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Factorizations of k-nonnegative matrices

Sunita Chepuri, Neeraja Kulkarni, Joe Suk, and Ewin Tang

A matrix is k-*nonnegative* if all its minors of size k or less are nonnegative. We give a parametrized set of generators and relations for the semigroup of $(n-1)$-nonnegative $n \times n$ invertible matrices and $(n-2)$-nonnegative $n \times n$ unitriangular matrices. For these two cases, we prove that the set of k-nonnegative matrices can be partitioned into cells based on their factorizations into generators, generalizing the notion of Bruhat cells from totally nonnegative matrices. Like Bruhat cells, these cells are homeomorphic to open balls and have a topological structure that neatly relates closure of cells to subwords of factorizations. In the case of $(n-2)$-nonnegative unitriangular matrices, we show that the link of the identity forms a Bruhat-like CW complex, as in the Bruhat decomposition of unitriangular totally nonnegative matrices. Unlike the totally nonnegative case, we show this CW complex is not regular.

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1. Introduction

A totally nonnegative (respectively totally positive) matrix is a matrix where all minors are nonnegative (respectively positive). Total positivity and nonnegativity are well-studied phenomena and arise in diverse areas such as planar networks, combinatorics, cluster algebras, and stochastic processes [9, 12]. We generalize the notion of total nonnegativity and positivity as follows. A k-*nonnegative* (resp. k-*positive*) matrix is a matrix where all minors of order k or less are nonnegative (resp. positive).

Our investigation of k-nonnegative matrices follows the path of previous work in parametrizing totally nonnegative matrices. In their 1998 paper [11], Fomin and Zelevinsky partitioned the semigroup of invertible totally nonnegative matrices into cells based on their factorizations into parametrized

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generators. Lusztig was able to extend the theory of total positivity to Lie groups by using canonical bases [14]. A question which arose soon afterwards was whether the Bruhat order of a Weyl group W of a semisimple algebraic group G, split over $\mathbb{R}$, can arise as the face poset of a regular CW complex. Of particular interest was the case where $G = \mathrm{SL}_n$ and $W = S_n$. Björner constructed a "synthetic" regular CW complex answering this question in [2]. Fomin and Shapiro asked in [10] whether such a regular CW complex exists "in nature", observing that the space of upper-triangular unipotent matrices with all minors nonnegative has a cell decomposition with face poset $(W, \leq)$. This space is not compact (and thus cannot be a finite CW complex), but Fomin and Shapiro conjectured that taking the link of the identity element in this set, which also has $(W, \leq)$ as its face poset, gives the desired construction. Specifically, this is the set of upper unitriangular totally nonnegative matrices whose entries immediately above the diagonal sum to a positive constant. Hersh proved this in the affirmative in [13]. In this work, we investigate the extent to which a similar structure can be found in semigroups of k-nonnegative matrices.

The idea of k-nonnegativity is not new; in fact, k-nonnegative and k-positive matrices have been the subject of study in several papers by Fallat, Johnson, and Sokal ([7, 6]). However, these works are largely unconcerned with the semigroup structure of k-nonnegative matrices and take considerably different directions than our own approach to understanding these matrices. This work is, to the authors' knowledge, the first attempt to fully characterize the generators of this semigroup and find an analogous Bruhat cell decomposition as done by Fomin and Zelevinsky in [11] for the case of totally nonnegative matrices.

Our results are as follows. The Loewner-Whitney Theorem (Theorem 2.2.2 of [8]) gives a set of generators for the semigroup of invertible totally nonnegative matrices. We begin by generalizing this theorem to k-nonnegative $n \times n$ matrices (2.4). This allows us to compute generators for the semigroup of $(n-2)$-nonnegative unitriangular matrices, which consist of the unitriangular Chevalley generators as well as a new class of T-generators. We will show that it is possible to factor any matrix in the semigroup as a product of these generators, and we also give relations by which we can move between any two factorizations of the same matrix. Analogous results for the semigroup of $(n-1)$-nonnegative invertible matrices can be found in Appendix D. We mostly focus on the $(n-2)$-nonnegative unitriangular case since this is where the topology of the semigroup (under the standard topology of $\mathrm{GL}_n(\mathbb{R})$) is more interesting.

As done by Fomin and Zelevinsky [12], we proceed to group matrices into cells based on their factorizations into generators (4.6, E.1). We show that these cells partition the space (4.8, E.3) and behave well with respect to taking the closure of their parameter space; namely, the parametrizations of these cells are homeomorphisms from open Euclidean balls (4.11, E.4) and we can obtain all of the elements in the closure of a cell by setting elements in the parametrization to zero (or equivalently, by taking subwords of our factorization word, via a subword order we define) (4.14, E.7). In fact, in the $(n-2)$ case, the link of the identity element forms a CW complex, and correspondingly, its closure poset is graded (4.16). However, the poset is not Eulerian (4.17), meaning it is not a regular CW complex as in the totally nonnegative unitriangular case.

Surprisingly, throughout this process, the generators, relations, cells, and resulting closure poset can all be simply described and follow fairly naturally from the restrictions on the space. This suggests that such structure might exist for more cases and even for k-nonnegative matrices for more general values of k.

Our paper is structured as follows. In Section 2, we detail some relevant background and proceed to our most general results on factorizations of k-nonnegative matrices. We describe partial factorizations of k-nonnegative matrices and give some lemmas necessary for future sections. Section 3 describes the factorizations of the $(n-2)$-nonnegative unitriangular matrices. We give specific generating sets for the semigroup as well as sets of relations in Appendix A. Appendix D contains the analogous results for the $(n-1)$-nonnegative matrices, with the appropriate background in Appendix B. In Section 4, we describe the cell decomposition of the semigroup of $(n-2)$-nonnegative unitriangular matrices and discuss the topological properties that these cells share with the standard Bruhat cells of totally nonnegative matrices. Appendix E describes the cell decomposition of the semigroup of $(n-1)$-nonnegative matrices.

2. Preliminaries

2.1. Background

We begin by establishing some conventions and notation that will be used throughout the paper. We use $[n]$ to refer to the set $\{1, \dots, n\}$. For any matrix X, $X_{I,J}$ refers to the submatrix of X indexed by a subset of its rows I and a subset of its columns J, and $|X_{I,J}|$ will refer to the minor, that is, the determinant of this submatrix. We say a minor $|X_{I,J}|$ is of *order* k if

$|I| = |J| = k$. A minor $|X_{I,J}|$ is called *solid* if both I and J are intervals. $|X_{I,J}|$ is called *column-solid* if J is an interval. Unless stated otherwise, all matrices discussed will belong to $GL_n(\mathbb{R})$. We identify S_n with a subgroup of $GL_n(\mathbb{R})$ by identifying an $\omega \in S_n$ with the matrix sending the basis vector e_i to the basis vector $e_{\omega(i)}$.

The set of all invertible k-nonnegative $n \times n$ matrices forms a semigroup: it is closed under multiplication by the Cauchy-Binet formula

$$|(AB)_{I,J}| = \sum_{\substack{S \subset [n] \\ |S| = \ell}} |A_{I,S}||B_{S,J}|. \tag{2.1}$$

Similarly, the set of all upper unitriangular k-nonnegative matrices forms a semigroup (and analogously for lower unitriangular). We would like to study the structure and topology of these semigroups. What are the generators and relations of the semigroup? What topological features does it have when endowed with the standard topology on $GL_n(\mathbb{R})$? We first summarize here the known answers to these questions for the case of $k = n$ (i.e. totally nonnegative or TNN matrices). We focus on the upper unitriangular semigroup since this is the basis for what is established in Sections 3 and 4. The analogous details for the larger semigroup of totally nonnegative matrices is mentioned briefly in Appendix E. Most of the following summary can be found in [12].

We first discuss the generators of the semigroup of totally nonnegative upper unitriangular matrices. We note the totally nonnegative lower unitriangular matrices also forms a semigroup and all of the following results will hold for this semigroup as well upon making the proper adjustments (which is usually taking the transpose). A *Chevalley generator* is defined as a matrix which differs from the identity by having some $a > 0$ in the $(i, i+1)$-st entry, and is by denoted $e_i(a)$. The Chevalley generators $e_i(a)$ generate the semigroup of upper unitriangular totally nonnegative matrices. Thus, any upper unitriangular matrix X that is totally nonnegative can be factored into Chevalley generators $e_i(a)$ with nonnegative parameters $a \geq 0$. These factorizations will allow us to parametrize the entire semigroup.

To do this, we need to discuss Bruhat cells. The following will come primarily from [11] § 4. Let us establish some notation. We will let B^+ be the subgroup of upper-triangular matrices in $\mathrm{GL}_n(\mathbb{R})$.

For any $u \in S_n$, let B^+uB^+ denote the corresponding double coset. We call $B_u^+ := B^+uB^+$ the Bruhat cell associated to u and we have the decomposition

$$\mathrm{GL}_n(\mathbb{R}) = \bigsqcup_{u \in S_n} B^+uB^+.$$

This decomposition allows us to parametrize the upper unitriangular totally nonnegative matrices. Recall that in the Coxeter presentation of the symmetric group, any permutation $w \in S_n$ can be written as a reduced word, or as a product of adjacent transpositions $w = w_1 \cdots w_{\ell(w)}$ where $\ell(\cdot)$ denotes the length function of S_n.

Theorem 2.1 (Theorems 2.2.3, 5.1.1, 5.1.4, and 5.4.1 of [1])**.** *Let $N_{\geq 0}$ be the set of $n \times n$ upper unitriangular totally-nonnegative matrices. Then, $N_{\geq 0} \cap B_w^+$ partition $N_{\geq 0}$ as w ranges over S_n. Furthermore, each $N_{\geq 0} \cap B_w^+$ is in bijective correspondence with an $\ell(w)$-tuple of positive real numbers via the map $(t_1, \ldots, t_{\ell(w)}) \mapsto e_{w_1}(t_1) \cdots e_{w_{\ell(w)}}(t_{\ell(w)})$ where $(w_1, \ldots, w_{\ell(w)})$ is a reduced word for w.*

We shall later refer to the image of the product map $(t_1, \ldots, t_{\ell(w)}) \mapsto e_{w_1}(t_1) \cdots e_{w_{\ell(w)}}(t_{\ell(w)})$ as $U(w)$.

We do not include the proof of the above theorem here, but one key to proving that the parameter map is a bijection is understanding the commutation relations between the generators of the semigroup, which tell us how to move between two different factorizations of a totally nonnegative matrix into Chevalley generators. Here, the relations obeyed by the e_i's are similar to the braid relations between adjacent transpositions in the Coxeter presentation of the symmetric group. These are

$$e_i(a)e_{i+1}(b)e_i(c) = e_{i+1}\left(\frac{bc}{a+c}\right) e_i(a+c) e_{i+1}\left(\frac{ab}{a+c}\right) \tag{2.2}$$

$$e_i(a)e_j(b) = e_j(b)e_i(a), |i-j| > 1 \tag{2.3}$$

$$e_i(a)e_i(b) = e_i(a+b) \tag{2.4}$$

Thus, any factorization of an upper unitriangular totally nonnegative matrix into Chevalley generators $e_{i_1}(t_1) \cdots e_{i_\ell}(t_\ell)$ is equal to a factorization $e_{i'_1}(t'_1) \cdots e_{i'_\ell}(t'_\ell)$ where the corresponding words $(i_1, \ldots, i_\ell)$ and $(i'_1, \ldots, i'_\ell)$ differ by a braid move and the parameters $t'_1, \ldots, t'_\ell$ can be given by invertible, rational, subtraction-free expressions in $t_1, \ldots, t_\ell$.

Next, we discuss the topology of the semigroups. The Bruhat cells give a stratification of the semigroup of upper unitriangular totally nonnegative matrices. The corresponding poset of closure relations is Bruhat order on S_n (Example 2.1.3 of [3]).

As a result, many of the properties of these Bruhat order posets transfer to the Bruhat decomposition of unitriangular totally nonnegative matrices. We will use several classic properties of this poset to deduce analogous results for the decomposition of the semigroup of unitriangular $(n-2)$-nonnegative matrices in Section 4.2. First, underlying all of our theory is the idea that we can compare elements of the group in the Bruhat order using their corresponding reduced words; two words $u, w \in S_n$ satisfy $u \leq w$ if and only if a subword of any reduced expression for w is a reduced expression for u (this is known as the Subword Property, or Theorem 2.2.2 in [3]). Another important fact is the Exchange Property (Theorem 1.5.1 of [3]), which states that, for any $s = s_m$ with $m \in [n-1]$ and $w = s_1 s_2 \cdots s_k \in S_n$, $\ell(sw) \leq \ell(w) \implies sw = s_1 \ldots \hat{s}_i \ldots s_k$ for some $i \in [k]$.

The posets of closure relations on cells of unitriangular totally nonnegative matrices has many special properties: it has a top and bottom element (Proposition 2.3.1 of [3]), it is ranked (Theorem 2.2.6 of [3]), and it is Eulerian (Corollary 7.4 of [15]). As discussed before, it is also a regular CW complex, i.e. the closure of each Bruhat cell is homeomorphic to a closed ball [13]. Our work will show how far these properties extend to cells of unitriangular $(n-2)$-nonnegative matrices.

2.2. Equivalent conditions and elementary generalizations

When discussing k-nonnegative matrices, it is useful to ask first whether we need to check all minors for nonnegativity (usually an intractable computation), or just some subset of minors. For example, a well-known result, from [12], is that total nonnegativity can be determined by checking only column-solid minors. Brosowsky and Mason study necessary and sufficient conditions for *k-positivity*, in their work in Section 4 of [4].

The following statement, which follows from Fallat and Johnson [8], provides a sufficient condition for k-nonnegativity.

Proposition 2.2. *An invertible matrix X is k-nonnegative if all column-solid (or alternatively, row-solid) minors of X of order k or less are nonnegative.*

Proof. Let $Q_n(q) = (q^{(i-j)^2})_{i,j=1}^n$ for $q \in (0,1)$. This matrix is totally positive since its positivity is equivalent to the positivity of $(q^{-2ij})_{i,j=1}^n$ which is the well-known totally positive Pólya matrix (cf. Example 8.2.11 in [8]). $Q_n(q)$ satisfies $\lim_{q\to 0^+} Q_n(q) = I_n$. Consider a matrix X whose column-solid minors of order at most k are nonnegative. Let $X_q = Q_n(q)X$, and apply the Cauchy-Binet formula on an order $r \leq k$ column-solid minor:

$$|(X_q)_{I,J}| = \sum_{\substack{S \subset [n] \\ |S|=r}} |Q_n(q)_{I,S}|\, |X_{S,J}|\,.$$

The sum on the right hand side must be positive, since the column-solid minors of X are nonnegative and X is invertible. By Corollary 3.1.6 of [8] (which gives an equivalent condition for k-positivity), X_q must be k-positive. Taking the limit $q \to 0^+$, we see that X is k-nonnegative. To get the analogous statement for row-solid minors, we can use $X_q = XQ_n(q)$. $\square$

We use this condition to discuss factorizations of k-nonnegative matrices. To begin, we would like to know the extent to which these matrices can be factored into elementary Jacobi matrices. We can consider this as an algebraic problem of factoring divisors in the semigroup. This leads to the notion of *k-irreducibility.*

Definition. A k-nonnegative matrix M is *k-irreducible* if $M = RS$ in the semigroup of invertible k-nonnegative matrices implies $R, S \notin \{f_i(a), e_i(a) \mid a > 0\}$.

This definition will lead the reader to expect the following theorem.

