Journal of Combinatorics
Volume 14, Number 2, 257–270, 2023

A Cantor-Bernstein theorem for infinite matroids

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We give a common matroidal generalisation of ‘A Cantor-Bernstein theorem for paths in graphs’ by Diestel and Thomassen and ‘A Cantor-Bernstein-type theorem for spanning trees in infinite graphs’ by ourselves.

AMS 2000 subject classifications: Primary 05B35, 05C63, 05C38; secondary 03E35.
Keywords and phrases: Infinite matroid, Cantor-Bernstein theorem, packing and covering.

1. Introduction

Let us reformulate the Cantor-Bernstein theorem in the language of graph theory:

Theorem 1.1 (Cantor-Bernstein, [1]). *If $G = (V_0, V_1; E)$ is a bipartite graph and matching I_i covers V_i for $i \in \{0, 1\}$, then G admits a perfect matching.*

Ore discovered the following generalisation of the Cantor-Bernstein theorem which is the extension of the Mendelsohn-Dulmage theorem [2, Theorem 1] to infinite graphs:

Theorem 1.2 (Ore, [3, Theorem 7.4.1]). *Let $G = (V_0, V_1; E)$ be a bipartite graph and let $I_0, I_1 \subseteq E$ be matchings in G. Then there exists a matching I such that $V(I) \cap V_i \supseteq V(I_i) \cap V_i$ for $i \in \{0, 1\}$.*

Diestel and Thomassen examined in their paper ‘A Cantor-Bernstein theorem for paths in graphs’ a more general graph-theoretic setting in which disjoint paths are used to connect two vertex sets. We call a finite path that meets the vertex sets V_0 and V_1 and subgraph-minimal with respect to this property a V_0V_1-path.

arXiv: 2009.08439
*The author would like to thank the generous support of the Alexander von Humboldt Foundation and NKFIH OTKA-129211.

Theorem 1.3 (Diestel and Thomassen, [4]). *Assume that $G = (V, E)$ is a graph, $V_0, V_1 \subseteq V$ and $\mathcal{P}_i$ is a system of disjoint V_0V_1-paths in G for $i \in \{0, 1\}$. Then there exists a system of disjoint V_0V_1-paths $\mathcal{P}$ with $V(\mathcal{P}) \cap V_i \supseteq V(\mathcal{P}_i) \cap V_i$ for $i \in \{0, 1\}$.*

Note that Theorem 1.2 is the special case of Theorem 1.3 where G is bipartite and the sets V_i are its vertex classes.

In our paper entitled 'A Cantor-Bernstein-type theorem for spanning trees in infinite graphs' we investigated if the existence of a κ-packing and a κ-covering by spanning trees implies the existence of a κ-family of spanning trees which is both, i.e. a κ-partition:

Theorem 1.4 (Erde et al. [5, Theorem 1.1]). *Let $G = (V, E)$ be a graph and let κ be a cardinal. If there are κ many pairwise edge-disjoint spanning trees in G and E can be covered by κ many spanning trees, then E can be partitioned into κ many spanning trees.*

At first sight the connection between Theorems 1.3 and 1.4 seems to be only analogical. In this paper, we show that the connection is actually stronger. There is an abstract matroidal "Cantor-Bernstein"-type phenomenon behind these theorems. Let us first state a special case of our main result which is the generalisation of a theorem by Kundu and Lawler (see [6]) to finitary matroids[1]:

Theorem 1.5. *For $i \in \{0, 1\}$, let M_i be a finitary matroid on E and let $I_i \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$. Then there is an $I \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ with $I_i \subseteq \mathsf{span}_{M_i}(I)$ for $i \in \{0, 1\}$.*

The proof for finite matroids by Kundu and Lawler in [6] is quite short: If I_0 spans I_1 in M_1, then $I := I_0$ is as desired. Otherwise we add an $e \in I_1 \setminus \mathsf{span}_{M_1}(I_0)$ to I_0 and if $I_0 + e \notin \mathcal{I}_{M_0}$, then delete a suitable $f \in I_0 \setminus I_1$ in order to restore the M_0-independence. This can be done because the fundamental circuit $C_{M_0}(e, I_0)$ (if exists) cannot be entirely in I_1. The resulting set $I_0 + e - f$ (or $I_0 + e$) still spans I_0 in M_0 and has strictly more edges in I_1 than I_0. After finitely many iterations of this step the desired I is obtained.

