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FOLD-FORMS FOR FOUR-FOLDS

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This paper explains an application of Gromov's h-principle to prove the existence, on any orientable four-manifold, of a folded symplectic form. That is a closed two-form which is symplectic except on a separating hypersurface where the form singularities are like the pullback of a symplectic form by a folding map. We use the h-principle for folding maps (a theorem of Eliashberg) and the h-principle for symplectic forms on open manifolds (a theorem of Gromov) to show that, for orientable even-dimensional manifolds, the existence of a stable almost complex structure is necessary and sufficient to warrant the existence of a folded symplectic form.

1. Introduction

One says that a differential problem satisfies the h-principle if any formal solution (i.e., a solution for the associated algebraic problem) is homotopic to a genuine (i.e., differential) solution. Therefore, when the h-principle holds, one may concentrate on a purely topological question in order to prove the existence of a differential solution.

Differential problems are equations, inequalities or, more generally, relations [**13**] involving derivatives of maps. The following are examples of problems known to satisfy the h-principle: existence of immersions in strictly positive codimension (theorems of Whitney [**30**], Nash [**24**], Kuiper [**16**], Smale [**26**], Hirsch [**14**] and Poénaru [**25**]), existence of symplectic forms on open manifolds (theorem of Gromov [**12**], who built the general machinery of the h-principle as an obstruction theory for the sheaves of germs of maps) and existence of maps whose only singularities are folds (theorem of Eliashberg [**6**, **7**]).

This paper explains an application of the h-principle to prove the existence, on any compact orientable four-manifold, of a folded symplectic form, that is, a closed two-form with only fold singularities as defined below. According to the h-principle philosophy, this proof is divided in two steps:

(1) show that the h-principle holds for this problem, and
(2) show that a formal solution exists.

For the first step, the basic ingredients are the h-principle for maps whose only singularities are folds [**6**, **7**] and the h-principle for symplectic forms on open manifolds [**12**]. This combination is a shortcut based on an idea contained in a book by Eliashberg and Mishachev [**9**]. We thus avoid dealing with the h-principle in its generality.

Here is the flavor of Eliashberg's result. Let Z be a hypersurface in a manifold M, that is, a codimension 1 embedded submanifold (this is the meaning of *hypersurface* throughout this paper). A map $f : M \to N$ between manifolds of the same dimension is called a *Z-immersion* (or said to *fold along the submanifold* Z) if it is regular (i.e., its derivative is invertible) on $M \setminus Z$, and if near any $p \in Z$ and near its image $f(p)$ there are coordinates centered at those points where f becomes

$$(x_1, x_2, \dots, x_n) \longmapsto (x_1^2, x_2, \dots, x_n).$$

A homomorphism $F : TM \to TN$ between tangent bundles is called a *Z-monomorphism*, if it is injective on $T(M \setminus Z)$ and on TZ, and if there exists a fiber involution $\tau : \mathcal{T} \to \mathcal{T}$ on a tubular neighborhood $\mathcal{T}$ of Z whose set of fixed points is Z and such that $F \circ d\tau = F$. The differential $df : TM \to TN$ of a Z-immersion is a Z-monomorphism. Eliashberg [**6**] proved that, if every connected component of $M \setminus Z$ is open, then any Z-monomorphism $TM \to TN$ is homotopic (within Z-monomorphisms $TM \to TN$) to the differential of a Z-immersion. In the language of [**13**], the theorem says that, when $M \setminus Z$ is open, Z-immersions satisfy the (everywhere C^0-dense) h-principle; a Z-monomorphism is then called a *formal solution*. For the present application, we require a more general statement [**7**] dealing with foliated target manifolds.

A *folded symplectic form* on a $2n$-dimensional manifold M is a closed two-form ω which is nondegenerate except on a hypersurface Z called the *folding hypersurface* where, centered at every point $p \in Z$, there are coordinates for M adapted to Z where the form ω becomes

$$x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

The pullback of a symplectic form by a Z-immersion is a folded symplectic form with folding hypersurface Z.

A formal solution for the problem of existence of a folded symplectic form turns out to be a stable almost complex structure. Let M be a $2n$-dimensional manifold with a structure of complex vector bundle on $TM \oplus \mathbb{R}^2$, where $\mathbb{R}^2$ denotes the trivial rank 2 real vector bundle over M. We will show that M admits folded symplectic forms.

