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NON-KÄHLER SOLVMANIFOLDS WITH GENERALIZED KÄHLER STRUCTURE

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We construct a compact 6-dimensional solvmanifold endowed with a non-trivial invariant generalized Kähler structure and which does not admit any Kähler metric. This is in contrast with the case of nilmanifolds which cannot admit any invariant generalized Kähler structure unless they are tori.

1. Introduction

The generalized Kähler structures were introduced and studied by Gualtieri in his PhD thesis [**16**] in the more general context of generalized geometry started by Hitchin in [**20**].

There are many explicit constructions of non-trivial generalized-Kähler structures [**1**, **2**, **4**, **7**, **21**, **24**, **25**]. For instance Gualtieri proved that all compact-even dimensional semisimple Lie groups are generalized Kähler. In [**25**], the generalized Kähler quotient construction is considered in relation with the hyperkähler quotient construction and generalized Kähler structures are given on $\mathbb{CP}^n$, on some toric varieties and on the complex Grassmannian.

Some obstructions and conditions on the underlying complex manifolds were found (see [**2**, **5**, **16**] and related references).

By [**2**, **16**] it turns out that a generalized Kähler structure on a $2n$-dimensional manifold M is equivalent to a pair of Hermitian structures (J_+, g) and (J_-, g), where $J_\pm$ are two integrable almost complex structures on M and g is a Hermitian metric with respect to $J_\pm$, such that the 3-form $H = d^c_+ F_+ = -d^c_- F_-$ is closed, where $F_\pm(\cdot,\cdot) = g(J_\pm \cdot, \cdot)$ are the fundamental 2-forms associated with the Hermitian structures $(J_\pm, g)$ and $d^c_\pm = i(\overline{\partial}_\pm - \partial_\pm)$. In particular, any Kähler metric (J, g) gives rise to a generalized Kähler structure by taking $J_+ = J$ and $J_- = \pm J$.

In the context of Hermitian geometry, the closed 3-form H is called the *torsion* of the generalized Kähler structure and it can be also identified with the torsion of the Bismut connection associated with the Hermitian structure $(J_\pm, g)$ (see [**3**, **14**]). The generalized Kähler structure is called *untwisted* or *twisted* according to the fact that the cohomology class $[H] \in H^3(M, \mathbb{R})$ vanishes or not. In this paper we will give a homogeneous example of twisted generalized Kähler manifold which does not admit any Kähler structure.

If $(J_\pm, g)$ is a generalized Kähler structure, then the fundamental 2-forms $F_\pm$ are $\partial_\pm\overline{\partial}_\pm$-closed. Therefore, the Hermitian structures $(J_\pm, g)$ are *strong Kähler with torsion* (SKT). Such structures have been studied by many authors in [**9**, **11**, **15**, **23**, **30**].

In dimension 4 a Hermitian metric g which satisfies the SKT condition is *standard* in the terminology of Gauduchon [**13**] and if, in addition, M is compact, then any Hermitian conformal class contains a standard metric. Compact examples in six dimensions are given in [**11**] where the Hermitian manifolds are provided by *nilmanifolds*, i.e. compact quotients of nilpotent Lie groups endowed with a Hermitian structure (J, g) in which J and g arise from left-invariant tensors. In [**5**] it was proved that these manifolds cannot admit an invariant generalized Kähler structure, since they are not formal unless they are tori. Nevertheless, all 6-dimensional nilmanifolds admit invariant generalized complex structures [**6**].

No general restrictions are known in the case of *solvmanifolds*, i.e., on compact quotients of solvable Lie groups by uniform discrete subgroups. By [**18**] a solvmanifold has a Kähler structure if and only it is covered by a complex torus which has a structure of a complex torus bundle over a complex torus.

As far as we know, the only known solvmanifold carrying a generalized Kähler structure is the Inoue surface. In [**2**] the complex solvmanifold from [**8**] was considered and it was shown that this manifold does not admit any left-invariant SKT metric compatible with the natural left-invariant complex structure.

Other examples are given in [**9**], where the SKT metrics are called *pluriclosed* in its terminology. By [**10**] there are compact complex manifolds of dimension higher than four which do no admit any SKT Hermitian metric.