A naive proof-idea for Theorem 1.5 would be to iterate the step above via transfinite recursion. Unfortunately it does not work. To demonstrate this we define a graph $G = (V, E)$ as a ray (one-way infinite path) $v_0, v_1, v_2, \ldots$

[1] A matroid is called finitary if all of its circuits are finite. In the older papers of Higgs, Oxley and others it is also called 'independence space'. For a brief introduction to the concept of infinite matroids see Section 2.

together with an additional vertex w connected to each vertex of the ray (see Figure 1). Let M_0 be the cycle matroid on E corresponding to G (i.e. the circuits are the edge sets of the graph-theoretic cycles) and let M_1 be the free matroid on E (i.e. every set is independent in M_1). We define I_0 as the set of edges incident with w and let $I_1 := E \setminus I_0$. The naive approach might proceed as:

$$I_0,\ I_0 + v_0v_1 - wv_0,\ I_0 \cup \{v_0v_1, v_1v_2\} \setminus \{wv_0, w_1\}, \dots$$

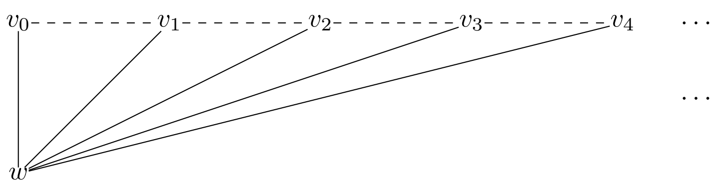

Figure 1: The failure of the naive approach for infinite matroids.

It terminates after ω steps and transforms I_0 into I_1. Since I_1 does not span I_0 in M_0, it fails to provide a desired I. It is easy to see that if we keep wv_0 and delete only $wv_1, wv_2, \dots$ (while the incoming edges are in the same order), then we end up with the same ray together with the edge wv_0 which is suitable as I. In order to prove Theorem 1.5, we are going to show in Section 3 that it is always possible to choose the leaving edge in each step in such a way that we obtain a solution at the end. The proof of Theorem 1.5 makes possible to understand quickly the main ideas without dealing with technicalities arising in the general form. Basic knowledge about finite matroids is already sufficient to understand the paper, all the necessary matroidal background is given in Section 2.

In Section 4 we discuss the general form of our main result. Let us denote the class of finitary matroids by $\mathfrak{F}$, the class of their duals (i.e. cofinitary matroids) by $\mathfrak{F}^*$ and let $\mathfrak{F} \oplus \mathfrak{F}^*$ be the class of matroids that are the direct sums of a finitary and a cofinitary matroid (equivalently the matroids with only finitary and cofinitary components). For a matroid class $\mathfrak{C}$, let $\mathfrak{C}(E)$ be the set of matroids on edge set E that are in class $\mathfrak{C}$.

Our main result generalises Theorem 1.5 in two ways. On the one hand, we replace $\mathfrak{F}$ by $\mathfrak{F} \oplus \mathfrak{F}^*$. On the other hand, we allow arbitrary edge sets instead of common independent sets (this possibility was conjectured by Bowler) in the following sense:

Theorem 1.6. *For $i \in \{0,1\}$, let $M_i \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$ and $F_i \subseteq E$. Then there exists an $F \subseteq E$ such that $\mathsf{span}_{M_i}(F) \supseteq F_i$ and $\mathsf{span}_{M_i^*}(E \setminus F) \supseteq E \setminus F_{1-i}$ for $i \in \{0,1\}$.*

We are going to prove the following family variant of Theorem 1.6 as well:

Theorem 1.7. *For $i \in \Theta$, let $M_i \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$, $P_i, R_i \subseteq E$ and for $e \in E$, let $N_e \in (\mathfrak{F} \oplus \mathfrak{F}^*)(\Theta)$. Then there are $T_i \subseteq P_i \cup R_i$ for $i \in \Theta$ such that*

1. $\mathsf{span}_{M_i}(T_i) \supseteq P_i$;
2. $\mathsf{span}_{M_i^*}(E \setminus T_i) \supseteq E \setminus R_i$;
3. *For every $e \in E$, the set $\{i \in \Theta : \ e \in T_i\}$ spans $\{i \in \Theta : \ e \in R_i\}$ in N_e;*
4. *For every $e \in E$, the set $\{i \in \Theta : \ e \notin T_i\}$ spans $\{i \in \Theta : \ e \notin P_i\}$ in N_e^*.*

The connection between the Theorems 1.6 and 1.7 is far from obvious. It worths to mention that it is impossible to extend our results above to arbitrary matroids working in set theory ZFC. Indeed, the analogue of Theorem 1.5 for arbitrary matroids fails under the Continuum Hypothesis even if E is countable, M_i is uniform and I_i is a base of M_i (take U and U^* in [7, Theorem 5.1]).

