Journal of Combinatorics
Volume 10, Number 2, 327–337, 2019

Graphs without large bicliques and well-quasi-orderability by the induced subgraph relation*

Aistis Atminas, Vadim Lozin, and Igor Razgon

Recently, Daligault, Rao and Thomassé asked in [3] if every hereditary class which is well-quasi-ordered by the induced subgraph relation is of bounded clique-width. While the question has been shown to have a negative answer in general [9], in the present paper we show that the statement is true for a family of hereditary classes of graphs that exclude large bicliques as subgraphs. In particular, this implies (through the use of Courcelle theorem [2]) that any problem definable in Monadic Second Order Logic can be solved in a polynomial time for all well-quasi-ordered hereditary classes of graphs that exclude large bicliques.

1. Introduction

Well-quasi-ordering is a highly desirable property and a frequently discovered concept in mathematics and theoretical computer science [6, 8]. One of the most remarkable recent results in this area is the proof of Wagner's conjecture stating that the set of all finite graphs is well-quasi-ordered by the minor relation [12]. However, the subgraph or induced subgraph relation is not a well-quasi-order. On the other hand, each of these relations may become a well-quasi-order when restricted to graphs with some special properties.

A *graph property* (or a *class of graphs*) is a set of graphs closed under isomorphism. A property is *hereditary* if it is closed under taking induced subgraphs. It is well-known (and not difficult to see) that a graph property X is hereditary if and only if X can be described in terms of forbidden induced subgraphs. More formally, X is hereditary if and only if there is a set M of graphs such that no graph in X contains any graph from M as an induced subgraph. We call M the set of *forbidden induced subgraphs* for X and say that the graphs in X are M-free.

*This works was supported by EPSRC grant EP/L020408/1.

Of our particular interest in this paper are graphs *without large bicliques*. We say that the graphs in a hereditary class X are *without large bicliques* if there is a natural number t such that no graph in X contains $K_{t,t}$ as a (not necessarily induced) subgraph. Equivalently, there are q and r such $K_{q,q}$ and K_r appear in the set of forbidden induced subgraphs for X. According to [11], these are precisely the graphs with a subquadratic number of edges. This family of properties includes many important classes, such as graphs of bounded vertex degree, of bounded tree-width, all proper minor closed graph classes. In all these examples, the number of edges is bounded by a linear function in the number of vertices and all of the listed properties are rather small (see e.g. [10] for the number of graphs in proper minor closed graph classes). In the terminology of [1], they all are at most factorial. In fact, the family of classes without large bicliques is much richer and contains classes with a superfactorial speed of growth, such as projective plane graphs (or more generally C_4-free bipartite graphs), in which case the number of edges is $\Theta(n^{\frac{3}{2}})$.

Recently, Daligault, Rao and Thomassé asked in [3] if every hereditary class which is well-quasi-ordered by the induced subgraph relation is of bounded clique-width. While the question has been shown to have a negative answer in general [9], the relationship holds true for some families of hereditary graph classes. Investigating such families is interesting because it connects two seemingly unrelated notions and leads to a strong algorithmic consequence. Indeed, it follows (through the use of Courcelle theorem [2]) that for such families any problem definable in Monadic Second Order Logic can be solved in a polynomial time on any class well-quasi-ordered by the induced subgraph relation.

In the present paper, we establish the relationship between well-quasi-ordering and boundedness of clique-width for graphs without large bicliques. More precisely, we prove that if a class X without large bicliques is well-quasi-ordered by the induced subgraph relation, then the graphs in X have bounded path-width, i.e. there is a constant c such that the path-width of any graph in X is at most c. Since bounded path-width implies bounded clique-width, the result affirmatively answers the question in [3] for graphs without large bicliques. Thus the above algorithmic consequence is confirmed e.g. for classes of graphs of bounded degree.

