

**LAGRANGIAN BOUNDARY CONDITIONS FOR
ANTI-SELF-DUAL INSTANTONS AND THE
ATIYAH–FLOER CONJECTURE**

KATRIN WEHRHEIM

The purpose of this survey is to explain an approach to the Atiyah–Floer conjecture via a new instanton Floer homology with Lagrangian boundary conditions. This is a joint project with Dietmar Salamon; see [14] for an earlier exposition. This paper also provides a rough guide to the analysis of anti-self-dual instantons with Lagrangian boundary conditions in [21, 22], which is the crucial ingredient of our approach.

1. Introduction

Atiyah [1] and Floer conjectured a natural isomorphism between the instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y)$ of a homology 3-sphere Y and the symplectic Floer homology $\mathrm{HF}_*^{\mathrm{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1})$ of a pair of Lagrangians L_{H_0}, L_{H_1} in the symplectic moduli space \mathcal{R}_Σ of flat $\mathrm{SU}(2)$ -connections associated to a Heegard splitting $Y = H_0 \cup_\Sigma H_1$. Both homologies were introduced by Floer [7, 8], but the symplectic Floer homology is not strictly defined in this case due to singularities of \mathcal{R}_Σ . Taubes [16] proved that the Euler characteristics both agree with the Casson invariant of Y . The main task in identifying the homology groups is a comparison between the trajectories: pseudoholomorphic curves in \mathcal{R}_Σ with Lagrangian boundary conditions and anti-self-dual instantons on $\mathbb{R} \times Y$ (which has no boundary).

The basic idea of our approach is to introduce a third Floer homology¹ $\mathrm{HF}_*^{\mathrm{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$ whose trajectory equation couples the anti-self-duality equation on $\mathbb{R} \times [0, 1] \times \Sigma$ with Lagrangian boundary conditions. We expect that two different degenerations of the metric on $[0, 1] \times \Sigma$ will

¹This is a special case of the invariant $\mathrm{HF}_*^{\mathrm{inst}}(Y, \mathcal{L})$ introduced below for a 3-manifold Y with boundary and a Lagrangian submanifold \mathcal{L} in the space of connections over ∂Y .

give rise to isomorphisms that would prove the Atiyah–Floer conjecture²

$$(1) \quad \mathrm{HF}_*^{\mathrm{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1}) \cong \mathrm{HF}_*^{\mathrm{inst}}(H_0 \cup_{\Sigma} H_1),$$

$$(2) \quad \mathrm{HF}_*^{\mathrm{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1}) \cong \mathrm{HF}_*^{\mathrm{symp}}(\mathcal{R}_{\Sigma}, L_{H_0}, L_{H_1}).$$

This approach separates the difficulties: The first isomorphism is a purely gauge theoretic comparison between anti-self-dual instantons over domains with and without boundary. The second isomorphism requires a comparison between anti-self-dual instantons and pseudoholomorphic curves (both with Lagrangian boundary conditions), which would be a generalization of the adiabatic limit of Dostoglou–Salamon [6], which they used to prove an analogon of the Atiyah–Floer conjecture for mapping tori. The mapping torus case does not involve boundary conditions. Moreover, the underlying bundle is nontrivial so that the moduli space of flat connections is smooth. In contrast, the Heegard splitting case deals with trivial bundles for which the moduli space \mathcal{R}_{Σ} and its Lagrangian submanifolds are always singular.

So the Atiyah–Floer conjecture poses as a first task (which we do not approach here) the construction of a symplectic Floer homology for symplectic and Lagrangian manifolds with quotient singularities. In fact, the singular symplectic space \mathcal{R}_{Σ} is the symplectic quotient (in the sense of Atiyah and Bott [2]) of a Hamiltonian group action (the infinite dimensional gauge group) on an infinite dimensional symplectic space (the space of connections over a Riemann surface). In the case of a finite dimensional Hamiltonian group action with smooth and monotone symplectic quotient, Gaio and Salamon [11] have identified the Gromov–Witten invariants of the symplectic quotient with new invariants arising from the symplectic vortex equations.

The anti-self-duality equation on $\mathbb{R} \times [0, 1] \times \Sigma$ is the exact analogue of the symplectic vortex equations for \mathcal{R}_{Σ} . We will show in Section 3 that the analytic behaviour of these trajectories of the new Floer homology is a mixture of local effects in the interior—as they are expected for anti-self-dual instantons—and surprising semiglobal effects near the boundary that resemble to the behaviour of pseudoholomorphic curves in \mathcal{R}_{Σ} . This shows that the new Floer homology indeed provides a good interpolation between the two Floer homologies in the Atiyah–Floer conjecture.

More generally, an instanton type Floer homology for 3-manifolds with boundary should naturally use Lagrangian boundary conditions. Fukaya [9] gives such a setup in the case of a nontrivial bundle: The anti-self-duality equation is coupled via a degeneration of the metric to the pseudoholomorphic curve equation in the moduli space of flat connections (which is

²There are moreover product structures on all three Floer homologies that should be intertwined by the isomorphisms, as sketched in [14]. Our analytic setup should allow for their definition and identification, but we do not discuss this topic here.

smooth in this case). Our new trajectory equation is a different setting that arises naturally from the Chern–Simons functional—the Morse function in the instanton Floer theory. It works in the gauge theoretic setting up to the boundary, which has the advantage that the Lagrangians are smooth Banach submanifolds of a symplectic Banach space, although the quotients might be singular. We thus give a setup for an instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y, \mathcal{L})$ of a compact 3-manifold Y with boundary and a gauge invariant Lagrangian submanifold \mathcal{L} in the space of $\mathrm{SU}(2)$ -connections over ∂Y .

This program is carried through in [15] for the case where $\mathcal{L} = \mathcal{L}_H$ arises from a disjoint union of handle bodies H with boundary $\partial H = \partial Y = \Sigma$ such that $Y \cup_\Sigma H$ is a homology 3-sphere. We expect that the isomorphism (1) will be true in this more general setting,

$$(1') \quad \mathrm{HF}_*^{\mathrm{inst}}(Y, \mathcal{L}_H) \cong \mathrm{HF}_*^{\mathrm{inst}}(Y \cup_\Sigma H).$$

The assumption $\mathcal{L} = \mathcal{L}_H$ is more of technical nature and is not required for the basic compactness in Theorem 3.4. We also have an approach to removing this assumption in Theorems 3.5 and 3.7, based on Mrowka’s understanding of the gauge group in borderline Sobolev cases. More essentially, we need the nontrivial flat connections on Y with boundary condition in \mathcal{L} to be irreducible (i.e., to have discrete isotropy in the gauge group). For the same reason, the original instanton Floer homology is only defined for homology 3-spheres. Now starting from a Heegard splitting $H_0 \cup_\Sigma H_1$ of a general closed 3-manifold, the irreducibility could be achieved by perturbing the Lagrangians \mathcal{L}_{H_0} and \mathcal{L}_{H_1} . Thus the problem of reducible connections can be transferred to transversality questions in our new instanton Floer homology with Lagrangian boundary conditions.

Section 2 provides an introduction to the gauge theoretic background. We explain the Chern–Simons functional and the moment map picture of the gauge group action and give the setup for an instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y, \mathcal{L})$ with Lagrangian boundary conditions. In Section 3, we specialize to the case $Y = [0, 1] \times \Sigma$ and the Lagrangian submanifold $\mathcal{L}_H = \mathcal{L}_{H_0} \times \mathcal{L}_{H_1}$ arising from two handle bodies $H = H_0 \sqcup H_1$ such that $H_0 \cup_\Sigma H_1 \cong Y \cup_{\Sigma \sqcup \Sigma} H$ is a homology 3-sphere. We give a detailed account of the new Floer homology $\mathrm{HF}_*^{\mathrm{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$, comparing its definition and the analytic properties of its trajectories to those of $\mathrm{HF}_*^{\mathrm{inst}}(H_0 \cup_\Sigma H_1)$ and $\mathrm{HF}_*^{\mathrm{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1})$ (what it would be if these quotients were smooth). In Section 4, we sketch the ideas for proofs of the isomorphisms (1) and (2).

The last two sections are a rough guide to the analysis of anti-self-dual instantons with Lagrangian boundary conditions, which was established in [20–23] in full technical detail. Section 5 provides an overview of the properties of gauge invariant Lagrangian submanifolds in the space of connections

over a Riemann surface. It moreover describes the special extension properties of Lagrangian submanifolds that arise from handle bodies. In Section 6, we sketch the proofs of the analytic results in Section 3. We put the proofs into context with the standard proofs of Uhlenbeck compactness (for anti-self-dual instantons) and Gromov compactness (for pseudoholomorphic curves) since—just as the results—each proof requires a subtle combination of the best techniques from both gauge theory and symplectic topology, which we hope the reader will find entertaining.

2. Gauge theory and symplectic topology

We give an introduction to some gauge theoretic concepts and notations. More details and proofs can be found in, e.g., [5, 19].

Let G be a compact Lie group. The Lie algebra $\mathfrak{g} = \mathbb{T}_1 G$ is equipped with a Lie bracket $[\cdot, \cdot]$ and with a G -invariant inner product $\langle \cdot, \cdot \rangle$. For the instanton Floer theories, we will be using $G = \mathrm{SU}(2)$ with the commutator $[\xi, \eta] = \xi\eta - \eta\xi$ and the trace $\langle \xi, \eta \rangle = -\mathrm{tr}(\xi\eta)$ for $\xi, \eta \in \mathfrak{su}(2)$. We describe a **connection** on the trivial G -bundle $G \times X \rightarrow X$ over a manifold X as a \mathfrak{g} -valued 1-form $A \in \Omega^1(X; \mathfrak{g})$ and thus denote the space of smooth connections by

$$\mathcal{A}(X) := \Omega^1(X; \mathfrak{g}).$$

(The discussion in this section generalizes to nontrivial bundles, where connections are given by 1-forms with values in an associated bundle.) On the trivial bundle, a 1-form $A \in \mathcal{A}(X)$ corresponds to an equivariant distribution $\{(-gA(Y), Y) \mid Y \in \mathbb{T}_x X\} \subset \mathbb{T}_{(g,x)}(G \times X)$ of horizontal subspaces. The corresponding covariant derivative on sections $s : X \rightarrow E$ of a trivial vector bundle with structure group $G \subset \mathrm{Hom}(E)$ is $\nabla_A s : Y \mapsto \nabla s(Y) + A(Y)s$.

The **curvature** of a connection $A \in \mathcal{A}(X)$ is given by the 2-form

$$F_A := dA + \frac{1}{2}[A \wedge A] \in \Omega^2(X; \mathfrak{g}).$$

Throughout $[\cdot \wedge \cdot]$ indicates that the values of the differential forms are paired by the Lie bracket. The differential of the map $A \mapsto F_A$ at a connection $A \in \mathcal{A}(X)$ is the “twisted” exterior derivative $d_A : \Omega^1(X; \mathfrak{g}) \rightarrow \Omega^2(X; \mathfrak{g})$. In general, $d_A : \Omega^k(X; \mathfrak{g}) \rightarrow \Omega^{k+1}(X; \mathfrak{g})$ acts on \mathfrak{g} -valued differential forms by

$$d_A \eta := d\eta + [A \wedge \eta].$$

One checks that $d_A d_A \eta = [F_A \wedge \eta]$, so $d_A^2 = 0$ iff the curvature vanishes. Such connections are called **flat** and we denote the set of flat connections by

$$\mathcal{A}_{\mathrm{flat}}(X) := \{A \in \mathcal{A}(X) \mid F_A = 0\}.$$

Moreover, a connection is flat iff the horizontal distribution is locally integrable. So parallel transport with respect to a flat connection around a loop

is given by an element in the group G that is invariant under homotopy of the loop with fixed base point $x \in X$. Thus the **holonomy** induces a map

$$\text{hol}_x : \mathcal{A}_{\text{flat}}(X) \longrightarrow \text{Hom}(\pi_1(X, x), G).$$

Next, connections that are the same up to a bundle isomorphism are called gauge equivalent. The bundle isomorphisms of the trivial bundle can be identified with maps $u : X \rightarrow G$ that are called **gauge transformations**. Composition of bundle isomorphisms corresponds to multiplication of gauge transformations, so the space of smooth gauge transformations has the structure of a group, called the **gauge group**

$$\mathcal{G}(X) := \mathcal{C}^\infty(X, G).$$

The action of $\mathcal{G}(X)$ on the space of connections $\mathcal{A}(X)$, called the **gauge action**, is given by pullback of the connection (i.e., the horizontal subspace or the covariant derivative), hence

$$u^*A := u^{-1}Au + u^{-1}du \quad \text{for } u \in \mathcal{G}(X), A \in \mathcal{A}(X).$$

The space of flat connections $\mathcal{A}_{\text{flat}}(X)$ is obviously invariant under $\mathcal{G}(X)$, and the curvature transforms by $F_{u^*A} = u^{-1}F_Au$. The holonomy of a flat connection $A \in \mathcal{A}_{\text{flat}}(X)$ transforms by conjugation under $u \in \mathcal{G}(X)$, more precisely $\text{hol}_x(u^*A) = u(x)^{-1}\text{hol}_x(A)u(x)$ for the holonomy based at $x \in X$. Similarly, a change of the base point also transforms the holonomy by conjugation. Hence the holonomy descends to a map

$$\text{hol} : \mathcal{A}_{\text{flat}}(X)/\mathcal{G}(X) \longrightarrow \text{Hom}(\pi_1(X), G)/G =: \mathcal{R}_X,$$

where the action of G is by conjugation. If there are no nontrivial G -bundles over X ,³ then this is in fact an isomorphism and we will identify the representation space \mathcal{R}_X with $\mathcal{A}_{\text{flat}}(X)/\mathcal{G}(X)$. In general, this is an isomorphism when taking the union over all isomorphism classes of bundles on the left hand side, see, e.g., [5, Proposition 2.2.3].

Uhlenbeck compactness. The observations above shows that the moduli space of flat connections is a compact subset of $\mathcal{A}(X)/\mathcal{G}(X)$ (in the \mathcal{C}^∞ -topology). Uhlenbeck's weak compactness theorem is a remarkable generalization of this compactness to connections with small curvature. It is the starting point for all analysis in gauge theory, so this is a good point to introduce the Sobolev completions of the spaces of connections and gauge transformations. For a compact manifold X and for $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$ let

$$\mathcal{A}^{k,p}(X) := W^{k,p}(X, T^*X \otimes \mathfrak{g}), \quad \mathcal{G}^{k,p}(X) := W^{k,p}(X, G).$$

³This is for example the case for $\dim X = 2$ or 3 and a connected, simply connected group as $G = \text{SU}(2)$. It also holds for a handle body X and any connected group G .

For $kp > \dim X$, the gauge group $\mathcal{G}^{k,p}(X)$ is a Banach manifold, on which multiplication and inversion are smooth, and it acts smoothly on $\mathcal{A}^{k-1,p}(X)$. We equip X with a metric, and then for any $p \geq 1$, the L^p -norm of the curvature,

$$\|F_A\|_p^p = \int_X |F_A|^p,$$

is a gauge invariant quantity. For $p = \frac{1}{2} \dim X$, this is the conformally invariant Yang–Mills energy of the connection, which can concentrate at single points. Thus for $p \leq \frac{1}{2} \dim X$, one cannot expect the compactness of a set of connections with bounded L^p -norm of the curvature. Uhlenbeck’s result [18] says that for $p > \frac{1}{2} \dim X$, however, every such set is compact in the weak $W^{1,p}$ -topology on the quotient $\mathcal{A}^{1,p}(X)/\mathcal{G}^{2,p}(X)$.

Theorem 2.1 (Weak Uhlenbeck Compactness). *Let X be a compact manifold and let $p > \frac{1}{2} \dim X$. Suppose that $A_i \in \mathcal{A}(X)$ is a sequence of connections such that $\|F_{A_i}\|_p$ is uniformly bounded. Then, after going to a subsequence, there exists a sequence of gauge transformations $u_i \in \mathcal{G}(X)$ such that $u_i^* A_i \rightarrow A_\infty$ converges in the weak $W^{1,p}$ -topology to a connection $A_\infty \in \mathcal{A}^{1,p}(X)$.*

In fact, one even has a weak $W^{1,\dim X/2}$ -compactness if one assumes that every point in X has a neighbourhood on which the Yang–Mills energy is bounded by a small constant. We will need the slightly stronger $W^{1,p}$ -compactness since it allows us to globally (not just locally over small balls in X) work in a local slice of the gauge action. The local slice theorem says that any connection A' that is suitably close to a fixed reference connection A can be put into relative Coulomb gauge, i.e., $u^* A' - A$ is L^2 -orthogonal to the gauge orbit through A . The linearized gauge action $T_{\mathbb{1}}\mathcal{G}(X) \rightarrow T_A\mathcal{A}(X)$ at $A \in \mathcal{A}(X)$ is given by $d_A : \Omega^0(X; \mathfrak{g}) \rightarrow \Omega^1(X; \mathfrak{g})$. Its formal adjoint is $d_A^* = - * d_A *$. More generally, with $m(X, k) = (\dim X - k)(k - 1)$, the twisted coderivative is

$$d_A^* := -(-1)^{m(X,k)} * d_A * : \Omega^k(X; \mathfrak{g}) \longrightarrow \Omega^{k-1}(X; \mathfrak{g}).$$

Here we only give the sequential form of the local slice theorem. A stronger statement and proof can be found, e.g., in [19, Theorem F].

Theorem 2.2 (Local Slice Theorem). *Let X be a compact Riemannian manifold with smooth boundary and let $p > \frac{1}{2} \dim X$. Suppose that $A_i \in \mathcal{A}^{1,p}(X)$ is a sequence of connections such that $A_i \rightarrow A \in \mathcal{A}(X)$ in the*

weak $W^{1,p}$ -topology. Then for sufficiently large i , there exist gauge transformations $u_i \in \mathcal{G}^{2,p}(X)$ such that $u_i^* A_i \rightarrow A$ and

$$\begin{cases} d_A^*(u_i^* A_i - A) = 0, \\ *(u_i^* A_i - A)|_{\partial X} = 0. \end{cases}$$

To see the strength of these two theorems consider the following example: Let X be a compact Riemannian 4-manifold. The extrema of the Yang–Mills energy $\int_X |F_A|^2$ are called **Yang–Mills instantons**. They satisfy the equation $d_A^* F_A = 0$ and the boundary condition $*F_A|_{\partial X} = 0$, which—augmented with the local slice conditions—pose an elliptic boundary value problem. Uhlenbeck’s compactness theorem for Yang–Mills instantons with L^p -bounded curvature then is a corollary of Theorems 2.1 and 2.2.

Theorem 2.3 (Strong Uhlenbeck Compactness). *Let $A_i \in \mathcal{A}(X)$ be a sequence of Yang–Mills instantons such that $\|F_{A_i}\|_p$ is uniformly bounded for some $p > 2$. Then, after going to a subsequence, there exists a sequence of gauge transformations $u_i \in \mathcal{G}(X)$ such that $u_i^* A_i \rightarrow A_\infty$ converges in the C^∞ -topology to another Yang–Mills instanton $A_\infty \in \mathcal{A}(X)$.*

Anti-self-dual instantons on an oriented Riemannian 4-manifold X are solutions $A \in \mathcal{A}(X)$ of the first order equation

$$F_A + *F_A = 0.$$

By the Bianchi identity $d_A F_A = 0$, these are special solutions of the Yang–Mills equation $d_A^* F_A = 0$. On a manifold with boundary, however, the anti-self-duality equation with boundary condition $*F_A|_{\partial X} = 0$ is an overdetermined boundary value problem comparable to Dirichlet boundary conditions for holomorphic maps. This is another reason why it is natural to consider (weaker) Lagrangian boundary conditions for anti-self-dual instantons.

