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Shrinking good coordinate systems associated to Kuranishi structures

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The notion of good coordinate system was introduced by Fukaya and Ono in [FOn] in their construction of virtual fundamental chain via Kuranishi structure which was also introduced therein. This notion was further clarified in [FOOO1] in some detail. In those papers no explicit ambient space was used and hence the process of gluing local Kuranishi charts in the given good coordinate system was not discussed there. In our more recent writing [FOOO2, FOOO3], we use an ambient space obtained by gluing the Kuranishi charts. In this note we prove in detail that we can always shrink the given good coordinate system so that the resulting 'ambient space' becomes Hausdorff. This note is self-contained and uses only standard facts in general topology.

1. Introduction

In [FOn, FOOO1] the present authors associated a virtual fundamental chain to a space with Kuranishi structure. For the construction we used the notion of *good coordinate system*. The process of constructing a good coordinate system out of Kuranishi structure corresponds to that of choosing and fixing an atlas consisting of a locally finite covering of coordinate charts in the manifold theory.

In [FOn, FOOO1] the process to associate the virtual fundamental chain to a space with good coordinate system, is described *without* using 'ambient space', that is, the space obtained by gluing Kuranishi charts by coordinate change. In our more recent writing, [FOOO2, FOOO3], which contains further detail of this construction, we describe the same process using 'ambient space', explicitly. For the description of the construction of virtual fundamental chain using ambient space, certain properties, especially Hausdorffness, of the ambient space is necessary.

In [FOn, FOOO1], the tools of Kuranishi structure and its associated good coordinate system are applied to study moduli spaces of stable maps. The moduli space of stable maps can be very singular in general but we can embed a small portion thereof at each point of the moduli space locally into an orbifold which is called a Kuranishi neighborhood. An element of a Kuranishi neighborhood appearing in such applications is a 'map' with domain a nodal curve satisfying a differential equation, that is, a slightly perturbed Cauchy-Riemann equation. To write down this perturbed Cauchy-Riemann equation, one needs to fix various extra data locally in our moduli space. Because of this reason, the union of Kuranishi neighborhoods cannot be globally regarded as a subset of certain well-defined set of maps, and gluing *the given* Kuranishi neighborhoods to construct an ambient space a priori may not make sense. The main result of the present article is to show that we can, however, always shrink the given Kuranishi neighborhoods and the domains of coordinate change and glue the resulting shrinked neighborhoods to obtain certain reasonable space, which one may call an 'ambient space' or a 'virtual neighborhood'. It also shows that we can always do so, *after some shrinking*, by employing only elementary general topology arguments, with the originally given definition of good coordinate system in [FOn, FOOO1].

Our purpose of writing this short note is to separate the abstract combinatorial general topology issue from other parts of the story of Kuranishi structure given in [FOOO3] and its implementations, and to clarify the parts of general topology. This note is self-contained and can be read independently of the previous knowledge of Kuranishi structures.

2. Statement

To make it clear that the arguments of this note do not involve the properties of orbifolds, vector bundles on them, the smoothness of the coordinate change and others, we introduce the following abstract notions that lie in the realm of general topology and not of manifold theory.

In this note, X is always assumed to be a locally compact separable metrizable space.

Definition 2.1. An *abstract K-chart* of X consists of $\mathcal{U} = (U, S, \psi)$ where U is a locally compact separable metrizable space, $S \subseteq U$ is a closed subset and $\psi : S \to X$ is a homeomorphism onto an open subset.

Definition 2.2. Let $\mathcal{U}_i = (U_i, S_i, \psi_i)$ $(i = 1, 2)$ be abstract K-charts of X. A *coordinate change* from $\mathcal{U}_1$ to $\mathcal{U}_2$ consists of $\Phi_{21} = (U_{21}, \varphi_{21})$ such that:

1) $U_{21} \subseteq U_1$ is an open set.

2) $\varphi_{21} : U_{21} \to U_2$ is a topological embedding, i.e., a continuous map which is a homeomorphism onto its image.

3) $S_1 \cap U_{21} = \varphi_{21}^{-1}(S_2)$. Moreover $\psi_2 \circ \varphi_{21} = \psi_1$ on $S_1 \cap U_{21}$ (i.e., whenever both are defined).

4) $\psi_1(S_1 \cap U_{21}) = \psi_1(S_1) \cap \psi_2(S_2)$.

Definition 2.3. Let $Z \subseteq X$ be a compact subset. An *abstract good coordinate system of Z in the weak sense* is $\widehat{\mathcal{U}} = (\mathfrak{P}, \{\mathcal{U}_\mathfrak{p}\}, \{\Phi_{\mathfrak{pq}}\})$ with the following properties.