In the last section (Section 5) we provide an application related to the following conjecture:

Conjecture 1.8 (Matroid Intersection Conjecture by Nash-Williams, [8, Conjecture 1.2])**.** *For every $M_0, M_1 \in \mathfrak{F}(E)$, there is an $I \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ and a partition $E = E_0 \sqcup E_1$ such that $I \cap E_i$ spans E_i in M_i for $i \in \{0,1\}$.*

The special case of the conjecture where E is assumed to be countable was proved in [9]. This was then generalised to the case where E is still countable but $\mathfrak{F}(E)$ is replaced by $(\mathfrak{F} \oplus \mathfrak{F}^*)(E)$ (see [10, Theorem 1.4]).

A maximal sized common independent set of two finite matroids can always be chosen in such a way that it spans a prescribed common independent set in both matroids. Indeed, if a common independent set is not a largest such a set, then the well-known ‘augmenting path’ method by Edmonds gives a new common independent set which is larger by one and spans the original in both matroids (see in [11]). Iterating such augmenting paths starting with the prescribed common independent set provides a desired largest common independent set.

The question can be phrased with respect to Conjecture 1.8 by replacing ‘maximal sized’ by ‘strongly maximal’ which we define as satisfying the property described in Conjecture 1.8. The same argument for the positive answer does not work because finitely many iteration of augmenting paths does not lead to a strongly maximal one in general. Even so, we can answer the question affirmatively based on our main results. Let us denote the set of strongly maximal common independent sets by $\mathsf{SM}(M_0, M_1)$. For $I, J \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$, let $J \trianglelefteq_{M_0,M_1} I$ iff $J \subseteq \mathsf{span}_{M_0}(I) \cap \mathsf{span}_{M_1}(I)$.

Theorem 1.9. *Let E be countable and let $M_i \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$ for $i \in \{0, 1\}$. Then $\mathsf{SM}(M_0, M_1)$ is cofinal but not necessarily upward closed in $(\mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}, \trianglelefteq_{M_0,M_1})$.*

2. Preliminaries

Rado asked in 1966 if there is an infinite generalisation of matroids preserving the key concepts (bases, circuits, duality and minors) of the finite theory. The positive answer was given by Higgs [12] (see also [13]). The same concept of infinite matroids was independently rediscovered by Bruhn, Diestel, Kriesell, Pendavingh and Wollan. They gave a set of cryptomorphic axioms for infinite matroids, generalising the usual independent set-, bases-, circuit-, closure- and rank-axioms of finite matroids (see [14]). They showed that several fundamental facts of the theory of finite matroids are preserved in the infinite case. It opened the door for a more systematic investigation of infinite matroids. An $M = (E, \mathcal{I})$ is a matroid (also called B-matroid) if $\mathcal{I} \subseteq \mathcal{P}(E)$ with

(I) $\varnothing \in \mathcal{I}$;
(II) $\mathcal{I}$ is downward closed;
(III) For every $I, J \in \mathcal{I}$ where J is $\subseteq$-maximal in $\mathcal{I}$ and I is not, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;
(IV) For every $X \subseteq E$, any $I \in \mathcal{I} \cap \mathcal{P}(X)$ can be extended to a $\subseteq$-maximal element of $\mathcal{I} \cap \mathcal{P}(X)$.

For a finite E, axioms (I)-(III) are equivalent to the usual axiomatization of finite matroids in terms of independent sets (while (IV) is redundant).

The terminology and the basic facts we will use are well-known for finite matroids. The elements of $\mathcal{I}$ are called *independent* sets while the sets in $\mathcal{P}(E) \setminus \mathcal{I}$ are *dependent*. The maximal independent sets are the *bases* and the minimal dependent sets are the *circuits* of the matroid. Every dependent set contains a circuit (which fact is not obvious if E is infinite). A singleton circuit is called a *loop*. The *components* of a matroid are the connected components of the hypergraph of its circuits on E. The *dual* of matroid M

is the matroid M^* on the same edge set whose bases are the complements of the bases of M. By the deletion of an $X \subseteq E$ we obtain the matroid $M - X := (E \setminus X, \{Y \in \mathcal{I} : Y \subseteq E \setminus X\})$ and the contraction of X gives $M/X := (M^* - X)^*$. If I is independent in M but $I + e$ is dependent for some $e \in E \setminus I$ then there is a unique circuit $C_M(e, I)$ of M through e contained in $I + e$ which is called the *fundamental circuit* of e on I in M. We say $X \subseteq E$ spans $e \in E$ in matroid M if either $e \in X$ or there exists a circuit $C \ni e$ with $C - e \subseteq X$. We denote the set of edges spanned by X in M by $\mathsf{span}_M(X)$. A matroid is called *finitary* if all of its circuits are finite. A matroid is *cofinitary* if its dual is finitary. If C_1 and C_2 are circuits with $e \in C_1 \setminus C_2$ and $f \in C_1 \cap C_2$, then there is a circuit C_3 with $e \in C_3 \subseteq C_1 \cup C_2 - f$. This fact is called (strong) *circuit elimination*. For more information about infinite matroids we refer to [15].