Proposition 2.3. *Every k-nonnegative matrix X can be factored into a product of finitely many Chevalley generators and a k-irreducible matrix.*

Proof. Suppose that X is not k-irreducible. Then, without loss of generality, $e_i(a)^{-1}X$ is k-nonnegative for some $i \in [n]$ and $a \in \mathbb{R}_{>0}$ (corresponding to removing a copies of row $i+1$ from row i). We claim it is possible to choose $b > 0$ so that $e_i(b)^{-1}X$ is k-nonnegative and $e_i(b+\varepsilon)^{-1}X$ is not k-nonnegative for any $\varepsilon > 0$.

We want to determine when $e_i(x)^{-1}X$ is k-nonnegative in terms of x. It suffices to consider row-solid order $\leq k$ minors containing row i and not row $i+1$, which we will think about as a collection of functions of x, $\{m_\gamma(x)\}_\gamma$. m_γ are multilinear functions in the rows of $e_i(x)^{-1}X$ so that $m_\gamma(x) = |A_\gamma| - x|B_\gamma|$ for order $\leq k$ submatrices A_γ, B_γ of X. Namely, $|A_\gamma|, |B_\gamma|$ are nonnegative constants. Thus, $m_\gamma^{-1}([0,\infty))$ is closed for any γ. Further, $|B_\gamma| > 0$ for some γ by invertibility of X. For this γ, $m_\gamma^{-1}([0,\infty))$ is bounded, so the intersection $\cap_\gamma m_\gamma^{-1}([0,\infty))$ is closed and compact. This intersection contains a, meaning it has a maximal element $b > 0$. This b has the desired property.

By inspection, $b = \min_\gamma\{|A_\gamma|/|B_\gamma| \mid |B_\gamma| \neq 0\}$. So, in this way, we factor out a Chevalley generator, leaving a matrix with one more zero minor of order at most k. We can iterate this process, which must stop eventually because the number of minors of size at most k is finite. The resulting matrix must be k-irreducible. $\square$

Thus, k-irreducible matrices and Chevalley matrices form a generating set for the semigroup of k-nonnegative matrices.[1] The general properties of k-irreducible matrices are investigated in our report [5], where we consider locations of zero minors and extend Section 7.2 of [8]. The remainder of this section is devoted to a number of useful observations or "factorization lemmas", which will later help to characterize all k-nonnegative matrix factorizations when $k = n-1$ and $k = n-2$ in the unitriangular case. Let X be a k-nonnegative matrix.

(F1) **(Zero entries in X)** If $k \geq 2$ and there is any zero entry $x_{ij} = 0$, then either $x_{i',j} = 0$ for all $i' \leq i$, or $x_{ij'} = 0$ for all $j' \geq j$. In other words either all column entries above x_{ij} must be 0, or all row entries to the left of x_{ij} must be 0, because all minors of size 2 containing x_{ij} must be nonnegative. Similarly, we have either $x_{i'j} = 0$ for all $i' \geq i$, or $x_{ij'} = 0$ for all $j' \leq j$.

(F2) In particular, if X is an invertible matrix and some $x_{ij} = 0$, we have

(a) $i \neq j$ (all diagonal entries are nonzero).

(b) if $i < j$, then $x_{i'j'} = 0$ for $i' \leq i$, $j' \geq j$ (all entries to the north-east of x_{ij} are 0).

(c) if $i > j$, then $x_{i'j'} = 0$ for $i' \geq i$, $j' \leq j$ (all entries to the south-west of x_{ij} are 0).

(F3) **(Change in minors after multiplication by a Chevalley generator)** After multiplying X by a Chevalley generator, we can use the Cauchy-Binet formula to see that the minors of the product matrix X' are the same as the minors of X except in the following cases.[2]

(a) If $X' = Xe_k(a)$ (adding a copies of column k to column $k+1$) then $|X'_{I,J}| = |X_{I,J}| + a|X_{I,J\setminus k+1\cup k}|$ when J contains $k+1$ but not k;

(b) If $X' = e_k(a)X$ (adding a copies of row $k+1$ to row k) then $|X'_{I,J}| = |X_{I,J}| + a|X_{I\setminus k\cup k+1,J}|$ when I contains k but not $k+1$.

(F4) **(Factoring out a Chevalley generator to get zero entries)** Suppose X is invertible and $i < j$ are indices such that (1) $i < k$ and (2) $x_{i',j'} = 0$ for $i' \leq i+1$ and $j' \geq j$, except for $x_{i+1,j}$ and $x_{i,j}$ which

[1] This does not give a minimal generating set, as there are k-irreducible matrices that can be factored as Xe_iY where X, Y are k-irreducible: see the case of tridiagonal matrices in [5].

[2] These results also hold for $e_k(a)$ replaced by $f_k(a)$, when $k+1$ and k are swapped.

may be positive (see Figure 1). That is, condition (2) states that all entries north-east of $x_{i+1,j}$ except for $x_{i+1,j}$ and $x_{i,j}$ are 0. Then either $x_{i,j} = 0$ or $e_i(-x_{i,j}/x_{i+1,j})X$ is k-nonnegative.

$$\begin{bmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \end{bmatrix}$$

Figure 1: A matrix that meets condition (2) of (F4) for $i = 3, j = 4$.

The above statement can be proved as follows. First note that if $x_{i,j} \neq 0$, we have $x_{i+1,j} \neq 0$ from (F2), so the row operation $e_i(-x_{i,j}/x_{i+1,j})X =: X'$ is defined. We need to verify that X' is k-nonnegative. By Proposition 2.2 and (F3), it is enough to check the row-solid minors containing row i but not row $i+1$. These are minors where $I = [h, i]$ for some $h \leq i$. From condition (2), it suffices to consider minors where $J \subset [1, j]$. Consider a minor $|X'_{I,J}|$ satisfying these properties. By (1), $|I| < k$, so these minors are of order less than k. Using (F3), we have

$$\begin{aligned} \left|X'_{I,J}\right| &= |X_{I,J}| - \frac{x_{i,j}}{x_{i+1,j}} \left|X_{I \setminus i \cup i+1, J}\right| \\ &= \frac{1}{x_{i+1,j}} \left(x_{i+1,j} \left|X_{I,J}\right| - x_{i,j} \left|X_{I \setminus i \cup i+1, J}\right|\right) \\ &= \frac{1}{x_{i+1,j}} \left|X_{I \cup i+1, J \cup j}\right| \geq 0. \end{aligned}$$

The last step follows because $|X_{I \cup i+1, J \cup j}|$ is order $\leq k$, and so must be nonnegative.

An statement analogous to that of (F4) holds for the transpose of X using f_is, and for X when factoring a Chevalley generator from the right of X. In the latter case, we can reduce our matrix to one where x_{ij} is zero by factoring out a Chevalley matrix.

(F5) Observation (F4) above immediately gives rise to a slightly more general statement about the zero entries in k-irreducible matrices. Suppose a matrix M is invertible and k-irreducible for some $k \geq 2$, and suppose that M has a zero entry $m_{ij} = 0$. Then we have

(a) $m_{i-1,j-1} = 0$ if $i \leq k$ or $j \leq k$

(b) $m_{i+1,j+1} = 0$ if $i > n - k$ or $j > n - k$.

Now we can show that k-irreducible matrices have staircases of zeroes in their northeast and southwest corners (see Figure 2).

$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

Figure 2: A matrix with staircases of zeros in the northeast and southwest corners.

Proposition 2.4. *If an invertible $n \times n$ matrix X is k-nonnegative, it is the product of Chevalley matrices (specifically, only e_i's) and a single k-nonnegative matrix where the ij-th entry is zero when $|j - i| > n - k$.*

In other words, if X is k-irreducible, $x_{ij} = 0$ whenever $|j - i| > n - k$. Note that the Loewner-Whitney theorem is obtained as a special case of this theorem by setting $k = n$.

Proof. The top-right entry of X satisfies the hypotheses of (F4), and for a matrix where that entry is zero, the entry directly below satisfies the criterion, and so on. Eliminate $k - 1$ entries in the last column, one by one top-down, then $k - 2$ entries in the second-to-last, and continue until all desired entries are zero. When $i > j$, one can consider the transpose of the matrix and use the above argument to get the zeros in the bottom-left corner of the original matrix. □

3. Factorizations

In this section, we describe generators for the semigroup of unitriangular $(n-2)$-nonnegative matrices. We also give sets of relations for these matrices, on the basis of which we will construct our Bruhat cell analogues.

As motivation for these choices of semigroups, we notice that, as a result of Proposition 2.4, the smaller k is, the less the semigroup of k-nonnegative matrices seems to resemble the structure of the semigroup of totally nonnegative matrices. Note that $(n - 1)$-nonnegative unitriangular matrices must be totally nonnegative.

By Proposition 2.4, we know that any invertible unitriangular $(n - 2)$-irreducible matrix M must only have three bands that can be nonzero: the diagonal (which consists of ones), the super-diagonal, and the super-super-diagonal. We will refer to matrices of this form as *pentadiagonal unitriangular* matrices. The statements (F2) and (F5) tell us that if the matrices are not diagonal, all of the entries in this band of three diagonals must be nonzero. In this case, we show that the matrices are parametrized by $2n-3$ entries that can vary. The entries are notated by $a_i = M_{i,i+1}$ and $b_i = M_{i,i+2}$. We will show that matrices of this form are $(n-2)$-irreducible precisely when certain key minors in the matrix are zero.

Some minors in a pentadiagonal unitriangular matrix (where entries on the subdiagonal are ones) can be expressed in terms of a continued fraction. We will notate continued fractions in the following way:

$$[a_0; a_1, \ldots, a_m; b_1, \ldots, b_m] := a_0 - \cfrac{b_1}{a_1 - \frac{b_2}{a_2 - \cdots}}$$

This is different from the standard notation, which adds recursively instead of subtracting. Let $C_i(r) := \left|M_{[i,i+r-1],[i+1,i+r]}\right|$. Then the following recursive relation is satisfied:

$$(\star) \qquad C_i(0) = 1,\ C_i(1) = a_i,\ C_i(r) = a_{i+r-1}C_i(r-1) - b_{i+r-2}C_i(r-2)$$

This is sometimes known as the recurrence defining the *generalized continuant*. The above statement is rephrased slightly in the following lemma.

Lemma 3.1. *If $C_i(s) \neq 0$ for $s < r$, then*

$$C_i(r) = C_i(r-1)[a_{i+r-1}; a_{i+r-2}, \ldots a_i; b_{i+r-2}, \ldots, b_i].$$

Proof. It is obviously true for the base cases of the recurrence. Rewrite the equation as follows:

$$\begin{aligned}\frac{C_i(r)}{C_i(r-1)} &= a_{i+r-1} - b_{i+r-2}\frac{C_i(r-2)}{C_i(r-1)} \\ &= a_{i+r-1} - \frac{b_{i+r-2}}{[a_{i+r-2}; a_{i+r-3}, \ldots, a_i; b_{i+r-3}, \ldots, b_i]} \\ &= [a_{i+r-1}; a_{i+r-2}, \ldots a_i; b_{i+r-2}, \ldots, b_i]\end{aligned}$$ □

These recurrences also hold if we take the base case to be at the bottom corner rather than the top corner, thus relating $C_i(r)$ to $C_{i+1}(r-1)$. To relate these recurrences to nonnegativity tests, we use the following theorem.

Theorem 3.2. *Let M be a pentadiagonal unitriangular matrix with nonzero entries on the super-diagonal and super-super-diagonal. Then M is $(n-2)$-nonnegative if and only if the following hold:*

$$a_i, b_i > 0 \tag{3.1}$$

$$[a_x; a_{x-1}, \dots a_1; b_{x-1}, \dots, b_1] > 0 \qquad \text{if } x < n-2 \tag{3.2}$$

$$[a_{n-2}; a_{n-3}, \dots a_1; b_{n-3}, \dots, b_1] \geq 0 \tag{3.3}$$

$$[a_{n-1}; a_{n-2}, \dots a_2; b_{n-2}, \dots, b_2] \geq 0 \tag{3.4}$$

Further, M is $(n-2)$-irreducible if and only if equality holds in (3.3) *and* (3.4).

Proof. By Proposition 2.2, M is $(n-2)$-nonnegative if and only if column-solid minors of order at most $n-2$ are nonnegative. Observe that in a pentadiagonal unitriangular matrix, all nonzero column-solid minors evaluate to a subtraction-free expression in solid minors. Thus the previous condition is equivalent to the condition that all minors of the form $C_i(r)$ are nonnegative for $r \leq n-2$, and all of the b_i's being nonnegative (that is, positive, since they are assumed to be nonzero).

We claim that for $r < n-2$, we cannot have $C_i(r) = 0$. To see this, consider the smallest such j. By our recurrence relation ($\star$), this must mean that $C_i(r+1)$ is negative, which breaks $(n-2)$-nonnegativity. If $C_i(r+1)$ is not a valid minor, then just take the recurrence in the opposite direction to get a contradiction for $C_{i-1}(r+1)$. Thus, we can use Lemma 3.1, and say that $C_i(r)$ are all nonnegative precisely when the base cases, that is a_i's, are positive, as well as all of the corresponding continued fractions. Among these continued fractions, notice that if $[a_k; a_{k-1}, \dots a_i; b_{k-1}, \dots, b_i] > 0$, then so are the continued fractions achieved by truncating on the right at any $j \in (i, k]$. This gives us the necessary and sufficient condition for $(n-2)$-nonnegativity as stated in the theorem.

If M is also $(n-2)$-irreducible, it is impossible to factor out e_{n-2}, e_{n-1}, f_1, f_2 (from the left) and $e_1, e_2, f_{n-2}, f_{n-1}$ (from the right). Then, if the order $n-2$ minors $C_2(n-2)$ and $C_1(n-2)$ are nonzero, either the matrix form would be different from the one described or the matrix is no longer $(n-1)$-nonnegative, which gives a contradiction. □

The above characterization for $(n-2)$-irreducible unitriangular matrices can be simplified into a $(2n-5)$-parameter family. This family, along with $e_i(a)$ and the identity matrix, generate all $(n-2)$-nonnegative unitriangular matrices. Matrices in our parameter family appear as follows:

$$T(\vec{a},\vec{b}) = \begin{bmatrix} 1 & a_1 & a_1b_1 & & & & & \\ & 1 & a_2+b_1 & a_2b_2 & & & & \\ & & 1 & \ddots & \ddots & & & \\ & & & \ddots & a_{n-3}+b_{n-4} & a_{n-3}b_{n-3} & \\ & & & & 1 & b_{n-3} & b_{n-2}\mu \\ & & & & & 1 & b_{n-2} \\ & & & & & & 1 \end{bmatrix}$$

where $a_1, \ldots, a_{n-3}, b_1, \ldots, b_{n-2}$ are positive numbers and

$$\mu = \frac{b_1 \cdots b_{n-3}}{\left|T_{[2,n-3],[3,n-2]}\right|} = \frac{b_1 \cdots b_{n-3}}{\sum_{k=1}^{n-3}\left(\prod_{\ell=1}^{k-1} b_\ell \prod_{\ell=k+1}^{n-3} a_\ell\right)}.$$

For example, for $n = 5$, we have

$$T(a_1,a_2,b_1,b_2,b_3) = \begin{bmatrix} 1 & a_1 & a_1b_1 & & \\ & 1 & a_2+b_1 & a_2b_2 & \\ & & 1 & b_2 & \frac{b_1b_2b_3}{a_2+b_1} \\ & & & 1 & b_3 \\ & & & & 1 \end{bmatrix}$$

Appendix A gives the expressions for the minors of this matrix. The following result implies that any generating set of the semigroup must include all elements of T.