1) $\mathfrak{P}$ is a partially ordered set. We assume $\mathfrak{P}$ is a finite set.

2) For $\mathfrak{p} \in \mathfrak{P}$, $\mathcal{U}_\mathfrak{p} = (U_\mathfrak{p}, S_\mathfrak{p}, \psi_\mathfrak{p})$ is an abstract K-chart.

3) If $\mathfrak{q} \le \mathfrak{p}$ then a coordinate change $\Phi_{\mathfrak{pq}} = (U_{\mathfrak{pq}}, \varphi_{\mathfrak{pq}})$ from $\mathcal{U}_\mathfrak{q}$ to $\mathcal{U}_\mathfrak{p}$ in the sense of Definition 2.2 is defined. We require $U_{\mathfrak{pp}} = U_\mathfrak{p}$ and $\varphi_{\mathfrak{pp}}$ to be the identity map.

4) If $\mathfrak{r} \le \mathfrak{q} \le \mathfrak{p}$ then $\varphi_{\mathfrak{pr}} = \varphi_{\mathfrak{pq}} \circ \varphi_{\mathfrak{qr}}$ on $U_{\mathfrak{pqr}} := \varphi_{\mathfrak{qr}}^{-1}(U_{\mathfrak{pq}}) \cap U_{\mathfrak{pr}}$ (i.e., whenever both are defined).

5) If $\psi_\mathfrak{p}(S_\mathfrak{p}) \cap \psi_\mathfrak{q}(S_\mathfrak{q}) \neq \emptyset$ then either $\mathfrak{p} \le \mathfrak{q}$ or $\mathfrak{q} \le \mathfrak{p}$ holds.

6) $\bigcup \psi_\mathfrak{p}(S_\mathfrak{p}) \supseteq Z$.

Definition 2.4. Let $\widehat{\mathcal{U}} = (\mathfrak{P}, \{\mathcal{U}_\mathfrak{p}\}, \{\Phi_{\mathfrak{pq}}\})$ be an abstract good coordinate system of Z in the weak sense. We consider the disjoint union $\coprod_\mathfrak{p} U_\mathfrak{p}$ and define a relation $\sim$ on it as follows. Let $x \in U_\mathfrak{p}, y \in U_\mathfrak{q}$. We say $x \sim y$ if one of the following holds. We put $\Phi_{\mathfrak{pq}} = (U_{\mathfrak{pq}}, \varphi_{\mathfrak{pq}})$.

(a) $\mathfrak{p} = \mathfrak{q}$ and $x = y$.

(b) $\mathfrak{p} \le \mathfrak{q}$, $x \in U_{\mathfrak{qp}}$ and $y = \varphi_{\mathfrak{qp}}(x)$.

(c) $\mathfrak{q} \le \mathfrak{p}$, $y \in U_{\mathfrak{pq}}$ and $x = \varphi_{\mathfrak{pq}}(y)$.

We remark that by the requirement given in the second half of Definition 2.3 3), the case (a) is redundant in that it is a special case of either of (b) or (c) but we state it separately for the semantic reason.

Definition 2.5. An abstract good coordinate system of Z in the weak sense $\widehat{\mathcal{U}} = (\mathfrak{P}, \{\mathcal{U}_\mathfrak{p}\}, \{\Phi_{\mathfrak{pq}}\})$ is said to be an *abstract good coordinate system of Z in the strong sense* if the following holds.

7) The relation $\sim$ is an equivalence relation.

8) The quotient space $(\coprod_{\mathfrak p} U_{\mathfrak p})/\sim$ is Hausdorff with respect to the quotient topology.

We denote by $|\widehat{\mathcal U}|$ the quotient space $(\coprod_{\mathfrak p} U_{\mathfrak p})/\sim$ equipped with quotient topology.

Remark 2.6. Suppose $\mathfrak p < \mathfrak q < \mathfrak r$ and $x \in U_{\mathfrak p}$, $y \in U_{\mathfrak q}$, $z \in U_{\mathfrak r}$. We assume $x \sim y$ and $x \sim z$. Then, by definition, $x \in U_{\mathfrak q \mathfrak p}$, $y = \varphi_{\mathfrak q \mathfrak p}(x)$. Moreover $x \in U_{\mathfrak r \mathfrak p}$, $z = \varphi_{\mathfrak r \mathfrak p}(x)$. Therefore if $y \in U_{\mathfrak r \mathfrak q}$ in addition, then Definition 2.3 4) implies $z = \varphi_{\mathfrak r \mathfrak q}(y)$, and hence $z \sim y$. Namely the transitivity holds in this case.

However the relation $y \in U_{\mathfrak r \mathfrak q}$ may not be satisfied in general. This is a reason why Definition 2.5 7) does not follow from Definition 2.3 1) – 6).

