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Combinatorial and arithmetical properties of the restricted and associated Bell and factorial numbers

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Set partitions and permutations with restrictions on the size of the blocks and cycles are important combinatorial sequences. Counting these objects lead to the sequences generalizing the classical Stirling and Bell numbers. The main focus of the present article is the analysis of combinatorial and arithmetical properties of them. The results include several combinatorial identities and recurrences as well as some properties of their p-adic valuations.

1. Introduction

The (unsigned) Stirling numbers of the first kind denoted by $c(n,k)$ or $\left[{n \atop k} \right]$ enumerate the number of permutations on n elements with k cycles. The corresponding Stirling numbers of the second kind, denoted by $S(n,k)$ or $\left\{ {n \atop k} \right\}$, enumerate the number of partitions of a set with n elements into k non-empty blocks; see [19] for general information about them. The recurrences

$$\left[{n+1 \atop k} \right] = \left[{n \atop k-1} \right] + n\left[{n \atop k} \right] \quad \text{and}$$
$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k\left\{ {n \atop k} \right\},$$

with the initial conditions

$$\left[{0 \atop 0} \right] = 1, \quad \left[{n \atop 0} \right] = \left[{0 \atop n} \right] = 0,$$
$$\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0,$$

hold for $n \geq 1$. They are related to each other by the orthogonality relation

$$\sum_{k \geq 0} \left\{ {n \atop k} \right\} \left[{k \atop m} \right] (-1)^{n-k} = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta function.

The Bell numbers, B_n, enumerate the set partitions of a set with n elements, so that $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$. The *Spivey's formula* [39]

$$B_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} \left\{ {m \atop j} \right\} \binom{n}{k} B_k, \tag{1}$$

gives a recurrence for them. Further properties of this sequence appear in [19, 28].

The literature contains several generalizations of Stirling numbers; see [29]. Among them, the so-called restricted and associated Stirling numbers of both kinds (cf. [10, 15, 16, 17, 19, 23, 24, 25, 26, 32]) constitute the central character of the work presented here.

The *restricted Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}_{\leq m}$ give the number of partitions of n elements into k subsets, with the additional restriction that none of the blocks contain more than m elements. Komatsu et al. [23] derived the recurrences

$$\begin{aligned} \left\{ {n+1 \atop k} \right\}_{\leq m} &= \sum_{j=0}^{m-1} \binom{n}{j} \left\{ {n-j \atop k-1} \right\}_{\leq m} \\ &= k \left\{ {n \atop k} \right\}_{\leq m} + \left\{ {n \atop k-1} \right\}_{\leq m} - \binom{n}{m} \left\{ {n-m \atop k-1} \right\}_{\leq m}, \end{aligned} \tag{2}$$

with initial conditions $\left\{ {0 \atop 0} \right\}_{\leq m} = 1$ and $\left\{ {n \atop 0} \right\}_{\leq m} = 0$, for $n \geq 1$. The *restricted Bell numbers* defined by [33]

$$B_{n,\leq m} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\leq m},$$

enumerate partitions of n elements into blocks, each one of them with at most m elements. For example, $B_{4,\leq 3} = 14$, the partitions being

$$\begin{array}{llll} \{\{1\},\{2\},\{3\},\{4\}\}, & \{\{1,2\},\{3\},\{4\}\}, & \{\{1,2\},\{3,4\}\}, & \{\{1,3\},\{2\},\{4\}\}, \\ \{\{1,3\},\{2,4\}\}, & \{\{1,4\},\{2\},\{3\}\}, & \{\{1,4\},\{2,3\}\}, & \{\{1,2,3\},\{4\}\}, \\ \{\{1,2,4\},\{3\}\}, & \{\{1,3,4\},\{2\}\}, & \{\{1\},\{2,3,4\}\}, & \{\{1\},\{2\},\{3,4\}\}, \\ \{\{1\},\{2,4\},\{3\}\}, & \{\{1\},\{2,3\},\{4\}\}. & & \end{array}$$

An associated sequence is the *restricted Stirling numbers of the first kind* $\left[{n \atop k} \right]_{\leq m}$. This gives the number of permutations on n elements with k cycles with the restriction that none of the cycles contain more than m items (see [32] for more information). Komatsu et al. [24] established the recurrence

$$\text{(3)} \quad \begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{\leq m} &= \sum_{j=0}^{m-1} \frac{n!}{(n-j)!} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{\leq m} \\ &= n \begin{bmatrix} n \\ k \end{bmatrix}_{\leq m} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{\leq m} - \frac{n!}{(n-m)!} \begin{bmatrix} n-m \\ k-1 \end{bmatrix}_{\leq m}, \end{aligned}$$

with initial conditions $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\leq m} = 1$ and $\begin{bmatrix} n \\ 0 \end{bmatrix}_{\leq m} = 0$. The *restricted factorial numbers*, see [32], are defined by

$$A_{n,\leq m} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\leq m}.$$

These enumerate all permutations of n elements into cycles with the condition that every cycle has at most m items. For example, $A_{4,\leq 3} = 18$ with the permutations being

(1)(2)(3)(4),	(1)(2)(43),	(1)(32)(4),	(1)(342),	(1)(432),
(1)(42)(3),	(21)(3)(4),	(21)(43),	(231)(4),	(241)(3),
(321)(4),	(31)(2)(4),	(341)(2),	(31)(42),	(421)(3),
(431)(2),	(41)(2)(3),	(41)(32).		

The outline of the paper is this: Section 2 contains some known identities of the restricted Bell numbers $B_{n,\leq 2}$. In this case, $m = 2$, the restricted Bell and restricted factorial numbers coincide, i.e., $B_{n,\leq 2} = A_{n,\leq 2}$. Information about their Hankel transform is included. Section 3 contains extensions of these properties to $m = 3$ and Sections 4 and 5 present the general case. Section 6 establishes the log-convexity of the restricted Bell and factorial sequences, extending classical results. Some conjectures on the roots of the restricted Bell polynomials are proposed here. Finally, Section 7 presents some preliminary results on the p-adic valuations of these sequences. Explicit expressions for the prime $p = 2$ are established. A more complete discussion of these issues is in preparation.

2. Restricted Bell numbers $B_{n,\leq 2}$ and restricted factorial numbers $A_{n,\leq 2}$

This section discusses the sequence $B_{n,\leq 2}$, which enumerates partitions of n elements into blocks of length at most 2. Then $B_{n,\leq 2} = A_{n,\leq 2}$ is precisely the number of *involutions* of the n elements, denoted in [5] by $\mathrm{Inv}_1(n)$. This sequence is also called *Bessel numbers of the second kind*, see [14] for further information.

The well-known recurrence

$$B_{n,\leq 2} = B_{n-1,\leq 2} + (n-1)B_{n-2,\leq 2}, \tag{4}$$

with initial conditions $B_{0,\leq 2} = B_{1,\leq 2} = 1$, yields the exponential generating function

$$\sum_{n=0}^{\infty} B_{n,\leq 2} \frac{x^n}{n!} = \exp\left(x + \tfrac{1}{2}x^2\right) \tag{5}$$

as well as the closed-form expression

$$B_{n,\leq 2} = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \frac{(2j)!}{2^j\, j!}. \tag{6}$$

The recurrence

$$B_{n_1+n_2,\leq 2} = \sum_{k\geq 0} k! \binom{n_1}{k}\binom{n_2}{k} B_{n_1-k,\leq 2} B_{n_2-k,\leq 2} \tag{7}$$

is established in [5].

