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A MAXIMAL RELATIVE SYMPLECTIC PACKING CONSTRUCTION

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In this paper we present an explicit construction of a relative symplectic packing. This confirms the sharpness of the upper bound for the relative packing of a ball into the pair $(\mathbb{CP}^2, \mathbb{T}^2_{\mathrm{Cliff}})$ of the standard complex projective plane and the Clifford torus, obtained by Biran and Cornea.

1. Introduction and main results

In this note we present an explicit construction of a relative packing. The subject of symplectic packing was introduced first in the seminal work of Gromov [**Gr**]. Gromov showed that looking at symplectic embeddings of a standard ball into a symplectic manifold, one may obtain an upper bound on the radius of a ball which is stronger than the obstruction coming from the volume. The first theorem in this direction is a non-squeezing theorem from [**Gr**]. This result has led to the definition of the Gromov capacity, which plays an important role in modern symplectic geometry. Later the subject of symplectic packing was treated by Biran, Karshon, Mc'Duff, Polterovich, Schlenk, Traynor and others, see [**Bi-1**, **Bi-2**, **Bi-3**, **Bi-4**, **K**, **M-P**, **Sch-1**, **Sch-2**, **Sch-3**, **Tr**]. New obstructions for symplectic packings of various domains were found. On the other hand, attempts were made to find explicit constructions of certain symplectic embeddings, in order to show that the obstructions, which were found, are tight (see, e.g., [**K**, **Sch-4**, **Tr**]). This note is devoted to proving a new result in this direction.

Recently, Biran and Cornea [**Bi-Co**] found new obstructions on the relative symplectic packing in a number of situations, which are stronger than

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those in the case of a usual packing. In this note we consider one specific example from [**Bi-Co**], and show that the corresponding obstruction is sharp.

Let us define first the notion of a relative symplectic packing. Consider a symplectic manifold (M^{2n}, ω) and a closed Lagrangian submanifold $L^n \subset M$. Take a standard open ball $B^{2n}(r) \subset (\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ of radius $r > 0$, where $\mathbb{R}^{2n}$ is endowed with coordinates (q, p), and set

$$B^{2n}_{\mathbb{R}}(r) = \{(q, p) \in B^{2n}(r) \mid p = 0\}.$$

A relative packing of $B^{2n}(r)$ into (M, L) is by definition a symplectic embedding

$$i : (B^{2n}(r), \omega_{\mathrm{std}}) \hookrightarrow (M^{2n}, \omega)$$

such that $i^{-1}(L) = B^{2n}_{\mathbb{R}}(r)$.

In this note we treat the situation where $(M, \omega) = (\mathbb{CP}^2, \omega_{\mathrm{FS}})$ and $L = \mathbb{T}^2 \hookrightarrow (\mathbb{CP}^2, \omega_{\mathrm{FS}})$ is the standard Clifford torus in M. Here, ω_{FS} is normalized so that $\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}} = \pi$. In [**Bi-Co**] it was shown that given a relative packing

$$\big(B^4(r), B^4_{\mathbb{R}}(r)\big) \hookrightarrow (\mathbb{CP}^2, \mathbb{T}^2),$$

there is an upper bound for the radius of this ball : $r \leqslant \sqrt{\frac{2}{3}}$. Our main result is the following.

Theorem 1.1. *For every $r < \sqrt{\frac{2}{3}}$ there exists a relative packing*

$$\big(B^4(r), B^4_{\mathbb{R}}(r)\big) \hookrightarrow (\mathbb{CP}^2, \mathbb{T}^2).$$

Let us mention that in [**Bi-Co**], the authors also consider relative packings into $(\mathbb{CP}^2, \mathbb{T}^2)$ by more than one ball, and obtain obstructions on their radii. The case of three balls was treated, and is strongly connected to the properties of the quantum cup-product in Floer homology. The hypothetical upper bound in this case was found, under the assumption of existence of pseudo-holomorphic discs with certain properties. It still remains to show the existence of such discs, and in the case that it will be proved, one can try to find the example which proves the tightness of this upper bound.

The rest of the paper is devoted to proving Theorem 1.1. In Section 2 we show how to reduce Theorem 1.1 to a two-dimensional problem. Section 3 contains the proof of Theorem 1.1.

