - C_{\mathrm{ht}(z_i)}$, n_i elements of $B_{l''} - C_{\mathrm{ht}(z_i)}$, and n_i elements of $B_{l''-1} - C_{\mathrm{ht}(z_i)}$ in $\beta[m]$, and thus $B_{l''}$ in $\beta[m]$, $B^{@}_{l''}$ in $\beta^{@}[m]$ are equinumerous and $B_{l''-1}$ in $\beta[m]$, $B^{@}_{l''-1}$ in $\beta^{@}[m]$ are equinumerous. If $|B_{l''-1} - C_{\mathrm{ht}(z_i)}| < n_i$ in $\beta[m]$, then Procedure 3.6.2 forms $\beta^{@}[m]$ by lengthening a_1 elements of $B_{l''+1} - C_{\mathrm{ht}(z_i)}$, a_2 elements of $B_{l''} - C_{\mathrm{ht}(z_i)}$, and all elements of $B_{l''-1} - C_{\mathrm{ht}(z_i)}$ in $\beta[m]$, where $a_2 \leq a_1$. Therefore, in all cases we have $|B^{@}_{l''}| \geq |B^{@}_{l''-1}|$ in $\beta^{@}[m]$. Since m was arbitrary, this completes the proof that $\beta^{@}$ satisfies the Lyndon property for all letters $l \leq p-1$.

Proof of Item 1

We show inductively that l-lanes of β and $\beta^{@}$ are identical for all $l \geq p$, by comparing β and $\beta^{@}$ one lower subinterval at a time, from left to right. In this case, given a lower subinterval or a portion of β, it will be obvious what we mean by the *corresponding lower subinterval* or *corresponding portion* of $\beta^{@}$, and vice versa.

Let G_1 denote the first lower subinterval of β to be lengthened, and let g'_1 be the lane number of the head p of G_1. By definition, the lanes of $\beta^{@}_{(G_1)_+}$ are identical to those of β_{G_1}. To determine the number of $(p-1)$-lanes in β_{G_1}, we need only determine the number of elements of B_{p-1} in β_{G_1}, since $\beta^{@}_{(G_1)_+}$ and β_{G_1} are identical, and since β_{G_1} is a $(p-1)$-plateau. By Lemma 3.2.4, any lower subinterval of $\beta^{@}_{(G_1)_+}$ whose entry p has depth one must have either p or $p-1$ as head, since $\beta^{@}_{(G_1)_+}$ is a $(p-1)$-plateau. Let J denote the rightmost element of $\widehat{B}_{p-1}$ in β_{G_1}. The entries $p-1, p$ of J have the same lane number, by Lemma 3.2.4. All elements of B_p in β_{G_1} right of J must be elements of $C_{\mathrm{ht}(z_i)}$ by definition, so they must be paired with the same number of elements of $B_{p-1} \cap C_{\mathrm{ht}(z_i)}$ in β_{G_1} (which by definition do not contain the head of any p-lane). Since β_{G_1} has the same number $g'_1 - 1$

of elements of B_{p-1} as those of B_p, it follows that $(G_1)_+$ has entry $p-1$ with lane number g'_1, so its entry p has lane number g'_1 as well.

Let G_d be a lower subinterval of $\beta^@$ and let G'_d denote the lower subinterval of β corresponding to G_d. For our inductive hypothesis suppose that the l-lanes of $\beta^@_{G_d}$ are identical to those of $\beta_{G'_d}$ for all $l \geq p$. We show that the action of G_d preserves this property; in fact, it suffices to show that entry p has the same lane number in G_d and G'_d. We treat separately the case that G'_d, G_d are identical and the case that $G_d = (G'_d)_+$. Recall that both β and $\beta^@$ contain a copy of $C_{\text{ht}(z_i)}$, the context will make it clear which copy we refer to.

Consider the case that $G_d = (G'_d)_+$ and $\min G'_d = p$. Let g^*_d be the lane number of the head p of G'_d. Since $\beta^@$ satisfies the Lyndon property, $\beta^@_{G_d}$ is a $(p-1)^*$-plateau by Lemma 3.7.9, so the number of $(p-1)$-lanes in $\beta^@_{G_d}$ is the number of elements of B_{p-1} in $\beta^@_{G_d}$. By Lemma 3.2.4, any lower subinterval of $\beta^@_{G_d}$ whose entry p has depth one must have either p or $p-1$ as head, since $\beta^@_{G_d}$ is a $(p-1)^*$-plateau. Let J_d denote the rightmost element of $\widehat{B}^@_{p-1}$ in $\beta^@_{G_d}$. The entries $p-1, p$ of J_d have the same lane number, by Lemma 3.2.4. All elements of $B^@_p$ in $\beta^@_{G_d}$ right of J_d must be elements of $C_{\text{ht}(z_i)}$ by definition (since they were fixed by Procedure 3.6.2), so they must be paired with the same number of elements of $B_{p-1} \cap C_{\text{ht}(z_i)}$ in $\beta^@_{G_d}$ (which by definition do not contain the head of any p-lane). Since $\beta^@_{G_d}$ has the same number $g^*_d - 1$ of $(p-1)$-lanes as p-lanes, it follows that G_d has entry $p-1$ with lane number g^*_d, so its entry p has lane number g^*_d as well by the inductive hypothesis.

Consider the case that $G_d = (G'_d)_+$ and $\min G'_d \leq p-1$. Let g_d be the lane number of the head of G'_d. Since $\beta^@$ satisfies the Lyndon property, $\beta^@_{G_d}$ is a $(p-1)^*$-plateau by Lemma 3.7.9, so the number of $(\min G'_d - 1)$-lanes in $\beta^@_{G_d}$ is the number of elements of $B_{\min G_d}$ in $\beta^@_{G_d}$. Similarly, the number of $(\min G'_d)$-lanes in $\beta_{G'_d}$ is the number of elements of $B_{\min G'_d}$ in $\beta_{G'_d}$. The elements of $B_{\min G'_d} - C_{\text{ht}(z_i)}$ in $\beta_{G'_d}$ are lengthened by Procedure 3.6.2, while the elements of $B_{\min G'_d} \cap C_{\text{ht}(z_i)}$ in $\beta^@_{G_d}$ are paired with the same number of elements of $B_{\min G'_d - 1} \cap C_{\text{ht}(z_i)}$ in $\beta^@_{G_d}$ by definition. In addition, all elements of $B_{\min G'_d - 1} - C_{\text{ht}(z_i)}$ in $\beta_{G'_d}$ are lengthened by Procedure 3.6.2 by Claim 3.7.12, since $|B_{\min G'_d} - C_{\text{ht}(z_i)}| < n_i$ in $\beta_{G'_d}$. Since $\beta^@_{G_d}$ has the same number $g_d - 1$ of elements of $B^@_{\min G'_d - 1}$ as elements of $B_{\min G'_d}$ in $\beta_{G'_d}$, it follows that G_d has entry $\min G_d = \min G'_d - 1$ with lane number g_d, so its entry $\min G'_d$ has lane number g_d as well by the inductive hypothesis.

Consider the case that G'_d, G_d are identical and $\min G_d < p - \text{ht}(z_i) - 1$. Since Procedure 3.6.2 lengthens no lower subintervals with head smaller than $p - \text{ht}(z_i)$, the $(\min G_d)$-lanes are identical in $\beta^@_{G_d}$ and $\beta_{G'_d}$. Thus, the entry $\min G_d$ has the same lane number h_d in G_d and G'_d. By Lemma 3.7.9, all entries of G_d not exceeding $p-1$ have lane number h_d, and all entries of G'_d not exceeding $p-1$ have lane number h_d. Since the p-lanes are identical in $\beta^{@}_{G_d}$ and $\beta_{G'_d}$ by the inductive hypothesis, entry p has the same lane number in G_d and G'_d.

Consider the case that G'_d, G_d are identical and $\min G_d \geq p - \mathrm{ht}(z_i) - 1$. By the inductive hypothesis, the l-lanes are identical in $\beta^{@}_{G_d}$ and $\beta_{G'_d}$ for all $l \geq p$, so we only need to consider the case $\min G_d \leq p-1$. We now apply Item 2, with $a' = \mathrm{ht}(z_i)$.

If $\min G'_d = p-1$, then entry p of G'_d must have depth one or two, by Lemma 3.2.4. Suppose that $\min G'_d = p-1$ and entry p has depth one. Then G'_d must lie inside $\beta[j_{p-\min G'_d}]$, and by Item 2 no element of $B_p \cup B_{p-1}$ in $\beta_{G'_d}$ can be lengthened by Procedure 3.6.2, so the entry $p-1$ has the same lane number in G_d and G'_d, and hence the entry p also has the same lane number in G_d and G'_d by the inductive hypothesis.