Congruences for the involution numbers appeared in Mező [32], in a problem on the distribution of last digits of related sequences. These include

$$B_{n,\leq 2} \equiv B_{n+5,\leq 2} \bmod 10 \text{ if } n > 1 \quad \text{and } B_{n,\leq 3} \equiv B_{n+5,\leq 2} \bmod 10 \text{ if } n > 3. \tag{8}$$

2.1. The Hankel transform of $B_{n,\leq 2}$

For a sequence $A = (a_n)_{n\in\mathbb{N}}$, its *Hankel matrix* H_n of order n is defined by

$$H_n = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{bmatrix}.$$

The *Hankel transform* of A is the sequence $(\det H_n)_{n\in\mathbb{N}}$. Aigner [1] showed that the Hankel transform of the Bell numbers is the sequence of the product of first n factorials, so-called *superfactorials*, i.e., $(1!, 1!2!, 1!2!3!, \dots)$. Theorem 2.2 below shows that the Hankel transform of $B_{n,\leq 2}$ is also given by superfactorials.

The first result gives the binomial transform of $B_{n,\leq 2}$. This involves the *double factorials*

$$(2n-1)!! = \prod_{k=1}^{n}(2k-1) = \frac{(2n)!}{n!2^n}.$$

Proposition 2.1. *The binomial transform of the sequence $B_{n,\leq 2}$ is*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}B_{i,\leq 2} = \begin{cases} (n-1)!!, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

The numbers on the right are called the aerated double factorial.

Proof. The exponential generating function $A(x)$ of a sequence $(a_n)_{n\geq 0}$ and that of its binomial transform $S(x)$ are related by $S(x) = e^{-x}A(x)$. The result now follows from (5). □

Combinatorial Proof of Proposition 2.1: Let $\mathcal{B}_{n,\leq 2}$ be the set of all partitions into blocks of length at most 2. Let $\mathcal{S}_{n,i} = \{\pi \in \mathcal{B}_{n,\leq 2} : \{i\} \in \pi\}$ be the set of partitions of $[n]$ in blocks of length less or equal to 2, where i is a singleton block. There are $B_{n-1,\leq 2}$ of them. Then

$$\mathcal{B}_{n,\leq 2} = \bigcup_{i=1}^{n}\mathcal{S}_{n,i} \bigcup \underbrace{(\mathcal{B}_{n,\leq 2} \setminus (\bigcup_{i=1}^{n}\mathcal{S}_{n,i}))}_{\text{Denote this by } L_n}.$$

The inclusion-exclusion principle gives

$$B_{n,\leq 2} = \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}B_{n-i,\leq 2} + |L_n|,$$

that yields

$$\begin{aligned} |L_n| &= B_{n,\leq 2} - \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}B_{n-i,\leq 2} \\ &= \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}B_{n-i,\leq 2} \end{aligned} \tag{9}$$

$$= \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}B_{i,\leq 2}.$$

On the other hand, $L_n = \{\pi \in \mathcal{B}_{n,\leq 2} :$ such that if $B \in \pi$ then $|B| = 2\}$, because it is the complement of the partitions with at least one singleton. Thus

$$|L_n| = \frac{\binom{n}{2,2,\ldots,2}}{\left(\frac{n}{2}\right)!} = \frac{n!}{2^{n/2}\left(\frac{n}{2}\right)!}$$

if n is even and 0 if n is odd. This establishes the identity.

Barry and Hennessy [7, Example 16] show that the Hankel transform of the aerated double factorial is the superfactorials. The fact that any integer sequence has the same Hankel transform as its binomial transform [27, 40], gives the next result.

Theorem 2.2. *The Hankel transform of the restricted Bell numbers $B_{n,\leq 2}$ is the superfactorials; that is, for any fixed n,*

$$\det\begin{bmatrix} B_{0,\leq 2} & B_{1,\leq 2} & B_{2,\leq 2} & \cdots & B_{n,\leq 2} \\ B_{1,\leq 2} & B_{2,\leq 2} & B_{3,\leq 2} & \cdots & B_{n+1,\leq 2} \\ \vdots & \vdots & \vdots & & \vdots \\ B_{n,\leq 2} & B_{n+1,\leq 2} & B_{n+2,\leq 2} & \cdots & B_{2n,\leq 2} \end{bmatrix} = \prod_{i=0}^{n} i!.$$

The Theorem 2.2 was proved by Ehrenborg [20, Theorem 3] for the restricted Bell polynomials.

3. The restricted Bell numbers $B_{n,\leq 3}$ and the restricted factorial numbers $A_{n,\leq 3}$

The goal of the current section is to extend some results in the previous section to the case $m = 3$. Recurrences established here are employed in Section 7 to discuss arithmetic properties of $B_{n,\leq 3}$ and $A_{n,\leq 3}$.

The first statement relates $B_{n,\leq 3}$ to the involution numbers $B_{n,\leq 2}$.

Theorem 3.1. *The restricted Bell numbers $B_{n,\leq 3}$ are given by*

$$B_{n,\leq 3} = \sum_{j=0}^{\lfloor n/3 \rfloor}\binom{n}{3j}\frac{(3j)!}{(3!)^j\, j!}B_{n-3j,\leq 2}. \tag{10}$$

Proof. Count the set of all partitions of $[n]$ into block of size at most 3, with exactly j blocks of size 3. To do so, first choose a subset of $[n]$ of size $3j$ to place the j blocks of size 3. This is done in $\binom{n}{3j}$ ways. Then, the number of set partitions of $[3j]$ such that each block has three elements is $\frac{(3j)!}{(3!)^j\, j!}$. The remaining $n-3j$ elements produce $B_{n-2j,\leq 2}$ partitions. Summing over j completes the argument. □

The next result gives a recurrence for $B_{n,\leq 3}$.

Theorem 3.2. *The restricted Bell numbers $B_{n,\leq 3}$ satisfy the recurrence*

$$B_{n,\leq 3} = B_{n-1,\leq 3} + \binom{n-1}{1} B_{n-2,\leq 3} + \binom{n-1}{2} B_{n-3,\leq 3}, \tag{11}$$

with initial conditions $B_{0,\leq 3}=1$, $B_{1,\leq 3}=1$, $B_{2,\leq 3}=2$.