Section 2 contains all preliminary information related to the topic. In this section we define an infinite family of graphs pairwise incomparable by the induced subgraph relation, which we call *canonical graphs*. In Section 3 we prove our main combinatorial result, Theorem 1, stating that a graph without large bicliques and having a large path-width has a large induced canonical graph. A consequence of this result is that if a class X without large bicliques has unbounded path-width, then X contains an infinite subset of canonical graphs, i.e. an infinite antichain. This implies that classes of graphs without large bicliques that are well quasi-ordered by the induced subgraph relation must have bounded path-width.

2. Notation and definitions

In this work we will be using standard graph theory terminology and notation consistent with the book of Diestel [4]. In particular, K_n and P_n denote the complete graph and the chordless path with n vertices, respectively, and $K_{n,m}$ stands for a complete bipartite graph with parts of size n and m.

Throughout the text, whenever we say that G contains H, we mean that H is a subgraph of G, unless we explicitly say that H is an *induced* subgraph of G (or G contains H as an *induced* subgraph). If H is not an induced subgraph of G, we say that G is H-free. By $R = R(k, r, m)$, we denote the Ramsey number, i.e. the minimum R such that in every colouring of k-subsets of an R-set with r colours there is a monochromatic m-set, i.e. a set of m elements all of whose k-subsets have the same colour.

According to the celebrated Graph Minor Theorem of Robertson and Seymour, the set of all graphs is well-quasi-ordered by the graph minor relation [12]. This, however, is not the case for the more restrictive relations such as subgraph or induced subgraph. Indeed, a sequence of graphs $H_1, H_2, \ldots$, creates an infinite antichain with respect to both relations, where H_i is the graph represented in Figure 1.

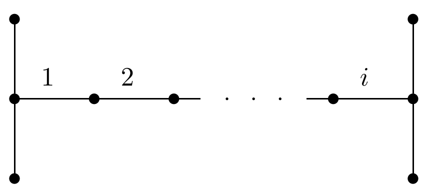

Figure 1: The graph H_i.

By connecting two vertices of degree one having a common neighbour in H_i, we obtain a graph represented on the left of Figure 2. Let us denote this graph by H'_i. By further connecting the other pair of vertices of degree one we obtain the graph H''_i represented on the right of Figure 2.

We call any graph of the form H_i, H'_i or H''_i an H-graph. Furthermore, we will refer to H''_i a *tight* H-graph and to H'_i a *semi-tight* H-graph. In an

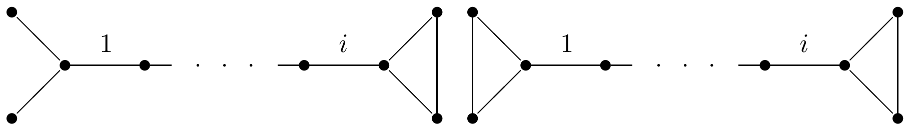

Figure 2: Graphs H_i' and H_i''.

H-graph, the path connecting two vertices of degree 3 will be called the *body* of the graph, and the vertices which are not in the body the *wings*.

Following standard graph theory terminology, we call a chordless cycle of length at least four a *hole*. Let us denote by

$\mathcal{C}$ the set of all holes and all H-graphs.

It is not difficult to see that any two distinct (i.e. non-isomorphic) graphs in $\mathcal{C}$ are incomparable with respect to the induced subgraph relation. In other words,

Claim 1. *$\mathcal{C}$ is an antichain with respect to the induced subgraph relation.*

Moreover, from the poof of Theorem 1 we will see that for classes of graphs without large bicliques which are of unbounded path-width this antichain is unavoidable, or *canonical*, in the terminology of [5]. Suggested by this observation, we introduce the following definition.

Definition 1. *The graphs in the set $\mathcal{C}$ will be called* CANONICAL.

The *order* of a canonical graph G is either the number of its vertices, if G is a hole, or the the number of vertices in its body, if G is an H-graph.

3. Main result

In this section we prove the following theorem which is the main result of the paper.