Theorem 3.3. *If $RS = T(\vec{a},\vec{b})$ in the semigroup of invertible $(n-2)$-nonnegative $n \times n$ matrices, one of R or S is the identity.*

Proof. Suppose we have $RS = T$. Since T is $(n-2)$-irreducible, neither of R, S can be TNN. However, by (F5), $R_{i,j}, S_{i,j} \neq 0$ for $j - i \in [0, 2]$. This means that $(RS)_{ij} \neq 0$ for some entry such that $j - i > 2$, contradicting with the form of T. □

We now list a set of relations involving generators of the form $T(\vec{a},\vec{b})$. It can be seen by direct computation that the following relations hold:

$$e_i(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_{i+2}(x'), \text{ where } 1 \leq i \leq n-3 \tag{3.5}$$

$$e_{n-2}(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_1(x') \tag{3.6}$$

$$e_{n-1}(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_2(x') \tag{3.7}$$

We also have a slightly different relation: a matrix $e_{n-1}(u)e_{n-2}(v)T(\vec{a},\vec{b})$ can be parametrized also in exactly one of three different ways, that are specified below.

(3.8)
$$e_{n-1}e_{n-2}T = e_{n-2}e_{n-1}T \sqcup e_{n-2}\cdots e_1 e_{n-1}\cdots e_2 \sqcup e_{n-2}\cdots e_1 e_{n-1}\cdots e_1$$

The proof that Equation 3.8 holds, as well as the parameters of all these relations, can be found in Appendix A. The expressions for new parameters are always subtraction-free rational expressions of the old parameters. As can be seen from the computations in Appendix 3.5, the fourth relation splits one cell into three based on the value of the $[1, n-1], [2, n]$ minor: the values of the minor in the three split factorizations are negative, zero, and positive respectively. Finally, we prove that one more relation exists between products of T's, and that it can be safely ignored.

Lemma 3.4. *$T(\vec{A},\vec{B})T(\vec{C},\vec{D})$ can always be written in a factorization that uses the same number of parameters or fewer, such that the factorization contains at most one instance of T.*

Proof. First, notice that $M := T(\vec{A},\vec{B})T(\vec{C},\vec{D})$ is TNN; the only minors to verify are those that are size $n-1$. A unitriangular matrix M has only one order $n-1$ minor, $|M_{[1,n-1],[2,n]}|$, that is not a minor of smaller order up to multiplication by entries of M, and this is nonnegative by Cauchy-Binet.

Further, M has only four nonzero diagonals: $M_{ij} = 0$ if $j - i \notin [0, 3]$. Any TNN matrix with this form can be factored using at most $4n - 10$ parameters (this can be seen via Lemma 4.2). M, thought of as a product of T's, has $4n - 10$ parameters, so we can always find a word for M using only Chevalley generators that is at least as short. □

4. Bruhat cells

The previous section gives a parametrized generating set for the semigroup of $(n-2)$-nonnegative unitriangular invertible matrices. In this section we will show that these semigroups can be partitioned into *cells* based on their factorizations into these generators. In the case of totally nonnegative matrices, the analogous cells reflect a nice interplay between the algebraic structure of relations between factorizations and the topological closure structure of the cells partitioning the space of TNN matrices.

With a view toward generalizing results from the TNN case to our case, we ask the following questions, which will motivate our sequence of study.

1. Is our list of relations complete?
2. Can we ignore parameters when studying factorizations? That is, do our relations respect the cell structure that naturally arises from considering factorizations?
3. Is our cell structure respected when adding or removing generators? Does this cell decomposition reflect the topological structure of our space?

The answer to all of these questions is yes.

4.1. Preliminaries

We first present combinatorial descriptions of Bruhat cell structure that will be used throughout the proofs that follow. First, recall that for $\omega \in S_n$, the length of ω is equal to the number of inversions of ω. We now give a characterization of the Bruhat order. For $\omega \in S_n$ and $i, j \in [n]$, let

$$\omega[i,j] := |\{a \in [i] : \omega(a) \geq j\}|$$

denote the number of non-zero entries between the southwest corner and the ijth entry of ω's permutation matrix. For example, for $n = 5$, the permutation $\omega = (4, 5, 1, 2, 3)$, given in one-line notation, has matrix

$$\omega = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \implies \omega[3,3] = 2$$

Lemma 4.1 (Theorem 2.1.5 of [3]). *Let $x, y \in S_n$. Then $x \leq y$ if and only if, for all $i, j \in [n]$, $x[i,j] \leq y[i,j]$.*

We define everything here for the B^- decomposition of $\mathrm{GL}_n(\mathbb{R})$ where B^- is the subgroup of lower triangular matrices, but taking the transpose will give the same results analogously for the B^+ decomposition, and taking both conditions will give descriptions for the double Bruhat cells. What follows, namely the characterization of a matrix being in a Bruhat cell by which of its minors are zero or non-zero, arises as a specific application of the results of Section 2.3 in [11].

Definition. Consider $\omega \in S_n$ as a permutation matrix. Call $I, J \subset [n]$ a *ω-NE-ideal* if $I = \omega(J)$ and, for all a, b such that $a < b$ and $\omega(a) > \omega(b)$, $(\omega(a), a) \in I \times J$ implies $(\omega(b), b) \in I \times J$. Call I, J a *shifted ω-NE-ideal* if $I \leq I'$ and $J \geq J'$ in termwise order for some ω-NE-ideal I', J' with $I, J \not\subset I', J'$.

Essentially we choose some set of entries that have ones in the permutation matrix ω, and have our ideal be those rows and columns, along with the rows and columns of any ones to the northeast of any of our existing ones. Shifted ideals are submatrices that are further to the NE than the ideals. Notice that shifted ω-NE-ideals cannot be ω-NE-ideals.

Definition. Call a matrix X *ω-NE-bounded* if the following two conditions hold:

- $|X_{I,J}| \neq 0$ for every I, J a ω-NE-ideal.
- $|X_{I,J}| = 0$ for every I, J a shifted ω-NE-ideal.

For B^+, the analogous definitions will be called *ω-SW-ideals* and *ω-SW-bounded* matrices. Notice that ω's permutation matrix is ω-NE-bounded (and ω-SW-bounded, in fact). The set of ω-NE-bounded matrices is precisely a Bruhat cell:

Lemma 4.2. *$M \in \mathrm{GL}_n$ is in B^-_ω iff it is ω-NE-bounded.*

Proof. If M is in B^-_ω, it is the product of ω's permutation matrix, which is ω-NE-bounded, with elements of B^-. The Cauchy-Binet formula shows that such multiplying by B^- preserves the ω-NE-bounded property.

Because all M are in some B^-_ψ and no matrix can be both ψ-NE-bounded and ω-NE-bounded for $\psi \neq \omega$, the other direction follows. $\square$

Finally, since we will be considering our new generator T, multiplied by e_i's, the following will be useful to distinguish factorizations.

Consider S_n as generated by the transpositions $s_i = (i, i+1)$. The resulting Coxeter group structure induces the *weak left Bruhat order* (*weak right Bruhat order*, respectively) on S_n, where $\alpha \leq \beta$ when β can be written as a reduced word $b = wa$ ($b = aw$, respectively) for some reduced word a of α and some w.

Lemma 4.3. *Let $\sigma \in S_n$ satisfy $\alpha \leq \sigma$ in the weak left Bruhat order and let w be a reduced word of $\sigma\alpha^{-1}$. If $M \in B^-_\alpha$, then $U(w)M \subset B^-_\sigma$.*

Proof. It suffices to show for $\sigma = s_i\alpha$; the general case follows from induction on the length of $\sigma\alpha^{-1}$. Now suppose we are taking some $M \in B^-_\alpha$ and $e_i(c)M$

such that $s_i\alpha$ is a reduced word; we want to show that $M' := e_i(c)M \in B^-_{s_i\alpha}$. This occurs precisely when $s_i\alpha$ has more inversions than α, i.e. when $\alpha^{-1}(i) < \alpha^{-1}(i+1)$.

Recall from Lemma (F3) that $|M'_{I,J}| = |M_{I,J}|$ unless I contains i but not $i+1$. So, $|M'_{I,J}| \neq 0$ for $s_i\beta$-NE-ideals that do not contain rows i and $i+1$ or contain both rows, since these are also β-NE-ideals and with unchanged minor values. When a $s_i\beta$-NE-ideal contains row $i+1$, it must contain row i, so the only remaining case is the set of $s_i\beta$-NE-ideals that contain only i. Again, using Lemma (F3), for such an $s_i\beta$-NE-ideal I, J,

$$|(e_i(c)M)_{I,J}| = |M_{I,J}| + c|M_{I\setminus i\cup i+1,J}|,$$

which is nonzero since the right hand side is the sum of a shifted β-NE-ideal and a β-NE-ideal.

Now, consider a shifted $s_i\beta$-NE-ideal I, J. We consider the I', J' from the definition (that is, the $s_i\beta$-NE-ideal such that $I \leq I'$ and $J \geq J'$). If I' contains neither i nor $i+1$, or if it contains both, then I', J' is a β-NE-ideal as well, and $|M'_{I,J}|$ can be written as a sum of ideals that shifted with respect to I', J'. If I' contains i but not $i+1$, then I is a shifted β-NE-ideal with respect to $I' \setminus i \cup i+1, J$. So, $|M'_{I,J}|$ can be expressed as a sum of shifted β-NE-ideal minors of M. $\square$

Finally, we will use the closure structure of the Bruhat decompositions of $GL_n(\mathbb{R})$.

Lemma 4.4. *Let S be a subset of a classical Bruhat cell $U(w)$. Then $\overline{S}$ is contained in the disjoint union of the cells $U(w')$, where $w' \leq w$.*

Proof. By Lemma 4.1, if $u \not\leq w$, there exists (i,j) with $u[i,j] > w[i,j]$. Consider the minimal u-NE-ideal R, S containing cell (i,j). Then $|X_{R,S}| \neq 0$ for $X \in U(u)$, by Lemma 4.2. But if $X \in U(w)$, then $X_{R,S}$ is not of full rank, because it is obtained by performing row operations on a matrix of rank less than $u[i,j]$. Thus $|X_{R,S}| = 0$, which means $X \notin U(u)$. Further, since all matrices $M \in \overline{U(w)}$ also satisfy $|M_{R,S}| = 0$, M is not in $U(u)$, and the statement follows. $\square$

Recall the signum function of a real number x is defined to be $\operatorname{sgn}(x) = |x|/x$ when $x \neq 0$ and 0 otherwise.

Corollary 4.5. *If X, Y are TNN unitriangular $n \times n$ matrices, they are in the same cell $U(w)$ if and only if $\operatorname{sgn}|X_{I,J}| = \operatorname{sgn}|Y_{I,J}|$ for all $I, J \subseteq [n]$ of the same size.*

Proof. For $X, Y \in U(w)$ where $X = x_w(\vec{a})$ and $Y = x_w(\vec{b})$, the value of a minor is a polynomial with nonnegative coefficients in the domain of x_w. If $|x_w(\vec{a})_{I,J}| = 0$, then this corresponding polynomial is identically zero, so $|x_w(\vec{b})_{I,J}| = 0$ as well.

If $X \in U(w)$ but $Y \in U(w')$, then X is w-NE-bounded and Y is w'-NE-bounded using Lemma 4.2. The result follows if $w \neq w'$ implies that there is a w-NE-ideal that is a shifted w'-NE-ideal, or vice versa. This can be seen by inspection. □

4.2. Cells of $(n-2)$-nonnegative unitriangular matrices

Denote the semigroup of $(n-2)$-nonnegative unitriangular matrices as G. In this section we will distinguish two different ways of defining cells of matrices in G, namely the *fine* and *coarse* cells. We also prove that our list of relations on G is complete by showing that the cells corresponding to factorizations are either disjoint or equal.

As in the background, we will associate factorizations of matrices in G to words in the free monoid

$$\mathbb{T} = \langle 1, 2, \ldots, n-1, T \rangle.$$

We define a length function $\ell : \mathbb{T} \to \mathbb{Z}_{\geq 0}$ which maps each letter of the alphabet to the number of parameters of the corresponding family: $\ell(T) = 2n-5$ and $\ell(i) = 1$ for every i (representing e_i). To abuse notation slightly, let $U(A)$ be the set of matrices that have a factorization corresponding to the word $A \in \mathbb{T}$. This set can also be defined as the image of the parameter map $x_A : \mathbb{R}_{>0}^{\ell(A)} \to G$ which "fills in" the parameters of a word. For example, $x_{1\,T\,2}(x_1, \ldots, x_{2n-3}) = e_1(x_1)T(x_2, \ldots, x_{2n-4})e_2(x_{2n-3})$.

Notice that the union of these cells is precisely G. We want to show how a subset of these cells, corresponding to the elements of an algebraic object, partition G. Considering $\mathbb{T}$ as free, a matrix M can belong to different cells since we can move between factorizations via the relations given by 3.5 and Theorem 4.9 of [11]. Further, because these relations only contain subtraction-free rational expressions, if a relation can be performed on some $M \in U(A)$, it can be performed on all matrices in $U(A)$, regardless of the values of the parameters. Thus, we can "lift" these relations on factorizations to relations on words in $\mathbb{T}$. For example, relation 3.5 corresponds to the word relation $i\,T \equiv T\,(i+2)$ for $1 \leq i \leq n-3$.

We want to say that two cells, say, $U(A)$ and $U(B)$, are either disjoint or equal (so we can use our relations to move from A to B). However, we need to resolve the issue arising in 3.8, since this is a case where one cell is the disjoint union of three others. There is a choice that can be made here: either we throw out the smaller cells and use the larger cell, or vice versa. Rigorously, we only allow the relation in one direction, and consider a word to be reduced if we can no longer perform the relation. We will call choosing the larger cells the *coarse* choice, and the smaller cells the *fine* choice.

Definition. Let $\equiv$ be the equivalence relation generated by:

$$\begin{aligned} i\,i &\equiv i & i\,T &\equiv T\,(i+2) \text{ for } i \in [n-3] \\ i\,j &\equiv j\,i \text{ when } |j-i| > 1 & (n-2)\,T &\equiv T\,1 \\ i\,(i+1)\,i &\equiv (i+1)\,i\,(i+1) & (n-1)\,T &\equiv T\,2 \end{aligned}$$

Let $\mathbb{T}_{\text{coarse}}$ be the set of words in $\mathbb{T}$ that is not equivalent to a word (1) with more than one T or (2) where one of the following one-way relations can be applied.

$$\begin{aligned} (n-1)\,(n-2)\,T &\leftarrow (n-2)\,(n-1)\,T \\ (n-1)\,(n-2)\,T &\leftarrow (n-2)\cdots(1)\,(n-1)\cdots(2) \\ (n-1)\,(n-2)\,T &\leftarrow (n-2)\cdots(1)\,(n-1)\cdots(1) \end{aligned}$$

Let $\bar{\mathbb{T}}_{\text{coarse}} := \mathbb{T}_{\text{coarse}}/\equiv$. Define $\mathbb{T}_{\text{fine}}$ and $\bar{\mathbb{T}}_{\text{fine}}$ in the same way, replacing $\leftarrow$ with $\rightarrow$.

Using Lemma 3.4, we can restrict to words with at most one T. The fine cells maintain the structure of the TNN cells, grafting on rest of the $(n-2)$-nonnegative cells. The coarse cells ignore the structure of minors of order $n-1$, thus representing more closely the process of Proposition 2.4. The coarse cells also have a top element, unlike the fine cells. Despite these differences, none of the proofs to come will depend on the choice used. Unless explicitly stated, all results will hold for both cases.

We will show that for two words A and B in $\mathbb{T}_{\text{coarse}}$ or $\mathbb{T}_{\text{fine}}$, $A = B$ in the equivalence relation from Definition 4.2 if and only if $U(A) = U(B)$. Next we will show that if $U(A) \neq U(B)$, then $U(A)$ and $U(B)$ are disjoint. The following theorem enumerates the elements of $\bar{\mathbb{T}}_{\text{coarse}}$ and $\bar{\mathbb{T}}_{\text{fine}}$ that, as we will later show, partition the set of $(n-2)$-nonnegative unitriangular matrices.