Here is how Gromov's theorem comes in. We embed M as level zero in $M \times \mathbb{R}$. The given stable almost complex structure on M yields a complex hyperplane field on $M \times \mathbb{R}$ and hence an almost complex structure on $M \times \mathbb{R}^2$. Since this manifold is open, Gromov's application of the h-principle [**12**] guarantees the existence of a symplectic form on $M \times \mathbb{R}^2$ inducing almost complex structures in the same homotopy class as the given one. Since $M \times \mathbb{R}$ sits here as a codimension one submanifold, the restriction ω_0 of the symplectic form to this submanifold has maximal rank, i.e., has exactly a one-dimensional kernel at every point. Let $\mathcal{L}$ be the one-dimensional foliation determined by the kernel L of ω_0. The projection of ω_0 to $T(M \times \mathbb{R})/L$ is well-defined and nondegenerate. Suppose that we could immerse M in $M \times \mathbb{R}$ in a *good* way, meaning that locally the composition of that immersion with the projection to the local leaf space of $\mathcal{L}$ is a Z-immersion, for some hypersurface Z in M. Since this leaf space is symplectic, by pullback we would obtain a folded symplectic form on M. Hence, we concentrate on deforming the initial embedding at level zero into a good immersion in order to prove:

Theorem A. *Let M be a $2n$-dimensional manifold with a stable almost complex structure J. Then M admits a folded symplectic form consistent with J in any degree 2 cohomology class.*

The notion of consistency is explained in Section 2. The existence of a stable almost complex structure is a necessary condition for the existence of a folded symplectic form on an orientable manifold (see Section 2). Theorem A is then saying that it is also sufficient. This contrasts with the case of a (honest) symplectic form, for whose existence an almost complex structure is necessary, but only sufficient if the manifold is open [**12**]. The sphere S^6 is a trivial example (thanks to Stokes' theorem) and $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ is an important example (thanks to Seiberg–Witten invariants [**28**]) of almost complex manifolds without any symplectic form.

To produce a formal solution for four-manifolds is easily accomplished. Hirzebruch and Hopf [**15**] showed that the integral Stiefel–Whitney class W_3 vanishes for any compact orientable four-manifold, or, in other words, such manifolds always have stable almost complex structures. (This is the same reason why such manifolds are spin-c [**17**, Theorem D.2].) Since we are in the stable range, it is enough to add a trivial $\mathbb{R}^2$ bundle to TM for this to admit a structure of complex vector bundle. All this is also true when M is not compact [**11**, Section 5.7]. We thus obtain the following relevant special case of Theorem A:

Theorem B. *Let M be an orientable four-manifold. Then M admits a folded symplectic form consistent with any given stable almost complex structure and in any degree 2 cohomology class.*

In higher dimensions, there are plenty of orientable manifolds that have no stable almost complex structures ($S^1 \times \mathrm{SU}(3)/\mathrm{SO}(3)$, for instance [**17**]), and hence cannot have folded symplectic forms. The condition $W_3(M) = 0$ is necessary and sufficient in dimensions 6 (since the next obstruction W_7 vanishes for dimensional reasons) and 8 (where Massey [**20**] proved that W_7 always vanishes). According to [**4**, **29**], until 1998 it was still not known general necessary and sufficient conditions (in terms of invariants such as characteristic classes and the cohomology ring) for the existence of a stable almost complex structure on manifolds of dimension ≥ 10.

As for the contents of this paper: Section 2 reviews folded symplectic manifolds and some *folded* tangent bundles associated to them; Section 3 describes the application of Gromov's theorem to guarantee a symplectic form starting from a structure of complex vector bundle; Section 4 proves the existence of an isomorphism between a *folded* tangent bundle and a suitable complex vector bundle; Section 5 describes the application of Eliashberg's theorem to produce folded symplectic forms; and Section 6 contains the conclusion of the proof of Theorems A and B.

2. Folded symplectic manifolds

Let M be an oriented manifold of dimension $2n$, and let ω be a closed two-form on M. The highest wedge power ω^n is a section of the (trivial) orientation bundle $\wedge^{2n}T^*M$.

Definition. A *folded symplectic form* is a closed two-form ω such that ω^n intersects the 0-section of $\wedge^{2n}T^*M$ transversally, and such that $\imath^*\omega$ has maximal rank everywhere, where $\imath : Z \hookrightarrow M$ is the inclusion of the zero-locus, Z, of ω^n.

By transversality, Z is a codimension-1 submanifold of M, called the *folding hypersurface*. A *folded symplectic manifold* is a pair (M, ω) where ω is a folded symplectic form on M. The folding hypersurface Z of a folded symplectic manifold (M, ω) separates M into the regions M^+ and M^-, where the form matches or is opposite to the given orientation, respectively. Hence, Z has a co-orientation depending on ω and on the choice of orientation on M. (The notion of folded symplectic form extends to arbitrary even-dimensional manifolds, not necessarily orientable, but we will not deal with those in this paper.)

The Darboux theorem for folded symplectic forms states that, if (M, ω) is a folded symplectic manifold and p is any point on the folding hypersurface Z, then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_{2n})$ centered at p such that on $\mathcal{U}$

$$\omega = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \ldots + dx_{2n-1} \wedge dx_{2n} \text{ and } Z \cap \mathcal{U} = \{x_1 = 0\}.$$

This follows, for instance, from a folded analog of Moser's trick [**3**].