The moduli space of flat connections over a Riemann surface Let Σ be a Riemann surface. The natural symplectic form on the space of connections $\mathcal{A}(\Sigma) = \Omega^1(\Sigma; \mathfrak{g})$ is

$$(3) \quad \omega(\alpha, \beta) := \int_\Sigma \langle \alpha \wedge \beta \rangle \quad \text{for } \alpha, \beta \in \Omega^1(\Sigma; \mathfrak{g}).$$

Here and throughout $\langle \cdot \wedge \cdot \rangle$ indicates that the values of the differential forms are paired by the inner product. Note that for any metric on Σ , the Hodge operator $*$ is a complex structure on $\mathcal{A}(\Sigma)$, which is compatible with ω and induces the L^2 -metric $\omega(\alpha, *\beta) = \langle \alpha, \beta \rangle_{L^2}$.

It was observed by Atiyah and Bott [2] that the action of the gauge group $\mathcal{G}(\Sigma)$ on $\mathcal{A}(\Sigma)$ can be viewed as Hamiltonian action of an infinite dimensional

Lie group. The Lie algebra of $\mathcal{G}(\Sigma)$ is $\Omega^0(\Sigma; \mathfrak{g})$ and the infinitesimal action of $\xi \in \Omega^0(\Sigma; \mathfrak{g})$ is given by the vector field

$$X_\xi : \mathcal{A}(\Sigma) \longrightarrow \Omega^1(\Sigma; \mathfrak{g}), \quad X_\xi(A) = d_A \xi.$$

This is the Hamiltonian vector field of the function $A \mapsto \int_\Sigma \langle \mu(A), \xi \rangle$, where

$$\mu : \mathcal{A}(\Sigma) \longrightarrow \Omega^0(\Sigma; \mathfrak{g}), \quad \mu(A) = *F_A$$

can be considered as a moment map. Its differential is $d\mu(A) = *d_A$, so one indeed has for all $\beta \in \Omega^1(\Sigma; \mathfrak{g})$

$$\omega(X_\xi(A), \beta) = \int_\Sigma \langle d_A \xi \wedge \beta \rangle = - \int_\Sigma \langle \xi, *d_A \beta \rangle = - \int_\Sigma \langle d\mu(A) \beta, \xi \rangle.$$

The zero set of μ is the set of flat connections. So the moduli space of flat connections on Σ can be seen as the symplectic quotient of the gauge action,

$$\mathcal{R}_\Sigma = \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma) = \mu^{-1}(0)/\mathcal{G}(\Sigma) = \mathcal{A}(\Sigma)//\mathcal{G}(\Sigma).$$

This quotient $\mathcal{R}_\Sigma \cong \text{Hom}(\pi_1(\Sigma), G)/G$ is singular at the reducible representations, but for irreducible⁴ $A \in \mathcal{A}_{\text{flat}}(\Sigma)$, it is a smooth manifold near the gauge equivalence class $[A]$. To understand its tangent space, notice that the linearized action $T_1\mathcal{G}(\Sigma) \rightarrow T_A\mathcal{A}(\Sigma)$ is given by $d_A : \Omega^0(\Sigma; \mathfrak{g}) \rightarrow \Omega^1(\Sigma; \mathfrak{g})$. At a flat connection $A \in \mathcal{A}_{\text{flat}}(\Sigma)$, it fits into a chain complex with the differential of the moment map (since $d_A \circ d_A = 0$),

$$\Omega^0(\Sigma; \mathfrak{g}) \xrightarrow{d_A} \Omega^1(\Sigma; \mathfrak{g}) \xrightarrow{*d_A} \Omega^0(\Sigma; \mathfrak{g}).$$

So the tangent space to \mathcal{R}_Σ at $[A]$ is the twisted first homology group, which can be identified with the harmonic 1-forms,

$$T_{[A]}\mathcal{R}_\Sigma = \ker *d_A / \text{im } d_A \cong \ker d_A \cap \ker d_A^* =: h_A^1.$$

Hodge theory gives a corresponding L^2 -orthogonal splitting

$$(4) \quad \Omega^1(\Sigma; \mathfrak{g}) = \text{im } d_A \oplus \text{im } (*d_A) \oplus h_A^1,$$

where $\text{im } d_A \oplus h_A^1 = \ker *d_A$ is the tangent space to $\mu^{-1}(0) = \mathcal{A}_{\text{flat}}(\Sigma)$ and $\text{im } (*d_A) \oplus h_A^1 = (\text{im } d_A)^\perp$ is the local slice of the gauge action through A .

We have seen that the moduli space of flat connections \mathcal{R}_Σ is a smooth manifold of dimension $(2g - 2) \dim G$ with singularities at the reducible connections. Moreover, the symplectic structure (3) on $\mathcal{A}(\Sigma)$ is $\mathcal{G}(\Sigma)$ -invariant and induces a symplectic structure on the smooth part of \mathcal{R}_Σ . For harmonic representatives $\alpha, \beta \in h_A^1 \cong T_{[A]}\mathcal{R}_\Sigma$, it is again given by $\omega(\alpha, \beta) = \int_\Sigma \langle \alpha \wedge \beta \rangle$. In this representation of the tangent space, we also

⁴A connection $A \in \mathcal{A}_{\text{flat}}(\Sigma)$ is called irreducible if its isotropy subgroup of $\mathcal{G}(\Sigma)$ (the group of gauge transformations that leave A fixed) is discrete, i.e., $d_A|_{\Omega^0}$ is injective. For a closed Riemann surface, this is equivalent to $d_A^*|_{\Omega^1}$ being surjective.

see that the Hodge operator $*$ descends to \mathcal{R}_Σ . So $(\mathcal{R}_\Sigma, \omega)$ is a (singular) symplectic manifold with compatible almost complex structure $*$.

The Chern–Simons functional and instanton Floer homology Let Y be a compact oriented 3-manifold. The **Chern–Simons 1-form** λ on the space of connections $\mathcal{A}(Y)$ is given by

$$\lambda_A(\alpha) := \int_Y \langle F_A \wedge \alpha \rangle \quad \text{for } \alpha \in T_A \mathcal{A}(Y) = \Omega^1(Y; \mathfrak{g}).$$

This 1-form is equivariant, $\lambda_{u^*A}(u^{-1}\alpha u) = \lambda_A(\alpha)$. If $\partial Y = \emptyset$ or $F_A|_{\partial Y} = 0$, then λ is also horizontal and thus descends to the (singular) moduli space⁵ $\mathcal{B}(Y) := \mathcal{A}(Y)/\mathcal{G}(Y)$. Indeed, a tangent vector to the gauge orbit through $A \in \mathcal{A}(Y)$ has the form $\alpha = d_A \xi$ with $\xi \in \Omega^0(Y; \mathfrak{g})$, and by Stokes’ theorem and the Bianchi identity

$$(5) \quad \lambda_A(d_A \xi) = - \int_Y \langle d_A F_A, \xi \rangle + \int_{\partial Y} \langle F_A, \xi \rangle = 0.$$

To calculate the differential of λ consider $\alpha, \beta \in \Omega^1(Y; \mathfrak{g})$ as (constant) vector fields on $\mathcal{A}(Y)$, then their Lie bracket vanishes and

$$(6) \quad \begin{aligned} d\lambda(\alpha, \beta) &= \nabla_\alpha(\lambda(\beta)) - \nabla_\beta(\lambda(\alpha)) \\ &= \int_Y \langle d_A \alpha \wedge \beta \rangle - \int_Y \langle d_A \beta \wedge \alpha \rangle = \int_{\partial Y} \langle \alpha \wedge \beta \rangle. \end{aligned}$$

So for $\partial Y = \emptyset$, the Chern–Simons 1-form descends to a closed 1-form on $\mathcal{B}(Y)$. In fact, λ is the differential of the **Chern–Simons functional**

$$\mathcal{CS}(A) := \frac{1}{2} \int_Y \langle A \wedge (F_A - \frac{1}{6}[A \wedge A]) \rangle.$$

For a more illuminating definition, let X be a compact 4-manifold with boundary $\partial X = Y$, then for any $\tilde{A} \in \mathcal{A}(X)$ with $\tilde{A}|_{\partial X} = A$

$$\mathcal{CS}(A) = \frac{1}{2} \int_X \langle F_{\tilde{A}} \wedge F_{\tilde{A}} \rangle.$$

For closed X , the right hand side is a topological invariant of the bundle. We fix $G = \text{SU}(2)$, then this invariant is $4\pi^2 c_2(P)$, where \tilde{A} is a connection on the bundle $P \rightarrow X$. From this, one can see that the Chern–Simons functional descends to an S^1 -valued functional $\mathcal{CS} : \mathcal{B}(Y) \rightarrow \mathbb{R}/4\pi^2\mathbb{Z}$ since it changes by $\mathcal{CS}(A) - \mathcal{CS}(u^*A) = 4\pi^2 \deg(u) \in 4\pi^2\mathbb{Z}$ under gauge transformations. If Y is a homology 3-sphere, then Floer [7] used the generalized Morse theory for this functional to define the instanton Floer homology $\text{HF}_*^{\text{inst}}(Y)$. Roughly speaking, the Floer complex is generated by the zeros of $d\mathcal{CS} = \lambda$, i.e., by the flat connections $A \in \mathcal{A}_{\text{flat}}(Y)$ modulo $\mathcal{G}(Y)$. The differential

⁵Note that $\mathcal{B}(Y)$ is not the moduli space of flat connections \mathcal{R}_Y , but the infinite dimensional and singular space of all connections modulo gauge equivalence.

on the complex is defined by counting negative gradient flow lines, so we choose a metric on Y and thus fix an L^2 -metric on $\mathcal{A}(Y)$. Then the gradient of \mathcal{CS} is $A \mapsto *F_A$ and a negative gradient flow line is a path $A : \mathbb{R} \rightarrow \mathcal{A}(Y)$ satisfying

$$\partial_s A = - * F_A.$$

Equivalently, one can view this path as connection $\Xi = \Phi ds + A \in \mathcal{A}(\mathbb{R} \times Y)$ in the special gauge $\Phi \equiv 0$. Then the above equation is the anti-self-duality equation $F_\Xi + *F_\Xi = 0$ for Ξ . For a general connection $\Xi \in \mathcal{A}(\mathbb{R} \times Y)$, this so-called temporal gauge can always be achieved by the solution $u \in \mathcal{G}(\mathbb{R} \times Y)$ of $\partial_s u = -\Phi u$ with $u|_{s=0} \equiv \mathbb{1}$. So the negative gradient flow lines of the Chern–Simons functional modulo $\mathcal{G}(Y)$ are in one-to-one correspondence with the anti-self-dual connections $\Xi \in \mathcal{A}(\mathbb{R} \times Y)$ modulo $\mathcal{G}(\mathbb{R} \times Y)$. An extensive discussion of instanton Floer homology for closed 3-manifolds can be found in Donaldson’s book [4].

If Y has nonempty boundary $\partial Y = \Sigma$, then the differential (6) is the symplectic form ω on $\alpha|_\Sigma, \beta|_\Sigma \in \mathcal{A}(\Sigma)$, compare (3). To render λ closed, it is natural⁶ to pick a Lagrangian submanifold $\mathcal{L} \subset \mathcal{A}(\Sigma)$ and restrict λ to

$$\mathcal{A}(Y, \mathcal{L}) := \{A \in \mathcal{A}(Y) \mid A|_\Sigma \in \mathcal{L}\}.$$

More precisely, we fix a $p > 2$ and make the following assumptions to ensure that λ defines a closed 1-form on $\mathcal{B}(Y, \mathcal{L}) := \mathcal{A}(Y, \mathcal{L})/\mathcal{G}(Y)$.

- (i) $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ is a Banach submanifold that is isotropic, $\omega|_{\mathcal{L}} \equiv 0$, and coisotropic in the sense of the following implication for all $\alpha \in \mathcal{A}^{0,p}(\Sigma)$:
If $\omega(\alpha, \beta) = 0$ for all $\beta \in \mathrm{T}_A \mathcal{L}$, then $\alpha \in \mathrm{T}_A \mathcal{L}$.
- (ii) \mathcal{L} is invariant under $\mathcal{G}^{1,p}(\Sigma)$.
- (iii) $\mathcal{L} \subset \mathcal{A}_{\mathrm{flat}}^{0,p}(\Sigma)$ lies in the space of weakly flat connections.⁷

Here (ii) ensures that $\mathcal{G}(Y)$ acts on $\mathcal{A}(Y, \mathcal{L})$, and (iii) implies that λ is horizontal by (5). These assumptions also imply that \mathcal{L} descends to a (singular) Lagrangian submanifold in the (singular) moduli space of flat connections,

$$L := \mathcal{L}/\mathcal{G}^{1,p}(\Sigma) \subset \mathcal{R}_\Sigma = \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma).$$

The assumptions (i)–(iii) also imply the orthogonal splitting, see Section 5,

$$\Omega^1(\Sigma; \mathfrak{g}) = \mathrm{T}_A \mathcal{L} \oplus * \mathrm{T}_A \mathcal{L} \quad \text{for all } A \in \mathcal{L}.$$

Compare this to (4) and note that $\mathrm{im} \, d_A \subset \mathrm{T}_A \mathcal{L} \subset \ker \, d_A$ due to (ii),(iii). So the $\mathrm{T}_A \mathcal{L}$ are determined up to a choice of Lagrangian subspaces in h_A^1 . Conversely, any Lagrangian $L \subset \mathcal{R}_\Sigma$ lifts to a (possibly nonsmooth) $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ as above. In order to obtain a well-defined Floer homology, one should moreover assume that L is simply connected (which ensures a monotonicity

⁶If $\mathcal{L} \subset \mathcal{A}(\Sigma)$ is any submanifold, then the closedness of $\lambda|_{\mathcal{A}(Y, \mathcal{L})}$ is equivalent to $\omega|_{\mathcal{L}} \equiv 0$, and the maximal such submanifolds are precisely the Lagrangian submanifolds.

⁷See [20, Section 3] or use the fact $\mathcal{A}_{\mathrm{flat}}^{0,p}(\Sigma) = \mathcal{G}^{1,p}(\Sigma) * \mathcal{A}_{\mathrm{flat}}(\Sigma)$ as definition. The conditions (ii) and (iii) are equivalent when G is connected and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, e.g., for $\mathrm{SU}(2)$.

property). In general, \mathcal{L} is not simply connected, but its fundamental group cancels with that of $\mathcal{G}(\Sigma)$. This is the reason why λ is not exact but can only be written as the differential of the multi-valued Chern–Simons functional

$$\mathcal{CS}_{\mathcal{L}}(A) = \frac{1}{2} \int_Y \langle A \wedge (F_A - \frac{1}{6}[A \wedge A]) \rangle + \int_0^1 \int_{\Sigma} \langle \tilde{A}(t) \wedge \partial_t \tilde{A}(t) \rangle dt.$$

This involves the choice of a path $\tilde{A} : [0, 1] \rightarrow \mathcal{L}$ with $\tilde{A}(1) = A|_{\Sigma}$ and $\tilde{A}(0) = A_0$ a fixed reference connection in \mathcal{L} . Again we fix $G = \text{SU}(2)$ to obtain a functional $\mathcal{CS}_{\mathcal{L}} : \mathcal{A}(Y, \mathcal{L}) \rightarrow \mathbb{R}/4\pi^2\mathbb{Z}$, which directly descends to $\mathcal{B}(Y, \mathcal{L})$ since the gauge group $\mathcal{G}(Y)$ is connected. We now propose to define a new Floer homology $\text{HF}_*^{\text{inst}}(Y, \mathcal{L})$ from the generalized Morse theory of the functional $\mathcal{CS}_{\mathcal{L}} : \mathcal{B}(Y, \mathcal{L}) \rightarrow \mathbb{R}/4\pi^2\mathbb{Z}$.

A critical point in this theory is a flat connection $A \in \mathcal{A}_{\text{flat}}(Y)$ with Lagrangian boundary condition $A|_{\Sigma} \in \mathcal{L}$ (modulo $\mathcal{G}(Y)$), and a negative gradient flow line is a path $A : \mathbb{R} \rightarrow \mathcal{A}(Y)$ (modulo $\mathcal{G}(Y)$) satisfying

$$(7) \quad \partial_s A = - * F_A, \quad A(s)|_{\Sigma} \in \mathcal{L} \quad \forall s \in \mathbb{R}.$$

Again, this is the anti-self-duality equation for $\Xi = A + \Phi ds \in \mathcal{A}(\mathbb{R} \times Y)$ in the temporal gauge $\Phi \equiv 0$. So the gauge equivalence classes of the gradient flow lines are in one-to-one correspondence with the gauge equivalence classes of anti-self-dual instantons with Lagrangian boundary conditions, i.e., solutions $\Xi \in \mathcal{A}(\mathbb{R} \times Y)$ of the boundary value problem

$$(8) \quad F_{\Xi} + *F_{\Xi} = 0, \quad \Xi|_{\{s\} \times \Sigma} \in \mathcal{L} \quad \forall s \in \mathbb{R}.$$

Lagrangians and handle bodies We have seen before how a Riemann surface Σ gives rise to a (singular) symplectic manifold $\mathcal{R}_{\Sigma} = \text{Hom}(\pi_1(\Sigma), G)/G$, which is a finite dimensional reduction of a symplectic Banach space $\mathcal{A}(\Sigma) = \Omega^1(\Sigma; \mathfrak{g})$ that arises from gauge theory on Σ . We will now discuss a class of examples of Lagrangian Banach submanifolds $\mathcal{L}_H \subset \mathcal{A}(\Sigma)$ that arise from gauge theory on a handle body H with $\partial H = \Sigma$ and that reduce to finite dimensional (singular) Lagrangian submanifolds $L_H \cong \text{Hom}(\pi_1(H), G)/G \subset \mathcal{R}_{\Sigma}$. Here and throughout a handle body is an oriented 3-manifold with boundary that is obtained from the 3-ball by attaching a finite number of 1-handles.