3. The infinite generalisation of the Kundu-Lawler theorem

Theorem 1.5. *For $i \in \{0, 1\}$, let M_i be a finitary matroid on E and let $I_i \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$. Then there is an $I \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ with $I_i \subseteq \mathsf{span}_{M_i}(I)$ for $i \in \{0, 1\}$.*

Proof. We may assume without loss of generality that E is the disjoint union of I_0 and I_1 since otherwise we can simply contract $I_0 \cap I_1$ and delete $E \setminus (I_0 \cap I_1)$ in both matroids. Let $<$ be a well-order on E in which I_1 is an initial segment, i.e. $e < f$ for every $e \in I_1$ and $f \in I_0$. From now on, the maximum of a finite subset of E is interpreted corresponding to $<$. We define a well-order $\prec$ on the set $E^{<\aleph_0}$ of finite subsets of E. For $X \neq Y \in E^{<\aleph_0}$ let $X \prec Y$ iff one of the following holds:

- $X = \varnothing$,
- $\max X < \max Y$,
- $\max X = \max Y =: z$ and $X - z \prec Y - z$.

It is not too hard to check that $\prec$ is indeed a well-order.

Observation 3.1. *If $X \prec Y$ then $X + z \prec Y + z$ for every $z \in I_0 \cup I_1$.*

Let $\langle E_\beta : \beta < \alpha \rangle$ be a sequence of subsets of E where α is a limit ordinal. If

$$\bigcup_{\gamma<\alpha} \bigcap_{\beta>\gamma} E_\beta = \bigcap_{\gamma<\alpha} \bigcup_{\beta>\gamma} E_\beta,$$

then we call this set the limit of the sequence and denote it by $\lim \langle E_\beta : \beta < \alpha \rangle$. We apply transfinite recursion starting with $J_0 := I_0$. Suppose that $J_\alpha \in$

$\mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ is defined and spans I_0 in M_0. If J_α spans I_1 in M_1 as well, then $I := J_\alpha$ is as desired. Otherwise let $e \in I_1 \setminus \mathsf{span}_{M_1}(J_\alpha)$ be arbitrary and let

$$J_{\alpha+1} := \begin{cases} J_\alpha + e & \text{if it is independent in } M_0 \\ J_\alpha + e - \max C_{M_0}(e, J_\alpha) & \text{otherwise.} \end{cases}$$

Note that $e \in I_1 \setminus I_0$ and $\max C_{M_0}(e, J_\alpha) \in I_0 \setminus I_1$. In limits steps we take the limit of the earlier members (which is well-defined). Clearly, $J_\alpha \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ remains true for limit ordinals because a finite circuit cannot show up first in a limit step. It is enough to show that $J_\beta \subseteq \mathsf{span}_{M_0}(J_\alpha)$ for $\beta < \alpha$. Let β and $g \in I_\beta$ be fixed and suppose for a contradiction that there is a (smallest) α with $g \notin \mathsf{span}_{M_0}(J_\alpha)$. It is obvious from the definition of successor steps that α must be a limit ordinal. For $\gamma \in [\beta, \alpha)$, let S_γ be the unique minimal subset of J_γ that spans g in M_0. It is enough to show that $S_{\gamma+1} \preceq S_\gamma$ for $\gamma \in [\beta, \alpha)$. Indeed, since there is no infinite $\prec$-decreasing sequence, S_γ is the same set S for every large enough γ. But then $S \subseteq J_\alpha$ and it spans g in M_0, a contradiction.

Let $\gamma \in [\beta, \alpha)$ be fixed. We may assume that $S_{\gamma+1} \neq S_\gamma$ since otherwise we are done. Suppose first that $S_\gamma = \{g\}$. Then $g \notin S_{\gamma+1}$ because otherwise $S_\gamma = S_{\gamma+1} = \{g\}$. But then there is an edge e such that $g = \max C_{M_0}(e, J_\alpha)$ and $J_{\gamma+1} = J_\gamma + e - g$. Therefore

$$S_{\gamma+1} = C_{M_0}(e, J_\gamma) - g \prec \{g\} = S_\gamma.$$

If $S_\gamma \neq \{g\}$, then $S_\gamma = C_{M_0}(g, J_\gamma) - g$ and there is an edge e such that $J_{\gamma+1} = J_\gamma + e - \max C_{M_0}(e, J_\gamma)$ with $\max C_{M_0}(e, J_\gamma) \in C_{M_0}(g, J_\gamma) - g$. By strong circuit elimination we know that

$$C_{M_0}(g, J_{\gamma+1}) \subseteq C_{M_0}(g, J_\gamma) \cup C_{M_0}(e, J_\gamma) - \max C_{M_0}(e, J_\gamma)$$

and therefore

$$S_{\gamma+1} \subseteq S_\gamma \cup C_{M_0}(e, J_\gamma) - \max C_{M_0}(e, J_\gamma).$$