In [**2**], generalized Kähler structures for which the corresponding complex structures $J_\pm$ commute are studied and a classification of compact 4-dimensional endowed with a generalized Kähler structure for which the induced complex structures yield opposite orientations is obtained. By [**2**], if J_+ and J_- commute, then the product J_+J_- is an involution of TM and the tangent bundle splits as $TM = T_-M \oplus T_+M$ direct sum of the ±1-eigenspaces of the involution. By their result, the holomorphic tangent bundle of a compact complex surface (M^4, J) admitting a generalized Kähler structure splits as a direct sum of two holomorphic sub-bundles and (M^4, J) is bi-holomorphic to one of the following:

(1) a geometrically ruled surface;
(2) a bi-elliptic complex surface;
(3) a compact complex surface of Kodaira dimension 1 and even first Betti number;
(4) a compact complex surface of general type;
(5) a Hopf surface;
(6) an Inoue surface in the family constructed in [**22**].

Moreover, in relation to the distinction between untwisted and twisted generalized Kähler structures, they prove that untwisted generalized Kähler structures on compact 4-dimensional manifolds may exist only if the first Betti number is even and in the twisted case the first Betti number must be odd.

The Inoue surface S^0 can be viewed also as the quotient $\mathcal{H}\times\mathbb{C}/G_M$, where $\mathcal{H}$ is the upper-half of the complex plane $\mathbb{C}$ and G_M is a group of analytic automorphisms of $\mathcal{H}\times\mathbb{C}$ (see [**22**, **32**]). The complex surface S^0 can be also obtained as a 4-dimensional solvmanifold [**17**]. Moreover, since by Hattori's theorem [**19**] its de Rham cohomology is given by the invariant one, it is easy to check that S^0 is formal.

Generalizing the description of the Inoue surface S^0 as solvmanifold, in Section 3, we will construct a compact 6-dimensional manifold with a twisted generalized Kähler structure. The 6-dimensional manifold is a solvmanifold, a compact quotient of a non-completely solvable Lie group and the generalized Kahler structure is left-invariant. Such a manifold does not admit any Kähler structure since its first Betti number is one and it is a total space of a $\mathbb{T}^2$-bundle over the Inoue surface. The construction can be extended in any even dimension bigger than six.

Moreover, in the last section, we give an example of a non-unimodular 6-dimensional Lie group endowed with a generalized Kähler structure. The corresponding Hermitian structures $(J_\pm, g)$ are locally conformal Kähler.

2. Preliminaries

In this section, we briefly recall the definition of generalized complex and Kähler structures, following [**2**, **5**, **16**, **20**]. Let M be an m-dimensional manifold. Denote by TM and T^*M the tangent and cotangent bundle of M, respectively.

Let H be a closed 3-form on M; the (twisted) *Courant bracket* on the sections of $TM\oplus T^*M$ is defined by

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(\eta(X)-\xi(Y)) + \iota_Y\iota_X H,$$

where $\mathcal{L}_X$ and ι_X denote, respectively, the Lie derivative and the contraction by X.

On $TM \oplus T^*M$ a natural symmetric pairing of signature (m,m) is given by:

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\eta(X)+\xi(Y)). \tag{2.1}$$

A *generalized complex structure* $\mathcal{J}$ on (M,H) is a complex structure on the bundle $TM \oplus T^*M$ which preserves the pairing and whose i-eigenspace L is involutive with respect to the Courant bracket.

A generalized complex structure $\mathcal{J}$ can be also viewed as an element of the orthogonal Lie algebra $\mathfrak{so}(TM \oplus T^*M)$ and thus, with respect to the splitting

$$\mathfrak{so}(TM \oplus T^*M) = \Lambda^2 TM \oplus \mathrm{End}(TM) \oplus \Lambda^2 T^*M,$$

it can be written as the block matrix

$$\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & A \end{pmatrix},$$

where π is a bi-vector field, A an endomorphism of TM and σ a 2-form.

For $H=0$, examples of generalized complex structures are given by complex and symplectic structures. In the case of a complex manifold, the i-eigenspace is given by $L = T^{0,1}M \oplus T^{*1,0}M$ and in the symplectic case $L = \{X - i\iota_X\omega, X \in T_{\mathbb{C}}M\}$, where ω is the symplectic form on M. In the block matrix form we may write:

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \quad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

A *generalized Kähler structure* on a $2n$-dimensional manifold M is a pair of commuting generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ on M which satisfy the following conditions:

(i) $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable with respect to the (twisted) Courant bracket on $TM \oplus T^*M$ and they are compatible with the natural inner product $\langle\cdot,\cdot\rangle$ of signature $(2n,2n)$ on $TM \oplus T^*M$ given by (2.1);

(ii) the quadratic form $\langle \mathcal{J}_1\cdot, \mathcal{J}_2\cdot\rangle$ is definite on $TM \oplus T^*M$.