For this purpose, let G be a compact, connected, and simply connected Lie group (e.g., $G = \text{SU}(2)$) and let Σ be a Riemann surface. For a start let H be any compact 3-manifold with boundary $\partial H = \Sigma$. Then

$$\mathcal{L}_H := \{ \tilde{A}|_{\Sigma} \mid \tilde{A} \in \mathcal{A}_{\text{flat}}(H) \} \subset \mathcal{A}(\Sigma)$$

satisfies the assumptions (ii) $\mathcal{L}_H \subset \mathcal{A}_{\text{flat}}(\Sigma)$, (iii) $\mathcal{G}(\Sigma)^* \mathcal{L}_H = \mathcal{L}_H$,⁸ and is isotropic: consider paths $\tilde{A}_1, \tilde{A}_2 : (-\varepsilon, \varepsilon) \rightarrow \mathcal{A}_{\text{flat}}(H)$ with $\tilde{A}_i(0) = \tilde{A}$, then

⁸This uses the assumption $\pi_1(G) = \{0\}$ and the fact that $\pi_2(G) = \{0\}$ for every compact Lie group, so that every gauge transformation $u : \Sigma \rightarrow G$ extends to $\tilde{u} : H \rightarrow G$.

$$\begin{aligned}
 d_{\tilde{A}}\partial_t\tilde{A}_i(0) &= \partial_t|_{t=0}F_{\tilde{A}_i} = 0 \text{ and hence with the symplectic form (3)} \\
 \omega(\partial_t\tilde{A}_1(0), \partial_t\tilde{A}_2(0)) &= \int_{\partial H} \langle \partial_t\tilde{A}_1(0) \wedge \partial_t\tilde{A}_2(0) \rangle \\
 &= \int_H \langle d_{\tilde{A}}\partial_t\tilde{A}_1(0) \wedge \partial_t\tilde{A}_2(0) \rangle - \langle \partial_t\tilde{A}_1(0) \wedge d_{\tilde{A}}\partial_t\tilde{A}_2(0) \rangle = 0.
 \end{aligned}$$

So \mathcal{L}_H descends to an isotropic subset in the symplectic quotient

$$L_H := \mathcal{L}_H/\mathcal{G}(\Sigma) \subset \mathcal{R}_\Sigma = \mathcal{A}(\Sigma)//\mathcal{G}(\Sigma).$$

The holonomy provides an isomorphism

$$L_H \cong \text{Hom}\left(\frac{\pi_1(\Sigma)}{\partial\pi_2(H, \Sigma)}, G\right)/G \subset \mathcal{R}_\Sigma \cong \text{Hom}(\pi_1(\Sigma), G)/G.$$

This is since the holonomy of a flat connection on H is trivial on the contractible loops in $\partial\pi_2(H, \Sigma)$; and all representations of $\pi_1(\Sigma)/\partial\pi_2(H, \Sigma)$ can be realized by a flat connection on H since that quotient embeds into $\pi_1(H)$ by the long exact sequence for homotopy

$$\dots \longrightarrow \pi_2(H) \longrightarrow \pi_2(H, \Sigma) \xrightarrow{\partial} \pi_1(\Sigma) \xrightarrow{\iota} \pi_1(H) \longrightarrow \pi_1(H, \Sigma) \longrightarrow \dots$$

This also shows that $\frac{\pi_1(\Sigma)}{\partial\pi_2(H, \Sigma)} \cong \pi_1(H)$ if H is a handle body (so $\pi_2(H)$ and $\pi_1(H, \Sigma)$ vanish). Now consider the commuting diagram of long exact sequences for homology and cohomology with the vertical Poincare duality:

$$\begin{array}{ccccc}
 H_2(H, \Sigma) & \xrightarrow{\partial} & H_1(\Sigma) & \xrightarrow{\iota} & H_1(H) \\
 \downarrow & & \downarrow & & \downarrow \\
 H^1(H) & \xrightarrow{\iota^*} & H^1(\Sigma) & \xrightarrow{\partial^*} & H^2(H, \Sigma)
 \end{array}$$

One can read off that $(\text{im } \partial)^\perp \cong (\text{im } \iota^*)^\perp = (\ker \partial^*)^\perp = \text{im } \partial$, and we obtain $\dim \frac{H_1(\Sigma)}{\partial H_2(H, \Sigma)} = \frac{1}{2} \dim H_1(\Sigma)$. Hence $\dim L_H = \frac{1}{2} \dim \mathcal{R}_\Sigma$ at smooth points. So a general compact 3-manifold H with $\partial H = \Sigma$ gives rise to a (singular) Lagrangian $\mathcal{L}_H \subset \mathcal{R}_\Sigma$, and in fact $\mathcal{L}_H \subset \mathcal{A}(\Sigma)$ is Lagrangian up to possible singularities. If H is a handle body, then one can prove that \mathcal{L}_H is in fact smooth,⁹ which is essentially due to the fact that $\frac{\pi_1(\Sigma)}{\partial\pi_2(H, \Sigma)} \cong \pi_1(H)$ is a free group.

This correspondence between low dimensional topology, symplectic topology, and gauge theory is summarized in a table on page 20. To round off this discussion, note that a Heegard splitting $H_0 \cup_\Sigma H_1$ of a 3-manifold into two handle bodies H_0, H_1 with common boundary $\partial H_i = \Sigma$ gives rise to a pair of (singular) Lagrangians in a symplectic manifold, $L_{H_0}, L_{H_1} \subset \mathcal{R}_\Sigma$. Now by the Atiyah–Floer conjecture there should be a natural isomorphism between the topological invariant $\text{HF}_*^{\text{inst}}(H_0 \cup_\Sigma H_1)$ and the symplectic invariant $\text{HF}_*^{\text{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1})$ —assuming that the first is defined, i.e., $H_0 \cup H_1$ is a homology 3-sphere, and that the second can be defined in spite of the singularities. On the gauge theoretic side one obtains two smooth (though infinite

⁹Its L^p -completion is a Banach submanifold of $\mathcal{A}^{0,p}(\Sigma)$, see Section 5.

dimensional) Lagrangian submanifolds $\mathcal{L}_{H_0}, \mathcal{L}_{H_1} \subset \mathcal{A}(\Sigma)$, to which we can associate the new invariant $\mathrm{HF}_*^{\mathrm{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$. This invariant is more generally defined in the setting below, where we again fix $G = \mathrm{SU}(2)$. Here we replace $[0, 1] \times \Sigma$ by a more general 3-manifold Y with boundary with boundary $\partial Y = \Sigma$. Then for a union of handle bodies $H = \bigsqcup H_i$ with boundary $\partial H = \bigsqcup \Sigma_i = \Sigma$, we denote by $\mathcal{L}_H \subset \mathcal{A}(\Sigma)$ the Lagrangian submanifold $\mathcal{L}_{H_0} \times \cdots \times \mathcal{L}_{H_N} \subset \mathcal{A}(\Sigma_0) \times \cdots \times \mathcal{A}(\Sigma_N)$.

Theorem 2.4. ([15]) *Let Y be a compact, oriented 3-manifold with boundary Σ . Let H be a disjoint union of handle bodies with $\partial H = \Sigma$, and suppose that $Y \cup_\Sigma H$ is a homology 3-sphere (with \mathbb{Z} -coefficients). Then the Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y, \mathcal{L}_H)$ is well defined and independent of the metric and perturbations of (7) and (8) used to define it.*

In this setting, Floer's original invariant $\mathrm{HF}_*^{\mathrm{inst}}(Y \cup_\Sigma H)$ is also defined, and we expect our invariant to carry the same information.

Conjecture 2.5. *There is a natural isomorphism*

$$\mathrm{HF}_*^{\mathrm{inst}}(Y, \mathcal{L}_H) \cong \mathrm{HF}_*^{\mathrm{inst}}(Y \cup_\Sigma H).$$

Hence the new Floer homology with Lagrangian boundary conditions fits into the Atiyah–Floer conjecture as well as for an approach to defining an invariant for more general 3-manifolds. In the next section, we explain its definition in more detail for the model case $Y = [0, 1] \times \Sigma$ and $\mathcal{L}_H = \mathcal{L}_{H_0} \times \mathcal{L}_{H_1}$, which also is the relevant case for the Atiyah–Floer conjecture.

3. Instanton and symplectic Floer homologies

This section sketches the instanton and symplectic versions of Floer theory and compares the analytic behaviour of the underlying trajectory equations. The purpose of this is to explain the definition of the new instanton Floer homology with Lagrangian boundary conditions (in-L) and to show how it fits between the instanton Floer homology (inst) and the symplectic Floer homology (symp) and thus provides an intermediate invariant for approaching the Atiyah–Floer conjecture. In fact, its trajectories exhibit this interpolation between anti-self-dual instantons (in their interior behaviour) and pseudoholomorphic curves (in their semiglobal behaviour at the boundary).

(inst): Let Y be a homology 3-sphere, i.e., a compact oriented 3-manifold with integer homology $H_*(Y, \mathbb{Z}) \cong H_*(S^3, \mathbb{Z})$. The instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y)$ was defined by Floer [7]. The basic analytic results for this setup that will be quoted below are mainly due to Uhlenbeck [17, 18].

(in-L): Let $Y = H_0 \cup_\Sigma H_1$ be the Heegard splitting of a homology 3-sphere into two handle bodies H_0, H_1 with common boundary $\partial H_i = \Sigma$.

We describe the special case $\mathrm{HF}_*^{\mathrm{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$ of the new instanton Floer homology with Lagrangian boundary conditions of Theorem 2.4. The analytic results for this case are established in [21, 22].

(symp): Let (M, ω) be a compact symplectic manifold and assume that it is simply connected, positive ($c_1(TM) = \lambda[\omega]$ with $\lambda > 0$), and has minimal Chern number $N \geq 2$ (where $\langle c_1, \pi_2(M) \rangle = N\mathbb{Z}$). Let $L_0, L_1 \subset M$ be two simply connected Lagrangian submanifolds. Then the symplectic Lagrangian intersection Floer homology $\mathrm{HF}_*^{\mathrm{symp}}(M, L_0, L_1)$ is defined by [8] and many other authors. The underlying analytic fact here is Gromov's compactness for pseudoholomorphic curves [10].

The instanton cases use the trivial $\mathrm{SU}(2)$ -bundle as before. In the third case one should think of $M = \mathcal{R}_\Sigma$ and $L_i = L_{H_i}$. However, their Floer homology is not yet well defined due to the quotient singularities. We do not give complete definitions of the Floer homologies here. More detailed expositions can be found in, e.g., [4, 13]. In particular, we do not mention the necessary perturbations of the equations for critical points and trajectories.

Definition 3.1. *A critical point is*

(inst): *A flat connection $A \in \mathcal{A}_{\mathrm{flat}}(Y)$.*

(in-L): *A flat connection $A + \Psi dt \in \mathcal{A}_{\mathrm{flat}}([0, 1] \times \Sigma)$ with Lagrangian boundary conditions $A(j) \in \mathcal{L}_{H_j}$ for $j = 0, 1$.*

(symp): *An intersection point $x \in L_0 \cap L_1$.*

In all three cases, the Floer chain complex is generated by the critical points,

$$\mathrm{CF}_* = \bigoplus_{x \text{ crit.pt.}} \mathbb{Z}\langle x \rangle.$$

(In the two instanton cases, the generators actually are gauge equivalence classes $x = [A]$ or $x = [A + \Psi dt]$, and the trivial connection is disregarded.) The boundary operator $\partial : \mathrm{CF}_* \rightarrow \mathrm{CF}_*$ is defined by counting trajectories,

$$\partial \langle x^- \rangle = \sum_{x^+ \text{ crit.pt.}} \# \mathcal{M}^0(x^-, x^+) \langle x^+ \rangle.$$

Here $\mathcal{M}^0(x^-, x^+)$ is the 0-dimensional part of the space of trajectories from x^- to x^+ . This will be a smooth, compact, oriented manifold, so its points can be counted with signs. The trajectory equations will be given below for the three cases. The main issue that we then discuss is the compactness of the space of trajectories, which will allow the definition of ∂ . To obtain a chain complex, one moreover has to establish $\partial \circ \partial = 0$ by identifying the boundary of the 1-dimensional part of the space of trajectories with the broken trajectories that contribute to $\partial \circ \partial$. The Floer homology in the different cases then is the homology $H_*(\mathrm{CF}, \partial)$ of the corresponding Floer

chain complex. It is graded modulo 8 in the instanton cases and modulo $2N$ in the symplectic case.

The trajectory equation depends on the choice of auxiliary data, which the Floer homology will not depend on. In the instanton cases, this is a metric on Y or $[0, 1] \times \Sigma$, respectively. (In the second case, we will give the equation for a product metric.) In the symplectic case, we fix an ω -compatible almost complex structure J on M . The moduli space of trajectories then is the space of solutions of the trajectory equation modulo time shift (in the \mathbb{R} -variable) and modulo gauge equivalence in the instanton cases.

Definition 3.2. *A trajectory is a solution of the trajectory equation (T).*

(inst): *An anti-self-dual instanton on $\mathbb{R} \times Y$:*

$$\begin{aligned} B : \mathbb{R} &\longrightarrow \mathcal{A}(Y) \quad \text{satisfying} \\ (T) \quad \partial_s B + *F_B &= 0 \end{aligned}$$

(in-L): *An anti-self-dual instanton on $\mathbb{R} \times [0, 1] \times \Sigma$ with Lagrangian boundary conditions:*

$$\begin{aligned} (A, \Psi) : \mathbb{R} \times [0, 1] &\longrightarrow \mathcal{A}(\Sigma) \times C^\infty(\Sigma, \mathfrak{su}(2)) \quad \text{satisfying} \\ (T) \quad \begin{cases} \partial_s A + *(\partial_t A - d_A \Psi) = 0 \\ \partial_s \Psi + *F_A = 0 \\ A(s, j) \in \mathcal{L}_{H_j} \quad \forall s \in \mathbb{R}, j \in \{0, 1\} \end{cases} \end{aligned}$$

(symp): *A J -holomorphic strip with Lagrangian boundary conditions:*

$$\begin{aligned} u : \mathbb{R} \times [0, 1] &\rightarrow M \quad \text{satisfying} \\ (T) \quad \begin{cases} \partial_s u + J\partial_t u = 0 \\ u(s, j) \in L_j \quad \forall s \in \mathbb{R}, j \in \{0, 1\} \end{cases} \end{aligned}$$

Pictures of these trajectories and a table that summarizes the definitions and results for the three Floer theories can be found on page 20 and 21. The equation in case (in-L) is $\partial_s B + *F_B = 0$ for $B = A + \Psi dt$, and in both instanton cases, this is the anti-self-duality equation for the connection $\Xi = 0 ds + B$ in temporal gauge; c.f. Section 2.

To ensure that the trajectories converge to critical points as the \mathbb{R} -variable tends to $\pm\infty$, one needs some a priori bound. This is provided by energy functionals given in the lemma below (a consequence of Theorems 3.4 and 3.5).

Lemma 3.3. *If a trajectory has finite energy \mathcal{E} , then it converges (exponentially) to critical points as $\mathbb{R} \ni s \rightarrow \pm\infty$.*

(inst):

$$\begin{aligned} \mathcal{E}(B) &= \int_{\mathbb{R} \times Y} |\partial_s B|^2 < \infty \\ \implies B(s) &\xrightarrow{s \rightarrow \pm\infty} B^\pm \in \mathcal{A}_{\text{flat}}(Y) \end{aligned}$$

(in-L):

$$\begin{aligned} \mathcal{E}(A, \Psi) &= \int_{\mathbb{R} \times [0,1] \times \Sigma} |\partial_s A_i|^2 + |F_{A_i}|^2 < \infty \\ \implies A(s) + \Psi(s) dt &\xrightarrow{s \rightarrow \pm\infty} A^\pm + \Psi^\pm dt \in \mathcal{A}_{\text{flat}}([0,1] \times Y); \\ &A^\pm(0) \in \mathcal{L}_{H_0}, \quad A^\pm(1) \in \mathcal{L}_{H_1} \end{aligned}$$

(symp):

$$\begin{aligned} \mathcal{E}(u) &= \int_{\mathbb{R} \times [0,1]} |\partial_s u|^2 < \infty \\ \implies u(s, \cdot) &\xrightarrow{s \rightarrow \pm\infty} x^\pm \in L_0 \cap L_1 \end{aligned}$$

In the two instanton cases, the energy of a trajectory equals to the Yang–Mills energy $\frac{1}{2} \int |F_\Xi|^2$ of the corresponding anti-self-dual connection. In all cases, the energy is conformally invariant, so by rescaling one solution one can obtain a sequence of solutions (on a ball) whose energy is bounded, but that blows up at one point – where all the energy concentrates. This effect can be excluded by assuming that the energy density does not blow up. For all three equations, this is enough to obtain $\mathcal{C}_{\text{loc}}^\infty$ -compactness.

Theorem 3.4 (Compactness). *Consider a sequence of trajectories and suppose that their energy density is locally uniformly bounded:*

(inst): $|\partial_s B_i|^2$ is locally uniformly bounded on $\mathbb{R} \times Y$.

(in-L): $\|\partial_s A_i\|_{L^2(\Sigma)}^2 + \|F_{A_i}\|_{L^2(\Sigma)}^2$ is locally uniformly bounded on $\mathbb{R} \times [0, 1]$.

(symp): $|\partial_s u_i|^2$ is locally uniformly bounded on $\mathbb{R} \times [0, 1]$.

Then, after going to a subsequence, and in the cases (inst), (in-L) applying a sequence of gauge transformations $g_i \in \mathcal{G}(\mathbb{R} \times Y)$ or $g_i \in \mathcal{G}(\mathbb{R} \times [0, 1] \times \Sigma)$, the trajectories converge uniformly with all derivatives on every compact subset (i.e., in the $\mathcal{C}_{\text{loc}}^\infty$ -topology) to a new trajectory.

The compactness statement in case (in-L) in fact also holds when the Lagrangians \mathcal{L}_{H_i} are replaced with general gauge invariant Lagrangians as on page 712. This result was proven in [21] under the (stronger) standard assumption from gauge theory that $|\partial_s B_i|^2 = |\partial_s A_i|^2 + |F_{A_i}|^2$ is locally uniformly bounded on $\mathbb{R} \times [0, 1] \times \Sigma$ (or is locally L^p -bounded for $p > 2$). The weaker assumption above implies pointwise bounds in the interior by a

mean value inequality. Near the boundary, this is not a direct consequence, but an extra argument [22, Lemma 2.4] provides local L^p -bounds for any $p < 3$. Thus we can state the compactness result in this form, which already hints at a similar behaviour to pseudoholomorphic curves on $\mathbb{R} \times [0, 1]$. This stronger statement becomes crucial in the bubbling analysis below.

The goal of our analytic discussion of the trajectory equation is to understand the compactness or compactification of the k -dimensional part $\mathcal{M}^k(x^-, x^+)$ of the space of trajectories with fixed limits x^\pm . (Here $k = 0$ and $k = 1$ are relevant for the definition of ∂ and for the proof of $\partial \circ \partial = 0$.)

The assumptions in Theorem 3.4 are too strong for that purpose since we only have a bound on the energy, not on the energy density, of trajectories in $\mathcal{M}^k(x^-, x^+)$. In fact, in the three present cases, the energy of a trajectory is uniquely determined by its limits x^-, x^+ and its index k via a monotonicity formula. So we need to consider a sequence of trajectories with fixed energy and analyse the possible divergence of the sequence when the uniform bounds in Theorem 3.4 do not hold. This divergence is usually described by the “bubbling off” of some part of the trajectory: in the case (inst) the “bubbles” are instantons on S^4 ; in the case (symp) they are pseudoholomorphic spheres or disks. In the new case (in-L), we also encounter instantons on S^4 “bubbling off” at both interior and boundary points. Additional “bubbles” in the form of anti-self-dual instantons on the half space were expected in [14]. Our result below now seems to indicate a semiglobal bubbling effect at the boundary, which conjecturally might be described as a holomorphic disk in the space of connections $\mathcal{A}(\Sigma)$. Fortunately, the geometric understanding of the bubbles is not necessary for the purpose of Floer theory in the monotone case. It can be replaced by an analytic understanding of the bubbling in the form of the following energy quantization result.

For the purpose of this statement, we abbreviate $Y = [0, 1] \times \Sigma$ in case (in-L) and $Y = [0, 1]$ in case (symp), so all trajectories are defined on $\mathbb{R} \times Y$.

Theorem 3.5 (Energy Quantization). *There exists a constant $\hbar > 0$ such that the following holds. Consider a sequence of trajectories whose energy is bounded by some $E < \infty$.*

Then, after going to a subsequence, the energy densities are locally uniformly bounded as in Theorem 3.4 on $(\mathbb{R} \times Y) \setminus \bigcup_{k=1}^N P_k$, the complement of a finite union of bubbling loci P_k as below. At each bubbling locus P_k , there is a concentration of energy of at least \hbar on neighbourhoods with radii $\varepsilon_i \rightarrow 0$.

(inst): *Each bubbling locus is a point $P_k = x_k \in \mathbb{R} \times Y$ with*

$$\int_{B_{\varepsilon_i}(x_k)} |\partial_s B_i|^2 \geq \hbar.$$

(in-L): Each bubbling locus is either an interior point $P_k = x_k \in \mathbb{R} \times (0, 1) \times \Sigma$ with

$$\int_{B_{\varepsilon_i}(x_k)} |\partial_s A_i|^2 + |F_{A_i}|^2 \geq \hbar,$$

or a boundary slice $P_k = \{(s_k, t_k)\} \times \Sigma$, $(s_k, t_k) \in \mathbb{R} \times \{0, 1\}$ with

$$\int_{B_{\varepsilon_i}(s_k, t_k)} \|\partial_s A_i\|_{L^2(\Sigma)}^2 + |F_{A_i}|_{L^2(\Sigma)}^2 \geq \hbar.$$

(symp): Each bubbling locus is a point $P_k = (s_k, t_k) \in \mathbb{R} \times [0, 1]$ with

$$\int_{B_{\varepsilon_i}(s_k, t_k)} |\partial_s u_i|^2 \geq \hbar.$$

In case (in-L), both an instanton on S^4 bubbling off at a boundary point and the conjectural holomorphic disk in $\mathcal{A}(\Sigma)$ are described by a boundary slice as bubbling locus. The proof in case (in-L) goes along the lines of an energy quantization principle explained in [23] but deals with some additional difficulties. In the cases (inst) and (symp), the above result can be obtained straight forward from this principle and a control on the Laplacian (and normal derivative) of the energy density. See Section 6 for details.