It follows that $S_{\gamma+1} \setminus S_\gamma \prec S_\gamma \setminus S_{\gamma+1}$ because $\max C_{M_0}(e, J_\gamma) \in S_{\gamma+1} \setminus S_\gamma$ is $<$-larger than any element of $S_\gamma \setminus S_{\gamma+1}$. Finally, this implies $S_{\gamma+1} \prec S_\gamma$ by applying Observation 3.1 repeatedly with the edges in $S_\gamma \cap S_{\gamma+1}$. $\square$

4. The proof of the main results

We are going to derive Theorems 1.6 and 1.7 from the following statement:

Proposition 4.1. *For $i \in \Theta$, let $M_i \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$ and $P_i, R_i \subseteq E$ such that the sets P_i form a packing and the sets R_i form a covering, i.e. $P_i \cap P_j = \varnothing$ for $i \neq j$ and $\bigcup_{i \in \Theta} R_i = E$. Then there are $T_i \subseteq P_i \cup R_i$ for $i \in \Theta$ forming a partition of E such that $\mathsf{span}_{M_i}(T_i) \supseteq P_i$ and $\mathsf{span}_{M_i^*}(E \setminus T_i) \supseteq E \setminus R_i$.*

Proof. We may assume without loss of generality by "trimming" that the sets R_i form a partition of E. We can also assume that $P_i \in \mathcal{I}_{M_i}$ since otherwise we replace P_i with a maximal M_i-independent subset of it. It is enough to consider the case where $P_i \cap R_i = \varnothing$ for $i \in \Theta$. Indeed, if it is not the case, then we contract $P_i \cap R_i$ and delete $P_j \cap R_j$ for $j \neq i$ in M_i. Finally, by decomposing each M_i into a finitary and a cofinitary matroid (which we extend to E by loops) and partition the sets R_i and P_i accordingly, it is enough to deal with matroid families where each M_i is either finitary or cofinitary.

Let $<_i$ be a well-order on $P_i \cup R_i$ where R_i is an initial segment. Then $<_i$ induces a well-order $\prec_i$ on the set $[P_i \cup R_i]^{<\aleph_0}$ the same way as in Section 3.

Observation 4.2. *Suppose that E_α is the limit of $\langle E_\beta : \beta < \alpha \rangle$.*

(i) If E_α contains an M_i-circuit $C \not\subseteq R_i$ where M_i is finitary, then so does E_β for every large enough $\beta < \alpha$;

(ii) If $g \in \mathsf{span}_{M_i}(E_\beta)$ for $\beta < \alpha$ where M_i is cofinitrary, then $g \in \mathsf{span}_{M_i}(E_\alpha)$.

To construct the desired partition $(T_i : i \in \Theta)$, we apply transfinite recursion. Let $T_i^0 := P_i$ for $i \in \Theta$. Suppose that T_i^β is defined for $\beta < \alpha$ and $i \in \Theta$ satisfying the following properties:

1. $T_i^\beta \cap T_j^\beta = \varnothing$ for $i \neq j \in \Theta$;
2. $T_i^\beta \subseteq P_i \cup R_i$;
3. $T_i^\beta \cap P_i$ is $\subseteq$-decreasing and $T_i^\beta \cap R_i$ is $\subseteq$-increasing in β;
4. $T_i^\beta = \lim \left\langle T_i^\delta : \delta < \beta \right\rangle$ if β is a limit ordinal;
5. $\mathsf{span}_{M_i}(T_i^\beta) \supseteq P_i$;
6. For every finitary M_i, each M_i-circuit $C \subseteq T_i^\beta$ is a subset of R_i;
7. For every finitary M_i and $g \in P_i$, the $\prec_i$-smallest finite $S_g^\beta \subseteq T_i^\beta$ that is witnessing $g \in \mathsf{span}_{M_i}(T_i^\beta)$ is a $\preceq_i$-decreasing function of β;
8. $(T_i^\delta : i \in \Theta) \neq (T_i^{\delta+1} : i \in \Theta)$ for $\delta + 1 < \alpha$.