Theorem 1. *If X is a hereditary subclass of $(K_t, K_{q,q})$-free graphs which is well-quasi-ordered by the induced subgraph relation, then graphs in X have a bounded path-width.*

To prove the theorem, we will show that a large path-width combined with the absence of large bicliques implies the existence of a large induced canonical graph, which is a much richer structural consequence than just the existence of a long induced path. An important part of showing the existence of a large canonical graph is verifying that its body (see Section 2 for the terminology) is induced. This will be done by application of the following theorem proved in [7].

Theorem 2. *For every s, t, and q, there is a number $Z = Z(s,t,q)$ such that every graph with a path of length at least Z contains either P_s or K_t or $K_{q,q}$ as an induced subgraph.*

A plan of the proof of Theorem 1 is outlined in Section 3.1. Sections 3.2, 3.3, 3.4, 3.5 contain various parts of the proof.

3.1. Plan of the proof

To prove Theorem 1 we will show that graphs of arbitrarily large path-width contain either arbitrarily large bicliques as subgraphs or arbitrarily large canonical graphs as induced subgraphs. The main notion in our proof is that of a *rake-graph*.

A *rake-graph* (or simply a *rake*) consists of a chordless path, the *base* of the rake, and a number of pendant vertices, called *teeth*, each having a private neighbour on the base. The only neighbour of a tooth on the base will be called the *root* of the tooth, and a rake with k teeth will be called a k-rake. We will say that a rake is *ℓ-dense* if any ℓ consecutive vertices of the base contain at least one root vertex. An example of a 1-dense 9-rake is given in Figure 3.

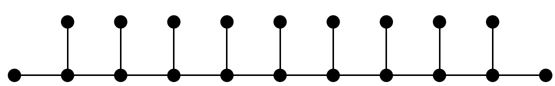

Figure 3: 1-dense 9-rake.

We will prove Theorem 1 through a number of intermediate steps as follows.

1. In Section 3.2, we observe that any graph of large path-width contains a rake with many teeth as a subgraph.
2. In Section 3.3, we show that any graph containing a rake with many teeth as a subgraph contains either
   - a *dense* rake with many teeth as a subgraph or
   - a large canonical graph as an *induced* subgraph.

3. In Section 3.4, we prove that dense rake subgraphs necessarily imply either
   - a large canonical graph as an *induced* subgraph or
   - a large biclique as a subgraph.
4. In Section 3.5, we use the results of sections 3.2–3.4 to deduce Theorem 1.

3.2. Rake subgraphs in graphs of large path-width

Lemma 1. *For any natural k, there is a number $f(k)$ such that every graph of path-width at least $f(k)$ contains a k-rake as a subgraph.*

Proof. In [13], Robertson and Seymour has shown that for any tree T there is a constant c_T such that any graph of path-width is at least c_T contains T as a minor. Taking T to be some fixed k-rake, we obtain that there exist a constant $f(k)$ such that any graph of path-width at most $f(k)$ contains a k-rake as a minor. Finally, it is not hard to see that if a graph contains a k-rake as a minor, then it also contains a k-rake as a subgraph. This observation completes the proof. □

3.3. From rake subgraphs to dense rake subgraphs

Lemma 2. *Let k and s be natural numbers. Every graph containing a $k+2$-rake as a subgraph contains either*

- *an $s+5$-dense k-rake as a subgraph or*
- *a canonical graph of order at least s as an induced subgraph.*

Proof. Consider a graph that contains a $k+2$-rake as a subgraph and choose such a $k+2$-rake with the minimal number of vertices. We denote the base of the rake by P. Let $\{u_1, u_2, \ldots, u_{k+2}\}$ denote the roots of the rake that are indexed respecting the linear order of the path P, i.e. so that u_1 and u_{k+2} are the endpoints of P and the subpaths of P from u_i to u_{i+1}, which we denote by P_i, are all mutually disjoint apart from the endpoints. Note that by minimality of the rake it follows that each endpoint of the path P is indeed a root vertex of the rake and that each P_i is an induced path. If each P_i for $i = 2, 3, \ldots, k$ has at most $s+5$ vertices, then we have an $s+5$-dense k-rake as required. So assume now that P_i for some $i = 2, 3, \ldots k$ has size more than $s+5$. To complete the proof we will show that this P_i gives rise to a canonical graph of order at least s as an induced subgraph. We proceed with some notation.