Throughout, notation is abused by identifying reduced words of $\mathbb{T}_{\text{coarse}}$ and $\mathbb{T}_{\text{fine}}$ that do not include T with S_n, since the relations for reduced words between them are the same. This means that for a word $w \in \mathbb{T}$, if it contains no T letters, we can think of w as both a word and as a permutation. Thus, we will talk about $U(w)$ as well as $w(i)$.

Theorem 4.6. *Define $w_{0,[n-2]} := n-2, n-3, \dots, 1, n-1, n$ in one-line notation and $\beta := (n-2)\cdots(1)(n-1)\cdots(2)$ as a product of transpositions.*

$$\bar{\mathbb{T}}_{\mathit{fine}} = \begin{cases} [\sigma\lambda] & \sigma \le w_{0,[n-2]}, \lambda \in \{T, (n-1)T, (n-2)T, (n-2)(n-1)T\}, \\ [\sigma] & \sigma \in S_n \end{cases}$$

$$\bar{\mathbb{T}}_{\mathit{coarse}} = \begin{cases} [\sigma\lambda] & \sigma \le w_{0,[n-2]}, \lambda \in \{T, (n-1)T, (n-2)T, (n-1)(n-2)T\}, \\ [\sigma] & \sigma \in S_n,\ \beta \not\le \sigma \end{cases}$$

All equivalence classes in each description are distinct, and the representatives chosen are minimal length (that is, the word is reduced).

Proof. We will consider the fine case; the coarse case follows similarly. First, we will give a set of words $S \subset \mathbb{T}_{\text{fine}}$ such that all $w \in S$ are reduced and the map $w \to [w]$ surjects onto $\bar{\mathbb{T}}_{\text{fine}}$. Second, we show that the quotient map is injective for S. If we quotient S by the equivalence relation generated by relations *not including* T, we get the result, since this factors the quotient map through a space where we can identify elements with elements of S_n via reduced word representatives. The full quotient map being bijective implies that the maps through the factored space are also bijective.

To find a list of elements $[A]$ of $\bar{\mathbb{T}}_{\text{fine}}$, it suffices to consider reduced $A \in \mathbb{T}_{\text{fine}}$; recall that $A \in \mathbb{T}_{\text{fine}}$ simply means that it is a word in $\mathbb{T}$ that has at most one T and is not equivalent to a word with $(n-1)(n-2)T$ as a substring. For the set of reduced words that contain *no* T, we know its quotient, since the relations that can act on it are precisely those for S_n. So, $S' = \{\text{reduced word for } \sigma \mid \sigma \in S_n\}$ gives the elements of $\bar{\mathbb{T}}_{\text{fine}}$ without T.

We devote the rest of the proof to showing that, for

$$\begin{aligned} S'' = \{(\text{reduced word for } \sigma)\lambda \mid \sigma \in S_n,\ \sigma(n-1) = n-1,\ \sigma(n) = n, \\ \lambda \in \{T, (n-1)T, (n-2)T, (n-2)(n-1)T\}\}, \end{aligned}$$

S'' is a set of reduced words and precisely enumerates the distinct equivalence classes of the elements of $\bar{\mathbb{T}}_{\text{fine}}$ with a T. The result will follow by taking $S = S' \cup S''$ (notice that $\{\sigma : \sigma \le w_{0,[n-2]}\} = \{\sigma : \sigma(n-1) = n-1,\ \sigma(n) = n\}$).

We will show that all words of $\mathbb{T}_{\text{fine}}$ with one T are equivalent to some element in S'', then show that no two elements of S'' are equivalent. A word of $\mathbb{T}_{\text{fine}}$ is equivalent to a reduced word of the form $w\lambda$, where w is a word without $(n-2), (n-1), T$ and λ is one of the above four options. From the relations, we can push T to the end of the word and $n-2$ and $n-1$ commute with everything except each other. For example,

$$[(n-2)(n-3)T] = [T(1)(n-1)] = [T(n-1)(1)] = [(n-3)(n-2)T].$$

This freely-moving portion of A is our λ. By the "fineness" restriction on $\mathbb{T}_{\text{fine}}$, λ can be one of only four options. Finally, we can interpret w as being a word factorization of some $\gamma \in S_n$. Since $w\lambda$ is reduced, w is reduced, so there is a sequence of relations sending w to the reduced word for γ chosen for S''. By applying these relations to $w\lambda$, we get that our word of $\mathbb{T}_{\text{fine}}$ is equivalent to an element of S''.

Now, suppose that $s\lambda \in S''$ is either not reduced or is equivalent to some other element of S''. In both cases, this means that $s\lambda \equiv t\lambda'$ where either $s \not\equiv t$, considered as elements of S_n, or $\lambda \neq \lambda'$. The case where $\lambda \neq \lambda'$ cannot occur: no relations send a word without a letter to a word with that letter, so if there is a sequence of relations between $s\lambda$ and $t\lambda'$, $s\lambda$ has a particular letter if and only if $t\lambda'$ has that letter. Since λ is determined by whether $(n-1)$ and $(n-2)$ is in the word, we have $\lambda = \lambda'$. The case where $s \not\equiv t$ also cannot occur: given a sequence of relations sending $s\lambda$ to $t\lambda$, we can get a sequence of relations sending s to t by deleting all relations that include letters from λ. Proving this just involves case analysis, showing that no relation involving $(n-2)$, $(n-1)$, or T can affect the structure of s. ☐

Theorem 4.7. *If X, Y are $(n-2)$-nonnegative unitriangular $n \times n$ matrices, they are in the same cell $U(w)$, for $w \in \bar{\mathbb{T}}_{\mathit{fine}}$, if and only if* $\operatorname{sgn}|X_{I,J}| = \operatorname{sgn}|Y_{I,J}|$ *for all $I, J \subseteq [n]$ of the same size.*

Proof. This proof follows similarly to Corollary 4.5. If $X, Y \in U(w)$, then the value of a minor is a rational function in the input of x_w. For minors of order at most $n-2$, these rational functions have nonnegative coefficients, and so the sign of the minor is constant over the image of x_w. For minors of order $n-1$, the only minor that can have negative coefficients is the $[1, n-1], [2, n]$ minor. However, the sign of this minor is still constant over the image of x_w, precisely because we only consider fine cells: a simple calculation shows that this minor can only take multiple signs when w can be written such that $(n-1)(n-2)T$ is a subword. This cannot occur for $w \in \bar{\mathbb{T}}_{\text{fine}}$.

In the other direction, it suffices to show the result for $X = w\lambda$ and $Y = w'\lambda'$, using the notation in Theorem 4.6. All matrices in $U(\lambda)$ are in the standard Bruhat cell B^-_α for $\alpha = (n-2)\cdots(1)(n-1)\cdots(1)$, and $w'\alpha$ is always reduced. That $w'\alpha$ is reduced follows from a simple Coxeter group argument: w' only contains letters 1 through $n-3$, which don't affect α's inversions; thus, $\ell(w'\alpha) = \ell(w') + \ell(\alpha)$, and so $w'\alpha$ is reduced if w' and α are.

Thus, by the same argument as Corollary 4.5 and Lemma 4.3, if $w \neq w'$, then $U(w\lambda)$ and $U(w'\lambda')$ are in different Bruhat cells, and there is a minor distinguishing them. So, we can distinguish our cells by signs of minors, up to containing $n-2$ and $n-1$. But we know how to distinguish these: they appear precisely when the minor indexed by $[1, n-2], [2, n-1]$ and the minor indexed by $[2, n-1], [3, n]$ are nonzero, respectively. $\square$

Using this, we can confirm that our list of relations is complete.

Corollary 4.8. *Suppose that $w \neq w'$ are reduced words which take the representative form $\sigma\lambda$ or σ for their equivalence class in $\bar{\mathbb{T}}_{coarse}, \bar{\mathbb{T}}_{fine}$ as described in Theorem 4.6. Then $U(w)$ and $U(w')$ are disjoint.*

Proof. It is enough to show this for the fine cells, since no two coarse cells contain the same fine cell. For the fine cells, $U(w)$ and $U(w')$ are disjoint by Theorem 4.7. $\square$

We can also see the core difference in behavior between the coarse and fine cells: the fine cells are concerned with the sign of all minors, while the coarse cell are only concerned with the sign of minors of order at most $n-2$.

Corollary 4.9. *If X, Y are $(n-2)$-nonnegative unitriangular $n \times n$ matrices, they are in the same cell $U(w)$, for $w \in \bar{\mathbb{T}}_{coarse}$, if and only if $\operatorname{sgn}|X_{I,J}| = \operatorname{sgn}|Y_{I,J}|$ for all $I, J \subseteq [n]$ where $|I| = |J| \leq n-2$.*

Proof. The proof for fine cells shows how to distinguish cells by minors of arbitrary size; the only thing we need to do to show that this distinguishing can be done by minors of order $\leq n-2$.

First, if $\{\sigma(1), \sigma(2)\} \neq \{n-1, n\}$ and $\sigma \neq \omega$, then B^-_σ and B^-_ω can be distinguished by a σ- or ω-NE-ideal of order at most $n-2$. This follows from noticing that, under these assumptions, the $(\sigma(i), i)$ such that $\sigma(i) \neq \omega(i)$ occurs outside of the 2×2 bottom-right corner, so the smallest ideal distinguishing them is order $\leq n-2$.

So, we only need to consider the case where $\{\sigma(1), \sigma(2)\} = \{\omega(1), \omega(2)\} = \{n-1, n\}$; this is equivalent to $\sigma, \omega \geq \beta$. Describing the σ such that $\sigma \geq \beta$ is straightforward; $\sigma = w\beta$ for some $w \leq w_{0,[n-2]}$ (recall this means w only uses letters 1 through $n-3$) or $\sigma = w\alpha$ for some $w \leq w_{0,[n-2]}$. By the same argument as before, all Bruhat cells B^-_σ and B^-_ω can be distinguished except when $\sigma = w\alpha$ and $\omega = w\beta$. After applying the one-way relation defining coarse cells, only four coarse cells lie in the Bruhat cells $B^-_{w\alpha}$ and $B^-_{w\beta}$: wT, $w(n-1)T$, $w(n-2)T$, and $w(n-1)(n-2)T$. These can be distinguished by the $[1, n-2], [2, n-1]$ and $[2, n-1], [3, n]$ order $n-2$ minors. $\square$

4.3. Topology of the cells of $(n-2)$-nonnegative unitriangular matrices

The aim of this section is to prove the following theorem, which states that the link of the identity of the semigroup of $(n-2)$-nonnegative unitriangular martices forms a CW complex.

Theorem 4.10. *Let $\mathcal{L} = \{M \in GL_n(\mathbb{R}) \mid \sum M_{i,i+1} = 1\}$. Then $\{U(w) \cap \mathcal{L}\}$ forms a CW complex, for w as in Theorem 4.6.*

Note that $\sum M_{i,i+1} = 1$ if and only if the parameters of w add up to 1. Thus, taking the intersection $U(w) \cap \mathcal{L}$ is equivalent to restricting our parameter space to a hyperplane. First we will show that the cells are homeomorphic to open balls. We take the standard topology on $GL_n(\mathbb{R})$.

Theorem 4.11. *For A a reduced word with at most one T, x_A, the map defined in Subsection 4.2, is a homeomorphism.*

Proof. First, notice that it is enough to prove the statement for a single representative of each equivalence class, since the relations give homeomorphisms between parameters. We will take this choice to be the one given by Theorem 4.6. We know the result for words without T, so we assume that w has a T, and so $w = w_1 w_2 \cdots w_k T$ for w_i non-T letters. Suppose we have two sets of parameters that map to the same matrix. Then

$$x_{w_1}(a_1) \cdots x_{w_k}(a_k) x_T(a_{k+1}, \ldots, a_{k+2n-5}) = \\ x_{w_1}(a'_1) \ldots x_{w_k}(a'_k) x_T(a'_{k+1}, \ldots, a'_{k+2n-5}).$$

By eliminating equal-valued generators, without loss of generality, we can assume that $a_1 \neq a'_1$, and also that $a_1 > a'_1$. If this parameter lies in the T, then by computation we know that $a_i = a'_i$ for all i. Otherwise,

$$x_{w_1}(a_1 - a'_1) \cdots x_{w_k}(a_k) x_T(a_{k+1}, \ldots, a_{k+2n-5}) = \\ x_{w_2}(a'_2) \ldots x_{w_i}(a'_k) x_T(a'_{k+1}, \ldots, a'_{k+2n-5})$$

and so this matrix is in two different cells, which gives a contradiction by Corollary 4.8.

Thus, the map is a bijection (that the map is surjective is obvious). We now only need to show that the map and its inverse are continuous. Clearly, the forward map is continuous, since we can express the matrix entries as polynomials in the parameters.

For the inverse map, first note that x_T is a homeomorphism onto its image, since we can give an explicit rational inverse map. We consider the functions that give the parameters of the factorization based on the word w from the matrix entries. If $w = w_1 \cdots w_k T$, we first determine the parameter a_1 of x_{w_1}. This must be the maximum value of a_1 that will leave the matrix $(n-2)$-nonnegative, since otherwise this would violate Lemma 4.2. Thus, from Lemma (F3), a_1 will be the minimum value of the set of a's that make any minor zero. Since a_1 is the minimum of a number of continuous functions, a_1 is itself determined by a continuous function in the entries. We can then recurse on the resulting matrix to obtain all of our parameters. □

As in the TNN case, for a cell $U(w)$, we can consider setting parameters of e_i and T to zero, which gives elements in the closure of $U(w)$. Further, if we can achieve a subword in this way, then the whole cell corresponding to the subword is in the closure. The question is whether this is everything that we can get in the closure.

Another way to consider this is via the subword order in $\bar{\mathbb{T}}_{\text{fine}}$ and $\bar{\mathbb{T}}_{\text{coarse}}$. We can describe the subword order by describing the subwords of each letter, and saying that $V \leq W$ if there is a representation of W (modulo the equivalence relation of $\bar{\mathbb{T}}_{\text{fine}}$) such that replacing every letter in W with a subword of that letter gives a representation of V. The typical conception of subword, and the one used in the TNN case, is by defining the subwords of a letter a as a itself and the empty word. Since we want this to reflect closure, for our alphabet, we will define subwords in this typical sense, except for T, whose subwords we base on the resulting matrix when setting parameters to zero.

Lemma 4.12. *By setting any parameters of T to zero, we get matrices that correspond to permutations that are below at least one of the permutations described below in the Bruhat order.*

(a) $T_1^i = e_{n-3} \cdots e_1 \, e_{n-1} \cdots \widehat{e_i} \cdots e_2$, *where* $2 \leq i \leq n-1$.
(b) $T_2^i = e_{n-2} \cdots \widehat{e_i} \cdots e_1 \, e_{n-1} \cdots \widehat{e_{i+1}} \cdots e_2$, *where* $1 \leq i \leq n-3$.

Generators with a cap represent missing generators.

Proof. First observe that if any positive minor of size less than $n-2$ in T goes to 0, then at least one of the parameters a_i or b_i for some i must go to 0 as well. This can be verified from the formulas for minors specified in Appendix A. Thus it is sufficient to consider the matrix factorizations that arise when one of the a_i's or b_i's are sent to 0. If one of the b_i's in the parametrization of T is 0, then it is straightforward to verify by computation that the word

$$e_{n-3}(a_{n-3}) \cdots e_1(a_1) \, e_{n-1}(b_{n-2}) e_{n-2}(b_{n-3}) \cdots \widehat{e_{i+1}(b_i)} \cdots e_3(b_2) e_2(b_1)$$

describes a factorization for the matrix $T(\vec{a},\vec{b})$. Conversely, every matrix of such a factorization, for $2 \le i \le n-1$, is in the closure of T and corresponds to the matrix $T(\vec{a},\vec{b})$ where $b_{i-1}=0$.