Note that condition (6) is a rephrasing of "$\mathsf{span}_{M_i^*}(E \setminus T_i^\beta) \supseteq E \setminus R_i$ for finitary M_i". Assume first that α is a limit ordinal. Then conditions (2) and (3) guarantee that $T_i^\alpha := \lim \left\langle T_i^\beta : \beta < \alpha \right\rangle$ is well-defined. Preservation of conditions (1)-(4) and (8) is straightforward. The restriction of condition (5) to cofinitary matroids and condition (6) are kept by Observation 4.2. To check condition (5) for a finitary M_i, let $g \in P_i$ be arbitrary. Since $\preceq_i$ is a well-order, it follows from condition (7) that there is an S_g such that $S_g^\beta = S_g$ for all large enough $\beta < \alpha$. But then $S_g \subseteq T_i^\alpha$ from which $g \in \mathsf{span}_{M_i}(T_i^\alpha)$ follows. Furthermore, clearly $S_g^\alpha = S_g$ since a finite set which is $\prec_i$-smaller than S_g and M_i-spans g would have appeared already before the limit.

Suppose now that $\alpha = \beta + 1$. If $\bigcup_{i \in \Theta} T_i^\beta \supseteq E$ and the analogue of condition (6) for the cofinitary M_i holds, then $(T_i^\beta : i \in \Theta)$ is a desired partition of E and we are done. Suppose it is not the case. If there is some T_j^β that contains an M_j-circuit C with $C \nsubseteq R_j$, then we take an $e \in P_j \cap C$ (see property (2)) and define $T_j^{\beta+1} := T_j^\beta - e$ and $T_i^{\beta+1} := T_i^\beta$ for $i \neq j$. The preservation of the conditions (1)-(8) is trivial. If there is no such a T_j^β, then there must be some $e \in E$ which is not covered by the sets T_i^β. Then there is a unique $k \in \Theta$ with $e \in R_k$. If M_k is cofinitary then let $T_k^{\beta+1} := T_k^\beta + e$ and $T_i^{\beta+1} := T_i^\beta$ for $i \neq k$. We proceed the same way if M_k is finitary and $T_k^\beta + e$ does not contain any M_k-circuit C with $C \nsubseteq R_k$. The preservation of the conditions is again straightforward in both cases.

Finally assume that M_k is finitary and $T_k^\beta + e$ contains an M_k-circuit C with $C \subsetneq R_k$. Let f be the $<_k$-maximal element of such a C and we define $T_k^{\beta+1} := T_k^\beta + e - f$ and $T_i^{\beta+1} := T_i^\beta$ for $i \neq k$. Since $C \cap P_k \neq \varnothing$ (because $C \nsubseteq R_k$) and the elements of P_k are $<_k$-larger than the elements of R_k, we have $f \in P_k$. Conditions (1)-(5) remain true for obvious reasons. Suppose for a contradiction that condition (6) fails and C' is an M_k-circuit in $T_k^{\beta+1}$ with $C' \nsubseteq R_k$. Then $f \notin C'$ and we must have $e \in C'$ since otherwise $C' \subseteq T_k^\beta$ and therefore this condition would have been already violated with respect to T_k^β. By applying strong circuit elimination with the M_k-circuits C and C', we obtain a circuit $C'' \subseteq C \cup C' - e$ through f. But then $C'' \subseteq T_k^\beta$ is an M_k-circuit and f witnesses $C'' \nsubseteq R_k$ in violation of condition (6) for β which is a contradiction. To check (7), we may assume that $f \in S_g^\beta$ since otherwise $S_g^\beta \subseteq T_k^{\beta+1}$ and thus $S_g^{\beta+1} \preceq_k S_g^\beta$. If $S_g^\beta = \{g\}$, then $f = g$ by $f \in S_g^\beta$ and by the choice of f we have $S_f^{\beta+1} \preceq_k C - f \prec_k \{f\}$. Otherwise there is an M_k-circuit $C' \ni f, g$ such that $S_g^\beta = C' - g \subseteq T_k^\beta$. By applying strong circuit elimination with C and C', we obtain a circuit $C'' \subseteq C \cup C' - f$ through g. Since $f \in C' \setminus C''$ and each element of $C'' \setminus C'$ is $\prec_k$-smaller than f (because $f = \max_{\prec_k} C$) we may conclude that $C'' \setminus C' \prec_k C' \setminus C''$. Thus by applying

Observation 3.1 iteratively we get $C'' - g \prec_k C' - g$. Therefore

$$S_g^{\beta+1} \preceq_k C'' - g \prec_k C' - g = S_g^{\beta}.$$

The recursion is done and it terminates at some ordinal since the constructed set families $(T_i^{\beta} : i \in \Theta)$ are pairwise distinct by conditions (2), (3) and (8). $\square$

Let us point out that the special case of Proposition 4.1 in which P_i and R_i are bases of M_i is exactly [7, Theorem 1.2]. Now we derive Theorems 1.6 and 1.7 from Proposition 4.1:

Theorem 1.6. *For $i \in \{0,1\}$, let $M_i \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$ and $F_i \subseteq E$. Then there exists an $F \subseteq E$ such that $\mathsf{span}_{M_i}(F) \supseteq F_i$ and $\mathsf{span}_{M_i^*}(E \setminus F) \supseteq E \setminus F_{1-i}$ for $i \in \{0,1\}$.*