By [**2**, **16**] it turns out that a generalized Kähler structure on a manifold M is equivalent to a triple (J_+, J_-, g), where $J_\pm$ are two integrable almost complex structures on M and g is a Hermitian metric with respect to $J_\pm$, which satisfy the equations:

$$d^cF_+ + d^c_-F_- = 0, \quad d(d^c_+F_+) = 0, \quad d(d^c_-F_-) = 0, \tag{2.2}$$

where $F_\pm(\cdot,\cdot) = g(J_\pm\cdot,\cdot)$ are the fundamental 2-forms associated with the Hermitian structures $(J_\pm, g)$ and $d^c_\pm = i(\overline{\partial}_\pm - \partial_\pm) = (-1)^r J_\pm dJ_\pm$. These conditions are equivalent to

$$J_+dF_+ + J_-dF_- = 0, \quad d(J_+dF_+) = 0, \quad d(J_-dF_-) = 0, \tag{2.3}$$

and in physics they appear in the target space geometry for a $(2,2)$ supersymmetric sigma model (see, e.g., [**12**]).

A trivial solution of equation (2.2) is given by a Kähler structure (g, J) on M, by taking $J_+ = J$ and $J_- = \pm J$. So the interesting case is when $J_- \neq \pm J_+$, i.e., when the generalized Kähler structure does not arise from a Kähler structure.

By (2.2), the fundamental 2-forms $F_\pm$ are $\partial_\pm\overline{\partial}_\pm$-closed. In general, a Hermitian structure (J, g, F) is called a strong Kähler structure with torsion (SKT) if $\partial\overline{\partial}F = 0$ and a Kähler structure satisfies this condition.

3. Compact example

In this section, we will describe explicitly a compact 6-dimensional example of generalized Kähler manifold.

Consider the 2-step solvable Lie algebra $\mathfrak{s}_{a,b}$ with structure equations:

$$
\begin{cases}
de^1 = ae^1 \wedge e^2,\\
de^2 = 0,\\
de^3 = \dfrac{1}{2}ae^2 \wedge e^3,\\
de^4 = \dfrac{1}{2}ae^2 \wedge e^4,\\
de^5 = be^2 \wedge e^6,\\
de^6 = -be^2 \wedge e^5,
\end{cases}
\tag{3.1}
$$

where a, b are non-zero real numbers. Let $S_{a,b}$ be the simply connected solvable Lie group with Lie algebra $\mathfrak{s}_{a,b}$ and $(t, x_1, x_2, x_3, x_4, x_5)$ be global coordinates on $\mathbb{R}^6$. Then the Lie group $S_{a,b}$ can be described using the following product:

$$
\begin{pmatrix} t\\ x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{pmatrix} \cdot \begin{pmatrix} t'\\ x_1'\\ x_2'\\ x_3'\\ x_4'\\ x_5' \end{pmatrix} = \begin{pmatrix} t + t'\\ e^{-at}x_1' + x_1\\ e^{\frac{a}{2}t}x_2' + x_2\\ e^{\frac{a}{2}t}x_3' + x_3\\ x_4'\cos(bt) - x_5'\sin(bt) + x_4\\ x_4'\sin(bt) + x_5'\cos(bt) + x_5 \end{pmatrix}.
\tag{3.2}
$$

Then the 1-forms

$$
e^1 = e^{at}dx_1, \quad e^2 = dt, \quad e^3 = e^{-\frac{a}{2}t}dx_2, \quad e^4 = e^{-\frac{a}{2}t}dx_3,
$$
$$
e^5 = \cos(bt)dx_4 + \sin(bt)dx_5, \quad e^6 = -\sin(bt)dx_4 + \cos(bt)dx_5.
$$

are left-invariant on $S_{a,b}$ and they satisfy the structure equations (3.1). It turns out that $S_{a,b}$ is a unimodular semidirect product

$$\mathbb{R} \ltimes_{\varphi} (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2),$$

where $\varphi = (\varphi_1, \varphi_2)$ is the diagonal action of $\mathbb{R}$ on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, given by (3.2).

We start with the following.