Doubles of symplectic manifolds with ω-convex [**8**] (or ω-concave) boundary are easy examples of manifolds with folded symplectic forms. Simplest instances are the spheres S^{2n}, where a folded symplectic form is obtained by pulling back the standard symplectic form on $\mathbb{R}^{2n}$ via the folding map $S^{2n} \to D^{2n}$.

Starting in dimension 4, folded symplectic forms are not generic in the set of closed two-forms. Let M be a (compact) oriented four-manifold, and let ω be a closed two-form on M. If γ is a given volume form on M, then $\omega \wedge \omega = f\gamma$ for some $f \in C^\infty(M)$. A generic ω [**18**] is never 0, has rank 2 on a (compact) codimension-1 submanifold, Z, and is nondegenerate elsewhere. The hypersurface Z is the 0-locus of f. Its complement $M \setminus Z$ is the disjoint union of the sets $M^+ = \{f > 0\}$ where ω matches the given orientation and $M^- = \{f < 0\}$ where ω induces the opposite orientation. For ω to be folded symplectic, we would need that TZ and the rank 2 bundle over Z given by ker ω intersect transversally as subbundles of $TM|_Z$. Yet generically ω is not folded symplectic, since its restriction to Z vanishes along some codimension-2 submanifold C (a union of circles), where ker ω is contained in TZ [**18**]. Although a generic two-form on a three-manifold vanishes only at isolated points, here the three-manifold already depends on the two-form. Moreover, generically there are isolated *parabolic* points on those lines (circles), where the tangent space to those lines is contained in ker ω. There is at least one continuous family of inequivalent neighborhoods of parabolic points [**1**, **10**].

Now let M be an m-dimensional manifold with a separating hypersurface Z. For instance, M could be an oriented manifold equipped with a folded symplectic form, and Z its folding hypersurface.

The complement $M \setminus Z$ is the disjoint union of open sets M^+ and M^-. Over Z, the tangent bundle has a trivial line subbundle V, spanned by a vector field transverse to Z pointing from M^- to M^+. The quotient TM/V is isomorphic to TZ, so that $TM|_Z \simeq TZ \oplus V$.

Definition. The *Z-tangent bundle* of M is the rank m real vector bundle ${}^Z TM$ over M obtained by gluing $TM|_{M \setminus M^-}$ to $TM|_{M \setminus M^+}$ by the constant diagonal map $\mathrm{Id} \oplus (-1) : Z \to \mathrm{GL}(TZ \oplus V)$.

There are analytic and algebraic approaches to ${}^{Z}TM$, which enhance its geometry [**3**]. From its definition it follows that:

Lemma 2.1. *Let M be an m-dimensional manifold with a separating hypersurface Z. Then there is an isomorphism of real vector bundles*

$$TM \oplus \mathbb{R} \simeq {}^{Z}TM \oplus \mathbb{R}.$$

A *complex structure* on a vector bundle E over a manifold M is a bundle homomorphism $J : E \to E$ such that $J^2 = -\mathrm{Id}$. If E is an orientable rank $2m$ vector bundle, the existence of a complex structure on E is equivalent to the existence of a section of the associated $(\mathrm{SO}(2m)/U(m))$-bundle. A *stable complex structure* on a vector bundle E over M is an equivalence class of complex structures on the vector bundles $E \oplus \mathbb{R}^k$ ($k \in \mathbb{Z}_0^+$), two complex structures, J_1 on $E \oplus \mathbb{R}^{k_1}$ and J_2 on $E \oplus \mathbb{R}^{k_2}$, being *equivalent* when there exist $m_1, m_2 \in \mathbb{Z}_0^+$ such that $((E \oplus \mathbb{R}^{k_1}) \oplus \mathbb{C}^{m_1}, J_1 \oplus \mathrm{i})$ and $((E \oplus \mathbb{R}^{k_2}) \oplus \mathbb{C}^{m_2}, J_2 \oplus \mathrm{i})$ are isomorphic complex vector bundles. A *stable almost complex structure* on M is a stable complex structure on TM.

The Z-tangent bundle for the folding hypersurface Z of a folded symplectic form ω has a canonical complex structure J_0 [**3**] *consistent* with ω. We say that a folded symplectic form ω is *consistent* with a stable almost complex structure on M if $({}^{Z}TM \oplus \mathbb{C}, J_0 \oplus i)$ belongs to the given equivalence class of complex structures on $TM \oplus \mathbb{R}^{2k}$, $k \in \mathbb{Z}_0^+$.

3. First instance of the h-principle

Let M be a $2n$-dimensional manifold with a stable almost complex structure. The homotopy groups $\Pi_q(\mathrm{SO}(2m)/U(m))$ are isomorphic for fixed q and variable m such that $q < 2m - 1$ (this is the so-called *stable range* [**19**]). Hence, if there exists a complex structure on $TM \oplus \mathbb{R}^{2k}$, then there exists a complex structure on $TM \oplus \mathbb{R}^2$.