Proof. We can assume by contracting $F_0 \cap F_1$ and deleting $E \setminus (F_0 \cup F_1)$ in both matroids that the sets F_i form a bipartition of E. We apply Proposition 4.1 with $\Theta = \{0,1\}$, matroids M_0 and M_1^* and sets $P_0 := R_1 := F_0$ and $P_1 := R_0 := F_1$. From the resulting bipartition $E = T_0 \sqcup T_1$ we take $F := T_0$. Then

1. $\mathsf{span}_{M_0}(F) \supseteq F_0$,
2. $\mathsf{span}_{M_1^*}(E \setminus F) \supseteq F_1$,
3. $\mathsf{span}_{M_0^*}(E \setminus F) \supseteq F_0$,
4. $\mathsf{span}_{M_1}(F) \supseteq F_1$.

$\square$

Theorem 1.7. *For $i \in \Theta$, let $M_i \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$, $P_i, R_i \subseteq E$ and for $e \in E$, let $N_e \in (\mathfrak{F} \oplus \mathfrak{F}^*)(\Theta)$. Then there are $T_i \subseteq P_i \cup R_i$ for $i \in \Theta$ such that*

1. *$\mathsf{span}_{M_i}(T_i) \supseteq P_i$;*
2. *$\mathsf{span}_{M_i^*}(E \setminus T_i) \supseteq E \setminus R_i$;*
3. *For every $e \in E$, the set $\{i \in \Theta : e \in T_i\}$ spans $\{i \in \Theta : e \in R_i\}$ in N_e;*
4. *For every $e \in E$, the set $\{i \in \Theta : e \notin T_i\}$ spans $\{i \in \Theta : e \notin P_i\}$ in N_e^*.*

Proof. We may assume that $\Theta \cap E = \varnothing$. For $i \in \Theta$, we construct a matroid M_i' by "copying" M_i to $\{i\} \times E$ and then extending to $\Theta \times E$ by loops. For $e \in E$, we construct a matroid N_e' by copying N_i^* to $\Theta \times \{e\}$ and then extending to $\Theta \times E$ by loops. The sets $R_i' := \{i\} \times R_i$ for $i \in \Theta$ together with the sets $R_e' := \{i \in \Theta : e \notin R_i\} \times \{e\}$ for $e \in E$ cover $\Theta \times E$.

Furthermore, the elements of the family consisting of $P_i' := \{i\} \times P_i$ for $i \in \Theta$ and $\{i \in \Theta : e \notin P_i\} \times \{e\}$ for $e \in E$ are pairwise disjoint. Let $\{T_i', T_e' : i \in \Theta, e \in E\}$ be a partition of $\Theta \times E$ obtained by applying Proposition 4.1 with the matroids M_i', N_e', covering R_i', R_e' and packing P_i', P_e' $(i \in \Theta, e \in E)$. It is easy to check that the family consisting of the projections T_i of T_i' to E for $i \in \Theta$ is as desired. $\square$

5. Applications

5.1. Cantor-Bernstein for path-systems

We derive Theorem 1.3 from Theorem 1.5.

Theorem 1.3 (Diestel and Thomassen, [4])**.** *Assume that $G = (V, E)$ is a graph, $V_0, V_1 \subseteq V$ and $\mathcal{P}_i$ is a system of disjoint V_0V_1-paths in G for $i \in \{0, 1\}$. Then there exists a system of disjoint V_0V_1-paths $\mathcal{P}$ with $V(\mathcal{P}) \cap V_i \supseteq V(\mathcal{P}_i) \cap V_i$ for $i \in \{0, 1\}$.*

Proof. For $i \in \{0, 1\}$, we define M_i to be the cycle matroid of the graph we obtain from G by contracting V_i to a single vertex. Then $E(\mathcal{P}_i) \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ for $i \in \{0, 1\}$. By applying Theorem 1.5 with $I_i := E(\mathcal{P}_i)$ and M_{1-i}, we can find an $I \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ with $E(\mathcal{P}_{1-i}) \subseteq \mathsf{span}_{M_i}(I)$ for $i \in \{0, 1\}$. Then $G[I]$ is a forest in which every tree meets each V_i at most once. Each connected component of $G[I]$ which meets both V_i contains a unique V_0V_1-path. We define $\mathcal{P}$ to be the set of these paths. It remains to show that $\mathcal{P}$ satisfies the requirements. Let $v_0 \in V(\mathcal{P}_i) \cap V_i$. It is enough to show that v_0 is reachable from V_{1-i} in $G[I]$ because then the (unique) path witnessing this is in $\mathcal{P}$. Consider the path $P \in \mathcal{P}_i$ through v_0. Let the vertices of P be $v_0, \dots, v_n$ enumerated in the path-order starting from V_i. It follows from $E(P) \subseteq \mathsf{span}_{M_{1-i}}(I)$ that for every $k < n$ either $G[I]$ contains a path between v_k and v_{k+1} or both of them are reachable from V_{1-i} in $G[I]$. Vertex v_n is obviously reachable from V_{1-i} because it is an element of it. If we already know that v_{k+1} is reachable from V_{1-i} in $G[I]$, then it follows that v_k is reachable as well. Thus by induction v_0 is reachable from V_{1-i} in $G[I]$ which completes the proof. $\square$