The combination of Theorems 3.4 and 3.5 can be rephrased as: “There is a $\mathcal{C}_{\text{loc}}^\infty$ -convergent subsequence if the energy is locally small.” In the cases (inst) and (symp), it is sufficient to assume that every point in $\mathbb{R} \times Y$ or $\mathbb{R} \times [0, 1]$ respectively has a neighbourhood on which the energy of each trajectory in the sequence is less than \hbar . In the case (in-L), this assumption is the same for points in the interior $\mathbb{R} \times (0, 1) \times \Sigma$. For a point $(s, j, z) \in \mathbb{R} \times \{0, 1\} \times \Sigma$ on the boundary, however, it is not enough to assume that the energies are small on a neighbourhood of that point, but one needs to assume that there is a neighbourhood of the whole boundary slice $\{(s, j)\} \times \Sigma$ on which the energy of each trajectory in the sequence is less than \hbar .

The full consequence of Theorems 3.5 and 3.4 is the following compactness.

Corollary 3.6. *Consider a sequence of trajectories with energy bounded by $E < \infty$. Then, after going to a subsequence, there exist finitely many bubbling loci P_1, \dots, P_N as in Theorem 3.5, and in the cases (inst) and (in-L), there exists a sequence of gauge transformations in $\mathcal{G}((\mathbb{R} \times Y) \setminus \bigcup_{i=1}^k P_k)$, such that the trajectories converge (after gauge transformation) in the $\mathcal{C}_{\text{loc}}^\infty$ -topology on $(\mathbb{R} \times Y) \setminus \bigcup_{i=1}^k P_k$ to a new solution of the trajectory equation (T) on $(\mathbb{R} \times Y) \setminus \bigcup_{i=1}^k P_k$ with energy $\mathcal{E} \leq E - N\hbar$.*

Keep in mind that the bubbling loci P_k and thus the singularities of the new solution obtained in Corollary 3.6 are always points, except for the case (in-L) where 2-dimensional singularities can occur at the boundary. The

next step in the compactification (or proof of compactness) of the spaces of trajectories is to remove these singularities. We give a general statement that is a consequence of the subsequent removable singularity theorems for the local models of the singularities.

Here B^n denotes the unit ball in \mathbb{R}^n centred at 0, and $D^2 := B^2 \cap \mathbb{H}^2$ is the unit half ball in the half space $\mathbb{H}^2 = \{(s, t) \in \mathbb{R}^2 \mid t \geq 0\}$ with centre 0. In the two boundary cases, the Lagrangian submanifold \mathcal{L}_H or L can be either of the two \mathcal{L}_{H_i} or L_i , respectively.

Theorem 3.7 (Removal of Singularities). *Consider a smooth solution of the trajectory equation (T) on $(\mathbb{R} \times Y) \setminus \bigcup_{k=1}^N P_k$ that has finite energy E . Then (in case (inst) and (in-L) after applying a gauge transformation in $\mathcal{G}((\mathbb{R} \times Y) \setminus \bigcup_{k=1}^N P_k)$), the solution extends to a trajectory on $\mathbb{R} \times Y$ with energy E .*

(inst), (in-L, interior): *Suppose that $\Xi \in \mathcal{A}(B^4 \setminus \{0\})$ satisfies*

$$F_\Xi + *F_\Xi = 0 \quad \text{and} \quad \int_{B^4 \setminus \{0\}} |F_\Xi|^2 < \infty.$$

Then there exists a gauge transformation $g \in \mathcal{G}(B^4 \setminus \{0\})$ such that g^Ξ extends to a solution $\tilde{\Xi} \in \mathcal{A}(B^4)$.*

(in-L, boundary): *Suppose that $\Xi \in \mathcal{A}((D^2 \setminus \{0\}) \times \Sigma)$ satisfies*

$$\begin{cases} F_\Xi + *F_\Xi = 0 \\ \Xi|_{\{(s,0)\} \times \Sigma} \in \mathcal{L}_H \quad \forall s \end{cases} \quad \text{and} \quad \int_{D^2 \setminus \{0\}} \int_\Sigma |F_\Xi|^2 < \infty.$$

Then there exists a gauge transformation $g \in \mathcal{G}((D^2 \setminus \{0\}) \times \Sigma)$ such that g^Ξ extends to a solution $\tilde{\Xi} \in \mathcal{A}(D^2 \times \Sigma)$.*

(symp, boundary): *Suppose that $u \in C^\infty(D^2 \setminus \{0\}, M)$ satisfies*

$$\begin{cases} \partial_s u + J\partial_t u = 0 \\ u(s, 0) \in L \quad \forall s \end{cases} \quad \text{and} \quad \int_{D^2 \setminus \{0\}} |\partial_s u|^2 < \infty.$$

Then u extends to a solution $\tilde{u} \in C^\infty(D^2, M)$.

(symp, interior): *Suppose that $u \in C^\infty(B^2 \setminus \{0\}, M)$ satisfies*

$$\partial_s u + J\partial_t u = 0 \quad \text{and} \quad \int_{B^2 \setminus \{0\}} |\partial_s u|^2 < \infty.$$

Then u extends to a solution $\tilde{u} \in C^\infty(B^2, M)$.

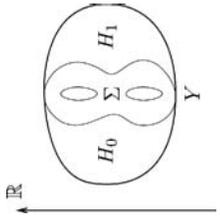
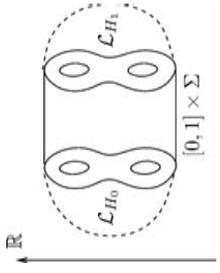
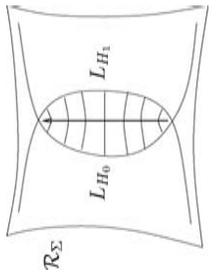
In the case (in-L), Uhlenbeck’s removable singularity theorem [17] applies to the bubbling loci in the interior. At the boundary, we have to remove 2-dimensional singularities of an anti-self-dual instanton. In the interior, there would be an obstruction to removing such singularities: the holonomies

of small loops around the singularity might have a nontrivial limit. So it is important to note that this “pseudoholomorphic behaviour” of the (in-L) trajectories only occurs at the boundary, where one does not have an obstruction since there are no loops around the singularity. One can then imitate the removal of the singularity of a pseudoholomorphic curve on $D^2 \setminus \{0\}$ with Lagrangian boundary conditions to remove the singularity of an anti-self-dual instanton on $(D^2 \times \{0\}) \times \Sigma$. This uses an isoperimetric inequality for a local Chern–Simons functional instead of the local symplectic action. So far, the definition of this local Chern–Simons functional crucially uses the fact that the Lagrangian boundary condition arises from a handle body.

The final result of the analysis of trajectories in Theorems 3.4, 3.5, and 3.7 is that the moduli spaces of trajectories are compact up to “bubbling” and “breaking of trajectories.” Here “bubbling” means the concentration of energy at a bubbling locus as in Theorem 3.5. The “breaking of trajectories” occurs when a sequence of trajectories with constant energy converges smoothly on every compact set to a new trajectory, but the limit has less energy. In that case, the energy difference must have moved out to $s \rightarrow \pm\infty$ and can be recaptured as the energy of a limit of shifted trajectories. A standard iteration of such shifts yields a finite collection of trajectories (a “broken trajectory”) whose total energy equals to the fixed energy of the sequence.

To proceed with the definition of ∂ and the proof of $\partial \circ \partial = 0$, one needs to perturb the trajectory equation (T) so that the moduli spaces $\mathcal{M}^k(x^-, x^+)$ of trajectories become smooth manifolds. Here a priori $k \in \mathbb{Z}$ is the index of a Fredholm operator (the linearization of (T)) associated to the trajectories. For a smooth moduli space, k equals to the dimension of the component, hence $\mathcal{M}^k(x^-, x^+)$ is empty for $k \leq -1$. By a monotonicity formula, k moreover determines the energy of the trajectories such that a trajectory of lower energy has to lie in a moduli space of lower dimension. From this one can deduce that $\mathcal{M}^0(x^-, x^+)$ is compact (and thus can be counted to define ∂): it consists of trajectories with the minimal energy that allows to connect x^- to x^+ . So bubbling can be ruled out since (after removal of the singularities) it would lead to a trajectory of even lower energy. The breaking of trajectories is ruled out by a similar index-energy argument.

Bubbling is also excluded in $\mathcal{M}^k(x^-, x^+)$ for $k \leq 7$ (or $2N - 1$ in the symplectic case) since x^- and x^+ determine the index k modulo 8 (or $2N$). So a loss of energy corresponds to a jump by 8 (or $2N$) in the dimension. The breaking of trajectories is no longer ruled out; on the contrary, $\partial \circ \partial = 0$ follows from the fact that the ends of the 1-dimensional moduli spaces exactly correspond to the broken trajectories, which are counted by $\partial \circ \partial$.

3-manifold topology	Gauge theory	Symplectic topology
Σ Riemann surface $H, \partial H = \Sigma$ handle body $Y = H_0 \cup_{\Sigma} H_1$ Heegard splitting $\text{HF}_*^{\text{inst}}(H_0 \cup_{\Sigma} H_1)$	$\mathcal{A}(\Sigma)$ symplectic Banach space $\mathcal{L}_H = \mathcal{A}_{\text{flat}}(H) _{\Sigma}$ Lagrangian Banach submanifold $\mathcal{L}_{H_0}, \mathcal{L}_{H_1} \subset \mathcal{A}(\Sigma)$ $\text{HF}_*^{\text{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$	$\mathcal{R}_{\Sigma} = \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma) = \mathcal{A}(\Sigma)//\mathcal{G}(\Sigma)$ $\cong \text{Hom}(\pi_1(\Sigma), \text{SU}(2))/\text{SU}(2)$ (singular) symplectic manifold $L_H = \mathcal{A}_{\text{flat}}(H) _{\Sigma}/\mathcal{G}(\Sigma)$ $\cong \text{Hom}(\pi_1(\Sigma)/\partial\pi_2(H, \Sigma), \text{SU}(2))/\text{SU}(2)$ (singular) Lagrangian submanifold $L_{H_0}, L_{H_1} \subset \mathcal{R}_{\Sigma}$ $\text{HF}_*^{\text{symp}}(\mathcal{R}_{\Sigma}, L_{H_0}, L_{H_1})$
		

	$\mathrm{HF}_*^{\mathrm{inst}}(Y)$	$\mathrm{HF}_*^{\mathrm{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$	$\mathrm{HF}_*^{\mathrm{symp}}(M, L_0, L_1)$
“Critical points”	$B \in \mathcal{A}_{\mathrm{flat}}(Y)$	$A + \Psi dt \in \mathcal{A}_{\mathrm{flat}}([0, 1] \times \Sigma),$ $A(j) _{\partial Y} \in \mathcal{L}_{H_j}$	$x \in L_0 \cap L_1$
Trajectories	$B : \mathbb{R} \rightarrow \mathcal{A}(Y)$ $\begin{cases} \partial_s B + *F_B = 0 \end{cases}$	$(A, \Psi) : \mathbb{R} \times [0, 1] \rightarrow \mathcal{A}(\Sigma) \times \mathcal{C}^\infty(\Sigma, \mathfrak{su}(2))$ $\begin{cases} \partial_s A + *(\partial_t A - d_A \Psi) = 0 \\ \partial_s \Psi + *F_A = 0 \end{cases}$ $A(s, j) \in \mathcal{L}_{H_j}$	$u : \mathbb{R} \times [0, 1] \rightarrow M$ $\begin{cases} \partial_s u + J \partial_t u = 0 \\ u(s, j) \in L_j \end{cases}$
Energy	$\int_{\mathbb{R} \times Y} \partial_s B ^2 \leq C$	$\int_{\mathbb{R} \times [0, 1] \times \Sigma} \partial_s A ^2 + F_A ^2 \leq C$	$\int_{\mathbb{R} \times [0, 1]} \partial_s u ^2 \leq C$
Uniform bounds for compactness	$\sup_{\mathbb{R} \times Y} \partial_s B ^2 < \infty$	$\sup_{\mathbb{R} \times [0, 1]} \ \partial_s A\ _{L^2(\Sigma)}^2 + \ F_A\ _{L^2(\Sigma)}^2 < \infty$	$\sup_{\mathbb{R} \times [0, 1]} \partial_s u ^2 < \infty$
Bubbling loci	Points $x \in \mathbb{R} \times Y$	Interior points $x \in \mathbb{R} \times (0, 1) \times \Sigma,$ boundary slices $\{(s, j)\} \times \Sigma$	Interior points $(s, t) \in \mathbb{R} \times (0, 1),$ boundary points $(s, j) \in \mathbb{R} \times \{0, 1\}$
Removable singularities	$B^4 \setminus \{0\}$	$B^4 \setminus \{0\}$ $(D^2 \times \Sigma) \setminus (\{0\} \times \Sigma)$	$B^2 \setminus \{0\}$ $D^2 \setminus \{0\}$

4. The Atiyah–Floer conjecture

To give a precise statement of the Atiyah–Floer conjecture, we need to refine the notion of handle bodies and Heegard splittings. A **handle body** is an oriented 3-manifold with boundary that is obtained by attaching finitely many 1-handles to a 3-ball. The **spine** of a handle body H is a graph $S \subset H$ embedded in its interior that arises from replacing the ball by a vertex and the handles by edges with ends on this vertex. Its significance is that $H \setminus S \cong [0, 1] \times \partial H$, so H retracts onto S . For each genus $g \in \mathbb{N}_0$, we fix a standard handle body and spine $S \subset H$.

Definition 4.1. A Heegard splitting of a closed oriented 3-manifold Y consists of two embeddings $\psi_i : H \hookrightarrow Y$ of a standard handle body H such that $\psi_0(\partial H) = \psi_1(\partial H) = \text{im } \psi_0 \cap \text{im } \psi_1$. We abbreviate the Heegard splitting by $Y = H_0 \cup_\Sigma H_1$, where $H_i := \psi_i(H) \subset Y$ and $\Sigma := \psi_i(\partial H) = H_0 \cap H_1$.

Next, a homology 3-sphere is a compact oriented 3-manifold Y whose integer homology is that of a 3-sphere, $H_*(Y, \mathbb{Z}) \cong H_*(S^3, \mathbb{Z})$.

Conjecture 4.2 (Atiyah–Floer). *Let Y be a homology 3-sphere. Then every Heegard splitting $Y = H_0 \cup_\Sigma H_1$ induces a natural isomorphism*

$$\text{HF}_*^{\text{inst}}(Y) \cong \text{HF}_*^{\text{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1}).$$

Here “natural” in particular means that the isomorphism should be invariant under isotopies of the Heegard splitting. Note that for nonisotopic Heegard splittings of the same genus, one can identify the \mathcal{R}_Σ , but the pairs of Lagrangians (and thus the conjectured isomorphism) will be different. The conjecture would then provide isomorphisms between the symplectic Floer homologies arising from different Heegard diagrams of the same 3-manifold.

The first task posed by this conjecture is to give a precise definition of the symplectic Floer homology for the Lagrangians L_{H_0}, L_{H_1} in the singular symplectic space \mathcal{R}_Σ . They can be viewed as symplectic quotients of the gauge action on the smooth Banach-manifolds $\mathcal{L}_{H_0}, \mathcal{L}_{H_1} \subset \mathcal{A}(\Sigma)$ [2 Section 2]. For finite dimensional Hamiltonian group actions, Salamon and coworkers [3] introduced invariants based on the symplectic vortex equations on the total space. Gaio and Salamon [11] identified these with the Gromov–Witten invariants for smooth and monotone symplectic quotients. In view of this result, a plausible definition of $\text{HF}_*^{\text{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1})$ could be to replace its ill-defined trajectories (pseudoholomorphic curves in the singular symplectic quotient) by solutions of the corresponding symplectic vortex equations: a triple of maps $A : \mathbb{R} \times [0, 1] \rightarrow \mathcal{A}(\Sigma)$ and

$\Phi, \Psi : \mathbb{R} \times [0, 1] \rightarrow \mathcal{C}^\infty(\Sigma, \mathfrak{su}(2)) \cong T_{\mathbb{1}}\mathcal{G}(\Sigma)$ that satisfy

$$(9) \quad \begin{cases} (\partial_s A - d_A \Phi) + *(\partial_t A - d_A \Psi) = 0, \\ \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + *F_A = 0, \\ A(s, i) \in \mathcal{L}_{H_i} \quad \forall s \in \mathbb{R}, \quad i \in \{0, 1\}. \end{cases}$$

Here $\Phi \mapsto d_A \Phi$ is the infinitesimal action and $A \mapsto *F_A$ is the moment map of the gauge action, where $*$ is the Hodge operator of a metric g_Σ on Σ . This system is the anti-self-duality equation with Lagrangian boundary conditions for the connection $\Phi ds + \Psi dt + A$ on $\mathbb{R} \times [0, 1] \times \Sigma$ with respect to the metric $ds^2 + dt^2 + g_\Sigma$, i.e., the trajectory equation of Definition 3.2 in temporal gauge $\Phi = 0$. So in this case the symplectic vortex equations lead directly to the new Floer homology $\text{HF}_*^{\text{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$, which is well defined since $([0, 1] \times \Sigma) \cup (H_0 \sqcup H_1) \cong H_0 \cup_\Sigma H_1$ is a homology 3-sphere. Defining $\text{HF}_*^{\text{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1})$ via (9) would reduce the Atiyah–Floer Conjecture 4.2 to the subsequent special case of Conjecture 2.5. We intend however to give a less far fetched definition of the symplectic Floer homology and use the following only as first step towards a proof of the Atiyah–Floer conjecture.

Conjecture 4.3. *Every Heegard splitting $Y = H_0 \cup_\Sigma H_1$ of a homology 3-sphere induces a natural isomorphism*

$$\text{HF}_*^{\text{inst}}(Y) \cong \text{HF}_*^{\text{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1}).$$

To prove this, one has to identify the critical points and trajectories of both Floer homologies. Our idea for a proof uses the following decomposition of Y . We restrict the embeddings ψ_i to the complement of the spine $H \setminus S \cong [\frac{1}{2}, 1) \times \Sigma$ and glue them at $\{\frac{1}{2}\} \times \Sigma$ to obtain an embedding $\psi : (0, 1) \times \Sigma \hookrightarrow Y$ such that $\psi(\frac{1}{2}, \cdot) = \text{id}_\Sigma$ and $\psi(t, \Sigma)$ converges to the spine $\psi_i(S) \subset H_i$ as $t \rightarrow i$ for $i = 0, 1$. Then

$$Y = H_0^\delta \sqcup Y_\delta \sqcup H_1^\delta; \quad Y_\delta := \psi([\delta, 1 - \delta] \times \Sigma).$$

Here the $H_i^\delta \subset Y$ are isotopic to the open handle bodies $\text{int}(H_i)$ and $Y_\delta \cong [0, 1] \times \Sigma$ via $\psi \circ \tau_\delta$, where $\tau_\delta : [0, 1] \times \Sigma \rightarrow [\delta, 1 - \delta] \times \Sigma$ is the obvious linear isomorphism. With this, the critical points can be identified elementary as follows: every $\tilde{A} \in \mathcal{A}_{\text{flat}}(Y)$ can be decomposed and pulled back to a triple $(A, \tilde{A}_0, \tilde{A}_1)$ of $A \in \mathcal{A}_{\text{flat}}([0, 1] \times \Sigma)$ and $\tilde{A}_i \in \mathcal{A}_{\text{flat}}(H_i)$ such that $A|_{\{i\} \times \Sigma} = \tilde{A}_i|_{\partial H_i}$. So every critical point $[\tilde{A}] \in \mathcal{R}_Y$ corresponds to the gauge equivalence class of a flat connection on $[0, 1] \times \Sigma$ with boundary values in \mathcal{L}_{H_0} and \mathcal{L}_{H_1} . One can check that this in fact gives a bijection between the critical points. In order to prove Conjecture 4.3, one needs to show that the induced map between the Floer complexes is a chain isomorphism.