Let J be a complex structure on $TM \oplus \mathbb{R}^2$. Let

$$\begin{array}{rccc} i: & M & \hookrightarrow & M \times \mathbb{R} \\ & p & \mapsto & (p,0) \end{array} \quad \text{and} \quad \begin{array}{rccc} \pi: & M \times \mathbb{R} & \twoheadrightarrow & M \\ & (p,t) & \mapsto & p \end{array}$$

be the embedding at level zero, and the projection to the first factor. By pullback, i induces an isomorphism in cohomology.

Via the identification $T(M \times \mathbb{R}) \simeq \pi^*(TM) \oplus \mathbb{R}$, the structure J induces a structure of complex vector bundle, still called J, on $T(M \times \mathbb{R}) \oplus \mathbb{R} \simeq \pi^*(TM) \oplus \mathbb{C}$. Then the complex subbundle

$$H_0 = T(M \times \mathbb{R}) \cap J(T(M \times \mathbb{R})) \subset T(M \times \mathbb{R}) \oplus \mathbb{R}$$

is a complex hyperplane field over $M \times \mathbb{R}$. Let ω_1 be a two-form of maximal rank in $M \times \mathbb{R}$ *compatible with* J, that is,

$$\omega_1(u,v) = g(Ju,v) \ , \ \forall u,v \in H_0 \ , \ \text{and } \omega_1(u,\cdot) = 0 \ , \ \forall u \in H_0^{\perp},$$

for some riemannian metric g on $TM \times \mathbb{R}$, where $H_0^{\perp}$ denotes the orthocomplement of H_0 with respect to g. A *regular homotopy* of 2 two-forms of maximal rank is a homotopy within two-forms of maximal rank.

Lemma 3.2. *Let M be a manifold with a structure J of complex vector bundle on $TM \oplus \mathbb{R}^2$. Then there exists in $M \times \mathbb{R}$ a closed two-form of maximal rank in any degree 2 cohomology class, which is regularly homotopic to any two-form of maximal rank compatible with J.*

This is an immediate consequence of the following proposition which was originally proved by McDuff [**21**]. The proof below is taken from Eliashberg-Mishachev [**9**]. We reproduce it since this result is not as widely known as the other applications of the h-principle and since the idea in this proof is crucial for the present paper's strategy. The key to this proof is Gromov's theorem [**12**] saying that, for every degree 2 cohomology class on any open manifold, any nondegenerate two-form is regularly homotopic to a symplectic form in that class; moreover, if two symplectic forms are regularly homotopic, then they are homotopic within symplectic forms. Recall that a manifold is *open* if there are no closed manifolds (i.e., compact and without boundary) among its connected components.

Proposition ([21]). *For any two-form of maximal rank on an odd-dimensional manifold and any degree 2 cohomology class, there exists a closed two-form of maximal rank in that class which is regularly homotopic to the given form.*

Proof. Let ω_1 be a two-form of maximal rank on a $(2n + 1)$-dimensional manifold N and let α be a degree 2 cohomology class in N. By homotopy, the projection to the first factor $\pi : N \times \mathbb{R} \to N$ induces an isomorphism in cohomology.

If N is orientable, then ω_1 extends in a homotopically unique way compatible with orientations to a nondegenerate two-form, ω_2, in $N \times \mathbb{R}$. Gromov's result [**12**] cited above guarantees the existence, in the class $\pi^*\alpha$, of a homotopically unique symplectic form ω_3 in $N \times \mathbb{R}$ regularly homotopic to ω_2. The restriction of ω_3 to the zero level M is a closed two-form of maximal rank.

If N is not orientable, we replace $N \times \mathbb{R}$ in the previous argument by the total space of the real line bundle given by the kernel of ω_1. □

4. Vector bundle isomorphism

Let $\widetilde{\omega}$ be a closed two-form of maximal rank in $M \times \mathbb{R}$, and let L be the line field on $M \times \mathbb{R}$ given by the kernel of $\widetilde{\omega}$ at each point. By orientability of M, the line bundle L is trivializable. Let $\mathcal{L}$ be the one-dimensional foliation corresponding to L. Choose a complementary hyperplane field H so that $T(M \times \mathbb{R}) \simeq H \oplus L$.