Example 2.7. Suppose $\mathfrak P = \{1, 2\}$ with $1 < 2$, $U_1 = U_2 = \mathbb R$, $U_{21} = (-1, 1)$. $\varphi_{21} : (-1, 1) \to \mathbb R$ is the inclusion map. We also take $S_1 = S_2 = X = Z = \{0\}$ and $\psi_1 = \psi_2$ is the identity map.

They satisfy Definition 2.3 1) – 6) and Definition 2.5 7). However the space $U_1 \sqcup U_2/\sim$ is not Hausdorff. In fact $1 \in U_1$ and $1 \in U_2$ do not have separating neighborhoods.

Definition 2.8. 1) Let V be an open subset of a separable metrizable space U. We say that V is a *shrinking* of U and write $V \Subset U$, if V is relatively compact in U, i.e., the closure $\overline{V}$ in U is compact.[1]

2) Let $\mathcal U = (U, S, \psi)$ be an abstract K chart and $U_0 \subseteq U$ be an open subset. We put $\mathcal U|_{U_0} = (U_0, S \cap U_0, \psi|_{S \cap U_0})$. This is an abstract K chart. If $U_0 \Subset U$, we say $\mathcal U|_{U_0}$ is a *shrinking* of $\mathcal U$.

3) Let $\widehat{\mathcal U} = (\mathfrak P, \{\mathcal U_{\mathfrak p}\}, \{\Phi_{\mathfrak p \mathfrak q}\})$ be an abstract good coordinate system of Z in the weak sense. We say an abstract good coordinate system $\widehat{\mathcal U}^0 = (\mathfrak P, \{\mathcal U^0_{\mathfrak p}\}, \{\Phi^0_{\mathfrak p \mathfrak q}\})$ of Z in the weak sense is a shrinking of $\widehat{\mathcal U}$ if the following hold:
   a) Each of $\mathcal U^0_{\mathfrak p}$ is a shrinking of $\mathcal U_{\mathfrak p}$.
   b) For $\mathfrak p \ge \mathfrak q$, the domain of $\Phi^0_{\mathfrak p \mathfrak q}$ is a shrinking of the domain of $\Phi_{\mathfrak p \mathfrak q}$ (namely $U^0_{\mathfrak p \mathfrak q} \Subset U_{\mathfrak p \mathfrak q}$) and $\Phi^0_{\mathfrak p \mathfrak q}$ is a restriction of $\Phi_{\mathfrak p \mathfrak q}$.

Theorem 2.9 (Shrinking Lemma). *Suppose $\widehat{\mathcal U} = (\mathfrak P, \{\mathcal U_{\mathfrak p}\}, \{\Phi_{\mathfrak p \mathfrak q}\})$ is an abstract good coordinate system of Z in the weak sense. Then there exists a shrinking $\widehat{\mathcal U}^0$ of $\widehat{\mathcal U}$ that becomes an abstract good coordinate system of Z in the strong sense.*

[1] We remark in a rare situation where V is both open and compact it may happen $V \Subset U$ and $V = U$.

Remark 2.10. Suppose (V, E, Γ, s, ψ) is a Kuranishi neighborhood in the sense of [FOOO1, Definition A1.1] or [FOn, Definition 6.1]. Then the triple $(V/\Gamma, s^{-1}(0)/\Gamma, \psi)$ is an abstract K-chart in the sense of Definition 2.1. It is easy to see that a coordinate change in the sense of [FOOO1, (A1.12)] or [FOn, Definition 6.1] induces a coordinate change in the sense of Definition 2.2.[2]

Thus a good coordinate system in the sense of [FOOO1, Lemma A1.11] or [FOn, Definition 6.1] induces an abstract good coordinate system in the weak sense (of X) of Definition 2.3.

The two conditions 7), 8) appearing in Definition 2.5 is exactly the same as the conditions 7), 8) in [FOOO3, Definition 3.14].

Thus Theorem 2.9 implies that we can always shrink a good coordinate system in the sense of [FOOO1, Lemma A1.11] or [FOn, Definition 6.1] to obtain one in the sense of [FOOO3, Definition 3.14].

Note Theorem 2.9 is used during the proof of [FOOO3, Theorem 3.30], which claims the existence of good coordinate system.