Suppose that $\min G'_d = p-1$ and entry p has depth two. Then $\beta_{G'_d}$ must have fewer $(p-1)$-lanes than p-lanes. By Claim 3.7.13, $|B_p - C_{\mathrm{ht}(z_i)}| \geq |B_{p-1} - C_{\mathrm{ht}(z_i)}|$ in $\beta_{G'_d}$. If $|B_{p-1} - C_{\mathrm{ht}(z_i)}| \geq n_i$ in $\beta_{G'_d}$, then Procedure 3.6.2 lengthens n_i elements of B_p and n_i elements of B_{p-1} in $\beta_{G'_d}$, so the number of $(p-1)$-lanes in $\beta^{@}_{G_d}$ equals that of $(p-1)$-lanes in $\beta_{G'_d}$ and remains less than that of p-lanes in $\beta^{@}_{G_d}$, and hence entry p of G_d has depth two. In the case $|B_{p-1} - C_{\mathrm{ht}(z_i)}| < n_i$ in $\beta_{G'_d}$, we must have $G'_d \in C_{\mathrm{ht}(z_i)}$. If $G'_d \in \widehat{B}_{p-1}$, then no element of B_p in $\beta_{G'_d}$ is lengthened, by Item 2, so the number of $(p-1)$-lanes does not increase. Suppose $G'_d \in (B_{p-1} \cap C_{\mathrm{ht}(z_i)}) - \widehat{B}_{p-1}$. Since each $I \in \widehat{B}_{p-1}$ in $\beta_{G'_d}$ contributes a new $(p-1)$-lane and a new p-lane, we can exclude $\widehat{B}_{p-1}$ from consideration. Procedure 3.6.2 fixes the elements of $C_{\mathrm{ht}(z_i)}$, and lengthens all elements of $B_{p-1} - C_{\mathrm{ht}(z_i)}$ in $\beta_{G'_d}$ and some elements of $B_p - C_{\mathrm{ht}(z_i)}$ in $\beta_{G'_d}$, and we have $|(B_{p-1} \cap C_{\mathrm{ht}(z_i)}) - \widehat{B}_{p-1}| < |B_p \cap C_{\mathrm{ht}(z_i)}|$ in $\beta_{G'_d}$ since $G'_d \in C_{\mathrm{ht}(z_i)}$ lies outside $\beta_{G'_d}$. It follows that the number of $(p-1)$-lanes in $\beta^{@}_{G_d}$ is again less than that of p-lanes in $\beta^{@}_{G_d}$, and hence entry p of G_d has depth two. Thus, in both cases entry p of G_d has depth two, so it has the same lane number in G_d and G'_d by the inductive hypothesis.

Suppose $\min G'_d < p-1$ and all elements of $B_{\min G'_d+1}$ in $\beta_{G'_d}$ belong to $C_{\mathrm{ht}(z_i)}$. Then all elements of $B_{\min G'_d}$ in $\beta_{G'_d}$ must belong to $C_{\mathrm{ht}(z_i)}$ as well, by Claim 3.7.12. It follows that no element of $B_{\min G'_d+1}$ in $\beta_{G'_d}$ can be lengthened by Procedure 3.6.2, and no element of $B_{\min G'_d}$ in $\beta_{G'_d}$ can be lengthened by Procedure 3.6.2, so the number of elements of $B_{\min G'_d}$ is the same in $\beta^{@}_{G_d}$ and $\beta_{G'_d}$. Thus, the entries $\min G'_d \leq l \leq p-1$ have the same lane number k_d in G_d and G'_d by Lemma 3.7.9, so the entry p also has the same lane number in G_d and G'_d by the inductive hypothesis.

Suppose $\min G'_d < p-1$ and $\beta_{G'_d}$ contains some element $H \in B_{\min G'_d+1} - C_{\mathrm{ht}(z_i)}$. Let H^* be the element of $B_{\min G'_d+1}$ left of G'_d. By Item 2, H must lie outside $\beta[j_{p-(\min G'_d+1)}] = \beta[j_{p-\min G'_d-1}]$, so H^* must lie outside $\beta[j_{p-\min G'_d-1}]$ as well. Applying the second part of Item 2, we know that all entries of H^* not exceeding p have the same lane number d^*, H^* has entry p of depth $p-(\min G'_d+1)+1 = p-\min G'_d$, and H^* has entry $p-1$ of depth $p-\min G'_d-1$. By Lemma 3.7.9, all entries of G'_d not exceeding $p-1$ have a common lane number l_1, and all entries of G_d not exceeding $p-1$ have a common lane number l_2. By Lemma 3.7.7, we have $l_1 \leq d^*$, so $l_1 < d^*+1$. By Lemma 3.2.4, in G'_d the depth of entry p exceeds that of entry $p-1$ by at most one, meaning that entry p of G'_d has depth at most $(p-\min G'_d)+1$ (since G'_d has entry $p-1$ of depth $(p-\min G'_d-1)+1 = p-\min G'_d$ by Lemma 3.7.9). By Claim 3.7.12, Procedure 3.6.2 lengthens no fewer elements of $B_{\min G'_d+1}$ than elements of $B_{\min G'_d}$ in $\beta_{G'_d}$. Looking at $\beta^{@}_{G_d}$, this means that $l_2 \geq l_1$, and hence the entry p of G_d has depth not exceeding that of the entry p of G'_d, by the inductive hypothesis. On the other hand, we now show that entry p of G_d has depth at least $p-\min G'_d+1$. If $|B_{\min G'_d} - C_{\mathrm{ht}(z_i)}| \geq n_i$ in $\beta_{G'_d}$, then Procedure 3.6.2 lengthens n_i elements of $B_{\min G'_d+1}$ and n_i elements of $B_{\min G'_d}$ in $\beta_{G'_d}$, so we must have $l_2 = l_1 \leq d^*$. If $|B_{\min G'_d} - C_{\mathrm{ht}(z_i)}| < n_i$ in $\beta_{G'_d}$, then $G'_d \in C_{\mathrm{ht}(z_i)}$, $|(B_{\min G'_d} \cap C_{\mathrm{ht}(z_i)})| < |B_{\min G'_d+1} \cap C_{\mathrm{ht}(z_i)}|$ in $\beta_{G'_d}$ since G'_d lies outside $\beta_{G'_d}$, and Procedure 3.6.2 lengthens all elements of $B_{\min G'_d} - C_{\mathrm{ht}(z_i)}$ in $\beta_{G'_d}$ and some elements of $B_{\min G'_d+1} - C_{\mathrm{ht}(z_i)}$ in $\beta_{G'_d}$, so we must have $l_2 - 1 < d^*$ as well. Thus, in both cases the entry p of G_d has depth exceeding $p-\min G'_d$. This shows that entry p has depth exactly $p-\min G'_d+1$ in both G'_d and G_d, so its lane number is the same in both G'_d and G_d by the inductive hypothesis. This completes the proof that the l-lanes are identical in β and $\beta^{@}$ for all $l \geq p$.

Since $\beta^{@}$ satisfies the Lyndon property, $\beta^{@}$ is a $(p-1)^*$-plateau. We now show that $\beta^{@}$ is indeed obtained from β by lengthening n_i $(p-1)$-lanes of length $\mathrm{ht}(z_i)$. Any initial segment $\beta^{@}[j]$ contains no fewer lower subintervals with head l than those with head $l-1$, for all $l \leq p-1$. Notice that, compared to β, $\beta^{@}$ has the same number of elements of B_m for all $m > p-\mathrm{ht}(z_i)$ and $m < p-\mathrm{ht}(z_i)-1$, but has n_i more elements of $B_{p-\mathrm{ht}(z_i)-1}$ and n_i fewer elements of $B_{p-\mathrm{ht}(z_i)}$. By Lemma 3.7.9, this shows that $\beta^{@}$ corresponds to the rigged configuration obtained after adding n_i noncontributing boxes beneath z_i. It follows that in particular $\beta^{@}$ is a $(p, \mathrm{ht}(z_i), r)$-plateau.

Proof of Item 2

Finally, we prove that Item 2 holds for $\beta^{@}$ for any integer $0 < s' < \min(\mathrm{ht}(z_i), r)$. Since β is a $(p-1)^*$-plateau, any lower subinterval containing the head of some p-lane must have head p or $p-1$. By Claim 3.7.13, if

$j_1^@ \geq j_1$ is minimal such that $\beta^@[j_1^@]$ contains all n_i added copies of $p-2$, then $\beta^@[j_1^@]$ contains all the lower subintervals with head $p-1$ containing the head of some p-lane, and $\beta^@[j_1^@]$ contains no more lower subintervals with head $p-1$ than lower subintervals with head $p-2$; by definition, any $I_1 \in B_{p-1} \cap C_{\mathrm{ht}(z_i)}$ inside $\beta[j_1^@]$ must be paired with an $I_2 \in B_{p-2} \cap C_{\mathrm{ht}(z_i)}$ after I_1 in $\beta[j_1^@]$, so that $\beta^@[j_1^@]$ contains both I_1, I_2. By Claim 3.7.12, for all $1 < d \leq s'$, if $j_d^@ \geq \max(j_d, j_{d-1}^@)$ is minimal such that $\beta^@[j_d^@]$ contains all n_i added copies of $p-d-1$, then $\beta^@[j_d^@]$ contains no more lower subintervals with head $p-d$ than lower subintervals with head $p-d-1$; by definition, any $J_1 \in B_{p-d} \cap C_{\mathrm{ht}(z_i)}$ inside $\beta[j_d^@]$ must be paired with a $J_2 \in B_{p-d-1} \cap C_{\mathrm{ht}(z_i)}$ after J_1 in $\beta[j_d^@]$, so that $\beta^@[j_d^@]$ contains both J_1, J_2.

Let I' be an element of $B_{p-h}^@$ outside $\beta^@[j_h^@]$, where $1 \leq h \leq s'$. Denote by I^* the lower subinterval of β corresponding to I'. By definition, Procedure 3.6.2 must have added all n_i copies of $p-h$ and all n_i copies of $p-h-1$ to $\beta[j_h^@]$, so I' must be identical to I^* (since I' was not obtained by lengthening I^*). Since I^* must lie outside $\beta[j_h]$, I^* has entry p of depth $p-(p-h)+1 = h+1$, entry $p-1$ of depth $p-1-(p-h)+1 = h$, and one common lane number i^* for all entries not exceeding p, by Item 2 for β. As already shown, $\beta^@$ is a $(p-1)^*$-plateau with identical l-lanes to those of β, for all $l \geq p$. Since $\beta^@[j_h^@]$ contains the same number of newly added copies of $p-h$ as newly added copies of $p-h-1$, the elements of $B_{p-h}^@$ in $\beta^@[j_h^@]$ must be equinumerous with the elements of B_{p-h} in $\beta[j_h^@]$, so $\min I'$ has the same lane number i^* as $\min I^*$. By Lemma 3.7.9 for $\beta^@$, all entries of I' not exceeding $p-1$ have lane number i^* and entry $p-1$ has depth h. Entry p of I' also has lane number i^* and has depth $h+1$ because $\beta_{I'}^@$ and β_{I^*} have identical p-lanes.