2. Overview of the construction

It is well-known that the open symplectic manifold $(\mathbb{CP}^2 \setminus \mathbb{CP}^1, \omega_{\mathrm{FS}})$ is symplectomorphic to the unit ball $(B^4(1), \omega_{\mathrm{std}}) \subset (\mathbb{R}^4, \omega_{\mathrm{std}})$. Identifying $\mathbb{R}^4 \cong \mathbb{C}^2$, we have a natural action of the torus $\mathbb{T}^2$ on $B^4(1)$. The moment map of this action is given by

$$B^4(1) \to \mathbb{R}^2,$$
$$(z_1, z_2) \mapsto (|z_1|^2, |z_2|^2).$$

The action of $\mathbb{T}^2$, restricted to the complement $B^4(1) \setminus \{z_1 z_2 = 0\}$ of the union of the two complex axes is free, and therefore, by a standard procedure, we obtain that $B^4(1) \setminus \{z_1 z_2 = 0\}$ is symplectomorphic to $\mathbb{T}^2 \times \triangle$. Here we use the notation

$$\triangle = \{(p_1, p_2) \mid p_1, p_2 > 0, p_1 + p_2 < 1\} \subset \mathbb{R}^2.$$

Look now at

$$K := \square \times \mathbb{R}^2(p_1, p_2) \subset \mathbb{T}^2 \times \mathbb{R}^2,$$

and its subset $K' := \square \times \triangle \subset K$, where

$$\square = \{(q_1, q_2) \mid 0 < q_1, q_2 < \pi\} \subset \mathbb{T}^2.$$

The above symplectomorphism between $B^4(1) \setminus \{z_1 z_2 = 0\}$ and $\mathbb{T}^2 \times \triangle$ induces a symplectic embedding

$$j : K' \hookrightarrow B^4(1).$$

From now on we will consider K, K' as

$$K = \{(q_1, p_1) \mid 0 < q_1 < \pi\} \times \{(q_2, p_2) \mid 0 < q_2 < \pi\} \subset \mathbb{R}^4,$$
$$K' = \{(q_1, p_1, q_2, p_2) \mid 0 < q_1 < \pi, 0 < q_2 < \pi, p_1, p_2 > 0, p_1 + p_2 < 1\} \subset K,$$

and the symplectic form is

$$dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

The Clifford torus lies entirely in $B^4(1)$, and its pre-image under the map j equals to

$$L' = \{(q_1, p_1, q_2, p_2) \mid 0 < q_1 < \pi, 0 < q_2 < \pi, p_1 = p_2 = 1/3\} \subset K'.$$

Fix $r < \sqrt{\frac{2}{3}}$ and consider

$$B^4(r) \subset B^2(r) \times B^2(r) \subset \mathbb{R}^4.$$

The construction is based on finding a certain area-preserving map

$$\sigma : B^2(r) \to \mathbb{R}^2.$$

Given such an σ, we define $\Phi : B^2(r) \times B^2(r) \to \mathbb{R}^4$ as

$$\Phi(z, w) = \big(\sigma(z), \sigma(w)\big).$$

Our aim is to find σ such that the image of $B^4(r) \hookrightarrow B^2(r) \times B^2(r)$ under the map Φ will be contained in K', and also

$$\Phi^{-1}(L') \cap B^4(r) = B^4_{\mathbb{R}}(r).$$

3. Proofs

Proof of Theorem 1.1. Given $r < \sqrt{\frac{2}{3}}$, we describe a construction of a relative symplectic embedding

$$\big(B^4(r), B^4_{\mathbb{R}}(r)\big) \hookrightarrow (\mathbb{CP}^2, \mathbb{T}^2).$$

As it was shown in Section 2, it is enough to find a symplectic embedding

$$\Phi : B^4(r) \to \mathbb{R}^4,$$

such that its image will be contained in the domain

$$\{(q_1, p_1, q_2, p_2) \mid 0 < q_1 < \pi, 0 < q_2 < \pi, 0 < p_1, 0 < p_2, 0 < p_1 + p_2 < 1\},$$

and the pre-image of

$$\{(q_1, p_1, q_2, p_2) \mid 0 < q_1 < \pi, 0 < q_2 < \pi, p_1 = p_2 = 1/3\}$$

will be equal to $B^4_{\mathbb{R}}(r) \subset B^4(r)$.