If one of the a_i's is 0, then the factorization is

$$(e_{n-2}(\mu)e_{n-3}(A_{n-3})\cdots e_{i+1}(A_{i+1})\widehat{e_i(a_i)}\cdots e_1(a_1))$$
$$(e_{n-1}(b_{n-2})\cdots e_{i+2}(B_{i+1})\widehat{e_{i+1}(b_i)}\cdots e_2(b_1)),$$

where $A_{i+1}=a_{i+1}+b_i$, $B_k=a_kb_k/A_k$ for $i+1\le k\le n-3$ and $A_k = a_k+b_{k-1}-B_{k-1}$ for $i+2\le k\le n-3$. Observe that the A_k's and B_k's thus defined are always positive, since we have $A_k > a_k$ for all k where A_k is defined, and $B_k < b_k$ for all k where B_k is defined. Next μ is the usual rational function calculated from $\vec{a}$ and $\vec{b}$, computed after setting $a_i=0$.

As a note, μ can also be expressed as $(b_{n-3}-B_{n-3})$. This can be seen by observing that μ as a rational functions equals $(b_{n-3}\cdots b_i)/(A_{n-4}\cdots A_{i+1})$, and for every $i+2\le k\le n-3$, we have $b_k-B_k=b_k(b_{k-1}-B_{k-1})/A_k$. This shows that the entries of the resulting matrix in the factorization above are exactly $\vec{a}$ and $\vec{b}$ as desired.

Finally, while setting parameters to 0, we need to consider the possibility that μ as a rational function may tend to infinity. But observe from the expressions for the numerator and denomenator of μ that whenever the denomenator tends to 0, it must be the case that one or more of the b_i's tends to 0. Thus it becomes clear that the numerator of μ will also tend to 0 in such a way that μ as a fraction always tends only to a finite number. □

The T_1^i's and T_2^i's are distinguished based on whether a b_i or an a_i is 0. However, for ease of notation, we also define T_2^{n-2} as the other T_2^i's. This gives us $T_2^{n-2} := T_1^{n-1}$.

We define the subwords of T to be the reduced words described above in Lemma 4.12. This naturally extends to a general subword order. By mapping the closure of the parameter space to the closure of the cell, we can conclude that, for this order, $A\le B \implies U(A)\subset \overline{U(B)}$. We want to say that $\overline{U(B)}$ contains exactly the cells $U(A)$ such that $A\le B$. Further, every element of the closure of the cell can be achieved by setting parameters to zero, which follows from Lemma 4.12.

We prove that this subword order exactly describes the closures of cells. To prove this, we will describe the closure of $\Lambda=\{T,(n-1)T,(n-2)T,(n-2)(n-1)T,(n-1)(n-2)T\}$ in two ways, through subwords and through determinants, and together these will give a straightforward characterization. We continue conflating words in $\bar{\mathbb{T}}_{\text{coarse}}$ and $\bar{\mathbb{T}}_{\text{fine}}$ that don't contain a T with elements of S_n by conflating Bruhat order on permutations with subword order for words without T.

Lemma 4.13. *The closure of matrices with factorizations in* $\Lambda = \{(n-1)T, (n-2)T, (n-2)(n-1)T, (n-1)(n-2)T\}$ *is given exactly by matrix factorizations beneath these elements in the subword order.*

Proof. The computations of minors in Appendix B, along with those in Appendix A, show that solid minors in the matrices corresponding to

$$\{T(\vec{a},\vec{b}), e_{n-1}(x)T(\vec{a},\vec{b}), e_{n-2}(y)T(\vec{a},\vec{b}), e_{n-2}(y)e_{n-1}(x)T(\vec{a},\vec{b})\}$$

are zero precisely when one or more of the parameters are zero. It follows through quick computations that the matrices in the closure of $e_{n-1}(x)T(\vec{a},\vec{b})$ are: $T(\vec{a}\vec{b})$, $e_{n-1}T_1^i = T_1^i$ with parameters

$$e_{n-3}(a_{n-3})\cdots e_1(a_1)\; e_{n-1}(b_{n-2}+x)e_{n-2}(b_{n-3})\cdots \widehat{e_{i+1}(b_i)}\cdots e_3(b_2)e_2(b_1)$$

for $2 \leq i \leq n-1$, and $e_{n-1}(x)T_2^i$, for $1 \leq i \leq n-3$, with the same parameters for T_2^i as the proof of Lemma 4.12 above. Similarly the matrices in the closure of $e_{n-2}(y)T(\vec{a},\vec{b})$ are $T(\vec{a},\vec{b})$, $e_{n-2}(y)T_1^i$ for $2 \leq i \leq n-1$, and $e_{n-2}(y)T_2^i$ for $1 \leq i \leq n-3$, along with all matrices in the closures of these matrices. Finally, matrices in the closure of $e_{n-2}(y)e_{n-1}(x)T(\vec{a},\vec{b})$ are $e_{n-1}(x)T(\vec{a},\vec{b})$, $e_{n-2}(y)T(\vec{a},\vec{b})$, $e_{n-2}(y)e_{n-1}(x)T_1^i$ for $2 \leq i \leq n-1$ and $e_{n-2}(y)e_{n-1}(x)T_2^i$ for $1 \leq i \leq n-3$.

Next we consider the closure of the matrix corresponding to the factorization

$$e_{n-1}(x)e_{n-2}(y)T(\vec{a},\vec{b}).$$

According to relation (3.8), matrices of this form can be factored in one of three ways. If the matrix can also be expressed in the form $(n-2)(n-1)T$, the above computations show that the matrices in the closure correspond exactly to subwords. The matrices in the closure of $(n-2)\cdots 1\,(n-1)\cdots 2$ consist of the following two factorizations. First, for $2 \leq i \leq n-1$ we have

$$(n-2)\cdots 1\,(n-1)\cdots \widehat{i}\cdots 2 = (n-2)T_1^i.$$

Second, for $1 \leq i \leq n-3$, we repeatedly use braid and commutation relations to see that

$$\begin{aligned}(n-2)\cdots\widehat{i}\cdots 1\,(n-1)\cdots 1 &= (n-1)(n-2)\cdots\widehat{i}\cdots 1\,(n-1)\cdots\widehat{i+1}\cdots 1\\ &= (n-1)T_2^i.\end{aligned}$$

Finally, for the matrices of the form $(n-2)\cdots 1\,(n-1)\cdots 1$, we make the following observations. First, for $1 \le i \le n-2$, we have

$$\begin{aligned}(n-2)\cdots\widehat{i}\cdots 1\;(n-1)\cdots 1 &= (n-1)\cdots 1\;(n-1)\cdots\widehat{i+1}\cdots 2\\ &= (n-1)(n-2)T_1^{i+1}.\end{aligned}$$

Second, for $2 \le i \le n-1$, we have

$$(n-2)\cdots 1\,(n-1)\cdots\widehat{i}\cdots 1 = (n-2)\cdots 1\,(n-1)\cdots\widehat{i}\cdots 2$$

which is in the closure of $(n-2)\cdots 1\,(n-1)\cdots 2$. Third, note that $(n-2)\cdots 1\,(n-1)\cdots 2$ is in the closure of $(n-2)\cdots 1\,(n-1)\cdots 1$, but both factorizations are contained in the coarse cell $(n-1)(n-2)T$, so the factorizations trivially correspond to subwords of $(n-1)(n-2)T$. This shows that the matrices in the three cells comprising $(n-1)(n-2)T$ are exactly words below $(n-1)(n-2)T$ in the subword order. $\square$

Now we will consider the closure of a general cell. Since any element in the closure of a cell must be $(n-2)$-nonnegative and unitriangular, the closure is clearly contained in the disjoint union of some cells. The following theorem shows that the closure of any cell is precisely the union of the cells of its subwords. That is, the poset given by subword order is equal to the poset given by closure order, which is defined by $A \le B \iff U(A) \subset \overline{U(B)}$.

Theorem 4.14. *The closure $\overline{U(B)}$ consists exactly of all $U(A)$ for all $A \le B$ in the subword order.*

Proof. Suppose that $A \le B$, then we already know that $U(A) \subset \overline{U(B)}$. This uses the fact that the x_B maps the closure of the parameter space into the closure of the image, and so we can set parameters to zero (that is, take subwords) to get full cells in the closure.

So, it suffices to show that if $U(A)$ intersects $\overline{U(B)}$, then $A \le B$ in the subword order. This is a known result if both A and B are TNN. The situation that A is not TNN but B is TNN cannot occur, since an element in the closure of B must be TNN.

Now suppose both B and A are not TNN. Using Theorem 4.6, B and A can be written as reduced words $b\lambda$ and $a\lambda'$ where b and a do not include $(n-1)$ or $(n-2)$ and $\lambda, \lambda' \in \Lambda$. Then Lemma 4.3 and properties of the standard Bruhat decomposition require that $a\alpha \le b\alpha$, where α is defined in the proof of Theorem 4.7. The non-TNN cells in standard Bruhat cells B_p^- for $p \le b\alpha$ are precisely $U(s_1 s_2)$, where $s_1 \le b$ and $s_2 \in \Lambda$.

By looking at the minors for different values of s_2, it is easy to see that we must have $\lambda' \leq \lambda$. For example, $\overline{(n-1)T}$ cannot intersect $(n-1)(n-2)T$, since elements of $(n-1)T$, and thus its closure, must have the top-left large minor be zero, which is always positive in $(n-1)(n-2)T$. The same argument with the minor description given by Lemma 4.2 also works to show that we must have $a \leq b$. From here, the possible cells that could intersect the closure of $U(B)$ are precisely $U(A)$ for A a subword of B.

Finally, suppose that B is not TNN but A is TNN. Again write B in the form $b\lambda$ where $\lambda \in \Lambda$. Elements in the closure of $U(B)$ can be written in the form $b'm$ where $b' \leq b$ and $U(m)$ is in the closure of $U(\lambda)$. This statement follows because matrix multiplication is continuous, and thus any limit point of the cell $U(b\lambda)$ corresponds to a product of limit points in some cells $U(b')$ and $U(m)$ which are in the closures of $U(b)$ and $U(\lambda)$ respectively. Thus, by our hypothesis, A is of the form $b'm$. Using Lemma 4.13, it follows that $A \leq B$. $\square$

So far we have showed that our coarse and fine cells induce a cell decomposition on the space of $(n-2)$-nonnegative unitriangular matrices. Finally we will prove that the link of the identity of this space forms a CW complex.

Proof of Theorem 4.10. The attaching maps for our CW complex will be the parameter maps x_A. Notice that the space $\mathcal{L}_\ell = \{v \in \mathbb{R}^\ell_{>0} \mid \|v\|_1 = 1\}$ is homeomorphic to an open $(\ell-1)$-dimensional ball, and $x_A(\mathcal{L}_\ell) = U(A) \cap \mathcal{L}$.

One way to see this is to show that if $\|v\|_1 = c$ then $\sum x_A(v)_{i,i+1} = c$ by inducting on ℓ. The inductive step follows from the formula for matrix multiplication. Thus, by Theorem 4.11, x_A is a homeomorphism $\mathcal{L}_\ell \to U(A) \cap \mathcal{L}$.

Next we need to show that x_A maps the closure $\bar{\mathcal{L}}_\ell$ onto a union of lower-dimensional cells. This map is clearly continuous. Further, notice that $x_A(\bar{\mathcal{L}}_\ell) \subset \mathcal{L}$, since $\|v\|_1 = 1$ for all $v \in \bar{\mathcal{L}}_\ell$ and x_A is continuous on $\bar{\mathcal{L}}_\ell$. Second, notice that the boundary of $\mathcal{L}$ is

$$\partial\mathcal{L} = \left\{ v \in \mathbb{R}^n_{\geq 0} \middle| \|v\|_1 = 1,\ v_i = 0 \text{ for some } i \right\}$$

and that for $v \in \partial\mathcal{L}$, $x_A(v)$ is an element in a lower-dimensional cell. Let $\hat{A}$ be the word formed by deleting letters corresponding to zero entries of v. Then $x_A(v) \in U(\hat{A})$, which is a lower-dimensional cell (because the length of the word must be smaller). Further, $\hat{A} \leq A$, and there are only finitely many such choices. So, x_A maps $\partial\mathcal{L}$ to the union of finitely many lower-dimensional cells. $\square$

Specifically, the closure poset corresponding to this CW complex is given by the subword order.

4.4. Properties of the closure poset

To conclude, we will prove that the poset on cells $U(A)$ is a graded poset. The choice of coarse or fine cells does not matter here, since the proof is based on the fact that the vast majority of the poset is lifted from the standard Bruhat order.

First, we generalize the exchange condition of S_n to our generating set, where we have the relation $e_i e_i = e_i$, as opposed to the Coxeter relation $s_i s_i = 1$.

Lemma 4.15. *Let w be a word in $\Theta = \{1, \ldots, n-1\}$ subject to Chevalley relations (2.2). If w is a reduced word and $t \in \Theta$, then either tw is reduced, so $\ell(tw) = \ell(w) + 1$, or $tw = w$, so $\ell(tw) = \ell(w)$.*

Proof. Suppose tw is not reduced. Let $M = (m_1, m_2, \ldots, m_r)$ be a sequence with $tw = m_1$, each m_k at most one local move away from the previous, with no $i \to i\ i$ moves, and m_r a reduced word. To see that such a sequence exists, note that such a sequence exists for Coxeter groups from [3] (cf. Theorem 3.3.1). We use this sequence after replacing the $s_i s_i = 1$ relation by the shortening relation $e_i e_i = e_i$.

We will use a function φ to indicate a sort of location for t as we move along the sequence M. Define $\varphi : M \to [\ell(w) + 1] \cup \varnothing$ recursively in the following way. Set $\varphi(m_1) = 1$. For $i > 1$, define

$$\varphi(m_i) = \begin{cases} \varnothing & t \text{ is in shortening relation from } m_{i-1} \text{ to } m_i \text{ or } \varphi(m_{i-1}) = \varnothing \\ \varphi(m_{i-1}) \pm 1 & t \text{ is in left/right position of nonadjacent relation used from } m_{i-1} \text{ to } m_i \\ \varphi(m_{i-1}) \pm 2 & t \text{ is in left/right position of adjacent relation used from } m_{i-1} \text{ to } m_i \\ \varphi(m_{i-1}) - 1 & t \text{ to the right of shortening relation used from } m_{i-1} \text{ to } m_i \\ \varphi(m_{i-1}) & \text{otherwise} \end{cases}$$

It is now enough to show that (a) φ is well-defined; (b) there are no length-shortening moves that do not involve t; and (c) m_r is a reduced word for w.

(a) Because we chose the sequence such that we never get a longer word than tw, our codomain is correctly stated. Thus, it only remains to check whether $\varphi(m_{i-1})$ can ever be in the middle of an adjacent relation.

Let n_i be m_i with the $\varphi(m_i)^{\text{th}}$ letter in the word removed, where we take out nothing if $\varphi(m_i) = \varnothing$. That is, we take out the t from the word, and $\varnothing$ signifies that the t no longer exists. Then notice that $w = n_1$, and each n_i is at most one local move away from n_{i-1}. The reason for this is that removing t does not affect any local moves not involving t, and the local moves that do involve t don't affect anything except t. Suppose we do have a move where the location of the t is in the center. Then if we consider the n_i up to that point, we get that there is an n_i with two adjacent identical letters:

$$m_i = \cdots j\ t\ j \cdots \implies n_i = \cdots j\ j \cdots$$

However, this would imply that w can be reduced to a word of shorter length. This is a contradiction.

(b) The same reasoning applies. Consider the n_i. If there was a length-shortening move then obviously we would get that n_i is a series of moves that shortens w, which is not possible.