Let $P_i = w_1w_2 \dots w_r$ with $w_1 = u_i$ and $w_r = u_{i+1}$. Extend P_i by adding the vertex w_0 of P_{i-1} that is adjacent to w_1 and the vertex w_{r+1} of P_{i+1} that is adjacent to w_r (unique choice as P_{i-1} and P_{i+1} are induced paths). Note that $w_0w_1w_2 \dots, w_rw_{r+1}$ is a subpath of P, the tooth v_i is adjacent to w_1 and the tooth v_{i+1} is adjacent to w_r. Let G be a graph induced by vertices $\{w_0, w_1, \dots, w_{r+1}\} \cup \{v_i, v_{i+1}\}$ and note that G contains an H-graph formed by edges $\{w_0w_1, w_1w_2, \dots, w_rw_{r+1}\} \cup \{v_iw_1, v_{i+1}w_r\}$ as a subgraph but not necessarily as an induced subgraph. Note that the body of the H-graph, spanned by vertices $\{w_1, w_2 \dots, w_r\}$, is a chordless path P_i. For the rest of the proof we will be arguing on the adjacencies of the wings of the H-graph in G, i.e. adjacencies of vertices w_0, w_r, v_i and v_{i+1} in G. It will follow G contains a canonical subgraph of order at least s as an induced subgraph.

We first claim that w_0 is not adjacent to w_l for any $l = 2, 3, \dots, r-1$. Indeed, suppose for contradiction that w_0 is adjacent to some w_l for $l = 2, 3, \dots, r-1$. Let a path P' be obtained from path P by replacing subpath $w_0w_1 \dots w_r$ of P by path $w_0w_lw_{l+1} \dots w_r$. The path P' has smaller number of vertices than path P, and note that the missing root vertex w_1 can be replaced by w_l with the new tooth being w_{l-1}. This gives us a $k+2$-rake that has smaller number of vertices than the original, which contradicts our minimality assumption.

Next, we show that v_i is not adjacent to $w_4, w_5, \dots, w_r$. Again, suppose for contradiction that v_i is adjacent to w_l for some $l = 4, 5, \dots, r$. Let the path P' be obtained from path P by replacing the subpath $w_1w_2 \dots w_r$ of P by path $w_1v_iw_lw_{l+1} \dots w_r$. Again, the path P' has fewer vertices than path P, all the root vertices of P remain in path P', but as v_i is now in the path P', we assign a new tooth w_2 to correspond to the root w_1. Again, we obtain a $k+2$-rake that has smaller number of vertices than the original, a contradiction.

By symmetry, we can show that w_{r+1} is not adjacent to w_l for any $l = 2, 3, \dots, r-1$ and v_{i+1} is not adjacent to any of $w_1, w_2, \dots, w_{r-3}$. We conclude that none of the wings of the H-graph are adjacent to any of $w_4, w_5, \dots, w_{r-3}$. In other words, vertices $w_4, w_5, \dots, w_{r-3}$ are of degree 2 in G. If w_4w_5 is a cut-edge of G, we have that no vertex of $\{w_0, w_1, w_2, w_3, v_i\}$ is adjacent to any of the vertex of $\{w_{r-2}, w_{r-1}, w_r, w_{r+1}, v_{i+1}\}$. Let $l \leq 3$ be the largest possible such that w_l has degree at least 3 in G, $p \geq r-2$ the smallest possible such that w_p has degree at least 3 in G. Taking the path $w_lw_{l+1} \dots w_p$ together with another two neighbours of w_l and w_p provides us with an induced H-graph whose base $w_lw_{l+1} \dots w_p$ has at least $s+1$ vertices. On the other hand, if w_4w_5 is not a cut-edge in G, then there is a chordless cycle in G containing the edge w_4w_5 and hence this cycle must contain $w_3 w_4 w_5 \dots w_{r-2}$ (because of vertices of degree 2). Therefore, we obtain an induced cycle of G with at least $r-4 \geq s+1$ vertices. Hence in both cases we obtain a canonical graph of order at least s as an induced subgraph. This finishes the proof. □