Proof. The expression for $B_{n,\leq 2}$ in (6) and (10) produce

$$B_{n,\leq 3} = \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{3i} \frac{(3i)!}{6^i i!} \binom{n-3i}{2j} \binom{2j}{j} \frac{j!}{2^j},$$

that may be written as

$$B_{n,\leq 3} = \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{3i+2j} \binom{3i+2j}{2j} \binom{2j}{j} \frac{(3i)!\, j!}{6^i\, i!\, 2^j}.$$

The recurrence is obtained as a routine application of the WZ-method [35, 36]. □

Combinatorial proof of Theorem 3.2: Suppose the first block is the size i with $i=1,2$ or 3. Since this block contains the minimal element, one only needs to choose l elements, with $l=0,1$ or 2. Therefore, the number of set partitions of $[n]$ with exactly i elements in the first block is given by $\binom{n-1}{i} B_{n-i,\leq 3}$ for $i=1,2,3$. Summing over i completes the argument.

The above recurrence produces the exponential generating function

$$\sum_{n=0}^{\infty} B_{n,\leq 3} \frac{x^n}{n!} = \exp\left(x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3\right). \tag{12}$$

The next result is an extension of (7) for the case of partitions with blocks of length at most 3. It is an analog of the Spivey-like formula (1).

Theorem 3.3. *Define* $a(i,j) = \frac{1}{3}(2i - j - k)$. *Then the restricted Bell numbers* $B_{n,\leq 3}$ *satisfy the relation*

$$\begin{aligned} B_{n+m,\leq 3} &= \sum_{i=0}^{n} \binom{n}{i} B_{n-i,\leq 3} \sum_{j=\lceil \frac{i}{2} \rceil}^{\min\{m,2i\}} \binom{m}{j} B_{m-j,\leq 3} \\ &\quad \times \sum_{\substack{k=0 \\ k \equiv -i-j \bmod 3}}^{\min\{i,j,2i-j,2j-i\}} \binom{i}{k}\binom{j}{k} k! \binom{i-k}{a(j,i)} \binom{j-k}{a(i,j)} \frac{(2a(i,j))!\,(2a(j,i))!}{2^{a(i,j)+a(j,i)}} \\ &= \sum_{i=0}^{n} \sum_{j=\lceil \frac{i}{2} \rceil}^{\min\{m,2i\}} \sum_{\substack{k=0 \\ k \equiv -i-j \bmod 3}}^{\min\{i,j,2i-j,2j-i\}} \frac{n!m!B_{n-i,\leq 3}B_{m-j,\leq 3}}{k!(n-i)!(m-j)!a(i,j)!a(j,i)!2^{\frac{i+j-2k}{3}}}. \end{aligned}$$

Proof. The set of $n+m$ elements whose partitions are enumerated by $B_{n+m,\leq 3}$ is split into two disjoint sets I_1 and I_2 of cardinality n and m, respectively. Any such partition π can be written uniquely in the form $\pi = \pi_1 \cup \pi_2 \cup \pi_3$, where π_1 is a partition of a subset of I_1, π_2 is a partition of a subset of I_2 and the blocks in π_3 contain elements of both I_1 and I_2. Denote by a_2 the number of blocks in π_1 and a_5 those in π_2.

The blocks in π_3 come in three different forms:

Type 1. The block is of the form $x = \{\alpha_1, \beta_1\}$ with $\alpha_1 \in I_1$ and $\beta_1 \in I_2$. Let a_1 be the number of them. The $n+m$ elements can be placed into these type of blocks in

$$\binom{n}{a_1}\binom{m}{a_1} a_1! \quad \text{ways.}$$

Type 2. The form is now $x = \{\alpha_1, \beta_1, \beta_2\}$ with $\alpha_1 \in I_1$ and $\beta_j \in I_2$, for $j = 1, 2$. Let a_3 denote the number of these type of blocks. These contributed

$$\binom{n}{2a_3}\binom{m}{a_3}\frac{(2a_3)!}{2^{a_3}} \quad \text{to the placement of the } n+m \text{ elements.}$$

Type 3. The final form is $x = \{\alpha_1, \alpha_2, \beta_1\}$ with $\alpha_j \in I_1$, $j = 1$, 2 and $\beta_1 \in I_2$. Denote by a_4 the number of such blocks. These contribute

$$\binom{n}{a_4}\binom{m}{2a_4}\frac{(2a_4)!}{2^{a_4}} \quad \text{to the count.}$$

Therefore the total number of partitions is given by

$$B_{n+m,\leq 3} = \sum \binom{n}{a_1, a_2, a_3, 2a_4} \binom{m}{a_1, a_5, a_4, 2a_3} a_1! \frac{(2a_3)!}{2^{a_3}} \frac{(2a_4)!}{2^{a_4}} B_{a_2,\leq n} B_{a_5,\leq m},$$

where the sum extends over all indices $0 \leq n_1, n_2, n_3, n_4, n_5$ such that

$$a_1 + a_2 + a_3 + 2a_4 = n \text{ and } a_1 + a_5 + 2a_3 + a_4 = m.$$

Introduce the notation $i = n - a_2$, $j = m - a_5$ and $k = a_1$ (so that i, j, $k \geq 0$) and solve for a_3 and a_4 from

$$\begin{aligned} a_3 + 2a_4 &= i - k, \\ 2a_3 + a_4 &= j - k \end{aligned}$$

to obtain

$$a_3 = \frac{2j - i - k}{3} \quad \text{and} \quad a_4 = \frac{2i - j - k}{3}. \tag{13}$$

The fact that a_3, $a_4 \in \mathbb{N}$ is equivalent to $i + j + k \equiv 0 \bmod 3$. This gives the result. □

The following theorem is the analog of Theorems 3.1, 3.2 and (12). The proof is similar, so it is omitted. The interested reader can find the proof of this theorem in [43].

Theorem 3.4. *The restricted factorial sequence $A_{n,\leq 3}$ is given by*

$$A_{n,\leq 3} = \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{n}{3j} \frac{(3j)!}{3^j\, j!} A_{n-3j,\leq 2}.$$

Moreover, it satisfies the recurrence

$$A_{n,\leq 3} = A_{n-1,\leq 3} + (n-1)A_{n-2,\leq 3} + (n-1)(n-2)A_{n-3,\leq 3},$$

with initial conditions $A_{0,\leq 3} = 1$, $A_{1,\leq 3} = 1$, $A_{2,\leq 3} = 2$. Its generating function is

$$\sum_{n=0}^{\infty} A_{n,\leq 3} \frac{x^n}{n!} = \exp\left(x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right).$$

4. The general case $B_{n,\leq m}$

In this section, some recurrences of the restricted Bell numbers are generalized. A relation between this sequence and the associated Bell numbers is established. The first statement generalizes (6) and (10).

Theorem 4.1. *The restricted Bell numbers $B_{n,\leq m}$ satisfy the recurrence*

$$B_{n,\leq m} = \sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} \frac{n!}{i!(m!)^i (n-im)!} B_{n-im,\leq m-1}. \tag{14}$$

Proof. Count the set of all partitions of $[n]$ with blocks of size at most k and contain exactly i blocks of size m. To do so, select $m \cdot i$ elements from n without any order. This is done in $\binom{n}{\underbrace{m, m, \ldots, m}_{i \text{ times}}} = \frac{n!}{m!^i (n-im)!}$ ways. Now divide by $i!$ to take into account the order of the blocks. The $n-im$ remaining elements of $[n]$ are placed in blocks of size $m-1$ or less elements, counted by $B_{n-im,\leq m-1}$. The result follows by summing over i. □

A direct argument generalizes Theorems 3.2 and (12). This result appears in [33].