Lemma 3.1. *The solvable Lie group $S_{1,\frac{\pi}{2}}$ (corresponding to $a = 1, b = \frac{\pi}{2}$) admits a compact quotient $M^6 = S_{1,\frac{\pi}{2}}/\Gamma$.*

Proof. In order to construct a uniform discrete subgroup of $S_{1,\frac{\pi}{2}}$, we will proceed as follows. The 4-dimensional solvable Lie group $\mathbb{R} \ltimes_{\varphi_1} (\mathbb{R} \times \mathbb{R}^2)$ with structure equations

$$\begin{cases} de^1 = e^1 \wedge e^2, \\ de^2 = 0, \\ de^3 = \dfrac{1}{2} e^2 \wedge e^3, \\ de^4 = \dfrac{1}{2} e^2 \wedge e^4, \end{cases}$$

admits a compact quotient by a uniform discrete subgroup of the form $\Gamma_1 = \mathbb{Z} \ltimes_{\varphi_1} \mathbb{Z}^3$, since it can be identified with the Inoue surface M^4 of type S^0 [**22**], described as in [**17**, **32**]. More precisely, the action φ_1 can be given by assigning a matrix $\varphi_1(1) = (m_{jk}) \in SL(3, \mathbb{Z})$, with two conjugate eigenvalues $\alpha, \overline{\alpha}$ and a irrational eigenvalue $c > 1$ such that $|\alpha|^2 c = 1$ and considering the product on $\mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C})$ defined by

$$(t, u, z) \cdot (t', u', z') = (t + t', c^t u' + u, \alpha^t z' + z), \quad t, t', u, u' \in \mathbb{R}, z, z' \in \mathbb{C}.$$

If we denote by $(\alpha_1, \alpha_2, \alpha_3)$ an eigenvector corresponding to α and by (c_1, c_2, c_3) a real eigenvector corresponding to c, then Γ_1 is generated by

$$\begin{aligned} &h_0 : (t, u, z) \longmapsto (t + 1, cu, \alpha z), \\ &h_j : (t, u, z) \longmapsto (t, u + c_j, z + \alpha_j), \quad j = 1, 2, 3. \end{aligned}$$

The vectors (c_j, α_j), $j = 1, 2, 3$ are linearly independent over $\mathbb{R}$ and

$$(cc_j, \alpha\alpha_j) = \sum_{k=1}^{3} m_{j_k} (c_k, \alpha_k), \tag{3.3}$$

for any $j = 1, 2, 3$.

The 3-dimensional solvable Lie group $\mathbb{R} \ltimes \mathbb{R}^2$ with structure equations

$$\begin{cases} de^2 = 0, \\ de^5 = 2\pi e^2 \wedge e^6, \\ de^6 = -2\pi e^2 \wedge e^5, \end{cases}$$

is a non-completely solvable Lie group which admits a compact quotient and the uniform discrete subgroup is of the form $\Gamma_2 = \mathbb{Z} \ltimes \mathbb{Z}^2$ (see [**31**, Theorem 1.9; **26**]). Indeed, the Lie group $\mathbb{R} \ltimes \mathbb{R}^2$ is the group of matrices

$$\begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) & 0 & x \\ -\sin(2\pi t) & \cos(2\pi t) & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and a lattice Γ_2 is generated by:

$$\begin{pmatrix} \cos\left(\dfrac{2\pi n}{p}\right) & \sin\left(\dfrac{2\pi n}{p}\right) & 0 & x \\ -\sin\left(\dfrac{2\pi n}{p}\right) & \cos\left(\dfrac{2\pi n}{p}\right) & 0 & y \\ 0 & 0 & 1 & \dfrac{n}{p} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & v_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & v_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where n is an integer, $p = 2, 3, 4, 6$ and

$$\det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \neq 0.$$

Therefore, $S_{1,\frac{\pi}{2}}$ is isomorphic to $(\mathbb{R}^6 = \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}), *)$, where the product $*$ is given by

$$(t, u, z, w) * (t', u', z', w') = (t + t', c^t u' + u, \alpha^t z' + z, e^{i\frac{\pi}{2}t} w' + w),$$

for any $t, u, t', u' \in \mathbb{R}$, $z, w, z', w' \in \mathbb{C}$.