Lastly, let $\widetilde{I}$ denote the rightmost element of $\widehat{B}_{p-1}$ in β. Let $C_{a''}^@$ denote the set of deleted elements of $\beta^@$, in the context of applying Procedure 3.6.2 to $\beta^@$, where the smallest entry to be added to $\beta^@$ is $p-a''-1 \geq p-s'-1$. By Item 2 for β, no element of B_p left of $\widetilde{I}$ can be lengthened by Procedure 3.6.2. It follows that Procedure 3.6.2 must lengthen the first n_i elements of $B_p - C_{\mathrm{ht}(z_i)}$ after $\widetilde{I}$. Therefore, if $I'' \in B_{p-1}^@$ contains the head of some p-lane, then all elements of $B_p^@$ in $\beta_{I''}^@$ must be elements of $C_{a''}^@$; we have $B_p \cap C_{\mathrm{ht}(z_i)} \subset B_p^@ \cap C_{a''}^@$, because $B_{p-a''-1}^@ \subset C_{a''}^@$, $B_{p-a''-1} \cap C_{\mathrm{ht}(z_i)} \subset B_{p-a''-1}^@$, $B_{b''} \cap C_{\mathrm{ht}(z_i)} \subset B_{b''}^@ \cap C_{a''}^@$ for all $p-a''-1 < b'' \leq p-1$, and $B_p^@ \subset B_p$. This completes the proof of Item 2. □

3.8. Growth algorithm

Convention 3.8.1. Given any partition ν, we will label its stretches from left to right by $\mathrm{S}[\nu,1]$, $\mathrm{S}[\nu,2]$, $\mathrm{S}[\nu,3]$, ... Let $\mathrm{s}[\nu,i] \in \mathbb{N}$ such that the row $\nu^{\mathrm{s}[\nu,i]}$

contains the stretch $\mathrm{S}[\nu, i]$; in other words, the $(\mathrm{s}[\nu, i])$th row of ν contains the ith stretch of ν.

Let $R' = (\mu_1, \mu_2, \ldots, \mu_n)$ be a $\mathcal{B}(\infty)$ rigged configuration of type A_n. Here we introduce notations and conventions that will be used in the growth algorithm.

Recall that $\mu_{d-1} \supset \overline{\mu_d}$. Following Convention 3.6.1, we label the stretches of $\overline{\mu_d}$ by $\mathrm{S}[\overline{\mu_d}, 1]$, $\mathrm{S}[\overline{\mu_d}, 2]$, $\mathrm{S}[\overline{\mu_d}, 3]$, …, where $\mathrm{S}[\overline{\mu_d}, i]$ is identical to $\mathrm{S}[\mu_d, i]$. By Theorem 3.3.11, μ_{d-1} has at most two rows beneath each stretch $\mathrm{S}[\overline{\mu_d}, j]$, where the first row consists of noncontributing boxes and contributing boxes, while the second row can consist of only contributing boxes.

Preliminaries 1: constraints imposed by riggings The following lemma determines how the riggings of μ_d constrain μ_{d-1}. By Theorem 3.2.7, to determine the riggings of μ_d it suffices to determine only the riggings of the rows $\mu_d^{\mathrm{s}[\mu_d,i]}$. If z is a stretch of μ_j intersecting the mth column y_m of μ_j, we say that z **spans** y_m. If y'_m is the mth column of any other partition, we also say that z **spans** y'_m.

Lemma 3.8.2. *The rigging* $\mathrm{r}^{[\mu_d,l]}$ *of the row* $\mu_d^{\mathrm{s}[\mu_d,l]}$ *containing* $\mathrm{S}[\mu_d, l]$ *can be written as*

$$\mathrm{r}^{[\mu_d,l]} = \sum_{i=1}^{l} -\mathrm{cb}[\mu_d, i] + \mathrm{acon}[\mu_d, i],$$

where $\mathrm{cb}[\mu_d, i]$ *is the number of contributing boxes in* $\mathrm{S}[\mu_d, i]$*, and* $0 \leq \mathrm{acon}[\mu_d, i] \leq |\mathrm{S}[\mu_d, i]|$ *is the number of columns of* μ_{d-1} *spanned by the stretch* $\mathrm{S}[\mu_d, i]$ *that end in a contributing box.*

Proof. Follows from Lemma 3.3.3. To be precise, the rigging of the row $\mu_d^{\mathrm{s}[\mu_d,l]}$ is determined by the number of contributing boxes added to the first $|\mu_d^{\mathrm{s}[\mu_d,l]}|$ columns of the dth partition and the number of contributing boxes that will be added to the corresponding columns of the $(d-1)$st partition; the former number corresponds to the sum $\sum_{i=1}^{l} -\mathrm{cb}[\mu_d, i]$, while the latter number corresponds to the sum $\sum_{i=1}^{l} \mathrm{acon}[\mu_d, i]$. The bounds $0 \leq \mathrm{acon}[\mu_d, i] \leq |\mathrm{S}[\mu_d, i]|$ follow because μ_d and $\overline{\mu_d}$ have identical stretches. ☐

Convention 3.8.3. As before, we follow the convention (used in Sage) of expressing R' by listing $\mu_1, \mu_2, \ldots, \mu_n$ from top to bottom, and we refer to μ_{d-1} as the **rigged partition above** μ_d. We will call the aforementioned positive integer $\mathrm{acon}[\mu_d, i]$ the **above contribution to** $\mathrm{S}[\mu_d, i]$ from μ_{d-1}, and we will call $\mathrm{cb}[\mu_d, i]$ the **contribution number of** $\mathrm{S}[\mu_d, i]$.

If given that μ_d has already been constructed (so in particular all $\mathrm{cb}[\mu_d, i]$ have been fixed), then fixing the riggings of μ_d is the same as fixing all $\mathrm{acon}[\mu_d, i]$. Once we fix $\mathrm{acon}[\mu_d, i]$ for all $i \le l$, then the number of columns of μ_{d-1} spanned by $\mathrm{S}[\mu_d, i]$ ending in a contributing box is completely determined (and must equal $\mathrm{acon}[\mu_d, i]$), for all $i \le l$. In what follows, we will always write the rigging of a given row in the form shown in Lemma 3.8.2, where we have partitioned the rigging $\mathrm{r}^{[\mu_d,l]}$ by stretches $\mathrm{S}[\mu_d, 1]$, $\mathrm{S}[\mu_d, 2]$, ..., $\mathrm{S}[\mu_d, l]$.

Preliminaries 2: rows allowed beneath a stretch

Lemma 3.8.4. *Fix a stretch* $\mathrm{S}[\overline{\mu_d}, j]$ *of* $\overline{\mu_d}$. μ_{d-1} *has at most two rows* $\eta_1^{[\overline{\mu_d},j]}$, $\eta_2^{[\overline{\mu_d},j]}$ *beneath* $\mathrm{S}[\overline{\mu_d}, j]$, *where* $0 \le |\eta_1^{[\overline{\mu_d},j]}| \le |\mathrm{S}[\overline{\mu_d}, j]|$ *and* $0 \le |\eta_2^{[\overline{\mu_d},j]}| \le |\eta_1^{[\overline{\mu_d},j]}|$.

Proof. Immediate from Theorem 3.3.11. □

Convention 3.8.5. We will always denote the first row of μ_{d-1} beneath $\mathrm{S}[\overline{\mu_d}, j]$ by $\eta_1^{[\overline{\mu_d},j]}$, and the second row of μ_{d-1} beneath $\mathrm{S}[\overline{\mu_d}, j]$ by $\eta_2^{[\overline{\mu_d},j]}$.

Lemma 3.8.6. $\eta_2^{[\overline{\mu_d},j]}$ *consists of* $|\eta_2^{[\overline{\mu_d},j]}| = \mathrm{cb}_2[\overline{\mu_d}, j]$ *contributing boxes, while* $\eta_1^{[\overline{\mu_d},j]}$ *consists of* $\mathrm{ncb}[\overline{\mu_d}, j]$ *noncontributing boxes followed by* $\mathrm{cb}_1[\overline{\mu_d}, j]$ *contributing boxes, where* $0 \le \mathrm{ncb}[\overline{\mu_d}, j] \le |\mathrm{S}[\overline{\mu_d}, j]|$, $0 \le \mathrm{cb}_2[\overline{\mu_d}, j] \le \mathrm{ncb}[\overline{\mu_d}, j]$, $0 \le \mathrm{cb}_1[\overline{\mu_d}, j] \le |\mathrm{S}[\overline{\mu_d}, j]| - \mathrm{ncb}[\overline{\mu_d}, j]$, *and* $\mathrm{cb}_2[\overline{\mu_d}, j] + \mathrm{cb}_1[\overline{\mu_d}, j] = \mathrm{acon}[\mu_d, j]$.

Proof. We must have $\mathrm{cb}_2[\overline{\mu_d}, j] + \mathrm{cb}_1[\overline{\mu_d}, j] = \mathrm{acon}[\mu_d, j]$ by the definition of $\mathrm{acon}[\mu_d, j]$, because $S[\overline{\mu_d}, j]$ and $S[\mu_d, j]$ span the same columns. The rest follow immediately from Theorem 3.3.11. □

Convention 3.8.7. We call $\mathrm{ncb}[\overline{\mu_d}, j]$ the **noncontribution number beneath** $S[\overline{\mu_d}, j]$, $\mathrm{cb}_1[\overline{\mu_d}, j]$ the **first contribution number beneath** $S[\overline{\mu_d}, j]$, and $\mathrm{cb}_2[\overline{\mu_d}, j]$ the **second contribution number beneath** $S[\overline{\mu_d}, j]$.