Theorem 4.2. *The restricted Bell numbers $B_{n,\leq m}$ satisfy the recurrence*

$$B_{n,\leq m} = \sum_{k=0}^{m-1} \binom{n-1}{k} B_{n-k-1,\leq m}.$$

Moreover, their exponential generating function is

$$\sum_{n=0}^{\infty} \frac{B_{n,\leq m} x^n}{n!} = \exp\left(\sum_{i=1}^{m} \frac{x^i}{i!} \right). \tag{15}$$

The next result generalizes Theorem 3.3.

Theorem 4.3. *Denote $f(i,j) = 2 + j + \binom{i-1}{2}$, then*

$$B_{n+m,\leq k} = n!m! \sum_{X} \frac{B_{a_1,\leq k} B_{a_2 \leq k}}{a_1! a_2! \prod_{i=2}^{k} \prod_{j=1}^{i-1} j!^{a_{f(i,j)}} (i-j)!^{a_{f(i,j)}} a_{f(i,j)}!},$$

where X stands for the following set of variables

$$X = \{(a_1, a_2, \ldots, a_{1+k+\binom{k-1}{2}}) : \\ a_1 + \sum_{i=2}^{k}\sum_{j=1}^{i-1} j a_{f(i,j)} = n \wedge a_2 + \sum_{i=2}^{k}\sum_{j=1}^{i-1}(i-j)a_{f(i,j)} = m\}.$$

Proof. The set of $n+m$ elements, whose partitions are enumerated by $B_{n+m,\leq 3}$, is split into two disjoint sets $I_1 = [n]$ and $I_2 = [n+m] \setminus [n]$ of cardinality n and m, respectively. Any such partition π can be written uniquely in the form $\pi = \pi_1 \cup \pi_2 \cup \pi_3$, where blocks in π_1 are subsets of I_1, blocks in π_2 are subsets of I_2 and the blocks in π_3 contain elements of I_1 and I_2. Denote by a_1 the number of elements that are going to be in blocks of π_1 and by a_2 the numbers of elements that are going to be in blocks of π_2. These are counted by $B_{a_1,\leq k}B_{a_2,\leq k}$.

The blocks in π_3 come in different forms depending in how many elements are in the blocks and how many come from $[n]$ and how many from $[n+m] \setminus [n]$. Denote by $a_{f(i,j)}$ the number of blocks in π_3 which have $j > 0$ elements of $[n]$ and $i-j > 0$ from $[n+m] \setminus [n]$. It is required to choose $ja_{f(i,j)}$ elements from $[n]$ and $(i-j)a_{f(i,j)}$ from $[n+m] \setminus [n]$. The total number of choices for grouping the $a_{f(i,j)}$ blocks is given by

$$\binom{(i-j)a_{f(i,j)}}{\underbrace{i-j, i-j, \ldots, i-j}_{a_{f(i,j)} \text{ times}}}\binom{ja_{f(i,j)}}{\underbrace{j, j, \ldots, j}_{a_{f(i,j)} \text{ times}}}\frac{1}{a_{f(i,j)}!} = \frac{(ja_{f(i,j)})!((i-j)a_{f(i,j)})!}{j!^{a_{f(i,j)}}(i-j)!^{a_{f(i,j)}}a_{f(i,j)}!}.$$

The multinomial coefficient accounts for the possible groups of each side and the factorial in the denominator accounts for the order of the blocks. Summing over all possible configurations gives the result. □

4.1. Relations between restricted and associated Bell numbers

The *associated Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}_{\geq m}$ give the number of partitions of n elements into k subsets under the restriction that every block contains *at least* m elements. Komatsu et al. [23] derived the recurrence

$$\begin{aligned} \left\{ {n+1 \atop k} \right\}_{\geq m} &= \sum_{j=m-1}^{n} \binom{n}{j} \left\{ {n-j \atop k-1} \right\}_{\geq m} \\ &= k\left\{ {n \atop k} \right\}_{\geq m} + \binom{n}{m-1}\left\{ {n-m+1 \atop k-1} \right\}_{\geq m}, \end{aligned}$$

with initial conditions $\left\{ {0 \atop 0} \right\}_{\geq m} = 1$ and $\left\{ {n \atop 0} \right\}_{\geq m} = 0$. The *associated Bell numbers* are defined by

$$B_{n,\geq m} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\geq m}.$$

They enumerate partitions of n elements into blocks with the condition that every block has at least the m elements. For example, $B_{4,\geq 3} = 1$ with the partition being $\{\{1,2,3,4\}\}$. Their generating function is

$$\sum_{n=0}^{\infty} \frac{B_{n,\geq m} x^n}{n!} = \exp\left(\exp(x) - \sum_{i=0}^{m-1} \frac{x^i}{i!}\right). \tag{16}$$

In the case $m = 2$, $B_{n,\geq 2}$ enumerate partitions of n elements without singleton blocks, it satisfies (cf. [10])

$$B_n = B_{n,\geq 2} + B_{n+1,\geq 2}, \tag{17}$$

and its exponential generating function is given by

$$\sum_{n=0}^{\infty} B_{n,\geq 2} \frac{x^n}{n!} = \exp\left(\exp(x) - 1 - x\right). \tag{18}$$

Therefore, the binomial transform of $B_{n,\geq 2}$ is the Bell sequence B_n, i.e.,

$$\sum_{i=0}^{n} \binom{n}{i} B_{i,\geq 2} = B_n. \tag{19}$$

The fact that integer sequences and their inverse binomial transform have the same Hankel transform [40], gives the following result. Note that Theorem 4.4 is a particular case of Theorem 3 in [20].

Theorem 4.4. *The Hankel transform of the associated Bell numbers $B_{n,\geq 2}$ is the superfactorials. That is, for any fixed n,*

$$\det \begin{bmatrix} B_{0,\geq 2} & B_{1,\geq 2} & B_{2,\geq 2} & \cdots & B_{n,\geq 2} \\ B_{1,\geq 2} & B_{2,\geq 2} & B_{3,\geq 2} & \cdots & B_{n+1,\geq 2} \\ \vdots & \vdots & \vdots & & \vdots \\ B_{n,\geq 2} & B_{n+1,\geq 2} & B_{n+2,\geq 2} & \cdots & B_{2n,\geq 2} \end{bmatrix} = \prod_{i=0}^{n} i!.$$

Theorem 4.5. *The associated Bell numbers $B_{n,\geq 2}$ and the Bell numbers B_n are related by*

$$B_{n,\geq 2} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} B_{n-i}.$$

Proof. Let $\mathcal{B}_n$ be the set of all partitions of $[n]$, and let $\mathcal{B}_{n,\geq 2}$ be the set of all partitions into blocks of length of at least 2. Denote by $\mathcal{S}_{n,i}$ the set of partitions where i is in a singleton block. Then

$$\mathcal{B}_{n,\geq 2} = \mathcal{B}_n \setminus \bigcup_{i\in[n]} \mathcal{S}_{n,i}, \tag{20}$$

and the inclusion-exclusion principle produces

$$B_{n,\geq 2} = B_n - \sum_{i=1}^{n} (-1)^{i-1} \sum_{a_1<a_2<\cdots<a_i} \left| \bigcap_{j=1}^{i} S_{n,a_j} \right|.$$

The identity now follows from $\left|\bigcap_{j=1}^{i} S_{n,a_j}\right| = B_{n-i}$. □

The next result gives a reduction for the associated Bell numbers $B_{n,\geq k}$, in the index k counting the minimal number of elements in a block.