Let Z_0 be a separating hypersurface in M with a coorientation. Since by Lemma 2.1 we have that

$$^{Z_0}TM \oplus \mathbb{R} \simeq TM \oplus \mathbb{R} \simeq i^*(H \oplus L),$$

the restriction i^*H is stably isomorphic to ^{Z_0}TM. The Stiefel–Whitney classes are stable invariants, and the mod 2 reduction of the Euler class of an orientable rank m real vector bundle E coincides with the mth Stiefel–Whitney class of E (see, for instance, [**23**]). Therefore, the Euler numbers (i.e., the evaluations of the Euler classes over the fundamental homology class) of i^*H and of ^{Z_0}TM differ by an even integer, let us say

$$\chi(i^*H) = \chi(^{Z_0}TM) + 2k.$$

If two stably isomorphic orientable rank $2n$ real vector bundles over an $2n$-dimensional connected manifold have the same Euler number, then they are isomorphic. This was contained in the work of Dold and Whitney when the base is a four-manifold [**5**]. In general, this follows from observing in the diagram

$$\begin{array}{ccl} & & S^{2n} \hookrightarrow \mathrm{SO}/\mathrm{SO}(2n) \\ & \nearrow & \downarrow \\ M^{2n} & \rightrightarrows & \mathrm{BSO}(2n) \\ & \searrow & \downarrow \\ & & \mathrm{BSO} \end{array}$$

that the fiber $\mathrm{SO}/\mathrm{SO}(2n)$ of $\mathrm{BSO}(2n) \to \mathrm{BSO}$ is $(2n-1)$-connected, that $[M^{2n}, S^{2n}] \simeq \mathbb{Z}$ where the homotopy type is detected by the degree, and that the pullback of the Euler class to S^{2n} is nontrivial (since $S^{2n} \to \mathrm{BSO}(2n)$ is the classifying map for TS^{2n}).

Consider the following operation on rank m real vector bundles over m-dimensional manifolds. If E is such a bundle and D^m is a small disk in the base manifold M, let $E\sharp TS^m$ be the bundle obtained by gluing $E|_{M\setminus \mathrm{Int}\, D^m}$ to the trivial bundle $\mathbb{R}^m$ over D^m by the characteristic map of TS^m, i.e., by the map $S^{m-1} \to \mathrm{SO}(m)$ which characterizes the tangent bundle of S^m as the gluing over the equator of northern and southern trivial bundles [**27**, Section 18.1]. For an integer k, the bundle $E\sharp kTS^m$ is built analogously by taking the kth power of the characteristic map of S^m. By counting with orientations the vanishing points of a section transverse to zero, we see that $E\sharp kTS^m$ has Euler characteristic $\chi(E) + 2k$. We conclude that

$$i^*H \simeq {}^{Z_0}TM\sharp kTS^{2n}.$$

For k positive, let Z be the union of Z_0 with k homologically trivial spheres S^n contained in the negative part of $M \setminus Z_0$ with respect to the given coorientation. For k negative, define Z similarly but with the spheres in the positive part of $M \setminus Z_0$. It follows from the computations in [**6**, Section 3.9] that i^*H and ${}^ZTM$ have the same Euler number, and hence are isomorphic. It is possible to start from the empty hypersurface, in which case a coorientation is not defined. Yet the same argument holds by taking Z to be a union of spheres (as many as half of the absolute value of the difference of the Euler numbers of TM and of i^*H) whose coorientation is determined by the sign of k above. We have thus proved the following:

Lemma 4.3. *Let H be a coorientable hyperplane field in $M \times \mathbb{R}$ and $i : M \hookrightarrow M \times \mathbb{R}$ the inclusion at level zero. The restriction i^*H is isomorphic to ${}^ZTM$, where Z is a separating hypersurface as described in the previous paragraph.*

5. Second instance of the h-principle

Throughout this section, let M be an m-dimensional manifold with a hypersurface Z, and let N be an $(m+1)$-dimensional manifold with a one-dimensional foliation $\mathcal{L}$. The following notions are due to Eliashberg [**7**].

Definition. A map $f : M \to N$ is a *Z-immersion relative to $\mathcal{L}$*, if near any point $p \in M \setminus Z$ there are coordinates $y_1, \ldots, y_{m+1}$ in N adapted to the foliation (i.e. each leaf is a level set of the first m coordinates) where the induced map to each level set of y_{m+1} is regular, and if near any $p \in Z$ and near its image there are coordinates centered at those points and adapted to the foliation where f becomes

$$(x_1, x_2, \ldots, x_m) \longmapsto (x_1^2, x_2, \ldots, x_m, 0).$$

In the adapted coordinates x_i, the hypersurface Z is given by $x_1 = 0$. Loosely speaking, a Z-immersion relative to $\mathcal{L}$ is a Z-immersion to the leaf space of $\mathcal{L}$. The definition extends to higher-dimensional foliations whose codimension is equal to the dimension of M.

Lemma 5.4. *Let $\widetilde{\omega}$ be a closed two-form of maximal rank in N whose kernel is the tangent space to the leaves of $\mathcal{L}$. If $f : M \to N$ is a Z-immersion relative to $\mathcal{L}$, then $f^*\widetilde{\omega}$ is a folded symplectic form on M with folding hypersurface Z.*

The reason is simply that the form $\widetilde{\omega}$ induces a symplectic form in the local leaf spaces and that the composition of f with the local quotient maps is a Z-immersion.