We will also prove the following:

Proposition 2.11. *Let $\widehat{\mathcal{U}} = (\mathfrak{P}, \{\mathcal{U}_{\mathfrak{p}}\}, \{\Phi_{\mathfrak{pq}}\})$ be an abstract good coordinate system in the strong sense of Z. Let $U'_{\mathfrak{p}} \Subset U_{\mathfrak{p}}$ chosen for each $\mathfrak{p}$. (Here $\mathcal{U}_{\mathfrak{p}} = (U_{\mathfrak{p}}, S_{\mathfrak{p}}, \psi_{\mathfrak{p}})$.) We consider the image $U'_{\mathfrak{p}} \to |\widehat{\mathcal{U}}|$ and denote it by the same symbol $U'_{\mathfrak{p}}$. Then the union*

$$U' = \bigcup_{\mathfrak{p} \in \mathfrak{P}} U'_{\mathfrak{p}} \subseteq |\widehat{\mathcal{U}}|$$

is separable and metrizable with respect to the induced topology.

Remark 2.12. Note U' can also be written as $\coprod U'_{\mathfrak{p}}/\sim$ for certain equivalence relation $\sim$. However we do not equip it with the quotient topology but with the subspace topology of the quotient topology on $|\widehat{\mathcal{U}}|$.

3. Proof of the main theorem

Lemma 3.1. *Let $\widehat{\mathcal{U}} = (\mathfrak{P}, \{\mathcal{U}_{\mathfrak{p}}\}, \{\Phi_{\mathfrak{pq}}\})$ be an abstract good coordinate system of Z in the weak sense and $U^0_{\mathfrak{p}} \subseteq U_{\mathfrak{p}}$, $U^0_{\mathfrak{pq}} \subseteq U_{\mathfrak{pq}}$ be open subsets. We*

[2] Note that Definition 2.2 4) is required for coordinate changes appearing in good coordinate systems.

assume

$$\varphi_{\mathfrak{pq}}^{-1}(U_{\mathfrak p}^0) \cap U_{\mathfrak q}^0 \cap S_{\mathfrak q} \subseteq U_{\mathfrak{pq}}^0 \subseteq \varphi_{\mathfrak{pq}}^{-1}(U_{\mathfrak p}^0) \cap U_{\mathfrak q}^0 \tag{3.1}$$

for $\mathfrak q \le \mathfrak p$ *and*

$$\bigcup_{\mathfrak p \in \mathfrak P} \psi_{\mathfrak p}(U_{\mathfrak p}^0) \supseteq Z. \tag{3.2}$$

Then $\widehat{\mathcal U_0} = (\mathfrak P, \{\mathcal U_{\mathfrak p}|_{U_{\mathfrak p}^0}\}, \{\Phi_{\mathfrak{pq}}|_{U_{\mathfrak{pq}}^0}\})$ *is an abstract good coordinate system of* Z *in the weak sense.*

Proof. We first show that $\Phi_{\mathfrak{pq}}|_{U_{\mathfrak{pq}}^0}$ is a coordinate change: $\mathcal U_{\mathfrak q}|_{U_{\mathfrak q}^0} \to \mathcal U_{\mathfrak p}|_{U_{\mathfrak p}^0}$. Definition 2.2 1), 2) are obvious. Definition 2.2 3) follows from

$$\begin{aligned}
\varphi_{\mathfrak{pq}}^{-1}(U_{\mathfrak p}^0 \cap S_{\mathfrak p}) \cap U_{\mathfrak{pq}}^0 &= \varphi_{\mathfrak{pq}}^{-1}(S_{\mathfrak p}) \cap \varphi_{\mathfrak{pq}}^{-1}(U_{\mathfrak p}^0) \cap U_{\mathfrak{pq}}^0 \\
&= S_{\mathfrak q} \cap U_{\mathfrak{pq}} \cap \varphi_{\mathfrak{pq}}^{-1}(U_{\mathfrak p}^0) \cap U_{\mathfrak{pq}}^0 \\
&= S_{\mathfrak q} \cap \varphi_{\mathfrak{pq}}^{-1}(U_{\mathfrak p}^0) \cap U_{\mathfrak{pq}}^0 \\
&= S_{\mathfrak q} \cap U_{\mathfrak q}^0 \cap \varphi_{\mathfrak{pq}}^{-1}(U_{\mathfrak p}^0) \cap U_{\mathfrak{pq}}^0 \\
&= S_{\mathfrak q} \cap U_{\mathfrak{pq}}^0.
\end{aligned}$$

The second equality is Definition 2.2 3) for $\Phi_{\mathfrak{pq}}$ and the last equality follows from the second inclusion of (3.1).

We next prove Definition 2.2 4). Let $\mathfrak q \le \mathfrak p$. (3.1) implies

$$S_{\mathfrak q} \cap U_{\mathfrak q}^0 \cap \varphi_{\mathfrak{pq}}^{-1}(U_{\mathfrak p}^0) = S_{\mathfrak q} \cap U_{\mathfrak{pq}}^0$$

Therefore using the fact $\varphi_{\mathfrak{pq}}^{-1}(S_{\mathfrak p}) \subseteq S_{\mathfrak q}$, we have

$$S_{\mathfrak q} \cap U_{\mathfrak q}^0 \cap \varphi_{\mathfrak{pq}}^{-1}(S_{\mathfrak p} \cap U_{\mathfrak p}^0) = S_{\mathfrak q} \cap U_{\mathfrak{pq}}^0.$$

Thus Definition 2.2 4) holds.