(c) We must have a shortening relation to get a reduced word. This relation must contain t, so there must be exactly one. Notice that once $\varphi(m_i) = \varnothing$, $m_i \equiv n_i$ modulo the Chevalley equivalence relations. We know that $w \equiv n_i$ for all i. Thus, $m_i \equiv w$. □

Theorem 4.16. *The closure poset of the cells of* $(n-2)$*-nonnegative unitriangular matrices is graded.*

Proof. For anything not containing a T, this is well-known, one proof being Theorem 2.2.6 of [3]. We know that when we only consider words containing T, the restriction gives a poset that is isomorphic to the product poset of an interval in the strong Bruhat poset with the Boolean algebra on 2 elements (corresponding to containment of $(n-1)$ and $(n-2)$). Both of these are graded, and it is easy to see that the rank function is equivalent to the sum of the rank functions of the individual posets. Thus, the Bruhat poset being graded implies that restricting to this case gives a graded poset.

By inspection of the subwords of T, the interval up to T is also a graded poset. Now, all that is left is to consider working between words with T and without T.

Now, suppose that wT is reduced but reducing T to some subword t makes wt not reduced. We want to show that there is a chain between wT and wt that behaves correctly with respect to our rank function.

Let $w = w_1 \cdots w_a$. Then consider $w_i \cdots w_a t$, starting from $i = a$ to $i = 1$. Using the exchange property, we can see that this reduces to some $w't$, where w' is a subword of w. Thus, this has the intermediary $w'T$, and from the lemma we get intermediaries as desired. $\square$

In addition to being ranked, the poset of the CW complex of the semigroup of unitriangular totally nonnegative matrices is Eulerian. However, the closure poset of our CW complex is not Eulerian in general.

Remark 4.17. The closure poset on cells of $(n-2)$-nonnegative unitriangular matrices is Eulerian for $n \leq 4$. For $n > 4$, the poset is not Eulerian: by computation using Lemma 4.1, the interval $[(n-2)\cdots(3)(n-1)\cdots(3), T]$ has only three elements, the middle one being T_2^1.

A CW complex is *regular* if the closure $\bar{C}$ of each cell C is homeomorphic to a closed ball and $\bar{C} \setminus C$ is homeomorphic to a sphere. Since the poset is not thin, our cell decomposition is not a regular CW complex by Lemma 2.8 in [15]. However, the poset could still be the face poset of some manifold, such as a ball, or have nice properties such as semi-Eulerianness.

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Appendix A. Minors and relations for $(n-2)$-nonnegative unitriangular matrices

The solid minors of $T(\vec{a}, \vec{b})$ are as follows.

(1) When $1 \leq i, j \leq n-3$,

$$\left| T(\vec{a}, \vec{b})_{[i,j],[i+2,j+2]} \right| = \prod_{k=i}^{j} a_k b_k.$$

(2) When $1 \leq i \leq n-2$,

$$\left|T(\vec{a},\vec{b})_{[i,n-2],[i+2,n]}\right| = b_{n-2}\mu \prod_{k=i}^{n-3} a_k b_k.$$

(3) When $1 \leq i, j \leq n$,

$$\left|T(\vec{a},\vec{b})_{[i,j],[i,j]}\right| = 1.$$

(4) When $1 \leq j \leq n-3$,

$$\left|T(\vec{a},\vec{b})_{[1,j],[2,j+1]}\right| = a_1 \cdots a_j.$$

(5) When $1 < i, j < n-1$ and $a_{n-1} = b_0 = 0$,

$$\left|T(\vec{a},\vec{b})_{[i,j],[i+1,j+1]}\right| = \sum_{k=i-1}^{j} \left(\prod_{\ell=i}^{k} b_{\ell-1} \prod_{\ell=k+1}^{j} a_\ell \right)$$

(6) When $2 \leq i \leq n-2$,

$$\left|T(\vec{a},\vec{b})_{[i,n-1],[i+1,n]}\right| = \frac{b_{n-2}\cdots b_{i-1}\left(\sum_{k=1}^{i-2}\left(\prod_{\ell=1}^{k-1} a_\ell \ \prod_{\ell=k+1}^{n-3} a_\ell\right)\right)}{|T(\vec{a},\vec{b})_{[2,n-3][]3,n-2]}|}.$$

(7) $\left|T(\vec{a},\vec{b})_{[1,n-1],[2,n]}\right| = -b_{n-2}\mu.$

All other minors are trivially zero.

Relations In these relations, the variables on the right-hand side are expressed in terms of the variables on the left-hand side.

(1) $e_i(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_{i+2}(x'),$ where $1 \leq i \leq n-3$.

$$\vec{A} = \left(a_1, \ldots, a_{i-1}, a_i + x, \frac{a_i a_{i+1}}{a_i + x}, a_{i+2}, \ldots, a_{n-3}\right)$$
$$\vec{B} = \left(b_1, \ldots, b_{i-1}, b_i + \frac{x a_{i+1}}{x + a_i}, \frac{b_i b_{i+1}(x + a_i)}{b_i(a_i + x) + x a_{i+1}}, b_{i+2}, \ldots, b_{n-2}\right)$$
$$x' = \frac{b_{i+1} a_{i+1} x}{b_i(a_i + x) + x a_{i+1}}$$

In the other direction, we have:

$$\begin{aligned}
\vec{a} &= \Bigg(A_1, \ldots, A_{i-1}, \frac{A_i A_{i+1} B_{i+1} + A_i A_{i+1} x'}{A_{i+1} B_{i+1} + A_{i+1} x' + B_i x'}, \\
&\quad A_{i+1} + \frac{B_i x'}{B_{i+1} + x'}, A_{i+2}, \ldots, A_{n-3} \Bigg) \\
\vec{b} &= \left(B_1, \ldots, B_{i-1}, \frac{B_i B_{i+1}}{B_{i+1} + x'}, B_{i+1} + x', B_{i+2}, \ldots, B_{n-2} \right) \\
x &= \frac{x' A_i B_i}{A_{i+1} B_{i+1} + A_{i+1} x' + B_i x'}
\end{aligned}$$

(2) $e_{n-2}(x) T(\vec{a}, \vec{b}) = T(\vec{A}, \vec{B}) e_1(x')$.
Here $\vec{A}$ and $\vec{B}$ satisfy the following recurrence:

$$\begin{aligned}
B_{n-3} &= b_{n-3} + x \\
A_i &= (a_i \cdot b_i)/B_i, \text{ where } 1 \leq i \leq n-3 \\
B_i &= a_{i+1} + b_i - A_{i+1}, \text{ where } 1 \leq i \leq n-4 \\
x' &= a_1 - A_1.
\end{aligned}$$

(Note that $B_{n-3} > b_{n-3}$, and consequently $A_{n-3} < a_{n-3}$. In turn, $B_{n-2} > b_{n-2}$, etc, so that in general $B_i > b_i$ and $A_i < a_i$.) In the other direction,

$$\begin{aligned}
a_1 &= x' + A_1 \\
c_i &= A_i C_i / a_i \text{ where } 1 \leq i \leq n-3 \\
a_i &= A_i + C_{i-1} - c_{i-1}, \text{ where } 1 \leq i \leq n-4 \\
x &= C_{n-3} - c_{n-3}.
\end{aligned}$$

(Similarly, $a_1 > A_1$, consequently $c_2 < C_2$. In turn, $a_2 > A_2$, etc, so that in general $a_i > A_i$ and $c_i < C_i$.)

(3) $e_{n-1}(x) T(\vec{a}, \vec{b}) = T(\vec{A}, \vec{B}) e_2(x')$

$$\begin{aligned}
\vec{A} &= \vec{a} \\
\vec{B} &= \left(b_1, \ldots, b_{n-3}, b_{n-2} + \frac{b_{n-2}}{b_1 x}, \right) \\
x' &= \frac{x}{\left| T(\vec{a}, \vec{b})_{[3,n-3],[4,n-2]} \right|}
\end{aligned}$$

In the other direction,

$$\vec{a} = \vec{A}$$
$$\vec{b} = \left(B_1, \dots, B_{n-3}, \frac{B_{n-2}B_1}{B_1 + x'},\right)$$
$$x = x' \left| T(\vec{A}, \vec{B})_{[3,n-3],[4,n-2]} \right|$$

(4) $e_{n-1}e_{n-2}T = e_{n-2}e_{n-1}T \sqcup e_{n-2} \cdots e_1 e_{n-1} \cdots e_2 \sqcup e_{n-2} \cdots e_1 e_{n-1} \cdots e_1$. The three factorizations on the right hand side of the equation arise from three possible values of the minor $\left|M_{[2,n][1,n-1]}\right|$, where M is the matrix $e_{n-1}(u)e_{n-2}(v)T(\vec{a}, \vec{b})$.

(a) When the minor is negative, then we have:

$$\begin{aligned} (a_1 \dots a_{n-3}) \cdot v \cdot (b_{n-2} + u) &< (a_1 \dots a_{n-3}) \cdot (b_{n-2}\mu + b_{n-2}v) \\ \Rightarrow\ v \cdot (b_{n-2} + u) &< b_{n-2}(\mu + v) \\ \Rightarrow\ vu &< b_{n-2}\mu. \end{aligned}$$

Then the matrix M can be factored as follows.

$$e_{n-1}(u)e_{n-2}(v)T(\vec{a}, \vec{b}) = e_{n-2}(v)e_{n-1}(u')T(\vec{A}, \vec{B}),$$

where $\vec{A} = \vec{a}$ and $\vec{B} = (b_1, \dots, b_{n-3}, b_{n-2} - uv/\mu)$ and $u' = b_{n-2} - B_{n-2} + u$. Note that u' is positive because $b_{n-2} > B_{n-2}$.

(b) When the minor is zero, then we have:

$$\begin{aligned} (a_1 \dots a_{n-3}) \cdot v \cdot (b_{n-2} + u) &= (a_1 \dots a_{n-3}) \cdot (b_{n-2}\mu + b_{n-2}v) \\ \Rightarrow\ vu &= b_{n-2}\mu. \end{aligned}$$

Then the matrix is totally nonnegative and can be factored as shown below.

$$\begin{aligned} e_{n-1}(u)e_{n-2}(v)T(\vec{a}, \vec{b}) = \ & e_{n-2}(v)e_{n-3}(a_{n-3}) \cdots e_1(a_1) \cdot \\ & e_{n-1}(b_{n-2} + u)e_{n-2}(b_{n-3}) \cdots e_2(b_1) \end{aligned}$$

(c) When the minor is positive, we similarly have $vu > b_{n-2}\mu$. Again the matrix is totally nonnegative and can be factored as written below.

$$e_{n-1}(u)e_{n-2}(v)T(\vec{a}, \vec{b}) = e_{n-2}(v')e_{n-3}(A_{n-3}) \cdots e_1(A_1) \cdot$$

$$e_{n-1}(b_{n-2}+u)e_{n-2}(B_{n-3})\cdots e_1(B_0),$$

where $\vec{A}$, $\vec{B}$ and v' can be determined from $\vec{a}$, $\vec{b}$, u and v, by recursive formulas:

$$\begin{aligned} v' &= \frac{vb_{n-2}}{b_{n-2}+u} \\ B_{n-3} &= b_{n-3}+v-v' \\ A_i &= (a_i \cdot b_i)/B_i, \text{ where } 1 \le i \le n-3 \\ B_i &= a_{i+1}+b_i-A_{i+1}, \text{ where } 0 \le i \le n-4 \end{aligned}$$

Note that since $v > v'$, we have $a_i > A_i$ and $b_i < B_i$ for all i, so all the new parameters are nonnegative.

Appendix B. Minors in $(n-1)T, (n-2)T$ and $(n-2)(n-1)T$

The matrix corresponding to the factorization $e_{n-1}(x)T$ has the following form:

$$e_{n-1}(x)T(\vec{a},\vec{b}) = \begin{bmatrix} 1 & a_1 & a_1b_1 & & & & \\ & 1 & a_2+b_1 & a_2b_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & a_{n-3}+b_{n-4} & a_{n-3}b_{n-3} & \\ & & & & 1 & b_{n-3} & b_{n-2}\mu \\ & & & & & 1 & b_{n-2}+x \\ & & & & & & 1 \end{bmatrix}$$

Clearly any minors not involving the matrix entry $b_{n-2}+x$ are given by the same expressions as those written for T above. It is sufficient to compute all solid minors involving $b_{n-2}+x$, as all other minors will be given by subtraction-free expressions in the solid minors. The solid minors are computed below. For $2 \le i \le n-2$ we have

$$\begin{aligned} &\left| e_{n-1}(x)T(\vec{a},\vec{b})_{[i,n-1],[i+1,n]} \right| \\ &\quad = \left| T(\vec{a},\vec{b})_{[i,n-1],[i+1,n]} \right| + xb_{n-2}\cdots b_{i-1} \\ &\quad = b_{n-2}\cdots b_{i-1}\left(\frac{\sum_{k=1}^{i-2}\left(\prod_{\ell=1}^{k-1} b_\ell \prod_{\ell=k+1}^{n-3} a_\ell\right)}{|T(\vec{a},\vec{b})_{[2,n-3][]3,n-2]}|} + x \right) \end{aligned}$$

And finally
$$\left|e_{n-1}(x)T(\vec{a},\vec{b})_{[1,n-1],[2,n]}\right| = -b_{n-2}\mu.$$
Next we compute the solid minors of the matrix corresponding to the factorization $e_{n-2}(y)T$ which has the following form.

$$e_{n-2}(y)T(\vec{a},\vec{b}) = \begin{bmatrix} 1 & a_1 & a_1b_1 & & & & & \\ & 1 & a_2+b_1 & a_2b_2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & 1 & a_{n-3}+b_{n-4} & a_{n-3}b_{n-3} & \\ & & & & 1 & b_{n-3}+y & b_{n-2}(\mu+y) \\ & & & & & 1 & b_{n-2} \\ & & & & & & 1 \end{bmatrix}$$

Observe that any solid minor that includes the submatrix $\begin{bmatrix} b_{n-3}+y & b_{n-2}(\mu+y) \\ 1 & b_{n-2} \end{bmatrix}$ remains unchanged from the corresponding minor in $T(\vec{a},\vec{b})$. Thus it is sufficient to consider solid minors which have the entry $b_{n-3}+y$ as the bottom right entry. Such minors are given by the following expression for $2 \leq i \leq n-3$.

$$\begin{aligned} \left|e_{n-2}(y)T(\vec{a},\vec{b})_{[i,n-2],[i+1,n-1]}\right| &= y\left|T(\vec{a},\vec{b})_{[i,n-3],[i+1,n-2]}\right| + b_{n-3}\cdots b_{i-1} \\ &= y\left(\sum_{k=i-1}^{n-3}\prod_{\ell=i}^{k} b_{\ell-1}\prod_{\ell=k+1}^{n-3} a_\ell\right) + b_{n-3}\cdots b_{i-1}. \end{aligned}$$

Finally, we have
$$\left|e_{n-2}(y)T(\vec{a},\vec{b})_{[1,n-2],[2,n-1]}\right| = a_1\cdots a_{n-3}\, y.$$

The matrix with factorization $e_{n-2}(y)e_{n-1}(x)T$ has the following form.

$$e_{n-2}(y)e_{n-1}(x)T(\vec{a},\vec{b}) = \begin{bmatrix} 1 & a_1 & a_1b_1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & a_{n-3}+b_{n-4} & a_{n-3}b_{n-3} & \\ & & & 1 & b_{n-3}+y & b_{n-2}\mu+b_{n-2}y+xy \\ & & & & 1 & b_{n-2}+x \\ & & & & & 1 \end{bmatrix}$$

Any solid minor with the entry $b_{n-3} + y$ in the bottom right corner is given by the same expression as the corresponding minor in the matrix $e_{n-2}(y)T(\vec{a}, \vec{b})$ above. Finally, any submatrix of $e_{n-2}(y)e_{n-1}(x)T$ containing the submatrix $\begin{bmatrix} b_{n-3}+y & b_{n-2}\mu + b_{n-2}y + xy \\ 1 & b_{n-2} \end{bmatrix}$ has the same determinant as the corresponding submatrix in $e_{n-1}(x)T(\vec{a}, \vec{b})$ above.