Proof. Let $p \in M$. There is a neighborhood $\mathcal{U}$ of $f(p)$ where we have a trivialization $\mathcal{U} \simeq \mathcal{F}_\mathcal{U} \times \mathcal{L}_\mathcal{U}$, given in local coordinates centered at $f(p)$ by $(x_1, \dots, x_{m+1}) \mapsto ((x_1, \dots, x_m), x_{m+1})$, the set $\mathcal{F}_\mathcal{U}$ being a leaf space (say the level zero of x_{m+1}), and $\mathcal{L}_\mathcal{U}$ a typical leaf (say the level zero of $(x_1, \dots, x_m)$). The restriction of $\widetilde{\omega}$ to $\mathcal{F}_\mathcal{U}$ is a symplectic form, $\omega_\mathcal{U}$. The composition $g_\mathcal{U} : f^{-1}(\mathcal{U}) \to \mathcal{F}_\mathcal{U}$ of f with the projection to $\mathcal{F}_\mathcal{U}$ is a $(Z \cap \mathcal{U})$-immersion, so that $g_\mathcal{U}^* \omega_\mathcal{U}$ is a folded symplectic form with folding hypersurface $Z \cap \mathcal{U}$. The result follows from the fact that $f^* \widetilde{\omega}$ on $f^{-1}(\mathcal{U})$ coincides with $g_\mathcal{U}^* \omega_\mathcal{U}$. □

We now turn to the formal analog of a Z-immersion.

Definition. A bundle map $F : TM \to TN$ is a *Z-monomorphism relative to $\mathcal{L}$*, if $F|_{T(M \setminus Z)}$ is transverse to $\mathcal{L}$, and if each $p \in Z$ admits a neighborhood $\mathcal{U}$ where $F|_{T\mathcal{U}}$ is the differential of some $(Z \cap \mathcal{U})$-immersion relative to $\mathcal{L}$.

The following lemma is a direct consequence of Eliashberg's result in [**7**, Section 6.3], where he extends to the case of foliations the result described in the introduction.

Lemma 5.5. *Let $N = M \times \mathbb{R}$ be equipped with a decomposition $TN \simeq H \oplus L$, where L is a line field, and let $\mathcal{L}$ be the corresponding one-dimensional foliation. Let the hypersurface Z be such that every connected component of $M \setminus Z$ is open. Then, for every Z-monomorphism $F : TM \to TN$ relative to $\mathcal{L}$, there exists a Z-immersion $f : M \to N$ relative to $\mathcal{L}$ whose differential df is homotopic to F through Z-monomorphisms relative to $\mathcal{L}$.*

Part of the work to prove Theorem A consists in showing a (general) procedure to deform by homotopy a weaker bundle map into a Z-monomorphism relative to $\mathcal{L}$. The weaker map is of the following type:

Definition. A bundle map $F : TM \to TN \simeq H \oplus L$ is a *Z-monomorphism relative to L*, if $\pi_L \circ F|_{T(M \setminus Z)}$ and $\pi_L \circ F|_{TZ}$ are fiberwise injective, $\pi_L : TN \to H$ being the projection along L, and if there is a tubular neighborhood $\mathcal{T}$ of Z in M, with a fiber involution $\tau : \mathcal{T} \to \mathcal{T}$ whose set of fixed points is Z, where $F \circ d\tau = F$.

6. Conclusion of the proof

Let M be a compact $2n$-dimensional manifold with a stable almost complex structure J. Then J is representable by a structure of complex vector bundle on $TM \oplus \mathbb{R}^2$, and any two such representatives are isomorphic, by Bott periodicity [**2**]. Let $N = M \times \mathbb{R}$ and denote still by J an induced structure of complex vector bundle on $TN \oplus \mathbb{R}$ as in Section 3.

By Lemma 3.2, there exists on N, in any degree 2 cohomology class, a closed two-form $\widetilde{\omega}$ of maximal rank compatible with J. Let $\widetilde{\omega}$ be such a form and let L be the line field given by its kernel, with associated foliation $\mathcal{L}$.

By Lemma 5.4, the existence of a folded symplectic form on M with some folding hypersurface Z is guaranteed by the existence of a Z-immersion $f : M \to N$ relative to $\mathcal{L}$. We will seek such a Z-immersion which is homotopic to the embedding at level zero $i : M \hookrightarrow N$, so that $f^* = i^*$ in cohomology. If M is connected and Z is nonempty, then $M \setminus Z$ is open.

By Lemma 5.5, in order to produce a Z-immersion f relative to $\mathcal{L}$ for $M \setminus Z$ open, it suffices to show that there exists a Z-monomorphism $F : TM \to TN$ relative to $\mathcal{L}$. So that f is homotopic to i, we search for an F covering a map $M \to N$ homotopic to i.