We thus checked Definition 2.3 3). Definition 2.3 1),2),4),5) follow from the corresponding properties of $\widehat{\mathcal U}$. Definition 2.3 6) is a consequence of (3.2). □

Remark 3.2. In fact, the converse of Lemma 3.1 also holds. More precisely, if $U_{\mathfrak p}^0 \subseteq U_{\mathfrak p}$, $U_{\mathfrak{pq}}^0 \subseteq U_{\mathfrak{pq}}$ are open subsets such that $\widehat{\mathcal U_0} = (\mathfrak P, \{\mathcal U_{\mathfrak p}|_{U_{\mathfrak p}^0}\}, \{\Phi_{\mathfrak{pq}}|_{U_{\mathfrak{pq}}^0}\})$ is an abstract good coordinate system of Z in the weak sense, then (3.1) and (3.2) hold.

Lemma 3.3. *Let* $\widehat{\mathcal{U}} = (\mathfrak{P}, \{\mathcal{U}_{\mathfrak{p}}\}, \{\Phi_{\mathfrak{pq}}\})$ *be an abstract good coordinate system of* Z *in the weak sense. Then there exist compact subsets* $K_{\mathfrak{p}} \subseteq X$ *such that*

$$\bigcup_{\mathfrak{p}\in\mathfrak{P}} K_{\mathfrak{p}} \supseteq Z, \qquad K_{\mathfrak{p}} \subseteq \psi_{\mathfrak{p}}(S_{\mathfrak{p}}). \tag{3.3}$$

Proof. Since $\bigcup_{\mathfrak{p}\in\mathfrak{P}} \psi_{\mathfrak{p}}(S_{\mathfrak{p}}) \supseteq Z$ is an open covering, for any $x \in Z$ there exist its neighborhood U_x and $\mathfrak{p}(x) \in \mathfrak{P}$ such that $U_x \Subset \psi_{\mathfrak{p}(x)}(S_{\mathfrak{p}(x)})$. We cover our compact set Z by finitely many $\{U_{x_\ell} \mid \ell = 1, \dots, L\}$ of them. Then $K_{\mathfrak{p}} := \bigcup_{\ell;\mathfrak{p}(x_\ell)=\mathfrak{p}} \overline{U}_{x_\ell}$ has the required properties. □

Proposition 3.4. *Any abstract good coordinate system of* Z *in the weak sense has a shrinking.*

Proof. Let $\widehat{\mathcal{U}} = (\mathfrak{P}, \{\mathcal{U}_{\mathfrak{p}}\}, \{\Phi_{\mathfrak{pq}}\})$ be an abstract good coordinate system of Z in the weak sense. We take compact subsets $K_{\mathfrak{p}}$ satisfying (3.3). Since $\psi_{\mathfrak{p}}$ is a topological embedding $\psi_{\mathfrak{p}}^{-1}(K_{\mathfrak{p}})$ is compact. There exists $U_{\mathfrak{p}}^0$ such that $\psi_{\mathfrak{p}}^{-1}(K_{\mathfrak{p}}) \subseteq U_{\mathfrak{p}}^0 \Subset U_{\mathfrak{p}}$, since $U_{\mathfrak{p}}$ is locally compact. Then (3.2) is satisfied. We put

$$A_{\mathfrak{pq}}^0 = S_{\mathfrak{q}} \cap \varphi_{\mathfrak{pq}}^{-1}(U_{\mathfrak{p}}^0) \cap U_{\mathfrak{q}}^0. \tag{3.4}$$

Let $A_{\mathfrak{pq}}$ be its closure in $U_{\mathfrak{q}}$.

Lemma 3.5. $A_{\mathfrak{pq}} \subseteq U_{\mathfrak{pq}}$ *and* $A_{\mathfrak{pq}}$ *is compact.*

Proof. Compactness of $A_{\mathfrak{pq}}$ immediately follows since $A_{\mathfrak{pq}} = \overline{A_{\mathfrak{pq}}^0} \subset \overline{U_{\mathfrak{q}}^0}$.