As it was shown by Schlenk ([**Sch-4**], Lemma 3.1.5), for any $\epsilon > 0$ there exists an area-preserving diffeomorphism

$$\sigma : B^2(r) \to (0, \pi) \times \left(0, \frac{2}{3}\right) \subset \mathbb{R}^2(Q, P)$$

with the following properties (Figure 1):

(1) For each $u \in (o, r^2)$ one has: if $p^2 + q^2 \leqslant u$, then $P(\sigma(q, p)) \leqslant \frac{1}{3} + \frac{u}{2} + \epsilon$.
(2) The line $\{p = 0\}$ in $B^2(r)$, and no other points of $B^2(r)$, is mapped to the line $\{(Q, \frac{1}{3}) \mid Q \in (0, \pi)\}$.

Take $\epsilon = \frac{1}{2}\left(\frac{1}{3} - \frac{r^2}{2}\right)$, and consider the corresponding map σ. Then for any $z, w \in \mathbb{R}^2$, such that $(z, w) \in B^4(r)$, define

$$\Phi(z, w) = \big(\sigma(z), \sigma(w)\big).$$

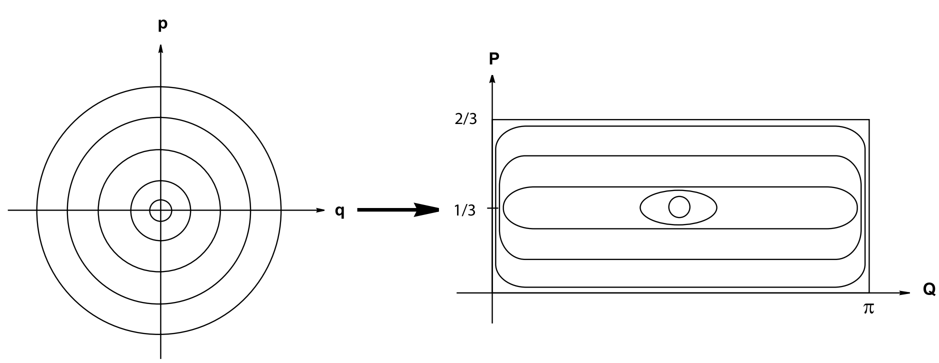

Figure 1. The map σ

We claim that the resulting map $\Phi : B^4(r) \to \mathbb{R}^4$ satisfies the desired properties. First of all, it is clear that the map Φ is a symplectic embedding, and the pre-image of

$$\{(q_1, p_1, q_2, p_2) \mid 0 < q_1 < \pi, 0 < q_2 < \pi, p_1 = p_2 = 1/3\}$$

equals $B^4_{\mathbb{R}}(r) \subset B^4(r)$. Let us show that, moreover, the image of Φ lies in the domain

$$\{(q_1, p_1, q_2, p_2) \mid 0 < q_1 < \pi, 0 < q_2 < \pi, 0 < p_1, 0 < p_2, 0 < p_1 + p_2 < 1\}.$$

Take any $(q_1, p_1, q_2, p_2) \in B^4(r)$, then we have

$$q_1^2 + p_1^2 + q_2^2 + p_2^2 < r^2.$$

Set $u = q_1^2 + p_1^2$. Then $q_2^2 + p_2^2 < r^2 - u$, and so

$$\begin{aligned} P_1(\sigma(q_1, p_1)) + P_2(\sigma(q_2, p_2)) &< \left(\frac{1}{3} + \frac{u}{2} + \epsilon\right) + \left(\frac{1}{3} + \frac{r^2 - u}{2} + \epsilon\right) \\ &= \frac{2}{3} + \frac{r^2}{2} + 2\epsilon \\ &= 1. \end{aligned}$$

□

Remark 3.1. The presented relative packing construction can be naturally generalized to the corresponding construction of a relative packing of a $2n$-dimensional ball $B^{2n}(r)$ into $(\mathbb{CP}^n, \mathbb{T}^n_{\mathrm{Cliff}})$, for any $n \geqslant 2$ and radius $r < \sqrt{\frac{2}{n+1}}$. This confirms the sharpness of the upper bound for the radius, obtained by Biran and Cornea [**Bi-Co**], for an arbitrary dimension.

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