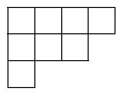
Convention 3.8.8. After the two new rows $\eta_1^{[\overline{\mu_d},j]}$, $\eta_2^{[\overline{\mu_d},j]}$ have been added beneath $\mathrm{S}[\overline{\mu_d}, j]$, at most three new stretches are formed:

1. $\eta_2^{[\overline{\mu_d},j]}$, which we denote by $\mathrm{S}_2[\overline{\mu_d}, j]$
2. the last $|\eta_1^{[\overline{\mu_d},j]}| - |\eta_2^{[\overline{\mu_d},j]}|$ boxes of $\eta_1^{[\overline{\mu_d},j]}$, which we denote by $\mathrm{S}_1[\overline{\mu_d}, j]$
3. the last $|\mathrm{S}[\overline{\mu_d}, j]| - |\eta_1^{[\overline{\mu_d},j]}|$ boxes of $\mathrm{S}[\overline{\mu_d}, j]$, which we denote by $\mathrm{S}_*[\overline{\mu_d}, j]$

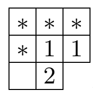
Convention 3.8.9. In the same vein as Convention 3.8.3,

1. denote by $\text{acon}_2[\overline{\mu_d}, j]$ the number of columns of μ_{d-2} spanned by $\text{S}_2[\overline{\mu_d}, j]$ ending in a contributing box
2. denote by $\text{acon}_1[\overline{\mu_d}, j]$ the number of columns of μ_{d-2} spanned by $\text{S}_1[\overline{\mu_d}, j]$ ending in a contributing box
3. denote by $\text{acon}_*[\overline{\mu_d}, j]$ the number of columns of μ_{d-2} spanned by $\text{S}_*[\overline{\mu_d}, j]$ ending in a contributing box

Example 3.8.10. Here we illustrate some of our introduced notations, by marking certain boxes in tableaux. Suppose that

$\mu_d =$ 

and

$\mu_{d-1} =$ 

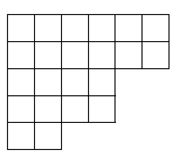
.

$\overline{\mu_d}$ consists of the boxes marked with * in μ_{d-1}. $\eta_1^{[\overline{\mu_d},2]}$ consists of the boxes marked with 1 in μ_{d-1}. $\eta_2^{[\overline{\mu_d},2]}$ consists of the box marked with 2 in μ_{d-1}, which must be a contributing box.

Definition 3.8.11. *The* ***inner cover of*** $\overline{\mu_d}$*, denoted* $\widetilde{\mu_d}$*, is the partition obtained from* $\overline{\mu_d}$ *by adding* $\eta_1^{[\overline{\mu_d},j]}$ *beneath* $\text{S}[\overline{\mu_d}, j]$ *such that* $|\eta_1^{[\overline{\mu_d},j]}| = \text{acon}[\mu_d, j]$ *for all* $j \geq 2$*, and adding* $\min(\text{maxr}_{d-1} - \max((\overline{\mu_d})^t), 1)$ *rows beneath* $\text{S}[\overline{\mu_d}, 1]$ *(set* $\eta_1^{[\overline{\mu_d},1]} = \emptyset$ *if no rows are allowed).*

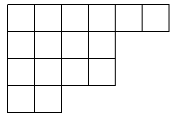
Definition 3.8.12. *The* ***outer cover of*** $\overline{\mu_d}$*, denoted* $\widehat{\mu_d}$*, is the partition obtained from* $\overline{\mu_d}$ *by adding the rows* $\eta_1^{[\overline{\mu_d},j]}$*,* $\eta_2^{[\overline{\mu_d},j]}$ *beneath* $\text{S}[\overline{\mu_d}, j]$ *such that* $|\eta_2^{[\overline{\mu_d},j]}| = \text{acon}[\mu_d, j]$ *and* $|\eta_1^{[\overline{\mu_d},j]}| = |\text{S}[\overline{\mu_d}, j]|$ *for all* j*, such that exactly* $\min(\text{maxr}_{d-1} - \max((\overline{\mu_d})^t), 2)$ *rows are added beneath* $\text{S}[\overline{\mu_d}, 1]$ *(set* $\eta_2^{[\overline{\mu_d},1]} = \emptyset$ *or* $\eta_1^{[\overline{\mu_d},1]} = \emptyset$ *if only one or zero rows is allowed, respectively) and exactly* $\min(|[(\widehat{\mu_d})^t]^{|\text{S}[\mu_d,1]|}| - \text{s}[\overline{\mu_d}, 2], 2)$ *rows are added beneath* $\text{S}[\overline{\mu_d}, 2]$*.*

Example 3.8.13. Suppose that

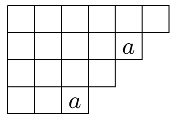
$\mu_d =$ 

with $\min(\text{maxr}_{d-1} - \max((\overline{\mu_d})^t), 2) = 0$, $\text{acon}[\mu_d, 1] = 0$, $\text{acon}[\mu_d, 2] = 1$,

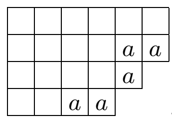
$\text{acon}[\mu_d, 3] = 1$. Then we have

$\overline{\mu_d} =$ 

and

$\widetilde{\mu_d} =$ 

and

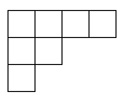
$\widehat{\mu_d} =$  ,

where "a" marks the added boxes.

Characterization by growth algorithm

Convention 3.8.14. For any rigged partition ν and any $a \in \mathbb{N}$ let $\text{U}[\nu, a]$ denote the uppermost row of ν with length at most a, and let $\text{r}^{\nu,a}$ denote the rigging of $\text{U}[\nu, a]$.

Example 3.8.15. Suppose

$\nu =$ 

and $a = 3$. Then $\text{U}[\nu, a]$ is the second row consisting of two boxes.

Here we give an algorithm for growing all A_n rigged configurations, which can be used to check inductively (starting with the last partition and going backward) whether a given tuple of rigged partitions is a legitimate A_n rigged configuration. Given μ_d and all $\text{acon}[\mu_d, i]$ fixed (equivalently, all riggings fixed), this algorithm produces μ_{d-1} by utilizing Procedures 3.6.2 and 3.6.8 to add noncontributing boxes and contributing boxes beneath the stretches of $\overline{\mu_d}$, one stretch at a time, from left to right. We will always follow Convention 3.6.1 on labeling the stretches of μ_d and the stretches of $\overline{\mu_d}$. By combining Theorem 3.3.11, Lemma 3.3.4, Lemma 3.8.2, and Lemma 3.7.10, we have our growth algorithm characterizing all A_n rigged configurations. Below, we give two versions of this **growth algorithm**. Version 1 is more explicit though technical. Version 2 is concise though less explicit regarding riggings.

Theorem 3.8.16 (Version 1)**.** *Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a tuple of rigged partitions. Then Λ is a $\mathcal{B}(\infty)$ rigged configuration of A-type if and only if Λ satisfies the following:*