Then, a uniform discrete subgroup $\Gamma \cong \mathbb{Z} \ltimes (\mathbb{Z}^3 \times \mathbb{Z}^2)$ of $S_{1,\frac{\pi}{2}}$ is the group generated by the transformations

$$\begin{aligned} g_0 &: (t, u, z, w) \longmapsto (t + 1, cu, \alpha z, iw), \\ g_j &: (t, u, z, w) \longmapsto (t, u + c_j, z + \alpha_j, w), \quad j = 1, 2, 3, \\ g_4 &: (t, u, z, w) \longmapsto (t, u, z, w + 1), \\ g_5 &: (t, u, z, w) \longmapsto (t, u, z, w + i). \end{aligned} \tag{3.4}$$

Indeed, Γ is a closed subgroup of $S_{1,\frac{\pi}{2}}$ and the action of Γ on $S_{1,\frac{\pi}{2}}$ is properly discontinuous and without fixed points. The compactness of the quotient can be checked. □

According to the Mostow structure theorem (see [**28**]), the solvmanifold M^6 can be fibered over S^1 with fiber a 5-dimensional torus $\mathbb{T}^5$, since the maximal connected nilpotent subgroup is the abelian Lie group N whose Lie algebra is spanned by $(e_1, e_3, e_4, e_5, e_6)$, where $(e_1, \ldots, e_6)$ denotes the dual frame of $(e^1, \ldots, e^6)$.

Proposition 3.2. *The compact manifold $M^6 = S_{1,\frac{\pi}{2}}/\Gamma$ is the total space of a $\mathbb{T}^2$-bundle over the Inoue surface and $b_1(M^6) = 1$.*

Proof. Since $\Gamma \cap (\mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}), *)$ is a uniform discrete subgroup of $(\mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C}), \cdot)$, the map

$$\begin{aligned} \pi : \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}) &\to \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C}), \\ (t, u, z, w) &\longmapsto (t, u, z), \end{aligned}$$

gives a fibration

$$\pi : M^6 \to M^4,$$

with fibre $\mathbb{T}^2$.

By (3.3) and (3.4), the generators of Γ satisfy the following relations:

$$g_j g_k = g_k g_j, \quad \forall j, k = 1, \dots, 5.$$

Moreover,

$$\begin{aligned} &[g_0, g_j] = g_0 g_j g_0^{-1} g_j^{-1} : (t, u, z, w) \longmapsto (t, u - c_j + c c_j, z - \alpha_j + \alpha\alpha_j, w), \\ &\qquad\qquad j = 1, 2, 3, \\ &[g_0, g_4] = g_0 g_4 g_0^{-1} g_4^{-1} : (t, u, z, w) \longmapsto (t, u, z, w - 1 + i), \\ &[g_0, g_5] = g_0 g_5 g_0^{-1} g_5^{-1} : (t, u, z, w) \longmapsto (t, u, z, w - 1 - i) \end{aligned}$$

and the other commutators $[g_j, g_k]$, for any $j, k = 1, \dots, 5$ are trivial. Hence,

$$\begin{aligned} &[g_0, g_j] = g_1^{m_{j1}} g_2^{m_{j2}} g_3^{m_{j3}} g_j^{-1}, \quad j = 1, 2, 3, \\ &[g_0, g_4] = {g_4}^{-1} g_5, \\ &[g_0, g_5] = {g_4}^{-1} {g_5}^{-1}. \end{aligned}$$

Since Γ is 2-step solvable, it follows that $[\Gamma, \Gamma]$ is a torsion-free abelian subgroup of Γ and the rank of $[\Gamma, \Gamma]$ is 5. By definition (see [**29**])

$$\operatorname{rank}\Gamma = \operatorname{rank}\Gamma/[\Gamma, \Gamma] + \operatorname{rank}[\Gamma, \Gamma];$$

therefore

$$\Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}$$

and consequently $b_1(M^6) = 1$. □

Remark 3.3. It has to be noted that we cannot apply Hattori's theorem [**19**] to compute the de Rham cohomology of the solvmanifold M^6 since the group $S_{1,\frac{\pi}{2}}$ is non-completely solvable.

As a direct consequence we obtain the following.