For that purpose, we fix a metric on Y and for a corresponding metric on $[0, 1] \times \Sigma$ try to establish a bijection between the trajectories that

contribute to the differential on the two Floer complexes. (Of course, we have to prove later that the isomorphism is independent of the choices.) A fixed metric on Y gives rise to a family of metrics g_δ on $[0, 1] \times \Sigma$ via pullback by $\psi \circ \tau_d : [0, 1] \times \Sigma \rightarrow Y_\delta \subset Y$. The metrics g_δ degenerate on $\{0\} \times \Sigma$ and $\{1\} \times \Sigma$ for $\delta \rightarrow 0$, but for sufficiently small $\delta > 0$, we expect to find a bijection between the trajectories of $\text{HF}_*^{\text{inst}}(Y)$ and those of $\text{HF}_*^{\text{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$ with respect to g_δ .

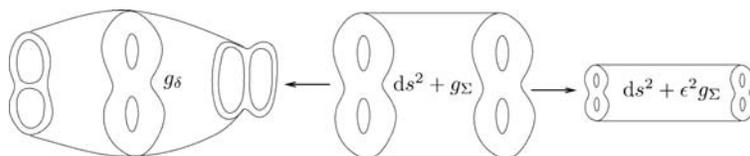
The first are anti-self-dual instantons (in temporal gauge) on $\mathbb{R} \times Y$, that is, $B : \mathbb{R} \rightarrow \mathcal{A}(Y)$ satisfying

$$\partial_s B + *F_B = 0 \quad \text{on } \mathbb{R} \times Y.$$

The latter are anti-self-dual instantons (in temporal gauge) $A + \Psi dt$ on $\mathbb{R} \times [0, 1] \times \Sigma$ with Lagrangian boundary conditions. Here the metric g_δ on $[0, 1] \times \Sigma$ is not of product form, so the equation (T) in Definition 3.2 has to be adjusted: the pair (A, Ψ) is a trajectory if $A + \Psi dt = \tau_\delta^* \psi^* B$, where $B : \mathbb{R} \rightarrow \mathcal{A}(Y_\delta)$ is anti-self-dual with respect to the fixed metric on Y and has boundary values in \mathcal{L}_{H_0} and \mathcal{L}_{H_1} , that is,

$$\begin{cases} \partial_s B + *F_B = 0 & \text{on } \mathbb{R} \times Y_\delta, \\ B|_{\psi(\{\delta\} \times \Sigma)} = \tilde{B}_0 & \text{for some } \tilde{B}_0 : \mathbb{R} \rightarrow \mathcal{A}_{\text{flat}}(H_0^\delta), \\ B|_{\psi(\{1-\delta\} \times \Sigma)} = \tilde{B}_1 & \text{for some } \tilde{B}_1 : \mathbb{R} \rightarrow \mathcal{A}_{\text{flat}}(H_1^\delta). \end{cases}$$

The task in identifying the trajectories is to consider anti-self-dual instantons on Y_δ and transfer between extensions $\tilde{B}_i : \mathbb{R} \rightarrow \mathcal{A}(H_i^\delta)$ that are slicewise flat ($F_{\tilde{B}_i} = 0$) and extensions that are anti-self-dual ($\partial_s \tilde{B}_i + *F_{\tilde{B}_i} = 0$). Here the handle bodies $H_i^\delta \subset Y$ are small tubes around their spines $\psi_i(S) \subset Y$. The restriction of given (anti-self-dual) connections \tilde{B}_i to the spines is up to gauge equivalence determined by their holonomies, i.e., $SU(2)$ -representations of $\pi_1(H_i)$. One can then pick flat connections on the H_i^δ that have the same holonomy and are close to the \tilde{B}_i (compared to their energy). For the converse, we will have to use special flat extensions \tilde{B}_i with a control on $\partial_s \tilde{B}_i$ as in lemma 5.3. Combined with the small volume of H_i^δ , this should make \tilde{B}_i close to anti-self-dual.



Degenerations of the metric on $[0, 1] \times \Sigma$.

The key to this plan of proof is the fact that one can degenerate the metric on $[0, 1] \times \Sigma$ (as sketched on the left in the above figure) without changing the invariant $\mathrm{HF}_*^{\mathrm{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1})$. In the limit of the degeneration, one should obtain the invariant $\mathrm{HF}_*^{\mathrm{inst}}(Y)$ for the closed manifold. The basic idea of the second step for the Atiyah–Floer conjecture is to use a second degeneration (on the right in the above sketch) to transfer from anti-self-dual instantons to pseudoholomorphic curves. This idea was successfully employed by Dostoglou and Salamon [6] in their proof of a mapping torus analogon of the Atiyah–Floer conjecture.

A trajectory of the symplectic Floer homology should be a pseudoholomorphic map $u : \mathbb{R} \times [0, 1] \rightarrow \mathcal{R}_\Sigma$ with boundary values in L_{H_0} and L_{H_1} ,

$$(10) \quad \partial_s u + J(u)\partial_t u = 0, \quad u(s, i) \in L_{H_i} \quad \forall s \in \mathbb{R}, \quad i = 0, 1.$$

Here we choose the almost complex structure J on \mathcal{R}_Σ that is induced by the Hodge operator of some fixed metric g_Σ on Σ . Let us first assume that u takes values in the irreducible representations, so the pseudoholomorphic equation for u actually makes sense since \mathcal{R}_Σ is smooth near its image. If we consider a lift $A : \mathbb{R} \times [0, 1] \rightarrow \mathcal{A}(\Sigma)$ of u , then this means that every $A(s, t)$ has stabilizer $\{\pm \mathbb{1}\} \subset \mathcal{G}(\Sigma)$, or equivalently $d_{A(s,t)}$ is injective on $\Omega^0(\Sigma; \mathfrak{su}(2))$. This lift is not unique, but it always takes values in $\mathcal{A}_{\mathrm{flat}}(\Sigma)$. So for every $A = A(s, t)$, one has the Hodge decomposition (4)

$$\Omega^1(\Sigma; \mathfrak{su}(2)) = d_A \Omega^0(\Sigma; \mathfrak{su}(2)) \oplus *d_A \Omega^0(\Sigma; \mathfrak{su}(2)) \oplus h_A^1.$$

Here $h_A^1 = \ker d_A \cap \ker d_A^* \cong T_{[A]} \mathcal{R}_\Sigma$ and $d_A \Omega^0(\Sigma; \mathfrak{su}(2))$ is the tangent space of the $\mathcal{G}(\Sigma)$ -orbit through A . So one can express $\partial_s u + J(u)\partial_t u = 0$ in terms of the lift: the projection of $\partial_s A + *\partial_t A$ onto $h_A^1 \cong T_{[A]} \mathcal{R}_\Sigma$ vanishes; i.e., $\partial_s A + *\partial_t A = d_A \Phi + *d_A \Psi$ for some $\Phi, \Psi : \mathbb{R} \times [0, 1] \rightarrow \Omega^0(\Sigma; \mathfrak{su}(2))$. More precisely, (10) for u mapping to the irreducible representations is equivalent to the existence of a lift $A : \mathbb{R} \times [0, 1] \rightarrow \mathcal{A}(\Sigma)$, $u(s, t) = [A(s, t)]$, and some $\Phi, \Psi : \mathbb{R} \times [0, 1] \rightarrow \Omega^0(\Sigma; \mathfrak{su}(2))$ such that

$$(11) \quad \begin{cases} \partial_s A - d_A \Phi + *(\partial_t A - d_A \Psi) = 0, \\ *F_A = 0, \\ A(s, i) \in \mathcal{L}_{H_i} \quad \forall s \in \mathbb{R}, \quad i = 0, 1. \end{cases}$$

One can also consider this as a boundary value problem for the connection $\Phi ds + \Psi dt + A$ on $\mathbb{R} \times [0, 1] \times \Sigma$. Just note that A determines Φ and Ψ uniquely since $\Delta_A \Phi = d_A \partial_s A$, $\Delta_A \Psi = d_A \partial_t A$, and $\Delta_A = d_A^* d_A$ are invertible for irreducible $A = A(s, t)$. If A is allowed to become reducible, then Φ and Ψ have some extra freedom. If for example $A \equiv 0$, then any two functions $\Phi, \Psi : \mathbb{R} \times [0, 1] \rightarrow \mathfrak{su}(2)$ would provide a solution of (11). Quotienting

out by the gauge action, this moduli space is still infinite dimensional. We expect however that one can use perturbations of (11) to obtain finite dimensional smooth moduli spaces of trajectories in the cases that are relevant for $\text{HF}_*^{\text{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1})$, i.e., when at least one critical point is irreducible. Once this symplectic Floer homology is defined via (11), one should be able to adapt the adiabatic limit in [6] to this boundary value problem and establish the following second step towards the Atiyah–Floer conjecture.

Conjecture 4.4. *If $Y = H_0 \cup_\Sigma H_1$ is a Heegard splitting of a homology 3-sphere, then there is a natural isomorphism*

$$\text{HF}_*^{\text{inst}}([0, 1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1}) \cong \text{HF}_*^{\text{symp}}(\mathcal{R}_\Sigma, \mathcal{L}_{H_0}, \mathcal{L}_{H_1}).$$

Again, the critical points of both Floer theories are naturally identified. In the instanton Floer homology, the critical points are flat connections on $A + \Psi dt$ on $[0, 1] \times \Sigma$ (where flatness means $F_A = 0$ and $\dot{A} - d_A \Psi = 0$) with boundary values $A(0) \in \mathcal{L}_{H_0}$ and $A(1) \in \mathcal{L}_{H_1}$. One can always make Ψ vanish by a gauge transformation, then A becomes t -independent, so $A(0) = A(1) \in \mathcal{L}_{H_0} \cap \mathcal{L}_{H_1}$. Thus the gauge equivalence classes of these critical points can be identified with intersection points of the Lagrangian submanifolds L_{H_0} and L_{H_1} in the moduli space \mathcal{R}_Σ —which are exactly the critical points of the symplectic Floer homology.

In order to identify the moduli spaces of trajectories, we can choose an appropriate metric on $[0, 1] \times \Sigma$ in the definition of the instanton Floer homology. Let us fix the metric g_Σ on Σ as in (11) and consider the family of metrics $dt^2 + \varepsilon^2 g_\Sigma$ for $\varepsilon > 0$. With respect to these metrics, the trajectory equation (9) of the instanton Floer homology becomes

$$(12) \quad \begin{cases} \partial_s A - d_A \Phi + *(\partial_t A - *d_A \Psi) = 0, \\ \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \varepsilon^{-2} * F_A = 0, \\ A(s, i) \in \mathcal{L}_{H_i} \quad \forall s \in \mathbb{R}, \quad i \in \{0, 1\}, \end{cases}$$

for the triple of $A : \mathbb{R} \times [0, 1] \rightarrow \mathcal{A}(\Sigma)$ and $\Phi, \Psi : \mathbb{R} \times [0, 1] \rightarrow \Omega^0(\Sigma; \mathfrak{su}(2))$. Their energy

$$\mathcal{E}(A, \Phi, \Psi) = \int_{\mathbb{R} \times [0, 1] \times \Sigma} |\partial_s A - d_A \Phi|^2 + \varepsilon^{-2} |F_A|^2$$

is determined, independently of ε , by the index and the limits at $\pm\infty$ (via a monotonicity formula). Analogously to [6], we expect that sequences of such anti-self-dual instantons for $\varepsilon \rightarrow 0$ converge (modulo gauge) to solutions of (11). Now the gauge equivalence classes of these solutions would exactly be the trajectories of the symplectic Floer homology. Conversely, an implicit function argument should show that for sufficiently small $\varepsilon > 0$ near every

solution of (11) one finds a solution of (12). This would give the required bijection between the trajectories of the symplectic and the instanton Floer homology.

Dostoglou and Salamon indeed dealt with the same equations. However, they considered a mapping torus $\mathbb{R} \times \Sigma / \sim$ (with $(t+1, z) \sim (t, f(z))$ for some diffeomorphism f of Σ) instead of our manifold with boundary $[0, 1] \times \Sigma$, so the boundary conditions in (11) and (12) are replaced by a twisting condition. The analytic setup for the definition of the new instanton Floer homology should also allow us to deal with the boundary conditions in this context. There are however additional difficulties due to reducible connections on the trivial $SU(2)$ -bundle over Σ , whereas [6] deals with the nontrivial $SO(3)$ -bundle over Σ that has no reducible connections.

5. Lagrangians in the space of connections

The purpose of this section is to describe some more properties of the Lagrangian submanifolds in the space of connections that were introduced in Section 2. We again consider more generally a trivial G -bundle over a Riemann surface Σ , where G is any compact Lie group with Lie algebra \mathfrak{g} . We fix $p > 2$, then the space of L^p -regular connections $\mathcal{A}^{0,p}(\Sigma)$ is a symplectic Banach space with symplectic form ω given by (3). The gauge group $\mathcal{G}^{1,p}(\Sigma)$ acts smoothly on $\mathcal{A}^{0,p}(\Sigma)$ and preserves ω . Moreover, recall that if we equip Σ with any Riemannian metric, then the corresponding Hodge $*$ operator induces an ω -compatible complex structure on $\mathcal{A}^{0,p}(\Sigma)$.

We have proven in [20, Theorem 3.1] that an L^p -connection is flat in the weak sense iff it is gauge equivalent to a smooth flat connection. So for our purposes, here we simply define the space of flat L^p -connections as $\mathcal{A}_{\text{flat}}^{0,p}(\Sigma) := \mathcal{G}^{1,p}(\Sigma) * \mathcal{A}_{\text{flat}}(\Sigma) \subset \mathcal{A}^{0,p}(\Sigma)$. With this definition, it is clear that the **based holonomy** at any $z \in \Sigma$ is well defined as a map

$$\text{hol}_z : \mathcal{A}_{\text{flat}}^{0,p}(\Sigma) \rightarrow \text{Hom}(\pi_1(\Sigma), G).$$

(Here and in the following, one actually has to fix one point z in each connected component of Σ .) It is invariant under the **based gauge group**

$$\mathcal{G}_z^{1,p}(\Sigma) := \{u \in \mathcal{G}^{1,p}(\Sigma) \mid u(z) = \mathbf{1}\}.$$

Next, we call a Banach submanifold $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ **Lagrangian** if it is isotropic, $\omega|_{\mathcal{L}} \equiv 0$, and coisotropic in the sense of the following implication for all $A \in \mathcal{L}$ and $\alpha \in \mathcal{A}^{0,p}(\Sigma)$: if $\omega(\alpha, \beta) = 0$ for all $\beta \in T_A \mathcal{L}$, then $\alpha \in T_A \mathcal{L}$. The main properties of gauge invariant Lagrangian submanifolds are summarized below. For proofs see [20, Lemma 4.2, 4.3]. (In the case $G = SU(2)$ and for any other connected, simply connected Lie group with discrete centre, the gauge invariance and Lagrangian property imply that \mathcal{L} lies in the flat connections; for general groups, we make this additional assumption.)

Lemma 5.1. *Let $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ be a Lagrangian submanifold. Suppose that $\mathcal{L} \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ and that \mathcal{L} is invariant under the action of $\mathcal{G}^{1,p}(\Sigma)$. Then the following holds:*

- (i) $\mathcal{L} \subset (\mathcal{A}(\Sigma), *)$ is totally real with respect to the Hodge $*$ operator for any metric on Σ . That is, $\Omega^1(\Sigma; \mathfrak{su}(2)) = \mathbb{T}_A \mathcal{L} \oplus * \mathbb{T}_A \mathcal{L}$ for all $A \in \mathcal{L}$.
- (ii) Fix any $z \in \Sigma$. Then \mathcal{L} has the structure of a principal $\mathcal{G}_z^{1,p}(\Sigma)$ -bundle

$$\mathcal{G}_z^{1,p}(\Sigma) \hookrightarrow \mathcal{L} \xrightarrow{\text{hol}_z} M.$$

Here $M \subset \text{Hom}(\pi_1(\Sigma), G)$ is a smooth manifold of dimension $g \cdot \dim G$.

Property (i) is crucial for the elliptic theory for the boundary value problem (8) in the proof of Theorem 3.4. Property (ii) gives rise to Banach submanifold coordinates for the Lagrangian that fit well with the Hodge decomposition of $\Omega^1(\Sigma; \mathfrak{su}(2))$. This also is the crucial point that forces us to work on L^p -spaces with $p > 2$. One does not have a corresponding statement for Lagrangians in $\mathcal{A}^{0,2}(\Sigma)$ unless one can find a generalization of the based gauge group in the $W^{1,2}$ -regular gauge transformations. This would have to be a subgroup that acts freely but has finite codimension.

Next, we consider the Lagrangians given by handle bodies. For that purpose, we suppose that G is connected and simply connected and that $\Sigma = \partial H$ is the boundary of a handle body H . (Both H and Σ might have several connected components, in which case “fixing $z \in \Sigma$ ” below should be replaced by “fixing a point in each component.”)

Let \mathcal{L}_H be the $L^p(\Sigma)$ -closure of the set of smooth flat connections on Σ that can be extended to a flat connection on H ,

$$\mathcal{L}_H := \text{cl} \{ A \in \mathcal{A}_{\text{flat}}(\Sigma) \mid \exists \tilde{A} \in \mathcal{A}_{\text{flat}}(H) : \tilde{A}|_{\Sigma} = A \} \subset \mathcal{A}^{0,p}(\Sigma).$$

Here again the assumption $p > 2$ is crucial for the subsequent properties. In particular, it is not clear whether the L^2 -closure is a smooth submanifold.

Lemma 5.2 [20, Lemma 4.6].

- (i) $\mathcal{L}_H = \{ u^*(A|_{\Sigma}) \mid A \in \mathcal{A}_{\text{flat}}(H), u \in \mathcal{G}^{1,p}(\Sigma) \}$
- (ii) $\mathcal{L}_H \subset \mathcal{A}^{0,p}(\Sigma)$ is a Lagrangian submanifold.
- (iii) $\mathcal{L}_H \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ and \mathcal{L}_H is invariant under the action of $\mathcal{G}^{1,p}(\Sigma)$.
- (iv) Fix any $z \in \Sigma$. Then¹⁰

$$\mathcal{L}_H = \{ A \in \mathcal{A}_{\text{flat}}^{0,p}(\Sigma) \mid \text{hol}_z(A) \in \text{Hom}(\pi_1(H), G) \subset \text{Hom}(\pi_1(\Sigma), G) \},$$

So \mathcal{L}_H obtains the structure of a $\mathcal{G}_z^{1,p}(\Sigma)$ -bundle over the g -fold product $M = G \times \cdots \times G \cong \text{Hom}(\pi_1(H), G)$,

$$\mathcal{G}_z^{1,p}(\Sigma) \hookrightarrow \mathcal{L}_H \xrightarrow{\text{hol}_z} \text{Hom}(\pi_1(H), G).$$

¹⁰Here we identify $\text{Hom}(\pi_1(H), G) \cong \{ \rho \in \text{Hom}(\pi_1(\Sigma), G) \mid \rho(\partial\pi_2(H, \Sigma)) = \{1\} \}$.