We now prove $A_{\mathfrak{pq}} \subseteq U_{\mathfrak{pq}}$. Let $x_a \in A_{\mathfrak{pq}}^0$ be a sequence. We will prove that it has a subsequence which converges to an element of $U_{\mathfrak{pq}}$. Since $x_a \in U_{\mathfrak{q}}^0 \Subset U_{\mathfrak{q}}$ we may assume that $x \in U_{\mathfrak{q}}$ is its limit. By definition of $A_{\mathfrak{pq}}^0$, $y_a := \varphi_{\mathfrak{pq}}(x_a) \in S_{\mathfrak{p}} \cap U_{\mathfrak{p}}^0$. Since $U_{\mathfrak{p}}^0$ is relatively compact in $U_{\mathfrak{p}}$, there is a subsequence of $\{y_a\}$ such that it converges to some $y \in U_{\mathfrak{p}}$. On the other hand, by Definition 2.2 3), $\psi_{\mathfrak{p}}(y_a) = \psi_{\mathfrak{q}}(x_a)$. Then by continuity of $\psi_{\mathfrak{p}} : S_{\mathfrak{p}} \to X$, $\psi_{\mathfrak{q}} : S_{\mathfrak{q}} \to X$, $\psi_{\mathfrak{q}}(x) = \psi_{\mathfrak{p}}(y)$. (We use the fact that X is Hausdorff here.) Obviously this point is contained in $\psi_{\mathfrak{p}}(S_{\mathfrak{p}}) \cap \psi_{\mathfrak{q}}(S_{\mathfrak{q}})$ which is equal to $\psi_{\mathfrak{q}}(S_{\mathfrak{q}} \cap U_{\mathfrak{pq}})$ by Definition 2.2 4). By the injectivity of $\psi_{\mathfrak{q}}$ on $S_{\mathfrak{q}}$, this implies $x \in U_{\mathfrak{pq}}$. This finishes the proof. □

Using Lemma 3.5 and the local compactness of $U_{\mathfrak{pq}}$, we then take $V^0_{\mathfrak{pq}}$ such that

$$A_{\mathfrak{pq}} \subseteq V^0_{\mathfrak{pq}} \Subset U_{\mathfrak{pq}} \tag{3.5}$$

and put

$$U^0_{\mathfrak{pq}} = V^0_{\mathfrak{pq}} \cap \varphi_{\mathfrak{pq}}^{-1}(U^0_{\mathfrak{p}}) \cap U^0_{\mathfrak{q}}.$$

Since $A^0_{\mathfrak{pq}} \subseteq \varphi_{\mathfrak{pq}}^{-1}(U^0_{\mathfrak{p}}) \cap U^0_{\mathfrak{q}}$, (3.5) implies $A^0_{\mathfrak{pq}} \subseteq U^0_{\mathfrak{pq}} \Subset U_{\mathfrak{pq}}$. Since $U^0_{\mathfrak{p}}$ and $U^0_{\mathfrak{pq}}$ satisfy (3.1) and (3.2), Proposition 3.4 follows from Lemma 3.1. □

We start the proof of the main theorem. We take a shrinking $\widehat{\mathcal{U}_1} = (\mathfrak{P}, \{\mathcal{U}_{\mathfrak{p}}|_{U^1_{\mathfrak{p}}}\}, \{\Phi_{\mathfrak{pq}}|_{U^1_{\mathfrak{pq}}}\})$ of given $\widehat{\mathcal{U}} = (\mathfrak{P}, \{\mathcal{U}_{\mathfrak{p}}\}, \{\Phi_{\mathfrak{pq}}\})$. We put

$$\varphi^1_{\mathfrak{pq}} = \varphi_{\mathfrak{pq}}|_{U^1_{\mathfrak{pq}}}. \tag{3.6}$$

We apply Lemma 3.3 to $\widehat{\mathcal{U}_1}$ to obtain $K_{\mathfrak{p}}$. We take a metric $d_{\mathfrak{p}}$ of $U_{\mathfrak{p}}$ and put:

$$U^\delta_{\mathfrak{p}} = \{x \in U^1_{\mathfrak{p}} \mid d_{\mathfrak{p}}(x, \psi_{\mathfrak{p}}^{-1}(K_{\mathfrak{p}})) < \delta\}. \tag{3.7}$$

Since $\psi_{\mathfrak{p}}^{-1}(K_{\mathfrak{p}})$ is compact and $U^1_{\mathfrak{p}}$ is locally compact, $U^\delta_{\mathfrak{p}} \Subset U^1_{\mathfrak{p}}$ for sufficiently small δ.

We use the next lemma several times in this section.