Corollary 3.4. *M^6 does not admit any Kähler metric.*

Now we can prove our main result:

Theorem 3.5. *The compact manifold $M^6 = S_{1,\frac{\pi}{2}}/\Gamma$ carries a left-invariant (non-trivial) twisted generalized Kähler structure.*

Proof. First of all we define the two almost complex structures $J_\pm$, by setting

$$\begin{aligned} &\omega_+^1 = e^1 + ie^2, \quad \omega_+^2 = e^3 + ie^4, \quad \omega_+^3 = e^5 + ie^6,\\ &\omega_-^1 = e^1 - ie^2, \quad \omega_-^2 = e^3 + ie^4, \quad \omega_-^3 = e^5 + ie^6. \end{aligned}$$

Then by definition $(\omega_\pm^1, \omega_\pm^2, \omega_\pm^3)$ are the $(1,0)$-forms associated with $J_\pm$. The almost complex structures $J_\pm$ are both integrable. Indeed:

$$\begin{aligned} d\omega_+^1 &= \frac{i}{2}\omega_+^1 \wedge \overline{\omega}_+^1,\\ d\omega_+^2 &= -\frac{i}{4}(\omega_+^1 \wedge \omega_+^2 + \omega_+^2 \wedge \overline{\omega}_+^1),\\ d\omega_+^3 &= -\frac{\pi}{4}(\omega_+^1 \wedge \omega_+^3 + \omega_+^3 \wedge \overline{\omega}_+^1), \end{aligned}$$

and

$$\begin{aligned} d\omega_-^1 &= -\frac{i}{2}\omega_-^1 \wedge \overline{\omega}_-^1,\\ d\omega_-^2 &= \frac{i}{4}(\omega_-^1 \wedge \omega_-^2 + \omega_-^2 \wedge \overline{\omega}_-^1)\\ d\omega_-^3 &= \frac{\pi}{4}(\omega_-^1 \wedge \omega_-^3 + \omega_-^3 \wedge \overline{\omega}_-^1). \end{aligned}$$

Moreover, it is easy to see that $J_\pm$ commute and $J_- \neq -J_+$. Consider the Riemannian metric g defined by

$$g = \sum_{i=1}^{6} e^i \otimes e^i. \tag{3.5}$$

Thus g is $J_\pm$-Hermitian. Denote by $F_\pm$ the fundamental 2-form associated with the Hermitian structures $(J_\pm, g)$; by a direct computation, we have

$$J_+ dF_+ = -e^1 \wedge e^3 \wedge e^4 = -J_- dF_-.$$

Since $e^1 \wedge e^3 \wedge e^4$ is a closed and non-exact 3-form, the conditions (2.3) are satisfied and $(J_\pm, g)$ define a non-trivial left-invariant twisted generalized Kähler structure on $M^6 = S/\Gamma$. □

The metric g given by (3.5) is not flat since the Ricci tensor is diagonal, with the only non-vanishing component $\mathrm{Ric}(e_2, e_2) = -\frac{3}{2}$ and the Hermitian structures $(J_\pm, g)$ are not locally conformally Kähler since

$$dF_\pm = e^2 \wedge e^3 \wedge e^4.$$

Remark 3.6. The previous construction can be extended in order to get a non-trivial generalized Kähler on a $\mathbb{T}^{2n}$-bundle over the Inoue surface, by considering the 2-step solvable Lie algebra

$$\begin{cases} de^1 = ae^1 \wedge e^2, \\ de^2 = 0, \\ de^3 = \dfrac{1}{2}ae^2 \wedge e^3, \\ de^4 = \dfrac{1}{2}ae^2 \wedge e^4, \\ de^{2k+3} = be^2 \wedge e^{2k+4}, \\ de^{2k+4} = -be^2 \wedge e^{2k+3}, \quad k = 1, \dots, n, \end{cases}$$

with $a = 1$ and $b = \frac{\pi}{2}$.

The two integrable complex structures $J_\pm$ are given by setting

$$\begin{array}{lll} \omega_+^1 = e^1 + ie^2, & \omega_+^2 = e^3 + ie^4, & \omega_+^{k+2} = e^{2k+3} + ie^{2k+4}, \\ \omega_-^1 = e^1 - ie^2, & \omega_-^2 = e^3 + ie^4, & \omega_-^{k+2} = e^{2k+3} + ie^{2k+4}, \end{array}$$

as the associated $(1,0)$-forms.

4. Non-compact homogeneous example

In this section, we construct a non-compact homogeneous example of 6-dimensional Lie group endowed with a non-trivial generalized Kähler structure.