3.4. Dense rake subgraphs

Lemma 3. *For every s, q and ℓ, there is a number $D = D(s, q, \ell)$ such that every graph containing an ℓ-dense D-rake as a subgraph contains either*

- *a canonical graph of order at least s as an induced subgraph or*
- *a biclique of order q as a subgraph.*

Proof. To define the number $D = D(s, q, \ell)$, we introduce intermediate notations as follows: $b := 2(q-1)s^q + 2sq + 4$ and $c := R(2, 2, \max(b, 2q))$, where R is the Ramsey number. With these notations the number D is defined as follows: $D = D(s, q, \ell) := Z(\ell c^2, 2q, q)$, where Z is the number defined in Theorem 2.

Consider a graph G containing an ℓ-dense D-rake R^0 as a subgraph. The base of this rake is a path P^0 of length at least D and hence, by Theorem 2, the subgraph of G induced by the base contains either a biclique of order at least q as a subgraph (in which case we are done) or an *induced* path P of length at least ℓc^2. Let us call any (inclusionwise) maximal sequence of consecutive vertices of P^0 that belong to P a *block*. Assume the number of blocks is more than c. Let P' be the subpath of P induced by the first c blocks. Let $w_1, \dots, w_c$ be the rightmost vertices of the blocks. Let $v_1, \dots, v_c$ be the vertices such that each v_i is the vertex of P_0 immediately following w_i. Then P' together with $v_1, \dots, v_c$ create a c-rake with P' being the induced base, $v_1, \dots, v_c$ being the teeth and $w_1, \dots, w_c$ being the respective roots. If the number of blocks is at most c, then P^0 must contain a block of size at least ℓc, in which case this block also forms an induced base of a c-rake (since R^0 is ℓ-dense). We see that in either case G has a c-rake with an induced base. According to the definition of c, the c teeth of this rake induce a graph which has either a clique of size $2q$ (and hence a biclique of order q in which case we are done), or an independent set of size b. By ignoring the teeth outside this set we obtain a b-rake R with an induced base and with teeth forming an independent set.

Let us denote the base of R by U, its vertices by $u_1, \dots, u_m$ (in the order of their appearances in the path), and the teeth of R by $t_1, \dots, t_b$ (following the order of their root vertices).

Denote $r := (q-1)s^q+2$ and consider two sets of teeth $T_1 = \{t_2, t_3, \dots, t_r\}$ and $T_2 = \{t_{b-1}, t_{b-2}, \dots, t_{b-r+1}\}$. By definition of r and b, there are $2sq$ other teeth between t_r and t_{b-r+1}, and hence there is a set M of $2sq$ consecutive vertices of U between the root of t_r and the root of t_{b-r+1}. We partition M into $2q$ subsets (of consecutive vertices of U) of size s each and for $i = 1, \dots, 2q$ denote the i-th subset by M_i.

If each vertex of T_1 has a neighbour in each of the first q sets M_i, then by the Pigeonhole Principle there is a biclique of order q with q vertices in T_1 and q vertices in M. Similarly, a biclique of order q arises if each vertex of T_2 has a neighbour in each of the last q sets M_i. Therefore, we assume that there are two vertices $t_a \in T_1$ and $t_b \in T_2$ and two sets M_x and M_y with $x < y$ such that t_a has no neighbours in M_x, while t_b has no neighbours in M_y.