1. *λ_n must consist of a single row with rigging* $\mathrm{r}^{[\lambda_n,1]} = -|\lambda_n| + \mathrm{acon}[\lambda_n, 1]$, *where* $0 \le \mathrm{acon}[\lambda_n, 1] \le |\lambda_n|$.
2. *In general, given that $\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n-i}$ have already been determined, we give the range of possible λ_{n-i-1}. Label the stretches of λ_{n-i} as* $\mathrm{S}[\lambda_{n-i}, 1]$, $\mathrm{S}[\lambda_{n-i}, 2]$, $\ldots$, $\mathrm{S}[\lambda_{n-i}, k_{n-i}]$ *where $k_{n-i} \in \mathbb{N}$. Write the rigging of* $\lambda_{n-i}^{\mathrm{s}[\lambda_{n-i},j]}$ *as* $\mathrm{r}^{[\lambda_{n-i},j]} = \sum_{m=1}^{j} -\mathrm{cb}[\lambda_{n-i}, m] + \mathrm{acon}[\lambda_{n-i}, m]$.
   To begin with, we have $\lambda_{n-i-1} \supset \overline{\lambda_{n-i}}$. *For $j \in [k_{n-i}]$ let* $Y_j := \sum_{u=1}^{j-1} |\mathrm{S}[\lambda_{n-i}, u]|$. *If* $\eta_1^{[\overline{\lambda_{n-i}},j]}$, $\eta_2^{[\overline{\lambda_{n-i}},j]}$ *exist, then we have* $|\eta_1^{[\overline{\lambda_{n-i}},j]}| = \mathrm{ncb}[\overline{\lambda_{n-i}}, j] + \mathrm{cb}_1[\overline{\lambda_{n-i}}, j]$ *and* $|\eta_2^{[\overline{\lambda_{n-i}},j]}| = \mathrm{cb}_2[\overline{\lambda_{n-i}}, j]$, *where we have* $0 \le \mathrm{ncb}[\overline{\lambda_{n-i}}, j] \le |\mathrm{S}[\overline{\lambda_{n-i}}, j]|$, $0 \le \mathrm{cb}_2[\overline{\lambda_{n-i}}, j] \le \mathrm{ncb}[\overline{\lambda_{n-i}}, j]$, $0 \le \mathrm{cb}_1[\overline{\lambda_{n-i}}, j] \le |\mathrm{S}[\overline{\lambda_{n-i}}, j]| - \mathrm{ncb}[\overline{\lambda_{n-i}}, j]$, *and* $\mathrm{cb}_2[\overline{\lambda_{n-i}}, j] + \mathrm{cb}_1[\overline{\lambda_{n-i}}, j] = \mathrm{acon}[\lambda_{n-i}, j]$. *We determine λ_{n-i-1} by specifying the exact number of rows allowed beneath* $\mathrm{S}[\overline{\lambda_{n-i}}, j]$ *and the riggings, for each* $j \in [k_{n-i}]$.
   (a) *First define* $\delta(\lambda_{n-i-1}) := \mathrm{maxr}_{n-i-1} - \max((\overline{\lambda_{n-i}})^t)$. *At most* $\min(\delta(\lambda_{n-i-1}), 2)$ *rows can exist beneath* $\mathrm{S}[\overline{\lambda_{n-i}}, 1]$ *in* λ_{n-i-1}. *Any row of length* $|\eta_2^{[\overline{\lambda_{n-i}},1]}|$ *has rigging* $\mathrm{r}_2^{[\overline{\lambda_{n-i}},1]} := \mathrm{cb}_2[\overline{\lambda_{n-i}}, 1] + \mathrm{acon}_2[\overline{\lambda_{n-i}}, 1]$, *any row of length* $|\eta_1^{[\overline{\lambda_{n-i}},1]}|$ *has rigging* $\mathrm{r}_1^{[\overline{\lambda_{n-i}},1]} := \mathrm{r}_2^{[\overline{\lambda_{n-i}},1]} - \mathrm{cb}_1[\overline{\lambda_{n-i}}, 1] + \mathrm{acon}_1[\overline{\lambda_{n-i}}, 1]$, *and any row whose length is* $|\mathrm{S}[\overline{\lambda_{n-i}}, 1]|$ *has rigging* $\mathrm{r}_*^{[\overline{\lambda_{n-i}},1]} := \mathrm{r}_1^{[\overline{\lambda_{n-i}},1]} + \mathrm{acon}_*[\overline{\lambda_{n-i}}, 1]$, *where* $0 \le \mathrm{acon}_2[\overline{\lambda_{n-i}}, 1] \le \mathrm{cb}_2[\overline{\lambda_{n-i}}, 1]$, $0 \le \mathrm{acon}_1[\overline{\lambda_{n-i}}, 1] \le |\eta_1^{[\overline{\lambda_{n-i}},1]}| - |\eta_2^{[\overline{\lambda_{n-i}},1]}|$, *and* $0 \le \mathrm{acon}_*[\overline{\lambda_{n-i}}, 1] \le \Upsilon[\overline{\lambda_{n-i}}, 1]$ *where*
   $$\Upsilon[\overline{\lambda_{n-i}}, 1] = \begin{cases} 0 & \text{if } \delta(\lambda_{n-i-1}) = 0 \\ |\mathrm{S}[\overline{\lambda_{n-i}}, 1]| - |\eta_1^{[\overline{\lambda_{n-i}},1]}| & \text{otherwise} \end{cases}$$
   (b) *Let* $2 \le m \le k_{n-i}$. *Assume we have already determined the rows and riggings allowed beneath* $\mathrm{S}[\overline{\lambda_{n-i}}, l]$ *for all* $l \le m-1$. *At most* $\min(|[(\lambda_{n-i-1})^t]^{Y_m}| - \mathrm{s}[\overline{\lambda_{n-i}}, m], 2)$ *rows can exist beneath* $\mathrm{S}[\overline{\lambda_{n-i}}, m]$. *There are three cases:*
      i. *If* $|\mathrm{U}[\lambda_{n-i-1}, Y_m]| < Y_m$ *and* $|\eta_2^{[\overline{\lambda_{n-i}},m]}| \neq 0$, *then any row of length* $Y_m + |\eta_2^{[\overline{\lambda_{n-i}},m]}|$ *has rigging* $\mathrm{r}_2^{[\overline{\lambda_{n-i}},m]} := \mathrm{r}^{\lambda_{n-i-1},Y_m} - \mathrm{cb}_1[\overline{\lambda_{n-i}}, m-1] - \mathrm{cb}_2[\overline{\lambda_{n-i}}, m] + \mathrm{acon}_2[\overline{\lambda_{n-i}}, m]$ *and any row of length* $Y_m + |\eta_1^{[\overline{\lambda_{n-i}},m]}|$ *has rigging* $\mathrm{r}_1^{[\overline{\lambda_{n-i}},m]} := \mathrm{r}_2^{[\overline{\lambda_{n-i}},m]} - \mathrm{cb}_1[\overline{\lambda_{n-i}}, m] + \mathrm{acon}_1[\overline{\lambda_{n-i}}, m]$, *where* $0 \le \mathrm{acon}_2[\overline{\lambda_{n-i}}, m] \le Y_m + |\eta_2^{[\overline{\lambda_{n-i}},m]}| - |\mathrm{U}[\lambda_{n-i-1}, Y_m]|$ *and* $0 \le \mathrm{acon}_1[\overline{\lambda_{n-i}}, m] \le |\eta_1^{[\overline{\lambda_{n-i}},m]}| - |\eta_2^{[\overline{\lambda_{n-i}},m]}|$.

ii. *If* $|\mathrm{U}[\lambda_{n-i-1}, Y_m]| < Y_m$ *and* $|\eta_2^{[\overline{\lambda_{n-i}},m]}| = 0$*, then any row of length* $Y_m + |\eta_1^{[\overline{\lambda_{n-i}},m]}|$ *has rigging* $\mathrm{r}_1^{[\overline{\lambda_{n-i}},m]} := \mathrm{r}^{\lambda_{n-i-1},Y_m} - \mathrm{cb}_1[\overline{\lambda_{n-i}}, m] + \mathrm{acon}_1[\overline{\lambda_{n-i}}, m]$*, where* $0 \leq \mathrm{acon}_1[\overline{\lambda_{n-i}}, m] \leq Y_m + |\eta_1^{[\overline{\lambda_{n-i}},m]}| - |\mathrm{U}[\lambda_{n-i-1}, Y_m]|$.

iii. *Otherwise, any row of length* $Y_m + |\eta_2^{[\overline{\lambda_{n-i}},m]}|$ *has rigging* $\mathrm{r}_2^{[\overline{\lambda_{n-i}},m]} := \mathrm{r}^{\lambda_{n-i-1},Y_m} - \mathrm{cb}_2[\overline{\lambda_{n-i}}, m] + \mathrm{acon}_2[\overline{\lambda_{n-i}}, m]$ *and any row of length* $Y_m + |\eta_1^{[\overline{\lambda_{n-i}},m]}|$ *has rigging* $\mathrm{r}_1^{[\overline{\lambda_{n-i}},m]} := \mathrm{r}_2^{[\overline{\lambda_{n-i}},m]} - \mathrm{cb}_1[\overline{\lambda_{n-i}}, m] + \mathrm{acon}_1[\overline{\lambda_{n-i}}, m]$*, where* $0 \leq \mathrm{acon}_2[\overline{\lambda_{n-i}}, m] \leq \mathrm{cb}_2[\overline{\lambda_{n-i}}, m]$ *and* $0 \leq \mathrm{acon}_1[\overline{\lambda_{n-i}}, m] \leq |\eta_1^{[\overline{\lambda_{n-i}},m]}| - |\eta_2^{[\overline{\lambda_{n-i}},m]}|$.

In all cases, any row whose length is $Y_m + |\mathrm{S}[\overline{\lambda_{n-i}}, m]|$ *has rigging* $\mathrm{r}_*^{[\overline{\lambda_{n-i}},m]} := \mathrm{r}_1^{[\overline{\lambda_{n-i}},m]} + \mathrm{acon}_*[\overline{\lambda_{n-i}}, m]$*, where* $0 \leq \mathrm{acon}_*[\overline{\lambda_{n-i}}, m] \leq |\mathrm{S}[\overline{\lambda_{n-i}}, m]| - |\eta_1^{[\overline{\lambda_{n-i}},m]}|$.

(c) *Finally, we determine the first row* λ^1_{n-i-1} *and its rigging.*

i. *If* $|\eta_1^{[\overline{\lambda_{n-i}},k_{n-i}]}| < |\mathrm{S}[\overline{\lambda_{n-i}}, k_{n-i}]|$ *and* $|\lambda^1_{n-i}| > |\lambda^2_{n-i}|$*, then* $|\lambda^1_{n-i-1}| = |\lambda^2_{n-i}| + |\eta_1^{[\overline{\lambda_{n-i}},k_{n-i}]}|$*, with rigging* $\mathrm{r}_1^{[\overline{\lambda_{n-i}},k_{n-i}]}$.

ii. *Otherwise, we have* $|\lambda^1_{n-i-1}| = |\lambda^1_{n-i}| + \mathrm{cb}_![\overline{\lambda_{n-i}}, k_{n-i}]$*, where* $\mathrm{cb}_![\overline{\lambda_{n-i}}, k_{n-i}]$ *can be any nonnegative integer. Let* $\mathrm{r}(\lambda^2_{n-i-1})$ *denote the rigging of* λ^2_{n-i-1}.

A. *If* $|\lambda^1_{n-i}| = |\lambda^2_{n-i}|$*, then* λ^1_{n-i-1} *has rigging* $\mathrm{r}_!^{[\overline{\lambda_{n-i}},k_{n-i}]} := \mathrm{r}(\lambda^2_{n-i-1}) - \mathrm{cb}_![\overline{\lambda_{n-i}}, k_{n-i}] + \mathrm{acon}_![\overline{\lambda_{n-i}}, k_{n-i}]$.

B. *If* $|\lambda^1_{n-i}| > |\lambda^2_{n-i}|$*, then the first row* λ^1_{n-i-1} *has rigging* $\mathrm{r}_!^{[\overline{\lambda_{n-i}},k_{n-i}]} := \mathrm{r}(\lambda^2_{n-i-1}) - \mathrm{cb}_1[\overline{\lambda_{n-i}}, k_{n-i}] - \mathrm{cb}_![\overline{\lambda_{n-i}}, k_{n-i}] + \mathrm{acon}_![\overline{\lambda_{n-i}}, k_{n-i}]$.

In both cases $0 \leq \mathrm{acon}_![\overline{\lambda_{n-i}}, k_{n-i}] \leq |\lambda^1_{n-i-1}| - |\lambda^2_{n-i-1}|$.

Remark 3.8.17. In short, this theorem states that the full range of boxes allowed under Theorem 3.3.11 can indeed be added to $\overline{\lambda_{n-i}}$ to form the $(n-i-1)$st partition, as long as the fixed above contributions $\mathrm{acon}[\cdot,\cdot]$ are respected; in other words, the constraints imposed by Theorem 3.3.11 are tight.

Proof. Starting with the empty rigged configuration $R_\emptyset$, the construction of Λ using Procedures 3.6.2 and 3.6.8 is described item by item as follows.