By Lemma 4.3, we have a vector bundle isomorphism $F_0 : {}^Z TM \to i^*H$ for some hypersurface Z, which may be chosen so that each connected component of $M \setminus Z$ is open.

The map F_0 may be translated into a fiberwise injective bundle map $F_1 : {}^Z TM \to H$ covering the immersion $i : M \to N$. This map guarantees the existence of a (canonically unique up to homotopy) almost Z-monomorphism $F_2 : TM \to H \oplus L$ relative to L, still covering i, defined by the following recipe:

Choose a trivial line bundle V over Z spanned by a vector field on M transverse to Z pointing from M^- to M^+. The quotient ${}^Z TM/V$ is isomorphic to TZ, so that ${}^Z TM|_Z \simeq TZ \oplus V$. We obtain TM by gluing ${}^Z TM|_{M \setminus M^-}$ to ${}^Z TM|_{M \setminus M^+}$ by the constant diagonal map $\mathrm{Id} \oplus (-1) : Z \to \mathrm{GL}(TZ \oplus V)$. Using this recovery of TM from ${}^Z TM$, we may define F_2 equal to $F_1 \oplus 0$ outside a tubular neighborhood $\mathcal{T}$ of Z in M, and on $\mathcal{T}$ set

$$F_2(u \oplus v) = F_1(u \oplus \psi v) \oplus 0,$$

with respect to the decomposition ${}^Z TM|_{\mathcal{T}} \simeq \pi^*(TZ) \oplus \pi^*V$, where $\pi : \mathcal{T} \to Z$ is the tubular projection, and $\psi : \mathcal{T} \to [0,1]$ is equal to 1 outside a narrower tubular neighborhood of Z and vanishes exactly over Z. By choosing ψ symmetric with respect to an involution $\tau : \mathcal{T} \to \mathcal{T}$ whose set of fixed points is Z, we obtain F_2 invariant under τ.

For each $p \in Z$, choose a connected neighborhood $\mathcal{U}$ whose image $i(\mathcal{U})$ is contained in a connected trivialization $\mathcal{N}_{\mathcal{U}} \simeq \mathcal{F}_{\mathcal{U}} \times \mathcal{L}_{\mathcal{U}}$ of the foliation $\mathcal{L}$, the set $\mathcal{F}_{\mathcal{U}}$ being a local leaf space and $\mathcal{L}_{\mathcal{U}}$ a leaf segment. Let $\pi_{\mathcal{U}} : \mathcal{N}_{\mathcal{U}} \to \mathcal{F}_{\mathcal{U}}$ be the projection to the first factor. The composition $F_{2,\mathcal{U}} = d\pi_{\mathcal{U}} \circ F_2|_{\mathcal{U}} : T\mathcal{U} \to T\mathcal{F}_{\mathcal{U}}$ is a $(Z \cap \mathcal{U})$-monomorphism.

$$\begin{array}{ccc} & & T\mathcal{N}_{\mathcal{U}} \\ & {\scriptstyle F_2} \nearrow & \downarrow{\scriptstyle d\pi_{\mathcal{U}}} \\ T\mathcal{U} & \xrightarrow{F_{2,\mathcal{U}}} & T\mathcal{F}_{\mathcal{U}} \end{array}$$

By Eliashberg [**6**, Section 2.2], the composition $F_{2,\mathcal{U}}$ is homotopic, through $(Z\cap\mathcal{U})$-monomorphisms, to the differential $dg_{\mathcal{U}}$ of a Z-immersion $g_{\mathcal{U}}:\mathcal{U}\to\mathcal{F}_{\mathcal{U}}$. Moreover, if over a closed subset $\mathcal{W}\subset\mathcal{U}$, the composition $F_{2,\mathcal{U}}$ was already the differential of a map, then there is a homotopy which is constant on $\mathcal{W}$. Let $G_t:T\mathcal{U}\to T\mathcal{F}_{\mathcal{U}}$, $1\le t\le 2$, be a homotopy such that $G_1=dg_{\mathcal{U}}$ and $G_2=F_{2,\mathcal{U}}$.

Choose a $(Z\cap\mathcal{U})$-immersion $\widetilde{g}_{\mathcal{U}}:\mathcal{U}\to\mathcal{N}_{\mathcal{U}}$ relative to $\mathcal{L}$ such that $\pi_{\mathcal{U}}\circ\widetilde{g}_{\mathcal{U}}=g_{\mathcal{U}}$. We can always pick a $\widetilde{g}_{\mathcal{U}}$ extending a sensible preassigned lift over a closed subset $\mathcal{W}$ of $\mathcal{U}$.