By definition, t_a has a neighbour in U (its root) on the left of M_x. If additionally t_a has a neighbour to the right of M_x, then a chordless cycle of length at least s arises (since $|M_x| = s$ and t_a has no neighbours in M_x), in which case the lemma is true. This restricts us to the case, when all neighbours of t_a in U are located to the left of M_x. By analogy, we assume that all neighbours of t_b in U are located to the right of M_y. Let u_i be the rightmost neighbour of t_a in U and u_j be the leftmost neighbour of t_b in U. According to the above discussion, $i < j$ and $j - j > 2s$. But then the vertices $t_a, t_b, u_{i-1}, u_i, \dots, u_j, u_{j+1}$ induce an H-graph (possibly tight or semi-tight) of order more than s (the existence of vertices u_{i-1} and u_{j+1} follows from the fact that T_1 does not include t_1, while T_2 does not include t_b). □

3.5. Proof of Theorem 1

Combining the results of Lemma 1, Lemma 2 and Lemma 3, we conclude that for every s, q, there is a number $X = X(s, q)$ such that every graph of path-width at least X contains either

- a canonical graph of order at least s as an induced subgraph or
- a biclique of order q as a subgraph.

From this it is not hard to conclude that a class of graphs with unbounded path-width that excludes a biclique of order q must contain an infinite family of distinct canonical graphs, hence the class must be not well-quasi-ordered. Therefore, well-quasi-ordered classes that exclude a biclique of order q for some q, must be of bounded path-width, as required.

References

[1] J. Balogh, B. Bollobás and D. Weinreich. The speed of hereditary properties of graphs. *Journal of Combinatorial Theory, Series B* 79(2): 131–156, 2000. MR1769217

[2] B. Courcelle, J. A. Makowsky and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory Comput. Syst.*, 33(2): 125–150, 2000. MR1739644

[3] J. Daligault, M. Rao and S. Thomassé. Well-Quasi-Order of Relabel Functions. *Order*, 27: 301–315, 2010. MR2728737

[4] R. Diestel, Graph Theory. Third edition. Graduate Texts in Mathematics, 173. Springer-Verlag, Berlin, 2005. xvi+411 pp. MR2159259

[5] G. Ding. On canonical antichains, *Discrete Mathematics*, 309: 1123–1134, 2009. MR2493532

[6] A. Finkel and Ph. Schnoebelen. Well-structured transition systems everywhere! *Theor. Comput. Sci.*, 256(1–2): 63–92, 2001. MR1821455

[7] F. Galvin, I. Rival and B. Sands. A Ramsey-Type Theorem for Traceable Graphs. *Journal of Combinatorial Theory, Series B*, 33: 7–16, 1982. MR0678167

[8] J. B. Kruskal. The theory of well-quasi-ordering: A frequently discovered concept. *J. Comb. Theory, Ser. A*, 13(3): 297–305, 1972. MR0306057

[9] V. Lozin, I. Razgon and V. Zamaraev. Well-quasi-ordering versus clique-width. *J. Comb. Theory, Ser. B*, 130: 1–18, 2018. MR3772732

[10] S. Norine, P. Seymour, R. Thomas and P. Wollan. Proper minor-closed families are small. *J. Comb. THeory Ser. B*, 96(5): 754–757, 2006. MR2236510

[11] T. Kövári, V.T. Sós and P. Turán. On a problem of K. Zarankiewicz. *Colloq. Math.*, 3: 50–57, 1954. MR0065617

[12] N. Robertson and P.D. Seymour. Graph Minors. XX. Wagner's conjecture, *Journal of Combinatorial Theory Ser. B*, 92: 325–357, 2004. MR2099147

[13] N. Robertson and P. Seymour. Graph Minors I. Excluding a Forest. *J. Comb. Theory Ser. B*, 35: 39–61, 1983. MR0723569

Aistis Atminas
Mathematics Institute
University of Warwick
Coventry CV4 7AL
UK
E-mail address: aistis.atminas@gmail.com

Vadim Lozin
Mathematics Institute
University of Warwick
Coventry CV4 7AL
UK
E-mail address: V.Lozin@warwick.ac.uk

Igor Razgon
Department of Computer Science and Information Systems
Birkbeck University of London
UK
E-mail address: igor@dcs.bbk.ac.uk

Received 26 January 2014