1. By Lemma 3.7.10, λ_n can be formed by using Procedure 3.6.8 to add $|\lambda_n|$ contributing boxes to the empty nth partition, and the resulting $(n-1)$st partition is $\overline{\lambda_n} = \emptyset$.

2. The base case of λ_n justifies the inductive hypothesis that the $(n-i-1)$st partition is $\overline{\lambda_{n-i}}$ (with riggings of zero) after the partitions $\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n-i}$ have been constructed, in that order, by Procedures 3.6.2 and 3.6.8. Lemma 3.7.10 ensures that Procedures 3.6.2 and 3.6.8 can add all the boxes Theorem 3.3.11 allows under $\mathrm{S}[\overline{\lambda_{n-i}}, j]$. To be precise, Procedure 3.6.2 will be used to add the $\mathrm{ncb}[\overline{\lambda_{n-i}}, j]$ noncontributing boxes to the first row beneath $\mathrm{S}[\overline{\lambda_{n-i}}, j]$, Procedure 3.6.8 will be used to add the $\mathrm{cb}_1[\overline{\lambda_{n-i}}, j]$ contributing boxes after these $\mathrm{ncb}[\overline{\lambda_{n-i}}, j]$ noncontributing boxes in the same row, and Procedure 3.6.8 will be used to add the $\mathrm{cb}_2[\overline{\lambda_{n-i}}, j]$ contributing boxes beneath these $\mathrm{ncb}[\overline{\lambda_{n-i}}, j]$ noncontributing boxes. Finally, by Lemma 3.8.2, a total of $\mathrm{acon}[\lambda_{n-i}, j]$ contributing boxes must be added beneath $\mathrm{S}[\overline{\lambda_{n-i}}, j]$ to account for the positive contribution to $\mathrm{S}[\lambda_{n-i}, j]$. In the cases (a) and (b), Theorem 3.3.11 determines how many rows can be added beneath $\mathrm{S}[\overline{\lambda_{n-i}}, j]$ and how many contributing boxes can be added beneath the corresponding stretch of the $(n-i-2)$nd partition.

   (a) We prove that $\min(\delta(\lambda_{n-i-1}), 2)$ rows can indeed be added beneath $\mathrm{S}[\overline{\lambda_{n-i}}, 1]$. Notice that, for either of the Procedures 3.6.2 and 3.6.8 to work, we must (using the notation from the statements of these two procedures) have $p - \mathrm{ht}(z_1) - 1 \geq 1$ or equivalently $(p-1) - \mathrm{ht}(z_1) > 0$. Indeed, by Lemma 3.7.9, $\min((p-1) - \mathrm{ht}(z_1), 2)$ is the number of rows that can be added beneath the first stretch by Procedures 3.6.2 and 3.6.8.

   By Theorem 3.3.11, at most $\min(\delta(\lambda_{n-i-1}), 2)$ rows can exist beneath $\mathrm{S}[\overline{\lambda_{n-i}}, 1]$ in λ_{n-i-1}. To prove the converse, let $H_l \leq \mathrm{maxr}_l$ denote the number of rows in λ_l. Then $\overline{\lambda_{n-i}}$ has $\overline{H_{n-i-1}} = H_{n-i} - 1 \leq \mathrm{maxr}_{n-i} - 1$ rows. By above, $\min(n-i-1-\overline{H_{n-i-1}}, 2) = \min(n-i-H_{n-i}, 2)$ is exactly the number of rows allowed to be added beneath $\mathrm{S}[\overline{\lambda_{n-i}}, 1]$ by Procedures 3.6.2 and 3.6.8. Consider the cases $n-i-1 > \frac{n+1}{2}$ and $n-i-1 \leq \frac{n+1}{2}$. If $n-i-1 \leq \frac{n+1}{2}$, then $\delta(\lambda_{n-i-1}) = n-i-1-\overline{H_{n-i-1}}$ since $\mathrm{maxr}_{n-i-1} = n-i-1$. Suppose $n-i-1 > \frac{n+1}{2}$. Then $n-i-1-\overline{H_{n-i-1}} \geq n-i-1-(\mathrm{maxr}_{n-i}-1) = n-i-1-(n-(n-i)+1-1) = n-i-1-i = n-2i-1 \geq 2$ by Lemma 3.3.5, since $i < \frac{n-3}{2}$. Also, $\delta(\lambda_{n-i-1}) = \mathrm{maxr}_{n-i-1} - \overline{H_{n-i-1}} = n-(n-i-1)+1-\overline{H_{n-i-1}} = i+3-H_{n-i} \geq i+3-\mathrm{maxr}_{n-i} = i+3-(i+1) = 2$ by Lemma 3.3.5. It follows that $\min(\delta(\lambda_{n-i-1}), 2) = 2 = \min(n-i-H_{n-i}, 2)$ as well. Thus, in both cases $\min(\delta(\lambda_{n-i-1}), 2)$ rows can indeed be added beneath $\mathrm{S}[\overline{\lambda_{n-i}}, 1]$.

The proof for the rigging formulas is straightforward but technical. The Examples 3.8.20, 3.8.21, 3.8.22 are more illustrative.

(b) The proof is straightforward but technical. The Examples 3.8.20, 3.8.21, 3.8.22 are more illustrative.

(c) The proof is straightforward but technical. The Examples 3.8.20, 3.8.21, 3.8.22 are more illustrative.

Finally, this procedure can be repeated to determine λ_{n-i-2}, because we have already determined the range for the above contribution to each stretch of λ_{n-i-1}; to repeat this procedure, just fix a number in each range. □

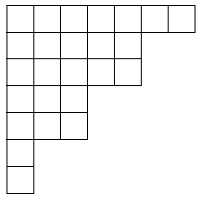
Theorem 3.8.18 (Version 2). *Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a tuple of rigged partitions. Then Λ is a $\mathcal{B}(\infty)$ rigged configuration of A-type if and only if Λ satisfies the following:*

1. *λ_n must consist of a single row with rigging $\mathrm{r}^{[\lambda_n,1]} = -|\lambda_n| + \mathrm{acon}[\lambda_n, 1]$, where $0 \le \mathrm{acon}[\lambda_n, 1] \le |\lambda_n|$.*
2. *In general, given that $\lambda_n, \lambda_{n-1}, \dots, \lambda_{n-i}$ have already been determined, we give the range of possible λ_{n-i-1}. Label the stretches of λ_{n-i} as $\mathrm{S}[\lambda_{n-i}, 1]$, $\mathrm{S}[\lambda_{n-i}, 2]$, …, $\mathrm{S}[\lambda_{n-i}, k_{n-i}]$ where $k_{n-i} \in \mathbb{N}$. Write the rigging of $\lambda_{n-i}^{\mathrm{s}[\lambda_{n-i},j]}$ as $\mathrm{r}^{[\lambda_{n-i},j]} = \sum_{m=1}^{j} -\mathrm{cb}[\lambda_{n-i}, m] + \mathrm{acon}[\lambda_{n-i}, m]$.*
   λ_{n-i-1} is any partition satisfying $\widetilde{\lambda_{n-i}} \subset \lambda_{n-i-1} \subset \widehat{\lambda_{n-i}}'$, where $\widehat{\lambda_{n-i}}'$ is a partition obtained by adding any nonnegative number of boxes to the first row of $\widehat{\lambda_{ni}}$.
   After the shape of λ_{n-i-1} has been determined, the number of contributing boxes in each stretch of λ_{n-i-1} can be determined as follows. Fix any stretch $\mathrm{S}[\overline{\lambda_{n-i}}, j]$ of $\overline{\lambda_{n-i}}$ in λ_{n-i-1}. If $\eta_1^{[\overline{\lambda_{n-i}},j]}$, $\eta_2^{[\overline{\lambda_{n-i}},j]}$ exist, then $\eta_1^{[\overline{\lambda_{n-i}},j]}$ consists of $\mathrm{ncb}[\overline{\lambda_{n-i}}, j]$ noncontributing boxes followed by $\mathrm{cb}_1[\overline{\lambda_{n-i}}, j]$ contributing boxes, and $\eta_2^{[\overline{\lambda_{n-i}},j]}$ consists of $\mathrm{cb}_2[\overline{\lambda_{n-i}}, j]$ contributing boxes, where $0 \le \mathrm{ncb}[\overline{\lambda_{n-i}}, j] \le |\mathrm{S}[\overline{\lambda_{n-i}}, j]|$, $0 \le \mathrm{cb}_2[\overline{\lambda_{n-i}}, j] \le \mathrm{ncb}[\overline{\lambda_{n-i}}, j]$, $0 \le \mathrm{cb}_1[\overline{\lambda_{n-i}}, j] \le |\mathrm{S}[\overline{\lambda_{n-i}}, j]| - \mathrm{ncb}[\overline{\lambda_{n-i}}, j]$, and as always $\mathrm{cb}_2[\overline{\lambda_{n-i}}, j] + \mathrm{cb}_1[\overline{\lambda_{n-i}}, j] = \mathrm{acon}[\lambda_{n-i}, j]$. We have $|\lambda_{n-i-1}^1| > |\lambda_{n-i}^1|$ if and only if $|\lambda_{n-i}^1| = |\lambda_{n-i}^2|$ or $|\eta_1^{[\overline{\lambda_{n-i}},k_{n-i}]}| = |\mathrm{S}[\overline{\lambda_{n-i}}, k_{n-i}]|$, in which case all boxes in λ_{n-i-1}^1 after $\mathrm{S}[\overline{\lambda_{n-i}}, k_{n-i}]$ are contributing boxes. Lastly, for any stretch $\mathrm{S}[\lambda_{n-i-1}, l]$ of λ_{n-i-1}, the above contribution $\mathrm{acon}[\lambda_{n-i-1}, l]$ has range $0 \le \mathrm{acon}[\lambda_{n-i-1}, l] \le |\mathrm{S}[\lambda_{n-i-1}, l]|$.