Appendix C. $(n-1)$-nonnegative matrix preliminaries

In Appendices C, D, E and F we will describe the factorizations and topological cell structures arising from $(n-1)$-nonnegative invertible matrices. We begin by describing the semigroup of totally nonnegative invertible matrices. Recall $e_i(a)$ generates the semigroup of upper unitriangular totally nonnegative matrices. The transpose form, denoted $f_i(a) := e_i(a)^T$ correspondingly generates the semigroup of lower unitriangular totally nonnegative matrices. More generally, *elementary Jacobi matrices* differ from the identity in exactly one positive entry either on, directly above, or directly below the main diagonal. Thus, an elementary Jacobi matrix is a Chevalley generator (i.e., $e_i(a)$ or $f_i(a)$) or a diagonal matrix which differs from the identity in exactly one entry on the main diagonal. Let this latter type of diagonal matrix be denoted by $h_i(a)$ which has (i,i)-th entry $a > 0$.

The Loewner-Whitney Theorem (Theorem 2.2.2 of [8]) states that the elementary Jacobi matrices generate the semigroup of invertible totally nonnegative matrices. We define *double Bruhat cells* as

$$B_{u,v} = B_u^+ \cap B_v^- := B^+uB^+ \cap B^-vB^-$$

so that since $\mathrm{GL}_n(\mathbb{R})$ is decomposed as

$$\mathrm{GL}_n(\mathbb{R}) = \bigcup_{u \in S_n} B^+uB^+ = \bigcup_{u \in S_n} B^-vB^-$$

we have that $\mathrm{GL}_n(\mathbb{R})$ is partitioned by $B_{u,v}$ for $(u,v) \in S_n \times S_n$. Next, consider the free monoid

$$\mathbb{A} = \langle 1, \dots, n-1, ①, \dots, ⓝ, \overline{1}, \dots, \overline{n-1} \rangle$$

and let

$$x_i(t) = e_i(t)$$

$$x_{\textcircled{i}}(t) = h_i(t)$$
$$x_{\bar{i}}(t) = f_i(t).$$

Thus, the generators of $\mathbb{A}$ correspond to the different types of generators for the semigroup. For any word $\mathbf{i} := (i_1, \ldots, i_\ell) \in \mathbb{A}$, there is a product map $x_{\mathbf{i}} : \mathbb{R}^\ell_{>0} \to \mathrm{GL}_n(\mathbb{R})$ defined by

$$x_{\mathbf{i}}(t_1, \ldots, t_\ell) := x_{i_1}(t_1) \cdots x_{i_\ell}(t_\ell).$$

With some conditions imposed on $\mathbf{i}$, the image of this map describes precisely the totally nonnegative matrices in a particular double Bruhat cell, allowing us to parametrize the double Bruhat cell and, consequently, the semigroup of invertible totally nonnegative matrices as we did in the unitriangular case. We describe this in more detail by introducing a definition:

Definition. Let $u, v \in S_n$. A *factorization scheme* of type (u, v) is a word $\mathbf{i}$ of length $n + \ell(u) + \ell(v)$ (where $\ell(u)$ denotes the Bruhat length of u in S_n) in $\mathbb{A}$ such that the subword of barred (resp. unbarred) entries of $\mathbf{i}$ form a reduced word for u (resp. v) and such that each circled entry $\textcircled{i}$ is contained exactly once in $\mathbf{i}$.

Theorem C.1 (Theorems 4.4 and 4.12 in [11]). *If $\boldsymbol{i} = (i_1, \ldots, i_\ell)$ is a factorization scheme of type (u, v), then the map $x_{\boldsymbol{i}}$ is a bijection between ℓ-tuples of positive real numbers and totally nonnegative matrices in the double Bruhat cell $B_{u,v}$.*

As before with unitriangular matrices, we would like to know the commutation relations between the h_i's, f_i's, and e_i's to move between equivalent factorizations. The commutation relations are given by

$$\begin{aligned}
e_i(a)e_{i+1}(b)e_i(c) &= e_{i+1}\left(\frac{bc}{a+c}\right) e_i(a+c)e_{i+1}\left(\frac{ab}{a+c}\right) \\
f_i(a)f_{i+1}(b)f_i(c) &= f_{i+1}\left(\frac{bc}{a+c}\right) f_i(a+c)f_{i+1}\left(\frac{ab}{a+c}\right) \\
e_i(t)f_j(t') &= f_j(t')e_i(t), i \neq j \\
e_i(a)h_j(b) &= h_j(b)e_i(a), j \notin \{i, i+1\} \\
e_i(a)h_i(b) &= h_i(b)e_i(a/b) \\
e_i(a)h_{i+1}(b) &= h_{i+1}(b)e_i(ab) \\
f_i(a)h_j(b) &= h_j(b)f_i(a), j \notin \{i, i+1\} \\
f_i(a)h_i(b) &= h_i(b)f_i(ab)
\end{aligned}$$

$$
\begin{aligned}
f_i(a)h_{i+1}(b) &= h_{i+1}(b)f_i(a/b)\\
e_i(a)e_i(b) &= e_i(a+b)\\
f_i(a)f_i(b) &= f_i(a+b)\\
e_i(a)e_j(b) &= e_j(b)e_i(a)\\
f_i(a)f_j(b) &= f_j(b)f_i(a)\\
e_i(a)f_i(b) &= f_i\left(\frac{b}{1+ab}\right)h_i(1+ab)h_{i+1}((1+ab)^{-1})e_i\left(\frac{a}{1+ab}\right)\\
f_i(a)e_i(b) &= e_i\left(\frac{b}{1+ab}\right)h_i((1+ab)^{-1})h_{i+1}(1+ab)f_i\left(\frac{a}{1+ab}\right)
\end{aligned}
$$

These relations can also be found in Section 2.2 and Theorem 1.9 or Theorem 4.9 of [11]. Finally, the poset of closure relations in the double Bruhat cells of the semigroup of totally nonnegative matrices is Bruhat order on the Coxeter group $S_n \times S_n$. It obeys many of the special properties of the poset of closure relations in the unitriangular case, such as being ranked and Eulerian. In addition, the cells of totally nonnegative invertible matrices form a CW complex, but it remains an open problem to show it is a regular CW complex.

Appendix D. Factorizations of $(n-1)$-nonnegative matrices

In this section we give a characterization of irreducible $(n-1)$-nonnegative matrices and a set of relations for the semigroup of $(n-1)$-nonnegative matrices. First we consider the possible values of entries and minors in these matrices.

Proposition D.1. *Let M be a tridiagonal matrix with 1's on the subdiagonal and nonzero entries on the diagonal and superdiagonal with $M_{i,i} = a_i$ and $M_{i,i+1} = b_i$. Then M is invertible and $(n-1)$-nonnegative if and only if the following hold:*

$$
\begin{aligned}
&\text{(D.1)} & b_i &> 0 && \textit{for all } i\\
&\text{(D.2)} & [a_x; a_{x-1}, \dots a_1; b_{x-1}, \dots, b_1] &> 0 && \textit{for } x < n-1\\
&\text{(D.3)} & [a_{n-1}; a_{n-2}, \dots a_1; b_{n-2}, \dots, b_1] &\geq 0 &&\\
&\text{(D.4)} & [a_n; a_{n-1}, \dots a_2; b_{n-1}, \dots, b_2] &\geq 0 &&
\end{aligned}
$$

Further, M is $(n-1)$-irreducible if and only if equality holds in (D.3) and (D.4).

Proof. The proof is entirely analogous to that of Theorem 3.2. We observe that M is invertible as the recurrence relation clearly shows that it has a negative determinant. □

The above criterion for $(n-1)$-irreducible matrices characterizes a $(2n-3)$-parameter family. This family, along with elementary Jacobi matrices, generates the semigroup of $(n-1)$-nonnegative matrices.[3] The $(2n-3)$-parameter family of generators appears as follows.

$$K(\vec{a},\vec{b}) = \begin{bmatrix} a_1 & a_1 b_1 & & & & \\ 1 & a_2+b_1 & a_2 b_2 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & a_{n-2}+b_{n-3} & a_{n-2}b_{n-2} & \\ & & & 1 & b_{n-2} & b_{n-1}\nu \\ & & & & 1 & b_{n-1}\mu \end{bmatrix}$$

where $a_1,\ldots,a_{n-2},b_1,\ldots,b_{n-1}$ are positive numbers, $\nu = b_1\cdots b_{n-2}$ and

$$\mu = \left|K_{[2,n-2],[2,n-2]}\right| = \sum_{k=1}^{n-2}\left(\prod_{\ell=2}^{k} b_{\ell-1}\prod_{\ell=k+1}^{n-2} a_\ell\right).$$

All the minors of these K-generators, except the determinant, can be written as subtraction-free expressions in the parameters. The full list of solid minors can be found in Appendix F, but in particular, we will use the following statements.

$$\left|K_{[1,n-1],[1,n-1]}\right| = \left|K_{[2,n],[2,n]}\right| = 0 \tag{D.5}$$

$$|K| = -a_1a_2\cdots a_{n-2}b_1b_2\cdots b_{n-1} \tag{D.6}$$

We can show that this set of generators is minimal, in the sense that every element in the set is necessary.

Theorem D.2. *Let $M = K(\vec{a},\vec{b})$ be a K-generator as defined above. Then if $RS = M$ in the semigroup of invertible $(n-1)$-nonnegative $n\times n$ matrices, one of R or S is a diagonal matrix.*

[3] Note that we can actually reduce this to an $(n-3)$-parameter family, just by scaling the superdiagonal to ones via diagonal matrices. However, we will lose relations between h_i generators if we do so, making the corresponding cell structure much more complicated.

Proof. Suppose we have $RS = M$. From (F2) we know that R and S have nonzero diagonals. Thus, we know that $r_{i,i+2}$, $s_{i,i+2}$ and their transpose analogues are all 0 from the formula for matrix multiplication. Further, we know that for every i, one of $r_{i,i+1}$ and $s_{i+1,i+2}$ is 0, and one of $r_{i,i+1}$ and $s_{i,i+1}$ is positive. Together, these show that R and S can only be as described above. □

We now turn to relations involving generators of the form $K(\vec{a},\vec{b})$. It can be seen by direct computation that the following relations hold.

(D.7) $e_i(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})e_{i+1}(x'),$ where $1 \le i \le n-2$

(D.8)
$$e_{n-1}(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})f_{n-1}(x')h_{n-1}(c)$$

(D.9)
$$f_{i+1}(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})h_{i+1}(1/w)f_i(x)h_i(w), \text{ where } 1 \le i \le n-2$$

(D.10) $f_1(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})e_1(x')h_1(c)$

(D.11) $h_i(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})h_{i-1}(x),$ where $2 \le i \le n$

(D.12) $h_1(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})$

(D.13) $K(\vec{a},\vec{b})h_n(x) = K(\vec{A},\vec{B})$

The values of the parameters of the relations can be found in Appendix F. The expressions for new parameters are always subtraction-free rational expressions of the old parameters. Finally, one relation is missing: this is the relation involving products of K-generators. The characterization of Bruhat cells in Lemma 4.2 shows that such a relation exists. However, the following lemma shows that ignoring this relation will not affect our discussion.

Lemma D.3. *$K(\vec{A},\vec{B})K(\vec{C},\vec{D})$ is totally nonnegative and admits a factorization into Chevalley generators that has length no more than than the number of parameters in $K \cdot K$.*

Proof. Since the only minor that is negative is the full determinant, the product of two Ks must be TNN. Further, it is pentadiagonal, meaning that the only nonzero entries are on the main diagonal, the superdiagonal, the super-superdiagonal, the subdiagonal, and the sub-subdiagonal; any pentadiagonal TNN matrix can be factored using at most $4n-6$ parameters (this can be seen via Lemma 4.2). KK has length $4n-6$, so we can always find a shorter word using only Chevalley generators. □

Appendix E. Cells of $(n-1)$-nonnegative matrices

Denote the semigroup of $(n-1)$-nonnegative invertible matrices as G. As done in $n-2$ case, we will associate factorizations of matrices in G to words in the free monoid

$$\mathbb{S} = \langle 1, \ldots, n-1, \textcircled{1}, \ldots, \textcircled{n}, \overline{1}, \ldots, \overline{n-1}, K\rangle.$$

We define a length function $\ell : \mathbb{S} \to \mathbb{Z}_{\geq 0}$ by giving each letter of the alphabet a value: (i) and $(\bar{i})$ are both one, $\textcircled{i}$ values to zero, and K to $2n-3$. That is, we are counting the number of parameters of the factorization, ignoring diagonal generators.[4] Let $U(A)$ be the matrices that have a factorization corresponding to the word $A \in \mathbb{S}$. This set can also be defined as the image of the parameter map $x_A : \mathbb{R}_{>0}^{\ell(A)} \to G$ associated to A which "fills in" the parameters of a word. Let $\mathbb{S}_{\text{cell}}$ be the set of words in $\mathbb{S}$ with at most one K and all diagonal generators (restricting to cells with interesting relational structure). Let $\bar{\mathbb{S}}_{\text{cell}}$ be $\mathbb{S}_{\text{cell}}/\equiv$, where $\equiv$ is the equivalence relation given by the equivalences among e_i, f_i, h_i, along with the relations involving K in D.7. Like the $n-2$ case, we wish to show that, for $A, B \in \bar{\mathbb{S}}$, $A = B$ if and only if $U(A) = U(B)$. Previous results show the forward direction, and we will now show the reverse direction. First, we will give a description of exactly what distinct cells we have under these relations via a characterization of their reduced words.

Theorem E.1. *Let $w_{0,[i,j]}$ denote the longest-length word in the set of permutations of $\{i, i+1, \ldots, j\}$ embedded in S_n. For example, $w_{0,[1,n-1]} = (n-1, n-2, \ldots, 1, n)$ (one-line notation).*

$$\bar{\mathbb{S}}_{cell} = \begin{cases} [\textit{factorization scheme of type } (\sigma,\omega)] & (\sigma,\omega) \in S_n \times S_n \\ [(\textit{factorization scheme of type } (\sigma,\omega))\gamma] & \sigma \leq w_{0,[1,n-1]}, \omega \leq w_{0,[2,n]}, \\ & \gamma \in \Gamma = \{K, (\overline{1})K, \\ & (n-1)K, (\overline{1})(n-1)K\}, \\ & \textit{scheme doesn't use } \textcircled{1} \end{cases}$$

Proof. The argument follows similarly to Theorem 4.6's. We show that all elements of $\mathbb{S}$ map to one of the described cells, and further, for the above-described lifts of elements of $\bar{\mathbb{S}}_{\text{cell}}$ into $\mathbb{S}_{\text{cell}}$, the lifted words are reduced and inject into $\bar{\mathbb{S}}_{\text{cell}}$.

[4]Ignoring these is motivated by trying to determine "distance" from the corresponding Weyl group.

Words without a K are equivalent to a factorization scheme, by Theorem C.1, so we consider a word with a K, which we can assume is reduced without loss of generality. We can move K to the end of the word. Further, our relations give us that f_1 and e_{n-1} (if present to the left of a K) commute with everything. For example,

$$(n-1)(n-2)K = K(\overline{n-1})(n-1) = K(n-1)(\overline{n-1}) = (n-2)(n-1)K.$$

As such, a reduced word can only have at most one of each of these; we account for this with our γ word, and the resulting e_is and f_is a word which is equivalent to a factorization scheme where σ and ω have no $n-1$ and 1, respectively. This is equivalent to the conditions in the theorem statement. (We use only $n-1$ generators in the second case, since this will produce the bijection of the parameter map x_w, as we will see in Theorem E.4.)