5.2. Matroid intersection

Theorem 1.9. *Let E be countable and let $M_i \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$ for $i \in \{0, 1\}$. Then $\mathsf{SM}(M_0, M_1)$ is cofinal but not necessarily upward closed in $(\mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}, \trianglelefteq_{M_0,M_1})$.*

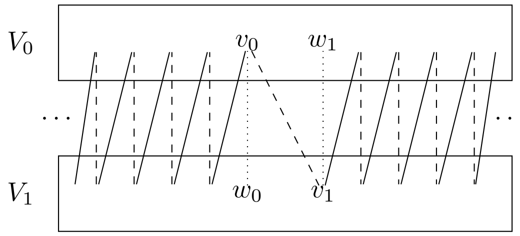

Figure 2: Matching I_0 consists of the dashed and I_1 consists of the normal edges.

Proof. We start with the 'cofinal' part of the statement. Let $J \in \mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ be given. We take an $I' \in \mathsf{SM}(M_0, M_1)$ and fix a partition $E = E_0 \sqcup E_1$ such that $I'_i := I' \cap E_i$ spans E_i in M_i for $i \in \{0, 1\}$. By applying Theorem 1.6 with the matroids $M_i \upharpoonright E_i$ and $M_{1-i}.E_i$ and sets I'_i and $J_i := J \cap E_i$, we obtain a base I_i of $M_i \upharpoonright E_i$ which is independent in $M_{1-i}.E_i$ and spans J_i in $M_{1-i}.E_i$. We claim that $I := I_0 \sqcup I_1$ is as desired. Indeed, $I \in \mathsf{SM}(M_0, M_1)$ because I_i is an $M_{1-i}.E_i$-independent base of $M_i \upharpoonright E_i$. Finally, I_{1-i} spans J_{1-i} in $M_i.E_{1-i} = M_i/E_i$ and $I_i \subseteq E_i$ spans E_i (which contains J_i) in M_i by construction thus $J \subseteq \mathsf{span}_{M_i}(I)$. Therefore $J \subseteq \mathsf{span}_{M_0}(I) \cap \mathsf{span}_{M_1}(I)$ which means $J \trianglelefteq_{M_0,M_1} I$.

In order to show the 'not necessarily upward closed' part we shall construct first a bipartite graph $G = (V_0, V_1; E)$. We start with a double ray $\dots, v_{-1}, v_0, v_1, \dots$ and add a new vertex w_i and new edge $v_i w_i$ for $i \in \{0, 1\}$ (see Figure 2). The bipartite graph G induces two partition matroids M_0 and M_1 on E in the way that $I \subseteq E$ is defined to be independent in M_i if no two edges in I have a common end-vertex in V_i. Then the elements of $\mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}$ are exactly the matchings, moreover, matching I is in $\mathsf{SM}(M_0, M_1)$ iff one can choose exactly one vertex from each $e \in I$ such that the resulting set is a vertex cover. Let

$$I_i := \{v_{2k+i} v_{2k+1+i} : \ k < \omega\} \text{ for } i \in \{0, 1\}.$$

On the one hand, the matchings I_i cover the same vertices thus

$$I_0 \trianglelefteq_{M_0,M_1} I_1 \trianglelefteq_{M_0,M_1} I_0.$$

On the other hand, we claim that I_1 is strongly maximal but I_0 is not. Indeed, $\{v_{-2k},\ v_{2k+1} : \ k < \omega\}$ is a vertex cover (upper-left and lower-right corners on Figure 2) that consists of choosing exactly one end-vertex of each edge in I_1 and therefore witnessing $I_1 \in \mathsf{SM}(M_0, M_1)$. But there is no such a vertex cover for I_0 because if we pick v_i from the edge v_0v_1, then we cannot choose any end-vertex of $v_{1-i}w_{1-i}$. Thus $\mathsf{SM}(M_0, M_1)$ is not upward closed in $(\mathcal{I}_{M_0} \cap \mathcal{I}_{M_1}, \trianglelefteq_{M_0,M_1})$. $\square$

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Received January 29, 2022