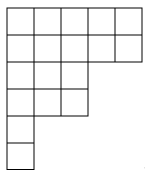
Remark 3.8.19. Note that, using this theorem and Lemma 3.8.2, the range of rigging of any row of λ_{n-i-1} is completely determined once the number of contributing boxes in each stretch of λ_{n-i-1} has been determined.

We now look at some examples of obtaining λ_{n-i-1} given predetermined $\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n-i}$. In the Young diagrams, we will mark each added contributing box with a "c" and each noncontributing box with an "n".

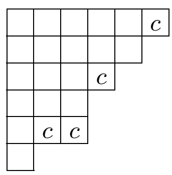
Example 3.8.20. Suppose that

$\lambda_{n-i} =$ 

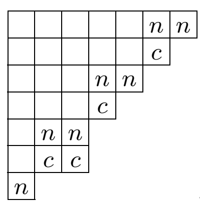
where $\min(\mathrm{maxr}_{n-i-1} - \max((\overline{\lambda_{n-i}})^t), 2) = 1$, $\mathrm{acon}[\lambda_{n-i}, 1] = 0$, $\mathrm{acon}[\lambda_{n-i}, 2] = 2$, $\mathrm{acon}[\lambda_{n-i}, 3] = 1$, and $\mathrm{acon}[\lambda_{n-i}, 4] = 1$ have been determined. We have

$\overline{\lambda_{n-i}} =$  .

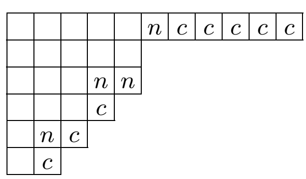
λ_{n-i-1} is any partition such that $\widetilde{\lambda_{n-i}} \subset \lambda_{n-i-1} \subset \widehat{\lambda_{n-i}}'$, where

$\widetilde{\lambda_{n-i}} =$ 

and

$\widehat{\lambda_{n-i}} =$  .

Recall that $\widehat{\lambda_{n-i}}'$ is obtained from $\widehat{\lambda_{n-i}}$ by adding any number of contributing boxes to the first row of $\widehat{\lambda_{n-i}}$. One possibility for λ_{n-i-1} is

$\lambda_{n-i-1} =$ 

whose riggings are given by $\mathrm{r}^{[\lambda_{n-i-1},1]} = -1 + \mathrm{acon}[\lambda_{n-i-1}, 1]$, $\mathrm{r}^{[\lambda_{n-i-1},2]} = \mathrm{r}^{[\lambda_{n-i-1},1]} - 1 + \mathrm{acon}[\lambda_{n-i-1}, 2]$, $\mathrm{r}^{[\lambda_{n-i-1},3]} = \mathrm{r}^{[\lambda_{n-i-1},2]} - 1 + \mathrm{acon}[\lambda_{n-i-1}, 3]$, $\mathrm{r}^{[\lambda_{n-i-1},4]} = \mathrm{r}^{[\lambda_{n-i-1},3]} + \mathrm{acon}[\lambda_{n-i-1}, 4]$, and $\mathrm{r}^{[\lambda_{n-i-1},5]} = \mathrm{r}^{[\lambda_{n-i-1},4]} - 5 + \mathrm{acon}[\lambda_{n-i-1}, 5]$, where $0 \le \mathrm{acon}[\lambda_{n-i-1}, 1] \le 2$, $0 \le \mathrm{acon}[\lambda_{n-i-1}, 2] \le 1$, $0 \le \mathrm{acon}[\lambda_{n-i-1}, 3] \le 1$, $0 \le \mathrm{acon}[\lambda_{n-i-1}, 4] \le 1$, and $0 \le \mathrm{acon}[\lambda_{n-i-1}, 5] \le 6$.

Example 3.8.21. In Example 3.8.20, another possibility for λ_{n-i-1} is

$$\lambda_{n-i-1} = \begin{ytableau} \ & \ & \ & \ & \ & n \\ \ & \ & \ & \ & \ & c \\ \ & \ & \ & n & c \\ \ & \ & \ \\ \ & c & c \\ \ \\ n \end{ytableau}$$

whose riggings are $\mathrm{r}^{[\lambda_{n-i-1},1]} = \mathrm{acon}[\lambda_{n-i-1}, 1]$, $\mathrm{r}^{[\lambda_{n-i-1},2]} = \mathrm{r}^{[\lambda_{n-i-1},1]} - 2 + \mathrm{acon}[\lambda_{n-i-1}, 2]$, $\mathrm{r}^{[\lambda_{n-i-1},3]} = \mathrm{r}^{[\lambda_{n-i-1},2]} - 1 + \mathrm{acon}[\lambda_{n-i-1}, 3]$, and $\mathrm{r}^{[\lambda_{n-i-1},4]} = \mathrm{r}^{[\lambda_{n-i-1},3]} - 1 + \mathrm{acon}[\lambda_{n-i-1}, 4]$, where we have $0 \le \mathrm{acon}[\lambda_{n-i-1}, 1] \le 1$, $0 \le \mathrm{acon}[\lambda_{n-i-1}, 2] \le 2$, $0 \le \mathrm{acon}[\lambda_{n-i-1}, 3] \le 2$, and $0 \le \mathrm{acon}[\lambda_{n-i-1}, 4] \le 1$. Note that no contributing boxes can be added to the first row of λ_{n-i-1} in this case, due to the predetermined restriction $\mathrm{acon}[\lambda_{n-i}, 4] = 1$; adding more contributing boxes would result in $\mathrm{acon}[\lambda_{n-i}, 4] = 2$, which violates the restriction.

Example 3.8.22. Let

$$\lambda_n = \begin{ytableau} \ & \ & \ \end{ytableau}$$

with rigging $-3 + \mathrm{acon}[\lambda_n, 1]$, and fix $\mathrm{acon}[\lambda_n, 1] = 2$. Then $\overline{\lambda_n}$ is an empty row of length 3. One possible choice of λ_{n-1} is

$$\lambda_{n-1} = \begin{ytableau} n & n & n & c & c \\ c & c \end{ytableau},$$

where the bottom row has rigging $-2 + \mathrm{acon}[\lambda_{n-1}, 1]$ and the top row has rigging $-4 + \mathrm{acon}[\lambda_{n-1}, 1] + \mathrm{acon}[\lambda_{n-1}, 2]$.

Suppose we now fix $\mathrm{acon}[\lambda_{n-1}, 1] = 2$ and $\mathrm{acon}[\lambda_{n-1}, 2] = 2$. Then one possible choice for λ_{n-2} is

$$\lambda_{n-2} = \begin{ytableau} \ & \ & n & n & c & c \\ n & n & c \\ c & c \end{ytableau}$$

where the third row has rigging $-2 + \mathrm{acon}[\lambda_{n-2}, 1]$, second row has rigging $-3 + \mathrm{acon}[\lambda_{n-2}, 1] + \mathrm{acon}[\lambda_{n-2}, 2]$, and first row has rigging $-5 + \mathrm{acon}[\lambda_{n-2}, 1] + \mathrm{acon}[\lambda_{n-2}, 2] + \mathrm{acon}[\lambda_{n-2}, 3]$. Here $0 \le \mathrm{acon}[\lambda_{n-2}, 1] \le 2$, $0 \le \mathrm{acon}[\lambda_{n-2}, 2] \le 1$, and $0 \le \mathrm{acon}[\lambda_{n-2}, 3] \le 3$.

3.9. Determining the cascading sequence of a rigged configuration

Based on Theorem 3.8.16, we now give the algorithm for determining the cascading sequence of a rigged configuration. Assume $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a $\mathcal{B}(\infty)$ rigged configuration of A-type.

Theorem 3.9.1. *The following algorithm constructs the cascading sequence α corresponding to Λ:*

1. *Start with the empty string α^0. Add $|\lambda_n|$ copies of lower subintervals (n) to α^0, obtaining α^1, which accounts for λ_n.*
2. *In general, suppose that we have constructed the cascading sequence α^i which accounts for $\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n-i}$. We want to construct α^{i+1} that accounts for λ_n, λ_{n-1}, …, λ_{n-i}, λ_{n-i-1}.*
   Label the stretches of λ_{n-i} by $\mathrm{S}[\lambda_{n-i}, 1]$, $\mathrm{S}[\lambda_{n-i}, 2]$, …, $\mathrm{S}[\lambda_{n-i}, k_{n-i}]$ where $k_{n-i} \in \mathbb{N}$. Write the rigging of $\lambda_{n-i}^{\mathrm{s}[\lambda_{n-i},j]}$ as

$$\mathrm{r}^{[\lambda_{n-i},j]} = \sum_{m=1}^{j} -\mathrm{cb}[\lambda_{n-i}, m] + \mathrm{acon}[\lambda_{n-i}, m].$$

   Let $\mathrm{S}'[\overline{\lambda_{n-i}}, 1]$, $\mathrm{S}'[\overline{\lambda_{n-i}}, 2]$, …, $\mathrm{S}'[\overline{\lambda_{n-i}}, k_{n-i}]$ denote the stretches of the copy of $\overline{\lambda_{n-i}}$ sitting inside λ_{n-i-1}.