Theorem 4.6. *The associated Bell numbers $B_{n,\geq k}$ satisfy*

$$B_{n,\geq k} = B_{n,\geq k-1} - \sum_{i=1}^{\lfloor \frac{n}{k-1} \rfloor} \frac{n!}{(k-1)!^i (n-(k-1)i)! i!} B_{n-(k-1)i,\geq k}.$$

Proof. Denote by $\mathcal{B}_{n,\geq k}$ the set of all partitions with blocks of length at least than k. Then $\mathcal{B}_{n,\geq k} \subseteq \mathcal{B}_{n,\geq k-1}$ and let $A = \mathcal{B}_{n,\geq k-1} \setminus \mathcal{B}_{n,\geq k}$ be the set difference. For $1 \leq k \leq n$, define

$$A_i = \{\pi \in \mathcal{B}_{n,\geq k-1} : \text{ the number of blocks of size } k-1 \text{ is } i\},$$

and observe that

$$A := \bigcup_{i=1}^{n} A_i = \mathcal{B}_{n,\geq k-1} \setminus \mathcal{B}_{n,\geq k}.$$

The sets $\{A_i\}$ form a partition of A with

$$|A_i| = \frac{1}{i!} \binom{n}{k-1, k-1, \ldots, k-1} B_{n-i(k-1),\geq k}. \tag{21}$$

The identity follows from this. □

Theorem 4.7. *The associated Bell numbers can be calculated from the Bell numbers and restricted Bell numbers via*

$$B_{n,\geq k} = B_n - \sum_{i=1}^{n} \binom{n}{i} B_{i,\leq k-1} B_{n-i,\geq k}. \tag{22}$$

Proof. Recall that $\mathcal{B}_n$ is the set of partitions of $[n]$. For any such partition, write $\pi = \{A, B\}$, where $A = \{\pi \in \mathcal{B}_n : \text{if } D \in \pi, \text{ then } |D| \geq k\}$, and B the complement of A in $\mathcal{B}_n$. Then $|A| + |B| = B_n$. Now $|A| = B_{n,\geq k}$ and B can be partitioned in $\{C_i\}_{i\in[n]}$ where C_i contains the partitions such that there are exactly i elements of $[n]$ that are in blocks with length less than k and the remaining $n-i$ are in blocks with length greater or equal to k. Therefore

$$|C_i| = \binom{n}{i} B_{i,\leq k-1} B_{n-i,\geq k},$$

and the result follows by summing over all partitions of $[n]$. □

The next result is the analog of Theorem 4.3 for the case of the associated Bell numbers.

Theorem 4.8. *Denote $f(i,j) = 2 + j + \binom{i-1}{2}$, then*

$$B_{n+m,\geq k} = n!m! \sum_{X} \frac{B_{a_1,\geq k} B_{a_2,\geq k}}{a_1! a_2! \prod_{i=k}^{n+m} \prod_{j=1}^{i-1} j!^{a_{f(i,j)}} (i-j)!^{a_{f(i,j)}} a_{f(i,j)}!},$$

where X stands for the following set of variables

$$X = \Bigg\{ (a_1, a_2, a_{3+\binom{k-1}{2}}, \ldots, a_{2+n+m-1+\binom{n+m-1}{2}}) : \\ a_1 + \sum_{i=k}^{n+m} \sum_{j=1}^{i-1} j a_{f(i,j)} = n \wedge a_2 + \sum_{i=k}^{n+m} \sum_{j=1}^{i-1} (i-j) a_{f(i,j)} = m \Bigg\}.$$

5. The general case $A_{n,\leq m}$

This section discusses the results presented in the previous section corresponding to the class $A_{n,\leq m}$.

The first statement generalizes Theorem 3.4 and is the analog of Theorem 4.1. The proof is similar to the one given above. Details are omitted.

Theorem 5.1. *The restricted factorial numbers $A_{n,\leq m}$ are given by*

$$A_{n,\leq m} = \sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} \frac{n!}{m^i i!(n-im)!} A_{n-im,\leq m-1}.$$

The next statement is found in [32].

Theorem 5.2. *The restricted factorial sequence $A_{n,\leq m}$ satisfies the recurrence*

$$A_{n,\leq m} = \sum_{j=0}^{m-1} \frac{(n-1)!}{(n-1-j)!} A_{n-1-j,\leq m},$$

with initial conditions $A_{0,\leq m} = 1$ and $A_{1,\leq m} = 1$. Its generating function is

$$\sum_{n=0}^{\infty} A_{n,\leq m} \frac{x^n}{n!} = \exp\left(x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots + \tfrac{1}{m}x^m\right).$$

The next reduction formula gives $A_{n+m,\leq k}$ in terms of lower value of the first index.

Theorem 5.3. *Denote $f(i,j) = 2 + j + \binom{i-1}{2}$, then*

$$A_{n+m,\leq k} = n!m! \sum_{X} \frac{A_{a_1,\leq k} A_{a_2 \leq k}}{a_1! a_2!} \prod_{i=2}^{k} \prod_{j=1}^{i-1} \binom{i}{j}^{a_{f(i,j)}} \frac{1}{i^{a_{f(i,j)}} \cdot a_{f(i,j)}!},$$

where X stands for the following set of variables

$$X = \{(a_1, a_2, \ldots, a_{1+k+\binom{k-1}{2}}) : \\ a_1 + \sum_{i=2}^{k} \sum_{j=1}^{i-1} j a_{f(i,j)} = n \wedge a_2 + \sum_{i=2}^{k} \sum_{j=1}^{i-1} (i-j) a_{f(i,j)} = m\}.$$

Example 5.4. The special case $k = 3$ gives

$$A_{n+m,\leq 3} = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{\substack{l=0 \\ l \equiv -n-m+i+j \bmod 3}}^{\min\{n-i,m-j\}} \frac{n!m! A_{i,\leq 3} A_{j,\leq 3}}{i!j!l! \left(\frac{2m-n+i-2j-l}{3}\right)! \left(\frac{2n-m-2i+j-l}{3}\right)!}.$$