Lemma 3.6. *Suppose $\mathfrak{q} \le \mathfrak{p}$, $\delta_n \to 0$ and $x_n \in U^{\delta_n}_{\mathfrak{q}} \cap (\varphi^1_{\mathfrak{pq}})^{-1}(U^{\delta_n}_{\mathfrak{p}})$. Then there exists a subsequence of x_n, still denoted by x_n, such that:*

1) *x_n converges to $x \in S_{\mathfrak{q}}$.*
2) *$\varphi^1_{\mathfrak{pq}}(x_n)$ converges to $y \in S_{\mathfrak{p}}$.*
3) *$\psi_{\mathfrak{q}}(x) = \psi_{\mathfrak{p}}(y) \in K_{\mathfrak{p}} \cap K_{\mathfrak{q}}$.*
4) *$x \in U^1_{\mathfrak{pq}}$ and $y = \varphi^1_{\mathfrak{pq}}(x)$.*

Proof. Let $\delta_0 > 0$ be a fixed sufficiently small constant and consider $\delta > 0$ with $\delta < \delta_0$. Since $U^\delta_{\mathfrak{p}} \Subset U^{\delta_0}_{\mathfrak{p}}$ and $U^\delta_{\mathfrak{q}} \Subset U^{\delta_0}_{\mathfrak{q}}$ for small δ, we may take a subsequence such that x_n and $\varphi^1_{\mathfrak{pq}}(x_n)$ converge to $x \in U^{\delta_0}_{\mathfrak{q}}$ and $y \in U^{\delta_0}_{\mathfrak{p}}$, respectively.

Then (3.7) implies $x \in \psi_{\mathfrak{q}}^{-1}(K_{\mathfrak{q}})$ and $y \in \psi_{\mathfrak{p}}^{-1}(K_{\mathfrak{p}})$. We have proved 1) and 2).

Since $x_n \in U^1_{\mathfrak{pq}} \Subset U_{\mathfrak{pq}}$, its limit x is in $U_{\mathfrak{pq}}$. Since $\varphi_{\mathfrak{pq}}$ is defined on $U_{\mathfrak{pq}}$ and is continuous, we have $\varphi_{\mathfrak{pq}}(x) = \varphi_{\mathfrak{pq}}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \varphi_{\mathfrak{pq}}(x_n) =$

y. Then by Definition 2.2 3) we have $\psi_{\mathfrak{q}}(x) = \psi_{\mathfrak{p}}(y)$. Note $\psi_{\mathfrak{q}}(x) \in K_{\mathfrak{q}}$ and $\psi_{\mathfrak{p}}(y) \in K_{\mathfrak{p}}$. Therefore 3) holds.

Then $x \in U^1_{\mathfrak{pq}}$ follows from Definition 2.2 4) and $K_{\mathfrak{p}} \subseteq \psi_{\mathfrak{p}}(S_{\mathfrak{p}} \cap U^1_{\mathfrak{p}})$, $K_{\mathfrak{q}} \subseteq \psi_{\mathfrak{q}}(S_{\mathfrak{q}} \cap U^1_{\mathfrak{q}})$. □

We take a decreasing sequence of positive numbers δ_n with $\lim_{n\to\infty} \delta_n = 0$ and put

$$U^n_{\mathfrak{p}} = U^{\delta_n}_{\mathfrak{p}}, \qquad U^n_{\mathfrak{pq}} = U^{\delta_n}_{\mathfrak{q}} \cap (\varphi^1_{\mathfrak{pq}})^{-1}(U^{\delta_n}_{\mathfrak{p}}). \tag{3.8}$$

We remark $U^n_{\mathfrak{pq}} \subseteq U^1_{\mathfrak{pq}}$ since $U^1_{\mathfrak{pq}}$ is the domain of $\varphi^1_{\mathfrak{pq}}$.

By Lemma 3.1, $\widehat{\mathcal{U}_n} = (\mathfrak{P}, \{\mathcal{U}_{\mathfrak{p}}|_{U^n_{\mathfrak{p}}}\}, \{\Phi_{\mathfrak{pq}}|_{U^n_{\mathfrak{pq}}}\})$ is an abstract good coordinate system of Z in the weak sense. Since $U^n_{\mathfrak{p}} \subseteq U^1_{\mathfrak{p}} \Subset U_{\mathfrak{p}}$ and $U^n_{\mathfrak{pq}} \subseteq U^1_{\mathfrak{pq}} \Subset U_{\mathfrak{pq}}$, $\widehat{\mathcal{U}_n}$ is a shrinking of $\widehat{\mathcal{U}}$.