Next, although the Lagrangian \mathcal{L}_H does not necessarily have a smooth L^2 -closure, the $L^2(\Sigma)$ -norm on \mathcal{L}_H can be used to control the corresponding flat connections on H in $L^3(H)$. This extension property is the crucial trick that circumvents dealing with the $W^{1,2}$ -topology on the gauge group.

Lemma 5.3. *There exists a constant C_H such that the following holds.*

- (i) *For every smooth path $A : (-\varepsilon, \varepsilon) \rightarrow \mathcal{L}_H \cap \mathcal{A}(\Sigma)$, there exists a path $\tilde{A} : (-\varepsilon, \varepsilon) \rightarrow \mathcal{A}_{\text{flat}}(H)$ with $\tilde{A}(s)|_{\partial H} = A(s)$ such that*

$$\|\partial_s \tilde{A}(0)\|_{L^3(H)} \leq C_H \|\partial_s A(0)\|_{L^2(\Sigma)}.$$

- (ii) *For all $A_0, A_1 \in \mathcal{L}_H \cap \mathcal{A}(\Sigma)$, there exist extensions $\tilde{A}_0, \tilde{A}_1 \in \mathcal{A}_{\text{flat}}(H)$ with $A_i = \tilde{A}_i|_{\partial H}$ such that*

$$(13) \quad \|\tilde{A}_0 - \tilde{A}_1\|_{L^3(H)} \leq C_H \|A_0 - A_1\|_{L^2(\Sigma)}.$$

The proof in [22, Lemma 1.6] uses the coordinates in Lemma 5.2 (iv). Extensions with the correct holonomy can be constructed by hand, and the estimates are immediate on this finite dimensional part. For dealing with the gauge transformations, the crucial fact is that there is a continuous extension operator from $W^{1,2}(\Sigma)$ to $W^{1,3}(H)$. In (i), this fact is used for functions with values in $\mathfrak{su}(2)$, whereas (ii) requires the nonlinear version for maps to $\text{SU}(2)$. The latter is a nontrivial construction of Hardt–Lin in this borderline Sobolev case (the maps are not automatically continuous).

6. Rough guide to the analysis

In this section, we give outlines of the proofs of Theorems 3.4, 3.5, and 3.7 for instantons with Lagrangian boundary conditions.¹¹ The detailed proofs can be found in [21, 22]. They actually hold for more general domains and metrics than considered here, which becomes important when proving the metric independence of the Floer homology and when defining products. We study the boundary value problem (8) for $\text{SU}(2)$ -connections $\Xi \in \mathcal{A}(\mathbb{H}^2 \times \Sigma)$,

$$(14) \quad F_{\Xi} + *F_{\Xi} = 0, \quad \Xi|_{\{(s,0)\} \times \Sigma} \in \mathcal{L}_H \quad \forall s \in \mathbb{R}.$$

Here $\mathbb{H}^2 = \{(s, t) \in \mathbb{R}^2 \mid t \geq 0\}$ denotes the half space and we equip $\mathbb{H}^2 \times \Sigma$ with a metric $ds^2 + dt^2 + g_{s,t}$, where the metric $g_{s,t}$ on Σ varies smoothly with $(s, t) \in \mathbb{H}^2$ and is constant outside of a compact subset.

6.1. Proof of Compactness. For all results in this subsection, the Lagrangian \mathcal{L}_H in (14) can be replaced by a general gauge invariant Lagrangian submanifold $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$. The compactness Theorem 3.4 in case (in-L) is a consequence of the following lemma and theorem. The lemma

¹¹The methods will be suitable for generalization to gauge invariant Lagrangians as on page 10. The special form of the Lagrangians arising from handle bodies is only used for the bound on $\frac{\partial}{\partial \nu} e$ in Lemma 6.4 and for the isoperimetric inequality in Proposition 6.6.

yields the local L^p -bounds that are assumed in the theorem. It is based on mean value inequalities and will thus be proven later in Section 6.2. Here $B_r(x) \subset \mathbb{R}^2$ is the closed 2-dimensional ball of radius $r > 0$ centred at $x \in \mathbb{R}^2$, and we denote $D_r(x) := B_r(x) \cap \mathbb{H}^2$. In particular, $D_r := D_r(0) \subset \mathbb{H}^2$ is the closed half ball of radius r .

Lemma 6.1 [22, Lemma 2.4]. *Let $\Xi^\nu \in \mathcal{A}(\mathbb{H}^2 \times \Sigma)$ be a sequence of anti-self-dual connections and suppose that for some $x_0 \in \mathbb{H}^2$ and $\delta > 0$*

$$\sup_{\nu} \sup_{x \in D_{2\delta}(x_0)} \|F_{\Xi^\nu}(x)\|_{L^2(\Sigma)} < \infty.$$

Then for every $2 < p < 3$,
$$\sup_{\nu} \|F_{\Xi^\nu}\|_{L^p(D_\delta(x_0) \times \Sigma)} < \infty.$$

If in fact $B_\delta(x_0) \cap \partial\mathbb{H} = \emptyset$, then moreover $\sup_{\nu} \|F_{\Xi^\nu}\|_{L^\infty(B_\delta(x_0) \times \Sigma)} < \infty$.

Theorem 6.2. *Let $p > 2$. Suppose that $\Xi^\nu \in \mathcal{A}(\mathbb{H}^2 \times \Sigma)$ is a sequence of solutions of (14) such that $\sup_{\nu} \|F_{\Xi^\nu}\|_{L^p(K)} < \infty$ for every compact subset $K \subset \mathbb{H} \times \Sigma$. Then there exists a subsequence (again denoted by Ξ^ν) and a sequence of gauge transformations $u^\nu \in \mathcal{G}(\mathbb{H}^2 \times \Sigma)$ such that $u^\nu * \Xi^\nu$ converges uniformly with all derivatives on every compact subset of $\mathbb{H}^2 \times \Sigma$.*

Note that it is crucial to establish this compactness for $2 < p < 3$ since the previous lemma only provides those curvature bounds near the boundary. Next, we outline the steps of the proof in [21, Theorem B] of Theorem 6.2. By standard gauge theoretic arguments, it boils down to the boundary regularity theory in 5(b)–(f) below. The crucial step is (f), where the Lagrangian enters as totally real boundary condition for a Cauchy–Riemann equation. The case $2 < p \leq 4$ requires a separate treatment described in (a’) and (f’).

- (1) **Reduction to compact domains:** By a Donaldson–Kronheimer trick [19, Prop. 7.6], it suffices to prove the assertion on $D_k \times \Sigma$ for every $k \in \mathbb{N}$. Then the gauge transformations on $D_k \times \Sigma$ can be extended to $\mathbb{H}^2 \times \Sigma$ and can be interpolated with gauge transformations obtained on larger domains. A diagonal subsequence then satisfies the claimed C_{loc}^∞ -convergence on $\mathbb{H}^2 \times \Sigma$.

So we consider a sequence $\Xi^\nu \in \mathcal{A}(\mathbb{H}^2 \times \Sigma)$ of solutions whose curvature is in particular L^p -bounded on $\mathcal{U} \times \Sigma$, where $\mathcal{U} \subset \mathbb{H}^2$ is some compact domain with smooth boundary and $D_k \subset \text{int}(\mathcal{U})$. Then we need to find gauge transformations and a convergent subsequence on $D_k \times \Sigma$.

In the subsequent steps, one frequently gets a new estimate only on a smaller domain $\mathcal{U}_i \subset \text{int}(\mathcal{U})$. (Note that the interior includes points on $\partial\mathbb{H}^2$.) However, we can always choose these such that $D_k \subset \text{int}(\mathcal{U}_i)$.

- (2) **Weak convergence:** We can apply Uhlenbeck’s weak compactness Theorem 2.1 on $\mathcal{U} \times \Sigma$. It provides a subsequence (still denoted Ξ^ν)

and gauge transformations $u^\nu \in \mathcal{G}^{2,p}(\mathcal{U} \times \Sigma)$ such that $u^\nu * \Xi^\nu \rightarrow \Xi^\infty$ in the weak $W^{1,p}$ -topology with a limit connection $\Xi^\infty \in \mathcal{A}^{1,p}(\mathcal{U} \times \Sigma)$.

- (3) **Regularity for limit solution:** The limit Ξ^∞ now also solves (14). For the boundary conditions, this is due to the compact Sobolev embedding $W^{1,p}(\mathcal{U} \times \Sigma) \hookrightarrow C^0(\mathcal{U}, L^p(\Sigma))$. In the nonstandard case $2 < p \leq 4$, this embedding is established in [21, Lemma 2.5].

Now one finds a gauge transformation $u \in \mathcal{G}(\mathcal{U}_1 \times \Sigma)$ such that $u^* \Xi^\infty$ is smooth on $\mathcal{U}_1 \times \Sigma$. (This is proven analogously to the iteration in 5), with estimates replaced by regularity statements. For the local slice theorem in (4), it suffices to pick a smooth connection Ξ_0 that is $W^{1,p}$ -close to $\Xi = \Xi^\infty$.) One thus finds that $(u^\nu u)^* \Xi^\nu \rightarrow \Xi_0$ in the weak $W^{1,p}$ -topology on $\mathcal{U}_1 \times \Sigma$, with a smooth limit $\Xi_0 = u^* \Xi^\infty$.

- (4) **Relative Coulomb gauge:** Next, the local slice Theorem 2.2 provides a sequence of gauge transformations $v^\nu \in \mathcal{G}(\mathcal{U}_1 \times \Sigma)$ such that still $v^\nu * \Xi^\nu \rightarrow \Xi_0$ converges $W^{1,p}$ -weakly, and in addition each $\Xi = v^\nu * \Xi^\nu$ satisfies

$$(15) \quad d_{\Xi_0}^*(\Xi - \Xi_0) = 0, \quad *(\Xi - \Xi_0)|_{\partial\mathcal{U}_1 \times \Sigma} = 0.$$

- (5) **Elliptic estimates for (14) and (15):** From (2)–(4) we have a subsequence and gauge transformations v^ν such that each $\Xi = v^\nu * \Xi^\nu$ satisfies (14), (15), and $\|\Xi\|_{W^{1,p}(\mathcal{U}_1 \times \Sigma)} \leq C_1$ for some uniform constant C_1 . By iterating the following steps (a)–(f), one next finds uniform constants C_ℓ such that $\|\Xi\|_{W^{\ell,p}(\mathcal{U}_\ell \times \Sigma)} \leq C_\ell$ for all $\ell \in \mathbb{N}$ and for all $\Xi = v^\nu * \Xi^\nu$. Finally, due to the compact Sobolev embeddings $W^{\ell,p}(D_k \times \Sigma) \hookrightarrow C^{\ell-2}(D_k \times \Sigma)$, one then finds a diagonal subsequence that converges with all derivatives on $D_k \times \Sigma$. This is what was to be shown according to (1).

For (a)–(f), we give the arguments in the case $\ell = 2$ and $p > 4$. This first step is considerably harder for $2 < p \leq 4$ and requires a separate iteration, which is roughly indicated in (a') and (f'). The iteration for $\ell \geq 3$ and any $p > 2$ then works completely analogous to the arguments below.

- (a) **Interior estimates:** From (14) and (15), we obtain the Hodge Laplacian

$$(16) \quad \Delta \Xi = -d^* \left(\frac{1}{2} [\Xi \wedge \Xi] + \frac{1}{2} * [\Xi \wedge \Xi] \right) + dd_{\Xi_0}^* \Xi_0 + d * [\Xi_0 \wedge * \Xi].$$

Here the right hand side is bounded in L^p , and the leading order of the left hand side in local coordinates is the Laplacian on the components of Ξ . Thus the elliptic estimate for the Laplace equation yields a $W^{2,p}$ -bound on Ξ in the interior of $\mathcal{U}_1 \setminus \partial\mathbb{H}^2$.

Going through the arguments up to this point also proves Theorem 3.4 in the instanton case (inst) without boundary.

- (a') **Special iteration for $W^{2,p}$ -bounds with $2 < p \leq 4$:** In this case, the right hand side of (16) lies in L^q for some $q < p$, so one only obtains a $W^{2,q}$ -bound. However, by a Sobolev embedding, this also gives a $W^{1,p'}$ -bound for some $p' > p$. Iteration of (a) then yields W^{2,q_i} -bounds for a strictly increasing sequence, which reaches $q_N \geq p$ after finitely many steps. An analogous iteration will work for steps (b)–(f).
- (b) **Splitting the equation near the boundary:** It remains to obtain a $W^{2,p}$ -bound on Ξ near $D_k \cap \partial\mathbb{H}^2$. For that purpose we rewrite (14) and (15) in the splitting $\Xi = \Phi ds + \Psi dt + A$ (and analogous for the smooth Ξ_0) with $\Phi, \Psi \in W^{1,p}(\mathcal{U}_1 \times \Sigma; \mathfrak{su}(2))$ and $A \in W^{1,p}(\mathcal{U}_1 \times \Sigma, T^*\Sigma \otimes \mathfrak{su}(2))$,

$$(17) \quad \begin{cases} (\partial_s A - d_A \Phi) + *(\partial_t A - d_A \Psi) = 0, \\ \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + *F_A = 0, \\ \nabla_s(\Phi - \Phi_0) + \nabla_t(\Psi - \Psi_0) - d_{A_0}^*(A - A_0) = 0, \\ (\Psi - \Psi_0)|_{t=0} = 0, \\ A(s, 0) \in \mathcal{L} \quad \forall s \in \mathbb{R}. \end{cases}$$

Here we use the notation $\nabla_s = \partial_s + [\Phi_0, \cdot]$ and $\nabla_t = \partial_t + [\Psi_0, \cdot]$.

- (c) **Estimates for Ψ :** From (16), we know that $\Delta\Psi$ is L^p -bounded. In addition, we have the inhomogeneous Dirichlet condition $\Psi|_{t=0} = \Psi_0|_{t=0}$. Thus the elliptic estimate for the Dirichlet boundary value problem implies a $W^{2,p}$ -bound on Ψ up to the boundary.
- (d) **Estimates for Φ :** Again, $\Delta\Phi$ is L^p -bounded due to (16). Moreover, we have an inhomogeneous Neumann condition $\partial_t \Phi|_{t=0} = (\partial_s \Psi_0 + [\Phi, \Psi_0])|_{t=0}$ since the Lagrangian boundary condition with $\mathcal{L} \subset \mathcal{A}_{\text{flat}}^{0,p}(\Sigma)$ in particular implies $F_A|_{t=0} = 0$. Then the elliptic estimate for the Neumann boundary value problem (e.g., [19, Theorem 3.2]) provides a $W^{2,p}$ -bound on Φ .
- (e) **Estimates for $\nabla_\Sigma A$:** We can now rewrite (17) to express the differential and codifferential of $A(s, t) \in \Omega^1(\Sigma; \mathfrak{su}(2))$ for every $(s, t) \in \mathcal{U}_1$ as

$$\begin{aligned} *d_\Sigma A &= -\frac{1}{2} * [A \wedge A] - \partial_s \Psi + \partial_t \Phi - [\Phi, \Psi], \\ d_\Sigma^* A &= \nabla_s(\Phi - \Phi_0) + \nabla_t(\Psi - \Psi_0) + *[A_0 \wedge *A] - d_{A_0}^* A_0. \end{aligned}$$

Due to the previously established bounds on Φ and Ψ , the right hand sides are bounded in $W^{1,p}(\mathcal{U}_2 \times \Sigma)$, that is, in $W^{1,p}(\mathcal{U}_2, L^p(\Sigma))$ and $L^p(\mathcal{U}_2, W^{1,p}(\Sigma))$. Now the elliptic estimates from the Hodge decomposition for each $(s, t) \in \mathcal{U}_2$ can be integrated to give bounds on $\nabla_\Sigma A$ in the same spaces, and hence a $W^{1,p}$ -bound. For a detailed statement and proof see [21, Lemma 2.9].

- (f) **Estimates for $\partial_s A, \partial_t A$:** So far A is bounded in $L^p(\mathcal{U}_2, W^{2,p}(\Sigma))$ and $W^{1,p}(\mathcal{U}_2, W^{1,p}(\Sigma))$. To achieve a $W^{2,p}$ -bound, it remains to find an estimate in $W^{2,p}(\mathcal{U}_2, L^p(\Sigma))$, that is, on $\partial_s A$ and $\partial_t A$. At this point, the full Lagrangian boundary condition needs to be used. Up to now, we only used its local part, the slice-wise flatness. The additional holonomy conditions are of global type (requiring knowledge of the connection on loops in Σ), so this information is lost when one localizes, i.e., goes to a coordinate chart in Σ . The solution is to consider A as map from \mathcal{U}_1 to the Banach space $\mathcal{A}^{0,p}(\Sigma)$. This is a complex space when equipped with the Hodge $*$ operator. So we can rewrite (17) and recall Lemma 5.1 (i) to see that A satisfies a Cauchy–Riemann equation with totally real boundary conditions:

$$(18) \quad \partial_s A + * \partial_t A = d_A \Phi + * d_A \Psi, \quad A(s, 0) \in \mathcal{L} \quad \forall s \in \mathbb{R}.$$

Now one basically has to go through the proof of Theorem 3.4 for the holomorphic curves in case (symp) with the extra difficulty that the target space is infinite dimensional. This would be fairly standard for a Hilbert space. However, the iteration only works for $p > 2$ and we also need to work with $p > 2$ to make sure that the Lagrangians are smooth submanifolds.

Here we use the general theory in [20] for maps to a complex Banach space X . The crucial assumption is that X is a closed subspace of an L^p -space for some $1 < p < \infty$ on a closed manifold (for example $X = \mathcal{A}^{0,p}(\Sigma)$). Then the elliptic L^p -estimates (with the same Sobolev exponent as in X) hold for the Dirichlet and Neumann problem. One can then use the usual argument for the Cauchy–Riemann equation with totally real boundary conditions — at the expense of the complex structure becoming u -dependent. From an L^p -bound on $(\partial_s^2 + \partial_t^2)u$, one then obtains a $W^{2,p}$ -estimate on u . Due to this nonlinearity from the complex structure, however, the L^p -estimate on $(\partial_s^2 + \partial_t^2)u$ requires $W^{1,2p}$ -bounds on u and $\partial_s u + J \partial_t u$.

In (18) the right hand side is bounded in $W^{1,p}(\mathcal{U}, L^p(\Sigma))$ due to the previous bounds on Φ and Ψ (on some domain $\mathcal{U} \subset \text{int}(\mathcal{U}_1)$ with $\mathcal{U}_2 \subset \text{int}(\mathcal{U})$). By the above discussion we now have to write $p = 2p'$ and we only obtain $W^{2,p'}$ -estimates for $A : \mathcal{U} \rightarrow X = \mathcal{A}^{0,p'}(\Sigma)$ with $\partial_s A + * \partial_t A \in W^{1,2p'}(\mathcal{U}, X)$. So this last step yields a bound on Ξ in $W^{2, \frac{p}{2}}(\mathcal{U}_2 \times \Sigma)$. For $p > 4$, we still have $\frac{p}{2} > 2$ and the further iteration yields $W^{\ell, \frac{p}{2}}$ -bounds for all $\ell \in \mathbb{N}$.