5.1. The associated factorial numbers $A_{n,\geq m}$

This section presents analogous results for sequence built from the *associated Stirling numbers of the first kind* $\left[{n \atop k}\right]_{\geq m}$. These numbers satisfy the following recurrence [24]

$$\begin{aligned}\begin{bmatrix} n+1 \\ k \end{bmatrix}_{\geq m} &= \sum_{j=m-1}^{n} \frac{n!}{(n-j)!} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{\geq m} \\ &= n \begin{bmatrix} n \\ k \end{bmatrix}_{\geq m} + \frac{n!}{(n-m+1)!} \begin{bmatrix} n-m+1 \\ k-1 \end{bmatrix}_{\geq m},\end{aligned}$$

with the initial conditions $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_{\geq m} = 1$ and $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right]_{\geq m} = 0$. The *associated factorial numbers* defined by

$$A_{n,\geq m} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\geq m},$$

enumerate all permutations of n elements into cycles with the condition that every cycle has at least the m items. Its generating function is given by [44]

$$\sum_{n=0}^{\infty} A_{n,\geq m} \frac{x^n}{n!} = \exp\left(\sum_{n=m}^{\infty} \frac{x^n}{n} \right) = \exp\left(\log \frac{1}{1-x} - \sum_{n=1}^{m-1} \frac{x^n}{n} \right).$$

In particular, if $m = 2$ we obtain the number of permutations of n elements with no fixed points, the classical derangements numbers. This sequence satisfies that (cf. [9])

$$A_{n,\geq 2} = nA_{n-1,\geq 2} + (-1)^n, \quad n \geq 1, \tag{23}$$

$$= (n-1)(A_{n-1,\geq 2} + A_{n-2,\geq 2}). \tag{24}$$

Radoux [37] has shown that the Hankel transform of the associated factorial numbers $A_{n,\geq 2}$ is given by $\prod_{i=1}^{n} i!^2$.

The following theorem is the analog of Theorem 4.6, with a similar proof. The details are omitted.

Theorem 5.5. *For n, $k \in \mathbb{N}$ with $k > 1$, the associated factorial numbers $A_{n,\geq k}$ satisfy*

$$A_{n,\geq k} = A_{n,\geq k-1} - \sum_{i=1}^{\lfloor \frac{n}{k-1} \rfloor} \frac{n!}{(k-1)^i (n-(k-1)i)! i!} A_{n-(k-1)i,\geq k}.$$

The following result corresponds to Theorem 5.3.

Theorem 5.6. *Denote $f(i,j) = 2 + j + \binom{i-1}{2}$, then*

$$A_{n+m,\geq k} = n! m! \sum_{X} \frac{A_{a_1,\geq k} A_{a_2 \geq k}}{a_1! a_2!} \prod_{i=k}^{n+m} \prod_{j=1}^{i-1} \binom{i}{j}^{a_{f(i,j)}} \frac{1}{i^{a_{f(i,j)}} \cdot a_{f(i,j)}!},$$

where X stands for the following set of variables

$$X = \{(a_1, a_2, \ldots, a_{1+k+\binom{k-1}{2}}) : \quad a_1 + \sum_{i=k}^{n+m} \sum_{j=1}^{i-1} j a_{f(i,j)} = n \wedge a_2 + \sum_{i=k}^{n+m} \sum_{j=1}^{i-1} (i-j) a_{f(i,j)} = m\}.$$

The next statement generalizes (24).

Theorem 5.7. *The associated factorial numbers $A_{n,\geq k}$ satisfy*

$$A_{n,\geq k} = (n-1)A_{n-1,\geq k} + (n-1)^{\underline{k-1}} A_{n-k,\geq k}, \quad n \geq 1,$$

where $n^{\underline{k}} := n(n-1)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}$ and $n^{\underline{0}} = 1$.

Proof. Denote by $\mathcal{A}_{n,\geq k}$ the permutations σ on n elements such that the length of every cycle in σ is not less than k (i.e., $\mathcal{A}_{n,\geq k} = \{\sigma \in \mathcal{S}_n : |\langle i \rangle| \geq k\}$. Here $\langle i \rangle$ denotes the cycle of $i \in [n]$ as a set). For $\sigma \in \mathcal{A}_{n,\geq k}$, there are two cases for $n \in [n]$:

- *Case 1*: here $|\langle n \rangle| = k$. It is required to construct a cycle of length k containing n. In order to do this, one must choose $k-1$ numbers from $[n-1]$ and place them in the same cycle. This can be done in $\binom{n-1}{k-1}$ ways and the total number of possible cycles is $\binom{n-1}{k-1}(k-1)! = (n-1)^{\underline{k-1}}$. The other cycles are counted by $A_{n-k,\geq k}$, for a total of $(n-1)^{\underline{k-1}} A_{n-k,\geq k}$.
- *Case 2*: now $|\langle n \rangle| > k$. Then one needs to place n in any cycle of a permutation $\sigma' \in \mathcal{A}_{n-1,\geq k}$. There are $(n-1)A_{n-1,\geq k}$ ways to do it.

The identity follows from this discussion. □

6. Log-convex and log-concavity properties

A sequence $(a_n)_{n\geq 0}$ of nonnegative real numbers is called *log-concave* if $a_n a_{n+2} \leq a_{n+1}^2$, for all $n \geq 0$. It is called *log-convex* if $a_n a_{n+2} \geq a_{n+1}^2$ for all $n \geq 0$. There is a large collection of results on log-concavity and log-convexity and its relation to combinatorial sequences. Some of these appear in [11], [30], [31], [38] and [44]. The Bell sequence is log-convex [6] and from a general result given by Bender and Canfield [8] it is not difficult to verify that the same is true for restricted Bell numbers and restricted factorial numbers.

Theorem 6.1 (Corollaries 1.1 and 1.2 of [8]). *The restricted Bell sequence* $(B_{n,\leq m})_{n\geq 0}$ *and the restricted factorial sequence* $(A_{n,\leq m})_{n\geq 0}$ *are log-convex and the sequences* $(B_{n,\leq m}/n!)_{n\geq 0}$ *and* $(A_{n,\leq m}/n!)_{n\geq 0}$ *are log-concave.*

6.1. Open questions

Some conjectured statements are collected here. The restricted Bell polynomials are defined by

$$B_{n,\leq m}(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\leq m} x^k.$$

The recurrence (2), produces

$$B_{n+1,\leq m}(x) = xB_{n,\leq m}(x) + xB'_{n,\leq m}(x) - \binom{n}{m} xB_{n-m,\leq m}(x).$$

This can be verified directly:

$$\begin{aligned} B_{n+1,\leq m}(x) &= x\sum_{k=0}^{n} k\left\{ {n \atop k} \right\}_{\leq m} x^{k-1} + x\sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\leq m} x^k \\ &\quad -\binom{n}{m} x \sum_{k=0}^{n-m} \left\{ {n-m \atop k} \right\}_{\leq m} x^k \\ &= xB'_{n,\leq m}(x) + xB_{n,\leq m}(x) - \binom{n}{m} xB_{n-m,\leq m}(x). \end{aligned}$$

The authors have tried, without success, to establish the next two statements:

Conjecture 6.2. *The roots of the polynomial* $B_{n,\leq m}(x)$ *are real and non-positive if* $m \neq 3, 4$.