Let $\mathfrak{l}$ be the Lie algebra with structure equations:

$$\begin{cases} df^1 = f^1 \wedge f^2, \\ df^2 = 0, \\ df^3 = \dfrac{1}{2}f^2 \wedge f^3, \\ de^4 = \dfrac{1}{2}f^2 \wedge f^4, \\ df^5 = \dfrac{1}{2}f^2 \wedge f^5, \\ df^6 = \dfrac{1}{2}f^2 \wedge f^6, \end{cases} \tag{4.1}$$

and L be the simply-connected Lie group with Lie algebra $\mathfrak{l}$.

The Lie group L can be described using the following product on $\mathbb{R}^6$ with global coordinates $(t, y_1, y_2, y_3, y_4, y_5)$:

$$\begin{pmatrix} t \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \cdot \begin{pmatrix} t' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{pmatrix} = \begin{pmatrix} t+t' \\ e^t y_1' + y_1 \\ e^{\frac{1}{2}t} y_2' + y_2 \\ e^{\frac{1}{2}t} y_3' + y_3 \\ e^{\frac{1}{2}t} y_4' + y_4 \\ e^{\frac{1}{2}t} y_5' + y_5 \end{pmatrix}.$$

The 1-forms

$$\begin{aligned} &f^1 = e^{-t} dx_1, \quad f^2 = dt, \quad f^3 = e^{-\frac{1}{2}t} dy_2, \\ &f^4 = e^{-\frac{1}{2}t} dy_3, \quad f^5 = e^{-\frac{1}{2}t} dy_4, \quad f^6 = e^{-\frac{1}{2}t} dy_5 \end{aligned}$$

are left-invariant on L and L is a 2-step completely solvable Lie group. Moreover, L is not-unimodular and consequently by [**27**] it does not admit any compact quotient.

Proposition 4.1. *The Lie group L admits a left-invariant (non-trivial) generalized Kähler structure.*

Proof. Consider the two almost complex structures $J_\pm$, whose $(1,0)$-forms are given by

$$\begin{aligned} &\theta^1_+ = f^1 + i f^2, \quad \theta^2_+ = f^3 + i f^4, \quad \theta^3_+ = f^5 + i f^6, \\ &\theta^1_- = f^1 - i f^2, \quad \theta^2_- = f^3 + i f^4, \quad \theta^3_- = f^5 + i f^6. \end{aligned}$$

We have

$$\begin{aligned} d\theta^1_+ &= \frac{i}{2} \theta^1_+ \wedge \overline{\theta}^1_+, \\ d\theta^2_+ &= -\frac{i}{4} (\theta^1_+ \wedge \theta^2_+ + \theta^2_+ \wedge \overline{\theta}^1_+), \\ d\theta^3_+ &= -\frac{i}{4} (\theta^1_+ \wedge \theta^3_+ + \theta^3_+ \wedge \overline{\theta}^1_+) \end{aligned}$$

and

$$\begin{aligned} d\theta^1_- &= -\frac{i}{2} \theta^1_- \wedge \overline{\theta}^1_-, \\ d\theta^2_- &= \frac{i}{4} (\theta^1_- \wedge \theta^2_- + \theta^2_- \wedge \overline{\theta}^1_-), \\ d\theta^3_- &= \frac{i}{4} (\theta^1_- \wedge \theta^3_- + \theta^3_- \wedge \overline{\theta}^1_-). \end{aligned}$$

Then, $J_{\pm}$ are both integrable, commute and $J_- \neq -J_+$. The Riemannian metric

$$g = \sum_{i=1}^{6} f^i \otimes f^i$$

is $J_{\pm}$-Hermitian and

$$J_+ dF_+ = -f^1 \wedge f^3 \wedge f^4 - f^1 \wedge f^5 \wedge f^6 = -J_- dF_-,$$

where $F_{\pm}$ the fundamental 2-form associated with the Hermitian structures $(J_{\pm}, g)$; since $f^1 \wedge f^3 \wedge f^4 + f^1 \wedge f^5 \wedge f^6$ is a closed 3-form, the conditions (2.3) are satisfied and $(J_{\pm}, g)$ define a non-trivial left-invariant generalized Kähler structure on L. □

The Ricci tensor is diagonal and given by

$$\mathrm{Ric}(g) = f^1 \otimes f^1 - 2f^2 \otimes f^2 - \frac{1}{2} \sum_{j=3}^{6} f^j \otimes f^j,$$

hence the metric g is not flat. Furthermore, in contrast with the previous example, the Hermitian structures $(J_{\pm}, g)$ are locally conformally Kähler since

$$dF_{\pm} = f^2 \wedge F_{\pm}.$$

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