By the covering homotopy property for the fibering $T\mathcal{N}_{\mathcal{U}}\to T\mathcal{F}_{\mathcal{U}}$, there is a lifted homotopy $\widetilde{G}_t:T\mathcal{U}\to T\mathcal{N}_{\mathcal{U}}$, $1\le t\le 2$, through Z-monomorphisms relative to L such that $\widetilde{G}_1=d\widetilde{g}_{\mathcal{U}}$ and $d\pi_{\mathcal{U}}\circ\widetilde{G}_t=G_t$ for all t. If G_t was constant on a closed subset $\mathcal{W}$, then we may choose $\widetilde{G}_t$ also constant on $\mathcal{W}$.

$$\begin{array}{ccc} & & T\mathcal{N}_{\mathcal{U}} \\ & {\scriptstyle \widetilde{G}_t}\nearrow & \downarrow{\scriptstyle d\pi_{\mathcal{U}}} \\ T\mathcal{U} & \xrightarrow{G_t} & T\mathcal{F}_{\mathcal{U}} \end{array}$$

Since $d\pi_{\mathcal{U}}\circ\widetilde{G}_2=G_2=F_{2,\mathcal{U}}=d\pi_{\mathcal{U}}\circ F_2$, the difference $\widetilde{G}_2-F_2$ takes values in $L=\ker d\pi_{\mathcal{U}}$. By fiberwise homotopy, we may deform the vertical component of $\widetilde{G}_2$ to make it equal to F_2. Without loss of generality, we hence assume that $\widetilde{G}_t$ also satisfies $\widetilde{G}_2=F_2$, and that all maps are invariant with respect to the same involution τ.

Take a riemannian metric symmetric with respect to τ. For a point $p\in Z$, choose spherical neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathcal{T}$, consisting of points at a riemannian distance less than ε and 4ε from p, with $\varepsilon>0$ small enough for the exponential map to be injective and for the closure of $\mathcal{U}_2$ to be contained in the neighborhood $\mathcal{U}$ above. Choose a smooth function $\rho:\mathcal{U}_2\to[1,2]$ satisfying $\rho(q)=2$ if the distance from p to q is greater than 3ε, and $\rho(q)=1$ if the distance from p to q is less than 2ε. Define $F_3:TM\to TN$ by

$$F_3=\begin{cases} F_2 & \text{on } M\setminus\mathcal{U}_2,\\ \widetilde{G}_{\rho(q)} & \text{over points } q\in\mathcal{U}_2\setminus\mathcal{U}_1,\\ d\widetilde{g}_{\mathcal{U}} & \text{on } \mathcal{U}_1.\end{cases}$$

Then F_3 is a Z-monomorphism with respect to L whose restriction to $\mathcal{U}_1$ is the differential of a $(Z\cap\mathcal{U}_1)$-immersion relative to $\mathcal{L}$.

Since Z is compact, take a subcover of Z in M by a finite number of the $\mathcal{U}_1$'s. Apply iteratively the construction of the previous paragraph to an ordering of the $\mathcal{U}_1$'s, starting first from F_2 and then from its replacements F_3, etc. At each stage, the homotopy should be taken constant over the closure $\mathcal{W}$ of the previous $\mathcal{U}_1$'s.

We have thus concluded the proof of Theorem A in the compact case by showing the existence of a Z-monomorphism relative to $\mathcal{L}$ covering a map homotopic to i.

Remark. If M is a compact oriented two-dimensional manifold, folded symplectic forms on M are generic two-forms. The cohomology class of a two-form is determined by its total integral. The isomorphism classes of complex structures on $TM \oplus \mathbb{R}^2$ are determined by the Euler number, which is an even integer. By changing Z as in Section 4, any even number may be obtained as Euler number for ${}^Z TM$, thus fitting any given stable complex structure. Let ω be a two-form which vanishes transversally on an appropriate Z. By changing the values of ω over $M \setminus Z$, any real number may be obtained as total integral of ω. Hence, Theorem A holds easily (and not interestingly) for compact two-manifolds.

For the noncompact case, a statement stronger than Theorem A is true. If a $2n$-dimensional manifold M is orientable, connected, not compact and $TM \oplus \mathbb{R}^2$ has a complex structure, then M has an almost complex structure because it retracts to a $(2n-1)$-dimensional cell complex [**22**, Theorem 8.1] and $\Pi_q(\mathrm{SO}(2n)/U(n)) \simeq \Pi_q(\mathrm{SO}(2n+2)/U(n+1))$ for $q \leq 2n-2$. By Gromov's theorem [**12**], M admits a compatible symplectic form in any degree 2 cohomology class.

Let E be a rank $2m$ oriented real bundle over M. The condition $W_3(E) = 0$ ensures the existence over the three-skeleton of M of a section for the associated $(\mathrm{SO}(2m)/U(m))$-bundle. By Bott's periodicity, $\Pi_q(\mathrm{SO}(6)/U(3)) = 0$ for $q < 5$. Therefore, the Hirzebruch–Hopf fact [**15**] that $W_3(M) = 0$ for any orientable four-manifold asserts the existence of a stable complex structure on any such manifold.

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