(f') **Special case $2 < p \leq 4$ for $W^{2,p}$ -bounds:** In this case, we only have $q < p$ in the $W^{2,q}$ - and $W^{1,q}$ -bounds on Φ, Ψ , and $\nabla_{\Sigma}A$ from (c)–(e). So the right hand side in (18) is of even lower regularity that will not fit in the above arguments. However, it is bounded in $L^r(\mathcal{U}, L^p(\Sigma))$ for some $r > p$. So one can use the submanifold charts for $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ to write A as a map $u : \mathcal{U} \rightarrow T_{A_0}\mathcal{L} \times T_{A_0}\mathcal{L}$, where $T_{A_0}\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ is a closed subspace. The two components of u then satisfy weak Dirichlet and Neumann equations with the weak Laplacian in $W^{-1,r}(\mathcal{U}, L^p(\Sigma))$. The previous general theory unfortunately only works when we replace the $r > p$ by p and it would then give a bound on u in $W^{1,p}(\mathcal{U}_2, L^p(\Sigma))$, which is what we started out with. However, one can use all the usual elliptic estimates when the target is a Hilbert space. So we consider u as map into $\mathcal{A}^{0,2}(\Sigma) \times \mathcal{A}^{0,2}(\Sigma)$ with a $W^{-1,r}$ -bound on its weak Laplacian. This yields a $W^{1,r}(\mathcal{U}_2, L^2(\Sigma))$ -bound on u with $r > p$. The previous bounds in (e) moreover imply a $W^{1,q}(\mathcal{U}_2, L^s(\Sigma))$ -bound on u , where $q < p$ but $s > p$ since it results from the Sobolev embedding $W^{1,q}(\Sigma) \hookrightarrow L^s(\Sigma)$. Now these two bounds can be interpolated to obtain a $W^{1,p'}(\mathcal{U}_2 \times \Sigma)$ -bound with $p' > p$. This bound on u also translates into a $W^{1,p'}$ -bound on A , which fits into the same iteration as in (a').

6.2. Mean value inequalities. The proof of Theorems 3.5 and 3.7 as well as Lemma 6.1 makes use of some mean value inequalities which we summarize here. These are based on a generalization of the mean value inequality for subharmonic functions. Here we state it for the Euclidean half space \mathbb{H}^n . In the interior case, this is well known for general metrics. In the case of balls intersecting the boundary, this was proven in [23, Theorem 1.3] for the Euclidean metric.

Proposition 6.3. *For every $n \geq 2$, there exists a constant C and for all $a, b \geq 0$ there exists $\hbar(a, b) > 0$ such that the following holds:*

Let $D_r(y) \subset \mathbb{H}^n$ be the partial ball of radius $r > 0$ and centre $y \in \mathbb{H}^n$. Suppose that $e \in C^2(D_r(y), [0, \infty))$ satisfies for some constants $A, B \geq 0$

$$\begin{cases} \Delta e & \leq Ae + ae^{(n+2)/n}, \\ \frac{\partial}{\partial \nu} e \Big|_{\partial \mathbb{H}^n} & \leq Be + be^{(n+1)/n}, \end{cases} \quad \text{and} \quad \int_{D_r(y)} e \leq \hbar(a, b).$$

Then $e(y) \leq C(A^{n/2} + B^n + r^{-n}) \int_{D_r(y)} e$.

For all three types of Floer theory that are discussed in Section 3, the energy densities satisfy the differential inequalities for Proposition 6.3 with exactly the critical nonlinearities. These estimates are summarized below.

Lemma 6.4. *Consider a solution of the trajectory equation (T) in Definition 3.2. Its energy density e satisfies the following nonlinear bounds on Δe and $\frac{\partial}{\partial v}e$ with constants a, b, C .*

$$\text{(inst): } e = |\partial_s B|^2 : \mathbb{R} \times Y \rightarrow [0, \infty) \text{ satisfies } \Delta e \leq C e + a e^{3/2}.$$

$$\text{(in-L, interior): } e = |\partial_s A|^2 + |F_A|^2 : \mathbb{R} \times [0, 1] \times \Sigma \rightarrow [0, \infty) \text{ satisfies}$$

$$\Delta e \leq C e + a e^{3/2}.$$

$$\text{(in-L, boundary): } e = \|\partial_s A\|_{L^2(\Sigma)}^2 + \|F_A\|_{L^2(\Sigma)}^2 : \mathbb{R} \times [0, 1] \rightarrow [0, \infty) \text{ satisfies}$$

$$\Delta e \leq C(1 + \|F_A\|_{L^\infty(\Sigma)})e, \quad \frac{\partial}{\partial v}e \leq C e + b e^{3/2}.$$

$$\text{(symp): } e = |\partial_s u|^2 : \mathbb{R} \times [0, 1] \rightarrow [0, \infty) \text{ satisfies}$$

$$\Delta e \leq a e^2, \quad \frac{\partial}{\partial v}e \leq b e^{3/2}.$$

Indications of Proofs of Lemma 6.4

For the holomorphic curves in case (symp) one picks up linear terms in the estimates if the almost complex structure J varies over the domain. The bound on the Laplacian can be found in, e.g., [12, Lemma 4.3.1]. The bound on the normal derivative was well known and is proven in [23, Lemma A.1] using Darboux-Weinstein coordinates near the Lagrangian.

For the anti-self-dual instantons in case (inst) this estimate is a direct consequence of a Bochner-Weitzenböck formula, see, e.g., [23, Lemma A.2]. It was used by Uhlenbeck [17, Lemma 3.1] in a slightly different formulation. For the anti-self-dual instantons with Lagrangian boundary conditions, one has the same bound on the Laplacian, as stated in (in-L, interior). However, this only provides estimates in the interior (on balls that do not intersect the boundary) since one does not have a bound on the normal derivative.

In view of the global methods in Section 6.1 (f) that were necessary for the proof of the basic compactness Theorem 3.4, it should not be surprising that we were not able to obtain any bound on $\frac{\partial}{\partial v}e$ in terms of e , let alone by $b e^{5/4}$. It is highly unclear how the (nonlocal) holonomy part of the Lagrangian boundary condition should be utilized for such a local estimate. On the other hand, there are examples showing that such an estimate cannot follow only from the (local) flatness part of the Lagrangian boundary condition.

Thus it seems natural that the full Lagrangian boundary condition is only captured by the 2-dimensional energy density given in (in-L, boundary). Indeed, we obtain the same bound on the normal derivative as in case (symp). The proof in [22, Lemma 2.3] works as follows: a simple calculation using the trajectory equation (T) in Definition 3.2 (that is (14) in temporal

gauge) gives the normal derivative at the $t = 0$ boundary component:

$$\begin{aligned} -\frac{1}{2} \frac{\partial}{\partial t} e \Big|_{t=0} &= - \int_{\Sigma} \langle \partial_s A \wedge * \partial_s (* \partial_s A) \rangle \Big|_{t=0} \\ &\leq \left(C \|\partial_s A\|_{L^2(\Sigma)}^2 + \int_{\Sigma} \langle \partial_s A \wedge \partial_s^2 A \rangle \right) \Big|_{t=0}. \end{aligned}$$

Recall that $e = \|\partial_s A\|_{L^2(\Sigma)}^2 + \|F_A\|_{L^2(\Sigma)}^2$ and $F_A|_{t=0} = 0$ by the boundary condition. So the first term on the right hand side is just Ce for a constant C . The crucial second term is $\omega(\partial_s A, \partial_s^2 A)$ for a path $A : (-\varepsilon, \varepsilon) \rightarrow \mathcal{L}_H$ in the Lagrangian and with the symplectic form (3).

This term would vanish if the Lagrangian was straight — as in Darboux–Weinstein coordinates. Otherwise, the curvature of the Lagrangian leads to a cubic term. For general infinite dimensional Lagrangians, the curvature might not be suitably bounded, and it is not clear whether Darboux–Weinstein coordinates even exist. Fortunately, we are dealing with Lagrangians that are compact modulo gauge transformations. A proof along this line would require a subtle linear estimate for gauge transformations in the critical Sobolev space $W^{1,2}(\Sigma)$, which has not been carried out yet. For the special Lagrangian \mathcal{L}_H arising from a handle body, we can use the following trick based on the extension property in Lemma 5.3 (i).

We have $A(s) = \tilde{A}(s)|_{\partial H}$ for a path of extensions $\tilde{A} : (-\varepsilon, \varepsilon) \rightarrow \mathcal{A}_{\text{flat}}(H)$ such that $\|\partial_s \tilde{A}\|_{L^3(H)} \leq C \|\partial_s A\|_{L^2(\Sigma)}$. Now Stokes' theorem gives

$$\begin{aligned} \int_{\Sigma} \langle \partial_s \tilde{A} \wedge \partial_s^2 \tilde{A} \rangle &= \int_H \langle d_{\tilde{A}} \partial_s \tilde{A} \wedge \partial_s^2 \tilde{A} \rangle - \int_H \langle \partial_s \tilde{A} \wedge d_{\tilde{A}} \partial_s^2 \tilde{A} \rangle \\ &= \int_H \langle \partial_s \tilde{A} \wedge [\partial_s \tilde{A} \wedge \partial_s \tilde{A}] \rangle \\ &\leq \|\partial_s \tilde{A}\|_{L^3(H)}^3 \leq C^3 \|\partial_s A\|_{L^2(\Sigma)}^3 = C^3 e^{3/2}. \end{aligned}$$

Here we used the fact that $F_{\tilde{A}} \equiv 0$, hence $d_{\tilde{A}} \partial_s \tilde{A} = \partial_s F_{\tilde{A}} = 0$, and moreover $0 = \partial_s^2 F_{\tilde{A}} = d_{\tilde{A}} \partial_s^2 \tilde{A} + [\partial_s \tilde{A} \wedge \partial_s \tilde{A}]$. This proves $\partial e / \partial \nu \leq C e + b e^{3/2}$.

The price for going to the more global energy density in (in-L, boundary) has to be paid when considering the Laplacian. The straight forward calculations in [22, Lemma 2.3] yield

$$\Delta e \leq C (\|\partial_s A\|_{L^2(\Sigma)}^2 + \|F_A\|_{L^2(\Sigma)}^2) - 20 \langle F_A, [\partial_s \tilde{A} \wedge \partial_s A] \rangle_{L^2(\Sigma)}.$$

The first term is just Ce . The second term should also be bounded in terms of the L^2 -norms of the curvature components $\partial_s A$ and F_A . However, the best bound that we can find is $\|F_A\|_{L^\infty(\Sigma)} \|\partial_s A\|_{L^2(\Sigma)}^2 \leq \|F_A\|_{L^\infty(\Sigma)} e$. Here we use the L^∞ -norm on F_A since this has better analytic properties, in particular Dirichlet boundary conditions $F_A|_{t=0} = 0$, whereas $\partial_s A$ only satisfies Lagrangian boundary conditions (of global type). This will be crucial in

the proof of the energy quantization Theorem 3.5, where we will find that $\Delta e \leq C(1 + \|F_A\|_{L^\infty(\Sigma)})e$ is essentially bounded by Ce^2 .

Proof of Lemma 6.1. This is a consequence of the mean value inequality in Proposition 6.3 applied to the energy densities $e_\nu = |\partial_s A^\nu|^2 + |F_A^\nu|^2 = \frac{1}{2}|F_{\Xi^\nu}|^2$ from case (in-L, interior) of Lemma 6.4. The assumption can be read as

$$(19) \quad \int_{\Sigma} e_\nu(x, \cdot) \leq K \quad \text{for all } x \in D_{2\delta}(x_0)$$

with some uniform constant K . On 4-dimensional balls $B_\varepsilon^4(y)$ that are entirely contained in $D_{2\delta}(x_0) \times \Sigma$, this implies $\int_{B_\varepsilon^4(y)} e_\nu \leq \pi K \varepsilon^2$. Now there is a maximal radius $\varepsilon_0 \in (0, \delta)$ such that for all $\varepsilon \leq \varepsilon_0$, this energy is less than $\hbar(a)$ and thus one has the mean value inequality

$$(20) \quad e_\nu(y) \leq C(1 + \varepsilon^{-4})\pi K \varepsilon^2.$$

In the interior case, one fixes a radius $0 < \varepsilon \leq \varepsilon_0$ less than the distance $\text{dist}(B_\delta(x_0), \partial\mathbb{H}^2) > 0$. Then all balls $B_\varepsilon^4(y)$ for $y \in B_\delta(x_0) \times \Sigma$ are contained in $D_{2\delta}(x_0) \times \Sigma$ and (20) is the claimed uniform bound.

In the boundary case $x_0 = (s_0, t_0)$ with $t_0 \leq \delta$, one cannot use a fixed radius for the balls near the boundary. At $y = (s, t, z) \in D_\delta(x_0) \times \Sigma$, the maximal ball that is entirely contained in $D_{2\delta}(x_0) \times \Sigma$ has radius $\varepsilon = \min(t, \delta)$. So for all $(s, t, z) \in D_\delta(x_0) \times \Sigma$ with $0 < t \leq \varepsilon_0$, the mean value inequality (20) gives

$$e_\nu(s, t, z) \leq C'(t^2 + t^{-2}).$$

Away from the boundary, for $t \geq \varepsilon_0$, this also holds with some modified constant C' by (20) with a fixed radius. Now this bound blows up as $t \rightarrow 0$, but it can be interpolated with (19) to give an L^p -bound on $|F_{\Xi^\nu}| = (e_\nu)^{1/2}$ by the following integral which is finite for $2 < p < 3$.

$$\int_{D_\delta(x_0) \times \Sigma} (e_\nu)^{p/2} \leq \int_{D_\delta(x_0)} (C'(t^2 + t^{-2}))^{p/2-1} \int_{\Sigma} e_\nu \leq C'' \left(1 + \int_0^{t_0+\delta} t^{2-p} dt\right).$$

□

6.3. Proof of Energy Quantization. The proof of Theorem 3.5 for anti-self-dual instantons without boundary and for the holomorphic curves is a direct consequence of the mean value inequality in Proposition 6.3 applied to the energy densities in Lemma 6.4. (See [23, Theorem 2.1] for this general energy quantization principle.) For (inst) and the interior of (in-L), this is the simplest version of the argument – on balls with no boundary condition in dimension $n = 4$. Here we give the argument for the holomorphic curves in (symp), more generally for a sequence of energy density functions $e_i : \mathbb{R} \times [0, 1] \rightarrow [0, \infty)$ satisfying

$$\Delta e_i \leq K e_i + a e_i^2, \quad \frac{\partial}{\partial \nu} e_i \leq B e_i + b e_i^{3/2}.$$

We need to prove that if the energy densities blow up at some $x \in \mathbb{R} \times [0, 1]$,

$$(21) \quad \sup_i \sup_{D_\delta(x)} e_i = \infty \quad \forall \delta > 0,$$

then (for a subsequence) a fixed energy quantum $\hbar > 0$ concentrates there,

$$(22) \quad \int_{D_\delta(x)} e_i > \hbar \quad \forall \delta > 0.$$

The same needs to be proven in case (in-L) for boundary points x . For these anti-self-dual instantons with Lagrangian boundary conditions, we use the energy density $e_i = \|\partial_s A_i\|_{L^2(\Sigma)}^2 + \|F_{A_i}\|_{L^2(\Sigma)}^2$ as in (in-L, boundary) of lemma 6.4. So the constant K above is replaced by the unbounded function $C(1 + \|F_{A_i}\|_{L^\infty(\Sigma)})$. Moreover, the assertion (22) in this case implies the concentration of energy near $\{x\} \times \Sigma \subset \mathbb{R} \times [0, 1] \times \Sigma$.

So let us assume (21). Then we find a subsequence and points $x_i \rightarrow x$ such that $e_i(x_i) = R_i^2$ blows up with a certain rate $R_i \rightarrow \infty$. We will now try to apply Proposition 6.3 on the balls $D_{\delta_i}(x_i)$ of radius $\delta_i := R_i^{-1/2} > 0$. For that purpose, we need to assume that $\int_{D_{\delta_i}(x_i)} e_i \leq \hbar = \hbar(a, b)$. If that is the case, then we obtain the mean value inequality

$$R_i^2 = e(x_i) \leq C(K + B^2 + \delta_i^{-2}) \int_{B_{\delta_i}(x_i)} e_i.$$

Multiplication by $R_i^{-2} = \delta_i^2 R_i^{-1}$ then implies

$$1 \leq C\hbar(KR_i^{-2} + B^2R_i^{-2} + R_i^{-1}).$$

First assume that K is constant. Then the right hand side converges to 0. Thus the assumption must have failed for all sufficiently large $i \in \mathbb{N}$, that is, $\int_{D_{\delta_i}(x_i)} e_i > \hbar$. This implies the energy concentration (22).

If K is not a constant, then this argument still works as long as $K \leq C'R_i^2$. In that case, the limit $i \rightarrow \infty$ implies $1 \leq CC'\hbar$. If one chooses $\hbar \leq (2CC')^{-1}$, then this gives a contradiction and thus proves the energy concentration.

So for anti-self-dual instantons with Lagrangian boundary conditions in case (in-L), we have to prove that if $e_i = R_i^2$ blows up, then the functions $K = C(1 + \|F_{A_i}\|_{L^\infty(\Sigma)})$ are bounded by $C'R_i^2$. This statement is slightly weaker than a direct bound $\|F_A\|_{L^\infty(\Sigma)} \leq C\|F_\Xi\|_{L^2(\Sigma)}^2 = Ce$ would be, but it still shows that $\Delta e \leq C(1 + \|F_A\|_{L^\infty(\Sigma)})e$ is essentially bounded by Ce^2 .

By using the Hofer trick within the previous argument, one can additionally control e_i by the blowup rate on small neighbourhoods. One then needs to establish the following as in [22, Proposition 2.7].

Lemma 6.5 (Crucial Estimate). *Let $\Xi_i = \Phi_i ds + \Psi_i dt + A_i \in \mathcal{A}(\mathbb{H}^2 \times \Sigma)$ be a sequence of solutions of (14). Consider a sequence of blowup points*

$\mathbb{H}^2 \ni x_i \rightarrow 0$ with the blowup speed $R_i \rightarrow \infty$. Assume an $L^2(\Sigma)$ -control on the full curvature on (partial) balls of radius $2\varepsilon_i \rightarrow 0$ such that $\varepsilon_i R_i \rightarrow \infty$,

$$\|F_{\Xi_i}(x, \cdot)\|_{L^2(\Sigma)} \leq R_i \quad \forall x \in D_{2\varepsilon_i}(x_i).$$

Then one obtains an L^∞ -control on the curvature component

$$\|F_{A_i}(x, \cdot)\|_{L^\infty(\Sigma)} \leq CR_i^2 \quad \forall x \in D_{\varepsilon_i}(x_i).$$

The proof combines all previous techniques to a subtle contradiction. This is what remains of the usual energy quantization proof via local rescaling:

- (1) **Assume the contrary:** Then one finds sequences of solutions Ξ_i , points $(x_i, z_i) \rightarrow (0, z) \in \mathbb{H}^2 \times \Sigma$, and $R_i \rightarrow \infty$, $\varepsilon_i \rightarrow 0$, $C_i \rightarrow \infty$ with $\varepsilon_i R_i \rightarrow \infty$,

$$\sup_{x \in D_{\varepsilon_i}(x_i)} \|F_{\Xi_i}(x, \cdot)\|_{L^2(\Sigma)} \leq R_i, \quad |F_{A_i}(x_i, z_i)| \geq (C_i R_i)^2.$$

- (2) **Local rescaling:** The crucial case is when $x_i = (s_i, t_i)$ converges to $\partial\mathbb{H}^2$ so fast that even $t_i R_i C_i \rightarrow 0$. So for simplicity, we assume here that $x_i \in \partial\mathbb{H}^2$. Then we can restrict Ξ_i to half balls of radius $\delta_i := (C_i R_i)^{-1} \leq \varepsilon_i$ and rescale them to connections $\tilde{\Xi}_i(y) := \Xi_i((x_i, z_i) + \delta_i y)$ on the half ball $D^4 \subset \mathbb{H}^4$ of radius 1 centred at 0. The rescaled connections then satisfy

$$(23) \quad |F_{\tilde{A}_i}(0)| \geq 1.$$

- (3) **L^p -decay of $F_{\tilde{\Xi}}$ for $p < 3$:** By a calculation similar to Lemma 6.1 for the curvature of the rescaled connections, one obtains for all $2 < p < 3$

$$(24) \quad \|F_{\tilde{\Xi}_i}\|_{L^p(D^4)} \rightarrow 0.$$

- (4) **\mathcal{C}^0 -estimates for $F_{\tilde{A}}$ in terms of L^p -bounds on $F_{\tilde{\Xi}}$ for $p > \frac{8}{3}$:** From (24) for $p > 2$ and Uhlenbeck’s weak compactness Theorem 2.1, we know that (up to gauge and taking a subsequence) the rescaled connections $\tilde{\Xi}_i \in \mathcal{A}(D^4)$ converge to a flat connection in the weak $W^{1,p}$ -topology. One obtains stronger estimates from the fact that the rescaling preserves the anti-self-duality equation. This implies \mathcal{C}^∞ -convergence of the $\tilde{\Xi}_i$ away from the boundary $\partial\mathbb{H}^4$. At the boundary, the local rescaling has lost the global part of the Lagrangian boundary condition, but the slice-wise flatness persists, $F_{\tilde{\Xi}_i}|_{\{(s,0)\} \times \mathbb{R}^2} = 0$. With this, one can go through the steps (b)–(e) in Section 6.1 to obtain $W^{2,q}$ -estimates on some components of the $\tilde{\Xi}_i$. One then feeds these back into (c) and (d) to obtain $W^{2,q}$ -bounds on $\nabla\tilde{\Phi}_i$ and $\nabla\tilde{\Psi}_i$, where the derivative ∇ is only in the \mathbb{R}^2 -directions corresponding to $T\Sigma$.