Recall that a finite sequence $\{a_j,\, 0 \leq j \leq n\}$ of non-negative real numbers is called *unimodal* if there is an index j^* such that $a_{j-1} \leq a_j$ for $1 \leq j \leq j^*$ and $a_{j-1} \geq a_j$ for $j^*+1 \leq j \leq n$. An elementary argument shows that a log-concave sequence must be unimodal. The unimodality of the restricted Stirling numbers $\left(\left\{ {n \atop k} \right\}_{\leq 2} \right)_{k\geq 0}$ was proved by Choi and Smith in [14]. Moreover, Han and Seo [21] gave a combinatorial proof of the log-concavity of this sequence. The log-concavity of the associated Stirling numbers of the first kind was studied by Brenti in [12]. Moreover, Bóna and Mező [10] proved that the associated Stirling numbers of the second kind $\left(\left\{ {n \atop k} \right\}_{\geq 2}\right)_{k \geq 0}$ are log-concave.

Conjecture 6.3. *The sequence of restricted Stirling numbers* $\left(\left\{ {n \atop k} \right\}_{\leq m}\right)_{k \geq 0}$ *is log-concave.*

One of the main sources of log-concave sequences comes from the following fact: if $P(x)$ is a polynomial all of whose zeros are real and negative, its coefficient sequence is log-concave. (See [44, Theorem 4.5.2] for a proof). Therefore the first conjecture implies the second one.

7. Some arithmetical properties

Given a prime p, the p-adic valuation of $x \in \mathbb{N}$, denoted by $\nu_p(x)$, is the highest power of p that divides x. For a given sequence of positive integers $(a_n)_{n \geq 0}$ a description of the sequence of valuations $\nu_p(a_n)$ often presents interesting questions. The classical formula of Legendre for factorials

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^r} \right\rfloor$$

is one of the earliest such descriptions. This may also be expressed in closed-form as

$$\nu_p(n!) = \frac{n - s_p(n)}{p - 1},$$

where $s_p(n)$ is the sum of the digits of n in its expansion in base p. The reader will find in [3, 4, 13, 18, 22, 34, 41, 42] a selection of results on this topic.

The 2-adic valuation of the Bell numbers has been described in [2].

Theorem 7.1. *The 2-adic valuation of the Bell numbers satisfy* $\nu_2(B_n) = 0$ *if* $n \equiv 0, 1 \bmod 3$. *In the missing case,* $n \equiv 2 \bmod 3$, $\nu_2(B_{3n+2})$ *is a periodic sequence of period 4. The repeating values are* $\{1, 2, 2, 1\}$.

The 2-adic valuation of the restricted Bell sequence $B_{n,\leq 2}$ was described in [5].

Theorem 7.2. *The 2-adic valuation of the restricted Bell numbers* $B_{n,\leq 2}$ *satisfy*

$$\nu_2(B_{n,\leq 2}) = \left\lfloor \frac{n}{2} \right\rfloor - 2\left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor = \begin{cases} k, & \textit{if } n = 4k; \\ k, & \textit{if } n = 4k+1; \\ k+1, & \textit{if } n = 4k+2; \\ k+2, & \textit{if } n = 4k+3. \end{cases}$$

This section discusses the 2-adic valuation of the numbers $B_{n,\geq 2}$ and $A_{n,\geq 2}$. Figure 1 shows the first 100 values.

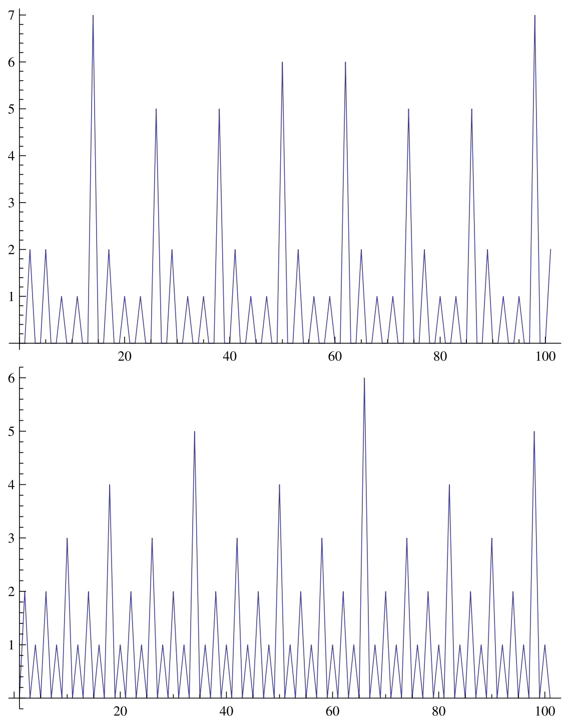

Figure 1: The 2-adic valuation of $B_{n,\geq 2}$ and $A_{n,\geq 2}$.

Theorem 7.3. *The 2-adic valuation of the associated Bell numbers* $B_{n,\geq 2}$ *is given by*

$$\nu_2(B_{n,\geq 2}) = 0 \textit{ if } n \equiv 0,\, 2 \bmod 3. \tag{25}$$

For $n \equiv 1 \bmod 3$, *the valuation satisfies* $\nu_2(B_{n,\geq 2}) \geq 1$.

Proof. The proof is by induction on n. Divide the analysis into three cases according to the residue of n modulo 3. If $n = 3k$ then (17) gives $B_{3k-1} =$

$B_{3k-1,\geq 2}+B_{3k,\geq 2}$. Theorem 7.1 shows that B_{3k-1} is even and by the induction hypothesis $B_{3k-1,\geq 2}$ is odd. Thus $B_{3k,\geq 2}$ is odd, so that $\nu_2(B_{3k,\geq 2})=0$. The proof is analogous for the case $3k+2$. The case $n \equiv 1 \bmod 3$ follows from the identity (17) in the form $B_{3k}=B_{3k,\geq 2}+B_{3k+1,\geq 2}$ and the fact that B_{3k} is odd (by Theorem 7.2) and so is $B_{3k,\geq 2}$ by the previous analysis. □

A partial description of the valuations of $B_{n,\geq 2}$ for $n \equiv 1 \bmod 3$ is given in the next conjecture.

Conjecture 7.4. *The sequence of valuations $\nu_2(B_{3k+1,\geq 2})$ satisfies the following pattern:*

$$\nu_2(B_{n,\geq 2}) = \begin{cases} 2, & \text{if } n \equiv 4 \quad \bmod 12; \\ 1, & \text{if } n \equiv 7,\ 10 \quad \bmod 12. \end{cases} \tag{26}$$

The remaining case $n \equiv 1 \bmod 12$, considered modulo 24, obeys the rule

$$\nu_2(B_{24n+1,\geq 2}) = 5+\nu_2(n), \qquad \text{for } n \geq 1, \tag{27}$$

with the case $n \equiv 13 \bmod 24$ remaining to be determined. Continuing this process yields the conjecture

$$\nu_2(B_{48n+37,\geq 2}) = 5 \text{ and } \nu_2(B_{96n+61,\geq 2}) = 6. \tag{28}$$

The details of this analysis will appear elsewhere.

A closed-form for the valuation $\nu_2(A_{n,\geq 2})$ is simpler to obtain.