To show that no two cells as described above are equal, again argue similarly to the $n-2$ case: a sequence of relations tranforming one into the other would imply a sequence of relations between two distinct factorization schemes, which gives a contradiction. $\square$

Theorem E.2. *For X, Y $(n-1)$-nonnegative, they are in the same cell $U(w)$, for $w \in \bar{\mathbb{T}}_{\mathit{fine}}$, iff for all minors I, J, $\operatorname{sgn}(|X_{I,J}|) = \operatorname{sgn}(|Y_{I,J}|)$.*

Proof. This follows similarly to Theorem 4.7: in the forward direction, the only minor of concern is the determinant, but since it is a group homomorphism on GL_n, it is clearly constant over a cell.

In the reverse direction, consider a cell characterized by (σ, ω, γ), where the variables are taken as their respective letters in Theorem E.1. As γ is in the double Bruhat cell corresponding to $(n-1)(n-2)\cdots(1)(\bar{1})\cdots(\overline{n-2})(\overline{n-1})$, we can distinguish cells with different σ or ω via their minors, as they must lie in different double Bruhat cells. To distinguish between the four options of γ, notice that $(n-1)$ appears in the cell word precisely when the minor indexed by $[1, n-1], [1, n-1]$ is nonzero and $\bar{1}$ appears precisely when the minor indexed by $[2, n], [2, n]$ is nonzero. $\square$

As a corollary, $\bar{\mathbb{S}}_{\text{cell}}$ reflects the topology of the semigroup.

Corollary E.3. *For reduced words A, B given by Theorem E.1, if $A \neq B$ then $U(A)$ and $U(B)$ are disjoint. As a result, these $U(A)$ partition the semigroup of $(n-1)$-nonnegative invertible matrices.*

Further, each of the cells $U(w)$ is homeomorphic to an open ball.

Theorem E.4. *For a reduced word $w \in \mathbb{S}_{\mathit{cell}}$, x_w is a homeomorphism.*

The argument is the same as Theorem 4.11, except we consider $L = K(1)\cdots(n-1)$ instead of T; that x_L is a homeomorphism follows from x_K being a homeomorphism.

The previous theorem also tells us about factorizations of $(n-1)$-positive matrices, which may be useful in trying to generalize results regarding total positivity, such as those in planar networks [12].

Remark E.5. $(n-1)$-positive matrices that are not totally positive, that is, $(n-1)$-positive matrices with a negative determinant, can be factored into the following form:

$$w_{0,[1,n-1]}(n-1)K(1)\overline{w_{0,[1,n-1]}}(1)(2)\cdots(n-1)$$

Here, $w_{0,[1,n-1]}$ signals any reduced word for the long word in $S_{n-1} \hookrightarrow S_n$; for example, in S_5, this could be $e_3e_2e_3e_1e_2e_3$. The parameter map is a bijection, so the $(n-1)(n-2)+2+(2n-3)+(n-1) = n^2$ parameters are a homeomorphic characterization of matrices of this form.

One statement that will help is noticing that the fact that the closure of cells with negative determinant has no elements with positive determinant. In fact, we can say slightly more.

Proposition E.6. *The set of invertible $(n-1)$-nonnegative matrices consists of two path-connected components, those with negative determinant and those with positive determinant.*

Proof. Take some $(n-1)$-nonnegative matrix M. Consider some minor that is 0. If we cannot affect this minor with Chevalley matrices, then it must be the case that that M is not invertible. Thus, we can always make minors nonzero. This means that by multiplying by Chevalley matrices, we can always force M to be $(n-1)$-positive while preserving the sign of the determinant. This describes a path from M to an $(n-1)$-positive matrix. We know that totally positive matrices are homeomorphic to a ball, so the component with positive determinant is path-connected. From Theorem E.4, the set of matrices that are $(n-1)$-positive with negative determinant is also homeomorphic to a ball. Thus, the negative component is path-connected as well. □

Now, consider the closure poset on $\bar{\mathbb{S}}_{\text{cell}}$ given by letting $\varnothing$ be a subword of i and $\bar{i}$, and having no nontrivial subwords of K or (i).

Theorem E.7. *The closure $\overline{U(B)}$ is exactly the disjoint union of $U(A)$ for $A \leq B$ in the defined subword order.*

Proof. By Proposition E.6, we can consider negative and positive determinant parts of the space separately. For the positive component, this is a known result. For the negative component, we simply use Lemma 4.4 to see that the only matrices in the cell of $B = (\sigma, \omega, \gamma)$ are those in the double Bruhat cells below $\sigma\omega\alpha$. Because elements in the closure of $(n-1)$-nonnegative matrices are $(n-1)$-nonnegative, these can only be the cells $A = (\sigma', \omega', \gamma')$ where $\sigma' \leq \sigma$ and $\omega' \leq \omega$. Further, notice that if a particular minor is zero for all elements in the cell, it must remain zero for elements in the closure of the cell. Thus, the same argument for disjointness works to show that $\gamma' \leq \gamma$ as well. Given these restrictions, all of these can be formed by taking subwords of B. □

Since we cannot decompose K into subwords, the poset that naturally results from taking closures is easy to describe: there are two connected components, corresponding to a positive and a negative determinant. The positive part is exactly the poset we get from the TNN case. The negative part is isomorphic to the Cartesian product of the interval between the identity and $(w_{0,[1,n-1]}, w_{0,[2,n]})$ in the strong Bruhat order with the Boolean algebra on two elements, corresponding to $\{K, (\overline{1})K, (n-1)K, (\overline{1})(n-1)K\}$. This gives us the following:

Proposition E.8. *Both parts of the poset are graded via the length function, and have a top and bottom element. The connected component corresponding to the matrices with positive determinant is Eulerian, making the poset as a whole trivially semi-Eulerian.*

Appendix F. Minors and relations for $(n-1)$-nonnegative invertible matrices

The solid minors of $K(\vec{a}, \vec{b})$ are as follows.

$$\left|K(\vec{a}, \vec{b})_{[i,j],[i+1,j+1]}\right| = \prod_{k=i}^{j} a_k b_k$$

$$\left|K(\vec{a}, \vec{b})_{[i,n-1],[i+1,n]}\right| = b_1 \cdots b_{n-1} \prod_{k=i}^{n-2} a_k b_k$$

$$\left|K(\vec{a}, \vec{b})_{[i+1,j+1],[i,j]}\right| = 1$$

Then, the principal minors:

$$\left|K(\vec{a},\vec{b})_{[i,j],[i,j]}\right| = \sum_{k=i-1}^{j}\left(\prod_{\ell=i}^{k} b_{\ell-1}\prod_{\ell=k+1}^{j} a_\ell\right) \text{ when } i,j<n \text{ and } a_{n-1},b_0=0$$

$$\begin{aligned}\left|K(\vec{a},\vec{b})_{[i,n],[i,n]}\right| &= \left(\prod_{k=i}^{n} b_{k-1}\prod_{k=i-1}^{n-2} a_k\right)\left|K(\vec{a},\vec{b})_{[2,i-2],[2,i-2]}\right| \\ &= \left(\prod_{k=i}^{n} b_{k-1}\prod_{k=i-1}^{n-2} a_k\right)\sum_{k=1}^{i-1}\left(\prod_{\ell=2}^{k} b_{\ell-1}\prod_{\ell=k+1}^{i-1} a_\ell\right) \quad \text{when } i>2 \\ \left|K(\vec{a},\vec{b})_{[2,n],[2,n]}\right| &= 0 \\ \left|K(\vec{a},\vec{b})\right| &= -a_1\cdots a_{n-2}b_1\cdots b_{n-1}\end{aligned}$$

All other minors are trivially zero.

Relations In these relations, the variables on the right-hand side are expressed in terms of the variables on the left-hand side.

(1) $e_i(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})e_{i+1}(x')$, where $1 \le i \le n-2$:
The following equalities hold for $i < n-2$.

$$\begin{aligned}\vec{A} &= \left(a_1,\ldots,a_{i-1},a_i+x,\frac{a_ia_{i+1}}{a_i+x},a_{i+2},\ldots,a_{n-2}\right) \\ \vec{B} &= \left(b_1,\ldots,b_{i-1},b_i+\frac{xa_{i+1}}{a_i+x},\frac{b_ib_{i+1}(a_i+x)}{b_i(a_i+x)+xa_{i+1}},b_{i+2},\ldots,b_{n-1}\right) \\ x' &= \frac{b_{i+1}a_{i+1}x}{b_i(a_i+x)+xa_{i+1}}.\end{aligned}$$

and in the other direction,

$$\begin{aligned}\vec{a} &= \left(A_1,\ldots,A_{i-1},\frac{A_iA_{i+1}B_{i+1}+A_iA_{i+1}x'}{A_{i+1}B_{i+1}+A_{i+1}x'+B_ix'},\right. \\ &\qquad \left. A_{i+1}+\frac{B_ix'}{B_{i+1}+x'},A_{i+2},\ldots,A_{n-3}\right) \\ \vec{b} &= \left(B_1,\ldots,B_{i-1},\frac{B_iB_{i+1}}{B_{i+1}+x'},B_{i+1}+x',B_{i+2},\ldots,B_{n-2}\right) \\ x &= \frac{x'A_iB_i}{A_{i+1}B_{i+1}+A_{i+1}x'+B_ix'}\end{aligned}$$

and when $i = n-2$, we have

$$\vec{A} = (a_1, \ldots, a_{n-3}, a_{n-2} + x)$$
$$\vec{B} = \left(b_1, \ldots, b_{n-2}, \frac{b_{n-1}a_{n-2}}{a_{n-2} + x}\right)$$
$$x' = \frac{b_{n-1} \cdots b_2 b_1 x}{b_{n-2}(a_{n-2} + x)}$$

and in the other direction,

$$\vec{a} = \left(A_1, \ldots, A_{n-3}, \frac{A_{n-2}B_1 \cdots B_{n-1}}{B_1 \cdots B_{n-1} + x'B_{n-2}}\right)$$
$$\vec{b} = \left(B_1, \ldots, B_{n-2}, B_{n-1} + \frac{x'}{B_1 \cdots B_{n-3}}\right)$$
$$x = \frac{A_{n-2}B_{n-2}x'}{B_1 \cdots B_{n-1} + x'B_{n-2}}$$

(2) $e_{n-1}(x)K(\vec{a}, \vec{b}) = K(\vec{A}, \vec{B})f_{n-1}(x')h_{n-1}(c)$:

$$c = \frac{\nu}{\nu + x\mu} = \frac{1}{1 + x'\mu}$$
$$\vec{A} = \vec{a}$$
$$\vec{B} = \left(b_1, \ldots, b_{n-3}, \frac{b_{n-2}}{c}, b_{n-1}\right)$$
$$x' = \frac{x}{\nu}$$

(3) $f_{i+1}(x)K(\vec{a}, \vec{b}) = h_{i+2}(1/w)K(\vec{A}, \vec{B})f_i(x)h_i(w)$, where $1 \leq i \leq n-2$:

when $1 \leq i < n-2$, we have:

$$w = \frac{1}{1 + xa_{i+1} + xb_i}$$
$$\vec{A} = \left(a_1, \ldots, a_{i-2}, a_{i-1}, a_i(xa_{i+1} + 1), \frac{a_{i+1}(xa_{i+1} + xb_i + 1)}{1 + xa_{i+1}}, \right.$$
$$\left. \frac{a_{i+2}}{xa_{i+1} + xb_{i+1} + 1}, a_{i+3}, \ldots, a_{n-2}\right)$$
$$\vec{B} = \left(b_1, \ldots, b_{i-2}, b_{i-1}(xa_{i+1} + xb_i + 1), \frac{b_i}{xa_{i+1} + 1}, \frac{b_{i+1}(1 + xa_{i+1})}{xa_{i+1} + xb_i + 1}, \right.$$
$$\left. b_{i+2}, \ldots, b_{n-1}\right)$$

and for the other direction:

$$\begin{aligned}
w &= \frac{1}{1 + xA_{i+1} + xB_i} \\
\vec{a} &= \Bigg(A_1, \ldots, A_{i-2}, A_{i-1}, \frac{A_i(1 + xB_i)}{xA_{i+1} + xB_i + 1}, \frac{A_{i+1}}{1 + xB_i}, \\
&\qquad A_{i+2}(xA_{i+1} + xB_i + 1), A_{i+3}, \ldots, A_{n-2}\Bigg) \\
\vec{b} &= \Bigg(B_1, \ldots, B_{i-2}, \frac{B_{i-1}}{xA_{i+1} + xB_i + 1}, \frac{B_i(xA_{i+1} + xB_i + 1)}{1 + xB_i}, \\
&\qquad B_{i+1}(1 + xB_i), B_{i+2}, \ldots, B_{n-1}\Bigg)
\end{aligned}$$

and when $i = n - 2$:

$$\begin{aligned}
w &= \frac{1}{1 + xb_{n-2}} \text{ and } \vec{A} = \vec{a} \\
\vec{B} &= \left(b_1, \ldots, b_{n-4}, b_{n-3}(xb_{n-2} + 1), b_{n-2}, \frac{b_{n-1}}{xb_{n-2} + 1}\right)
\end{aligned}$$

(4) $f_1(x)K(\vec{a}, \vec{b}) = K(\vec{A}, \vec{B})e_1(x')h_1(c)$:

$$\begin{aligned}
\vec{A} &= \left(\frac{a_1}{1 + xa_1}, a_2, \ldots, a_{n-2}\right) \\
\vec{B} &= \vec{b} \\
x' &= xb_1a_1 \\
c &= \frac{1}{1 + xa_1}
\end{aligned}$$

For the other direction, we use $a_1 = A_1 + \frac{A_1x'}{B_1}$ and $c = \frac{B_1}{B_1+x'}$.

(5) $h_i(x)K(\vec{a}, \vec{b}) = K(\vec{A}, \vec{B})h_{i-1}(x)$, where $2 \le i \le n$:

$$\begin{aligned}
\vec{A} &= \left(a_1, \ldots, a_{i-1}, xa_i, \frac{a_{i+1}}{x}, a_{i+2}, \ldots, a_{n-3}\right) \\
\vec{B} &= \left(b_1, \ldots, xb_{i-1}, \frac{b_i}{x}, b_{i+1}, b_{i+2}, \ldots, b_{n-2}\right)
\end{aligned}$$

(6) $h_1(x)K(\vec{a}, \vec{b}) = K(\vec{A}, \vec{B})$, where $\vec{A} = (xa_1, a_2, \ldots, a_{n-3})$ and $\vec{B} = \vec{b}$.

(7) $K(\vec{a}, \vec{b})h_n(x) = K(\vec{A}, \vec{B})$, where $\vec{A} = \vec{a}$ and $\vec{B} = (b_1, \ldots, b_{n-3}, xb_{n-2})$.

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Sunita Chepuri
Department of Mathematics
University of Minnesota
127 Vincent Hall
206 Church St. SE
Minneapolis, MN 55455
USA
E-mail address: chepu003@umn.edu

Neeraja Kulkarni
Department of Mathematics and Statistics
Carleton College
One North College Street
Northfield, MN 55057
USA
E-mail address: neeraja.r.kulkarni@gmail.com

Joe Suk
Department of Mathematics
Stony Brook University
Stony Brook, NY 11794
USA
E-mail address: ybjosuk@gmail.com

Ewin Tang
Department of Mathematics
University of Texas at Austin
2515 Speedway
Austin, TX 78712
USA
E-mail address: ewin@utexas.edu

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