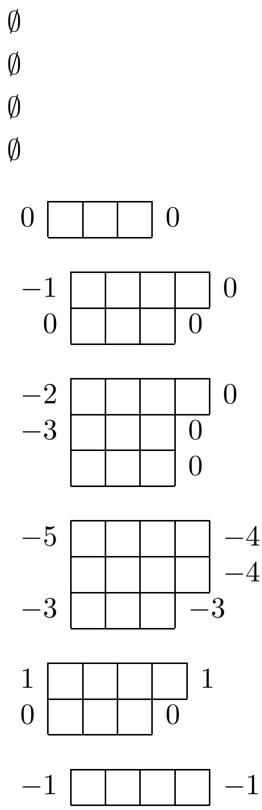
   (a) *For $m = 1, 2, \ldots, k_{n-i}$, let l_m denote the number of boxes in the second row beneath $\mathrm{S}'[\overline{\lambda_{n-i}}, m]$ and let $l_m \le u_m \le |\mathrm{S}'[\overline{\lambda_{n-i}}, m]|$ denote the number of boxes in the first row beneath $\mathrm{S}'[\overline{\lambda_{n-i}}, m]$. For m ranging through $1, 2, \ldots, k_{n-i}$ in that order, first apply Procedure 3.6.2 to add $l_m + u_m - \mathrm{acon}[\lambda_{n-i}, m]$ noncontributing boxes beneath $\mathrm{S}[\overline{\lambda_{n-i}}, m]$, then apply Procedure 3.6.8 to add l_m contributing boxes beneath these added noncontributing boxes, and finally apply Procedure 3.6.8 to add $\mathrm{acon}[\lambda_{n-i}, m] - l_m$ contributing boxes to the first row beneath $\mathrm{S}[\overline{\lambda_{n-i}}, m]$, updating the cascading sequence (starting from α^i) with each application of each procedure.*

   (b) *Suppose we have added all the boxes required beneath the stretches of $\overline{\lambda_{n-i}}$. Let g denote the resulting first row. Apply Procedure 3.6.8 to add $|\lambda_{n-i-1}^1| - |g|$ contributing boxes to g, updating the cascading sequence. This completes the construction of λ_{n-i-1}, and the resulting cascading sequence is the desired α^{i+1}.*

Proof. By assumption, Λ is a legitimate rigged configuration. This algorithm works by comparing Λ with the rigged configuration corresponding to the cascading sequence constructed so far, seeing what boxes need to be added to construct the next partition of Λ, and then applying Procedure 3.6.2 and Procedure 3.6.8 to add the boxes required. The full proof is similar to that of Theorem 3.8.16, and is a matter of bookkeeping. □

Now let us look at some examples of how to obtain the cascading sequence given a rigged configuration using the algorithm described above.

Example 3.9.2. Consider the following rigged configuration $R = (\nu_1, \nu_2, \ldots, \nu_{10})$ (in top-bottom order) of type A_{10} where ν_i is the ith rigged partition whose jth row has rigging rig_i^j:



From the viewpoint of its cascading sequence, R is constructed (by the growth algorithm) in the following process (where newly added letters or lower subintervals at each stage are marked with a prime (′)):

$$(10')(10')(10')(10') \to^{\textcircled{1}} (9', 10)(9', 10)(9', 10)(9', 10)$$

$\to^{②}$ $(9,10)(9,10)(9,10)(9,10)(8,9)'(8,9)'(8,9)'$

$\to^{③}$ $(8',9,10)(8',9,10)(8',9,10)(9,10)(7',8,9)(7',8,9)$
$(7',8,9)$

$\to^{④}$ $(8,9,10)(8,9,10)(8,9,10)(9,10)(7,8,9)(7,8,9)$
$(7,8,9)(6,7,8)'(6,7,8)'(6,7,8)'$

$\to^{⑤}$ $(8,9,10)(8,9,10)(8,9,10)(8',9,10)(7,8,9)(7,8,9)$
$(7,8,9)(6,7,8)(6,7,8)(6,7,8)$

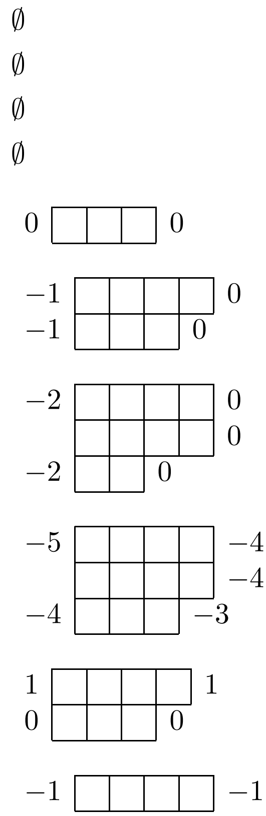
$\to^{⑥}$ $(8,9,10)(8,9,10)(8,9,10)(8,9,10)(7,8,9)(7,8,9)$
$(7,8,9)(6,7,8)(6,7,8)(6,7,8)(7,8)'$

$\to^{⑦}$ $(7',8,9,10)(7',8,9,10)(7',8,9,10)(8,9,10)$
$(6',7,8,9)(6',7,8,9)(6',7,8,9)(5',6,7,8)(5',6,7,8)$
$(5',6,7,8)(7,8)$

$\to^{⑧}$ $(7,8,9,10)(7,8,9,10)(7,8,9,10)(8,9,10)(6,7,8,9)$
$(6,7,8,9)(6,7,8,9)(5,6,7,8)(5,6,7,8)(5,6,7,8)$
$(6',7,8)$

Explanation of the above process: We started out by adding four 10-boxes, which completes Partition 10. Since $\mathrm{rig}^1_{10} = -1 = -4+3$, we first added four noncontributing 9-boxes in ①, and then added three contributing 9-boxes in ② beneath these noncontributing boxes, which completes Partition 9 and adds three noncontributing 8-boxes. Since $\mathrm{rig}^2_9 = 0 = -3+3$ and $\mathrm{rig}^1_9 = 1 = -3+4$, we first added three noncontributing 8-boxes beneath the first row in ③ (along with three noncontributing 7-boxes), and then added three contributing 8-boxes beneath the second row in ④ (along with three noncontributing 7-boxes and three noncontributing 6-boxes), and then added one noncontributing 8-box to the first row in ⑤, and then added one contributing 8-box beneath the first row in ⑥ (along with one noncontributing 7-box to the first row). This completes Partition 8. Now, Partitions 5-7 all have zero riggings, while the remaining partitions are empty. To complete Partition 7, we added three noncontributing 7-boxes beneath the second row in ⑦ (along with three noncontributing 6-boxes to the second row and three noncontributing 5-boxes to the first row). Finally, we added one noncontributing 6-box to the first row in ⑧ to complete Partition 6. This gives us the desired rigged configuration.

Example 3.9.3. Consider the following A_{10} rigged configuration $S = (\nu_1, \nu_2, \ldots, \nu_{10})$ (in top-bottom order) where ν_i is the ith rigged partition whose jth row has rigging rig_i^j:



From the viewpoint of cascading sequences, S is constructed in the following process:

$$
\begin{aligned}
(10')(10')(10')(10') &\rightarrow^{①} (9',10)(9',10)(9',10)(9',10)\\
&\rightarrow^{②} (9,10)(9,10)(9,10)(9,10)(8,9)'(8,9)'(8,9)'\\
&\rightarrow^{③} (8',9,10)(8',9,10)(8',9,10)(9,10)(7',8,9)(7',8,9)\\
&\quad (7',8,9)\\
&\rightarrow^{④} (8,9,10)(8,9,10)(8,9,10)(9,10)(7,8,9)(7,8,9)\\
&\quad (7,8,9)(6,7,8)'(6,7,8)'(6,7,8)'\\
&\rightarrow^{⑤} (8,9,10)(8,9,10)(8,9,10)(8',9,10)(7,8,9)(7,8,9)
\end{aligned}
$$

$$(7,8,9)(6,7,8)(6,7,8)(6,7,8)$$
$$\to^{⑥} (8,9,10)(8,9,10)(8,9,10)(8,9,10)(7,8,9)(7,8,9)$$
$$(7,8,9)(6,7,8)(6,7,8)(6,7,8)(7,8)'$$
$$\to^{⑦} (7',8,9,10)(7',8,9,10)(8,9,10)(8,9,10)(6',7,8,9)$$
$$(6',7,8,9)(7,8,9)(5',6,7,8)(5',6,7,8)(6,7,8)(7,8)$$
$$\to^{⑧} (7,8,9,10)(7,8,9,10)(7',8,9,10)(8,9,10)(6,7,8,9)$$
$$(6,7,8,9)(7,8,9)(5,6,7,8)(5,6,7,8)(6,7,8)(6',7,8)$$
$$\to^{⑨} (6',7,8,9,10)(7,8,9,10)(7,8,9,10)(8,9,10)$$
$$(6,7,8,9)(6,7,8,9)(7,8,9)(5,6,7,8)(5,6,7,8)$$
$$(5',6,7,8)(6,7,8)$$

Explanation of the above process:

We started out by adding four 10-boxes, which completes Partition 10. Since $\mathrm{rig}_{10}^1 = -1 = -4+3$, we first added four noncontributing 9-boxes in ①, and then added three contributing 9-boxes in ② beneath these noncontributing boxes, which completes Partition 9 and adds three noncontributing 8-boxes. Since $\mathrm{rig}_9^2 = 0 = -3+3$ and $\mathrm{rig}_9^1 = 1 = -3+4$, we first added three noncontributing 8-boxes beneath the first row in ③ (along with three noncontributing 7-boxes), and then added three contributing 8-boxes beneath the second row in ④ (along with three noncontributing 7-boxes and three noncontributing 6-boxes), and then added one noncontributing 8-box to the first row in ⑤, and then added one contributing 8-box beneath the first row in ⑥ (along with one noncontributing 7-box to the first row). This completes Partition 8. Since $\mathrm{rig}_8^3 = -3+0$ and $\mathrm{rig}_8^1 = \mathrm{rig}_8^2 = -4+0$, there are no contributing 7-boxes to add. In ⑦, we added two noncontributing 7-boxes to the third row. In ⑧, we added a noncontributing 7-box to the second row. In ⑨, we added a noncontributing 6-box to the second row. This completes Partition 6, and yields the desired rigged configuration.

4. Further discussions

One can try to characterize $\mathcal{B}(\infty)$ rigged configurations in the types B, C, D, G, by modifying or extending the methods used in this paper. One can also try to find a non-recursive characterization of $\mathcal{B}(\infty)$ rigged configurations, which describes the ith rigged partition without reference to the $(i+1)$st partition.

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The author used Sage ([7] and [8]) extensively to do computations with marginally large tableaux and rigged configurations that would have been forbidding by hand.

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