Theorem 7.5. *The 2-adic valuation of the associated factorial numbers $A_{n,\geq 2}$ is given by*

$$\nu_2(A_{n,\geq 2}) = \begin{cases} 0, & \text{if } n = 2k \quad \text{and } k \geq 0; \\ \nu_2(k)+1, & \text{if } n = 2k+1 \text{ and } k \geq 1. \end{cases}$$

Proof. If n is even, then (23) shows that $A_{n,\geq 2}$ is odd, so that $\nu_2(A_{n,\geq 2})=0$. If n is odd then (24) gives $\nu_2(A_{2k+1,\geq 2})=\nu_2(2k)=\nu_2(k)+1$. □

7.1. Some additional patterns

In this subsection we show some additional examples of the p-adic valuation of the restricted and associated Bell and factorial sequences.

Theorems 3.4 and 5.7 are now used to produce explicit formulas for the 2-adic valuation of the restricted and associated factorial numbers for $m=3$.

Theorem 7.6. *The 2-adic valuation of the restricted factorial numbers* $A_{n,\leq 3}$*, for* $n \geq 1$*, is given by*

$$\nu_2(A_{n,\leq 3}) = \begin{cases} k, & \text{if } n = 4k; \\ k, & \text{if } n = 4k+1; \\ k+1, & \text{if } n = 4k+2; \\ k+1, & \text{if } n = 4k+3. \end{cases}$$

Proof. The proof is by induction on n. It is divided into four cases according to the residue of n modulo 4. The symbols O_i denote an odd number. If $n = 4k$ then Theorem 3.4 and the induction hypothesis give

$$\begin{aligned} A_{4k,\leq 3} &= A_{4k-1,\leq 3} + (4k-1)A_{4k-2,\leq 3} + (4k-1)(4k-2)A_{4k-3,\leq 3} \\ &= 2^k O_1 + (4k-1)2^k O_2 + (4k-1)(4k-2)2^{k-1}O_3 \\ &= 2^k(O_1 + (4k-1)O_2 + (4k-1)(2k-1)O_3) \\ &= 2^k O_4. \end{aligned}$$

Therefore $\nu_2(A_{4k,\leq 3}) = k$. The remaining cases are analyzed in a similar manner. □

Theorem 7.7. *The 2-adic valuation of the associated factorial numbers* $A_{n,\geq 3}$*, for* $n \geq 1$*, is given by*

$$\nu_2(A_{n,\geq 3}) = \begin{cases} k, & \text{if } n = 4k; \\ \nu_2(k) + k + 2, & \text{if } n = 4k+1; \\ \nu_2(k) + k + 4, & \text{if } n = 4k+2; \\ k+1, & \text{if } n = 4k+3. \end{cases}$$

Proof. The proof is as in the previous theorem. If $n = 4k$ then Theorem 5.7 and the induction hypothesis give

$$\begin{aligned} A_{4k,\geq 3} &= (4k-1)A_{4k-1,\geq 3} + (4k-1)(4k-2)A_{4k-3,\geq 3} \\ &= 2^k O_1 + (4k-1)(4k-2)2^{\nu_2(k-1)+k+1}O_2 \\ &= 2^k(O_1 + (4k-1)(2k-1)2^{\nu_2(k-1)+2}O_2) \\ &= 2^k O_3. \end{aligned}$$

Therefore $\nu_2(A_{4k,\geq 3}) = k$.

If $n = 4k+1$ then Theorem 5.7 and the induction hypothesis now give

$$\begin{aligned} A_{4k+1,\geq 3} &= (4k)A_{4k,\geq 3} + (4k)(4k-1)A_{4k-2,\geq 3} \\ &= 2^{k+2}kO_1 + (4k)(4k-1)2^{\nu_2(k-1)+k+3} \\ &= 2^{k+2}kO_1 + k2^{\nu_2(k-1)+k+5}O_3) \\ &= 2^{k+2}k(O_1 + 2^{\nu_2(k-1)+3}O_3) \\ &= 2^{k+2}kO_4. \end{aligned}$$

Therefore $\nu_2(A_{4k+1,\geq 3}) = \nu_2(k) + k + 2$. The remaining cases are analyzed in a similar manner. □

Divisibility properties of the sequences $B_{n,\leq 2}$ and $B_{n,\leq 3}$ by the prime $p = 3$ turn out to be much simpler: 3 does not divide any element of this sequence. The proof is based on the recurrences (4) and (11).

Theorem 7.8. *The sequence of residues $B_{n,\leq 2}$ modulo 3 is a periodic sequence of period 3, with fundamental period $\{1, 1, 2\}$.*

Proof. Assume $n \equiv 0 \bmod 3$ and write $n = 3k$. Then (4) gives

$$\begin{aligned} B_{3k,\leq 2} &= B_{3k-1,\leq 2} + (3k-1)B_{3k-2,\leq 2} \\ &\equiv 2 - 1 = 1 \bmod 3, \end{aligned}$$

and $B_{n,\leq 2} \equiv 1 \bmod 3$. The remaining two cases for n modulo 3 are treated in the same form. □

Theorem 7.9. *The sequence of residues $B_{n,\leq 3}$ modulo 3 is a periodic sequence of period 6, with fundamental period $\{1, 1, 2, 2, 2, 1\}$.*

Proof. Assume $n \equiv 0 \bmod 6$ and write $n = 6k$. Then (11) gives

$$\begin{aligned} B_{6k,\leq 3} &= B_{6k-1,\leq 3} + (6k-1)B_{6k-2,\leq 3} + (3k-1)(6k-1)B_{6k-3,\leq 3} \\ &\equiv 1 - 2 + 2 = 1 \bmod 3, \end{aligned}$$

showing that $B_{n,\leq 3} \equiv 1 \bmod 3$. The remaining five cases for n modulo 6 are treated in the same form. □

Corollary 7.10. *The restricted Bell numbers $B_{n,\leq 2}$ and $B_{n,\leq 3}$ are not divisible by 3.*

Using this type of analysis it is possible to prove the following results:

- The 5-adic valuation of the sequence $B_{n,\leq 3}$ is given by

$$\nu_5(B_{n,\leq 3}) = \begin{cases} 1, & \text{if } n \equiv 3 \bmod 5; \\ 0, & \text{if } n \not\equiv 3 \bmod 5. \end{cases}$$

- The 7-adic valuation of the sequence $B_{n,\leq 3}$ satisfies $\nu_7(B_{n,\leq 3}) = 0$ if $n \not\equiv 4 \bmod 7$.
- The sequence of residues $B_{n,\leq 5}$ modulo 7 is a periodic sequence of period 7, with fundamental period $\{1, 1, 2, 5, 1, 3, 6\}$.
- The 3-adic valuation of the associated factorial numbers $A_{n,\geq 3}$ satisfy $\nu_3(A_{n,\geq 3}) = 0$ if $n \equiv 0 \bmod 3$. For $n = 3k+1$, the valuation is given by $\nu_3(A_{n,\geq 3}) = \nu_3(A_{n+1,\geq 3}) = \nu_3(n-1)$. This covers all cases.
- The sequence of residues $A_{n,\leq 5}$ modulo 7 is a periodic sequence of period 7, with fundamental period $\{1, 1, 2, 6, 3, 1, 5\}$.

Many other results of this type can be discovered experimentally. A discussion of a general theory is in preparation.

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