Now we need to assume (24) with $p > \frac{8}{3}$, then we can work with $q > 2$ and the above bounds are just strong enough to imply \mathcal{C}^0 -convergence of the curvature part $*F_{\tilde{A}} = \partial_t \tilde{\Phi} - \partial_s \tilde{\Psi} + [\tilde{\Psi}, \tilde{\Phi}]$. Since this convergence is to a flat

connection, it provides a contradiction to (23). Note that this contradiction between (3) and (4) crucially relies on the celebrated fact $\boxed{\frac{8}{3} < 3}$.

6.4. Proof of Removability of Singularities. The proof of Theorem 3.7 in case (in-L, boundary) proceeds through the subsequent three propositions. Throughout, we denote by $D_r := D_r(0) \subset \mathbb{H}^2$ the half ball of radius $r > 0$, by $D_r^* := D_r \setminus \{0\}$ the punctured half ball, and we will use polar coordinates $(r, \phi) \in D_1^*$ with $r \in (0, 1]$ and $\phi \in [0, \pi]$.

We will consider solutions of (14) on $D_1^* \times \Sigma$, which is anti-self-dual connections which satisfy the Lagrangian boundary condition on $\{(s, 0)\} \times \Sigma$ for $s \neq 0$. An important tool for a connection $\Xi \in \mathcal{A}(D_1^* \times \Sigma)$ with finite energy $\int_{D_1^* \times \Sigma} |F_\Xi|^2 < \infty$ is its **energy function** $\mathcal{E} : (0, 1] \rightarrow [0, \infty)$ given by

$$\mathcal{E}(r) := \frac{1}{2} \int_{D_r^* \times \Sigma} |F_\Xi|^2 \left[= \lim_{\delta \rightarrow 0} \frac{1}{2} \int_{(D_r \setminus D_\delta) \times \Sigma} |F_\Xi|^2 = \lim_{\delta \rightarrow 0} (\mathcal{E}(r) - \mathcal{E}(\delta)) \right].$$

The above calculation shows that finite energy directly implies $\mathcal{E}(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. For a finite energy solution of (14), one thus obtains mean value inequalities as in Section 6.2 on sufficiently small punctured balls.

Proposition 6.6 [22, Lemma 5.4]. *There are constants C and $\varepsilon > 0$ such that the following holds. Let $\Xi \in \mathcal{A}(D_1^* \times \Sigma)$ be a solution of (14) and suppose that $\mathcal{E}(2r) \leq \varepsilon$ for some $r \in (0, \frac{1}{2}]$. Then for all $\phi \in [0, \pi]$*

- (i) $\|F_\Xi(r, \phi)\|_{L^2(\Sigma)} \leq Cr^{-1} \sqrt{\mathcal{E}(2r)}$,
- (ii) $\|F_\Xi(r, \phi)\|_{L^\infty(\Sigma)} \leq C(r \sin \phi)^{-2} \sqrt{\mathcal{E}(2r)}$.

Sketch of Proof. The estimate (ii) is the mean value inequality for $e = |F_\Xi|^2$ that follows from Proposition 6.3. Since Lemma 6.4 does not provide a control on $\frac{\partial}{\partial v} e$, we can only work on balls that are entirely contained in $D_{2r}^* \times \Sigma$. When centred at $(r, \phi, z) \in D_1^* \times \Sigma$, their maximal radius is $r \sin \phi$.

Next, write the connection as $\Xi = \Phi ds + \Psi dt + A$. For the curvature component F_A , which vanishes at the boundary $\phi \in \{0, \pi\}$, we can improve (ii) to $\|F_A(r, \phi)\|_{L^\infty(\Sigma)} \leq Cr^{-2}$. This follows from $\|F_\Xi(r, \phi)\|_{L^2(\Sigma)} \leq Cr^{-1}$ similar to Lemma 6.5 ($\|F_A\|_{L^\infty(\Sigma)}$ is essentially bounded by $\|F_\Xi\|_{L^2(\Sigma)}^2$). The latter estimate is proven by an indirect argument as in Section 6.3. This uses the mean value inequality for $e = \|F_\Xi\|_{L^2(\Sigma)}^2$ from Proposition 6.3, based on Lemma 6.4 and again Lemma 6.5.

Once $\|F_A(r, \phi)\|_{L^\infty(\Sigma)} \leq Cr^{-2}$ is established that way, one can use it again in the mean value inequality for $e = \|F_\Xi\|_{L^2(\Sigma)}^2$. It provides $\Delta e \leq Cr^{-2}e$ on (partial) balls of radius $\frac{1}{2}r$ around (r, ϕ) . The claim (i) then follows directly.

The curvature decay established here is almost sufficient to remove the singularity. The exponent of r only has to be slightly improved to achieve the conditions in the following removable singularity result. This improvement

will finally be achieved in the crucial Proposition 6.8 by a control on the speed of convergence of the energy function $\mathcal{E}(r) \rightarrow 0$ as $r \rightarrow 0$. \square

Proposition 6.7 [22, Theorem 5.3]. *Let $\Xi \in \mathcal{A}(D_1^* \times \Sigma)$ and suppose that for some constants C and $\beta > 0$ and for all $(r, \phi) \in D_1^*$*

- (i) $\|F_\Xi(r, \phi)\|_{L^2(\Sigma)} \leq Cr^{\beta-1}$,
- (ii) $\|F_\Xi(r, \phi)\|_{L^\infty(\Sigma)} \leq C(\sin \phi)^{-2}r^{\beta-2}$.

Then there exists $p = p(\beta) > 2$ and a gauge transformation $u \in \mathcal{G}^{2,p}(D_1^ \times \Sigma)$ such that $u^*\Xi$ extends to a connection $\tilde{\Xi} \in \mathcal{A}^{1,p}(D_1 \times \Sigma)$.*

Moreover, if Ξ is a solution of (14), then $\tilde{\Xi}$ automatically solves (14) on $D_1 \times \Sigma$. A further gauge transformation then makes $\tilde{\Xi} \in \mathcal{A}(D_1 \times \Sigma)$ smooth.

Sketch of Proof. To control the connection in terms of its curvature, we fix a special gauge: trivializing the bundle along rays $0 < r \leq 1$ for fixed $\phi = \frac{\pi}{2}$ and $z \in \Sigma$ and then along $0 \leq \phi \leq \pi$ for fixed r and $z \in \Sigma$ we obtain

$$\Xi = A + R dr + 0 d\phi \quad \text{with } R|_{\phi=\frac{\pi}{2}} = 0.$$

Here $A : D_1^* \rightarrow \Omega^1(\Sigma; \mathfrak{su}(2))$ and $R : D_1^* \rightarrow \Omega^0(\Sigma; \mathfrak{su}(2))$. In this gauge we have $|\partial_r \Xi|_{\phi=\frac{\pi}{2}} \leq |F_\Xi|$ and $|\partial_\phi \Xi| \leq r|F_\Xi|$ since the curvature decomposes as

$$|F_\Xi|^2 = |F_A|^2 + |\partial_r A - d_A R|^2 + r^{-2}|\partial_\phi R|^2 + r^{-2}|\partial_\phi A|^2.$$

The bounds (i) and (ii) combine to $|F_\Xi| \in L^p(D_1 \times \Sigma)$ for some $p > 2$ that only depends on β . Roughly, they also imply $\Xi|_{\{(r, \frac{\pi}{2})\} \times \Sigma} \rightarrow A_0 \in \mathcal{A}^{0,p}(\Sigma)$ and $\Xi|_{\{r\} \times [0, \pi] \times \Sigma} \rightarrow A_0 \in \mathcal{C}^0([0, \pi], \mathcal{A}^{0,p}(\Sigma))$ as $r \rightarrow 0$, and A_0 provides the extension over $\{0\} \times \Sigma$. In practice, one constructs a family of connections $(\Xi_\varepsilon)_{\varepsilon \geq 0}$ on $D_1 \times \Sigma$ that coincide with Ξ outside of $D_{2\varepsilon} \times \Sigma$ and equal to $A(\varepsilon, \frac{\pi}{2})$ on $D_\varepsilon \times \Sigma$. Using (i) and (ii), this cutoff construction can be done such that $\|F_{\Xi_\varepsilon} - F_\Xi\|_{L^p(D_1 \times \Sigma)} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

By Uhlenbeck's compactness Theorem 2.1, one then finds a sequence $\varepsilon_i \rightarrow 0$ and gauge transformations $u_i \in \mathcal{G}(D_1 \times \Sigma)$ such that $u_i^* \Xi_{\varepsilon_i}$ converges $W^{1,p}$ -weakly to a limit connection $\tilde{\Xi} \in \mathcal{A}^{1,p}(D_1 \times \Sigma)$. Note that on every compact subset of $D_1^* \times \Sigma$, the sequence Ξ_{ε_i} eventually coincides with Ξ . So the above convergence also implies that (for a subsequence) the gauge transformations u_i converge to a limit $u \in \mathcal{G}_{\text{loc}}^{2,p}(D_1^* \times \Sigma)$ in the weak $W^{2,p}$ -topology on every compact set. Then by the uniqueness of the limit $u^* \Xi = \tilde{\Xi}|_{D_1^* \times \Sigma}$, so $\tilde{\Xi}$ is the claimed extension. If moreover Ξ and hence $\tilde{\Xi}$ are solutions of (14), then the regularity theorem [21, Theorem A] for this boundary value problem asserts that $\tilde{\Xi}$ is gauge equivalent to a smooth solution. \square

Proposition 6.8 [22, Lemma 4.1]. *Let $\Xi \in \mathcal{A}(D_1^* \times \Sigma)$ be a solution of (14) with finite energy $\mathcal{E}(1) < \infty$. Then for all $r \in (0, 1]$*

$$\mathcal{E}(r) \leq r^{1/\pi} \mathcal{E}(1).$$

Sketch of Proof. By the anti-self-duality equation $*F_{\Xi} = -F_{\Xi}$, we have

$$\frac{1}{2} \int_{(D_{r_0} \setminus D_{\delta}) \times \Sigma} \langle F_{\Xi} \wedge *F_{\Xi} \rangle = \frac{1}{2} \int_{(D_{r_0} \setminus D_{\delta}) \times \Sigma} d \langle \Xi \wedge (F_{\Xi} - \frac{1}{6}[\Xi \wedge \Xi]) \rangle.$$

This converges to $\mathcal{E}(r_0)$ as $\delta \rightarrow 0$. On the other hand, Stokes' theorem expresses this as integral over $\partial(D_{r_0} \setminus D_{\delta}) \times \Sigma$. Our goal is to rewrite it as $\mathcal{F}(A_{\delta}) - \mathcal{F}(A_{r_0})$ for a functional \mathcal{F} depending on $A_r := A(r, \cdot) : [0, \pi] \rightarrow \mathcal{A}(\Sigma)$. Here, as in Proposition 6.7, we work in the special gauge $\Xi = A + R dr$. Then Proposition 6.6 (i) gives $\|\partial_{\phi} A_r\|_{L^2(\Sigma)} \leq C\sqrt{\mathcal{E}(2r)} \rightarrow 0$ as $r \rightarrow 0$, so the paths A_r are L^2 -short paths connecting $A_r(0), A_r(\pi) \in \mathcal{L}_H$. These contribute to \mathcal{F} on the boundary components $\{r_0\} \times [0, \pi] \times \Sigma$ and $\{\delta\} \times [0, \pi] \times \Sigma$. So it remains to deal with the boundary components¹² at $\phi = 0$ and $\phi = \pi$. We identify these with $([-r_0, -\delta] \cup [\delta, r_0]) \times \Sigma$ and glue in the domain $([-r_0, -\delta] \cup [\delta, r_0]) \times H$. Now extending the families $A_r(0), A_r(\pi) \in \mathcal{L}_H$ by $\tilde{A}_r(0), \tilde{A}_r(\pi) \in \mathcal{A}_{\text{flat}}(H)$ preserves the value of $\int \langle F_{\Xi} \wedge F_{\Xi} \rangle$, and

$$(25) \quad \mathcal{E}(r_0) = \mathcal{CS}(A_{\delta}, \tilde{A}_{\delta}) - \mathcal{CS}(A_{r_0}, \tilde{A}_{r_0}).$$

Here we introduce the Chern–Simons functional for a path $A : [0, \pi] \rightarrow \mathcal{A}(\Sigma)$ with L^2 -close ends $\tilde{A}(0), \tilde{A}(\pi) \in \mathcal{L}_H$ and extensions $\tilde{A}(0), \tilde{A}(\pi) \in \mathcal{A}_{\text{flat}}(H)$,

$$\begin{aligned} \mathcal{CS}(A, \tilde{A}) &= -\frac{1}{2} \int_0^{\pi} \int_{\Sigma} \langle A \wedge \partial_{\phi} A \rangle + \frac{1}{12} \left[\int_H d \langle \tilde{A}(\phi) \wedge [\tilde{A}(\phi) \wedge \tilde{A}(\phi)] \rangle \right]_{\phi=0}^{\phi=\pi} \\ &= -\frac{1}{2} \int_0^{\pi} \int_0^{\phi} \int_{\Sigma} \langle \partial_{\theta} A(\theta) \wedge \partial_{\phi} A(\phi) \rangle d\theta d\phi \\ &\quad - \frac{1}{12} \int_H \langle [(\tilde{A}(0) - \tilde{A}(\pi)) \wedge (\tilde{A}(0) - \tilde{A}(\pi))] \wedge (\tilde{A}(0) - \tilde{A}(\pi)) \rangle. \end{aligned}$$

This magic identity together with the special choice of extensions as in Lemma 5.3 (ii) allow us to obtain the isoperimetric inequality

$$\begin{aligned} |\mathcal{CS}(A_r, \tilde{A}_r)| &\leq \frac{1}{2} \left(\int_0^{\pi} \|\partial_{\phi} A_r\|_{L^2(\Sigma)} d\phi \right)^2 + \frac{1}{12} \left(\|\tilde{A}_r(0) - \tilde{A}_r(\pi)\|_{L^3(Y)} \right)^3 \\ &\leq \left(\frac{1}{2} + \frac{C_H^3}{12} \|A_r(0) - A_r(\pi)\|_{L^2(\Sigma)} \right) \left(\int_0^{\pi} \|\partial_{\phi} A_r\|_{L^2(\Sigma)} d\phi \right)^2. \end{aligned}$$

¹²One could eliminate these by gluing in paths $A'_r : [0, \pi] \rightarrow \mathcal{L}_H$ in the Lagrangian connecting $A_r(0), A_r(\pi) \in \mathcal{L}_H$. This would reach the goal with a functional $\mathcal{F} = \mathcal{F}(A_r, A'_r)$. For the subsequent argument, however, the L^2 -length of the path A'_r has to be controlled by the L^2 -distance of its endpoints. The crucial point would be to establish this fact for paths in a fixed gauge orbit — a subtle nonlinear $W^{1,2}$ -estimate for gauge transformations.

For sufficiently short A_r , this implies $|\mathcal{CS}(A_r, \tilde{A}_r)| \leq \pi \int_0^\pi \|\partial_\phi A_r\|_{L^2(\Sigma)}^2$. As seen before, this converges to 0 as $r = \delta \rightarrow 0$, and moreover it is bounded $\pi r \dot{\mathcal{E}}(r)$. So (25) provides the differential inequality $\mathcal{E}(r) \leq \pi r \dot{\mathcal{E}}(r)$. Integrating $\frac{d}{dr} \ln \mathcal{E}(r) \geq (\pi r)^{-1}$ then proves the claimed decay of $\mathcal{E}(r)$. \square

References

- [1] M.F. Atiyah, *New invariants of three and four dimensional manifolds*, Proc. Symp. Pure Math. **48** (1988).
- [2] M.F. Atiyah and R. Bott, *The Yang–Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. Lond. A **308** (1982), 523–615.
- [3] K. Cieliebak, A.R. Gaio, I. Mundet, and D.A. Salamon, *The symplectic vortex equations and invariants of Hamiltonian group actions*, J. Symp. Geom. **1** (2002), 543–645.
- [4] S.K. Donaldson, *Floer Homology Groups in Yang–Mills Theory*, Cambridge University Press, 2002.
- [5] S.K. Donaldson and P.B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press, 1990.
- [6] S. Dostoglou and D.A. Salamon, *Self-dual instantons and holomorphic curves*, Ann. Math. **139** (1994), 581–640.
- [7] A. Floer, *An instanton invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), 215–240.
- [8] A. Floer, *Morse theory for Lagrangian intersections*, J. Diff. Geom. **28** (1988), 513–547.
- [9] K. Fukaya, *Floer homology for 3-manifolds with boundary I*, preprint, 1997.
- [10] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.
- [11] A.R. Gaio and D.A. Salamon, *Gromov–Witten invariants of symplectic quotients and adiabatic limits*, J. Symp. Geom., **3** (2005), 55–159.
- [12] D. McDuff and D. Salamon, *J-Holomorphic Curves and Symplectic Topology*, AMS Colloquium Publications, 2004.
- [13] D.A. Salamon, *Lectures on Floer Homology*, Park City Series **7** (1999), 145–229.
- [14] D.A. Salamon, *Lagrangian intersections, 3-manifolds with boundary, and the Atiyah–Floer conjecture*, Proc. ICM, Zürich 1994, **1**, 526–536.
- [15] D.A. Salamon and K. Wehrheim, *Instanton Floer homology with Lagrangian boundary conditions*, work in progress.
- [16] C.H. Taubes, *Casson’s invariant and gauge theory*, J. Diff. Geom. **31** (1990), 547–599.
- [17] K.K. Uhlenbeck, *Removable singularities in Yang–Mills fields*, Comm. Math. Phys. **83** (1982), 11–29.
- [18] K.K. Uhlenbeck, *Connections with L^p -bounds on curvature*, Comm. Math. Phys. **83** (1982), 31–42.
- [19] K. Wehrheim, *Uhlenbeck Compactness*, EMS Lectures in Mathematics, 2004.
- [20] K. Wehrheim, *Banach space valued Cauchy–Riemann equations with totally real boundary conditions*, Comm. Contemp. Math. **6**(4) (2004), 601–635.

- [21] K. Wehrheim, *Anti-self-dual instantons with Lagrangian boundary conditions I: Elliptic theory*, Comm. Math. Phys. **254**(1) (2005), 45–89.
- [22] K. Wehrheim, *Anti-self-dual instantons with Lagrangian boundary conditions II: Bubbling*, Comm. Math. Phys. **258**(2) (2005), 275–315.
- [23] K. Wehrheim, *Energy quantization and mean value inequalities for nonlinear boundary value problems*, J. Eur. Math. Soc. **7**(3) (2005), 305–318.

INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ 08540
E-mail address: wehrheim@princeton.edu

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