We will prove that $\widehat{\mathcal{U}_n}$ is an abstract good coordinate system of Z in the *strong* sense for sufficiently large n. The proof occupies the rest of this section. We put

$$C^n_{\mathfrak{p}} = \overline{U^n_{\mathfrak{p}}}, \qquad C^n_{\mathfrak{pq}} = \overline{U^n_{\mathfrak{q}}} \cap (\varphi^1_{\mathfrak{pq}})^{-1}(\overline{U^n_{\mathfrak{p}}}). \tag{3.9}$$

Here $\overline{U^n_{\mathfrak{p}}}$ is the closure of $U^n_{\mathfrak{p}}$ in $U_{\mathfrak{p}}$, which coincides with the closure of $U^n_{\mathfrak{p}}$ in $U^1_{\mathfrak{p}}$. (This is because $U^n_{\mathfrak{p}} \Subset U^1_{\mathfrak{p}}$.) Moreover $C^n_{\mathfrak{p}}$ is compact. We consider

$$\hat{U}^n = \coprod_{\mathfrak{p}\in\mathfrak{P}} U^n_{\mathfrak{p}}, \qquad \hat{C}^n = \coprod_{\mathfrak{p}\in\mathfrak{P}} C^n_{\mathfrak{p}}$$

where the right hand sides are disjoint union. Note $\hat{U}^n \subseteq \hat{C}^n$. We define a relation on $\hat{U}^n$ by applying Definition 2.4 to $\widehat{\mathcal{U}_n}$. We denote it by $\sim_n$. We also define a relation $\sim'_n$ on $\hat{C}^n$ as follows.

Definition 3.7. Let $x \in C^n_{\mathfrak{p}}$ and $y \in C^n_{\mathfrak{q}}$. We say $x \sim'_n y$ if one of the following holds.

(a) $\mathfrak{p} = \mathfrak{q}$ and $x = y$.

(b) $\mathfrak{p} \le \mathfrak{q}$, $x \in C^n_{\mathfrak{qp}}$ and $y = \varphi^1_{\mathfrak{qp}}(x)$.

(c) $\mathfrak{q} \le \mathfrak{p}$, $y \in C^n_{\mathfrak{pq}}$ and $x = \varphi^1_{\mathfrak{pq}}(y)$.

The next lemma is immediate from our choice (3.8) of $U^n_{\mathfrak{pq}}$.

Lemma 3.8. *Let $x, y \in \hat{U}^n \subseteq \hat{C}^n$. Then $x \sim_n y$ if and only if $x \sim'_n y$.*

We now prove:

Proposition 3.9. *The relations $\sim_n$ and $\sim'_n$ are equivalence relations for sufficiently large n.*

Proof. In view of Lemma 3.8 it suffices to show that $\sim'_n$ is an equivalence relation for sufficiently large n.

We assume that this is not the case. Note $\sim'_n$ satisfies all the property required for the equivalence relation possibly except transitivity. Therefore by taking a subsequence if necessary we may assume that there exist $x_n, y_n, z_n \in \hat{C}^n$ such that $x_n \sim'_n y_n$, $y_n \sim'_n z_n$ but $x_n \sim'_n z_n$ does not hold.

Let $x_n \in C^n_{\mathfrak{p}_n}$, $y_n \in C^n_{\mathfrak{q}_n}$, $z_n \in C^n_{\mathfrak{r}_n}$. Since $\mathfrak{P}$ is a finite set we may assume, by taking a subsequence if necessary, that $\mathfrak{p} = \mathfrak{p}_n$, $\mathfrak{q} = \mathfrak{q}_n$, $\mathfrak{r} = \mathfrak{r}_n$ are independent of n.

We first examine the consequence of $x_n \sim'_n y_n$. We remark that $C^n_{\mathfrak{p}} \subseteq U^{2\delta_n}_{\mathfrak{p}}$ and $C^n_{\mathfrak{q}} \subseteq U^{2\delta_n}_{\mathfrak{q}}$. Therefore, in Case (b) of Definition 3.7 we can apply Lemma 3.6 to x_n and take a convergence subsequence, still denoted by x_n. We denote $\lim_{n\to\infty} x_n = x$. Then by continuity of $\varphi_{\mathfrak{q}\mathfrak{p}}$, $y_n = \varphi_{\mathfrak{q}\mathfrak{p}}(x_n)$ also converges. We denote $y = \lim_{n\to\infty} y_n$. Then $x \in U^1_{\mathfrak{q}\mathfrak{p}}$ and $\psi_{\mathfrak{p}}(x) = \psi_{\mathfrak{q}}(y)$. In Case (c) of Definition 3.7, the same reasoning gives rise to a point $y \in U^1_{\mathfrak{p}\mathfrak{q}}$ with $x = \varphi_{\mathfrak{p}\mathfrak{q}}(y)$ and the equality $\psi_{\mathfrak{p}}(x) = \psi_{\mathfrak{q}}(y)$. (By the same reason as that of the remark after Definition 2.4, we do not need to examine the case (a) separately.)

Next we examine the consequence of $y_n \sim'_n z_n$. Starting from the subsequence we took above, we can again apply Lemma 3.6 with x_n or y_n, $\mathfrak{p}$, $\mathfrak{q}$ replaced by y_n or z_n, $\mathfrak{q}$, $\mathfrak{r}$, respectively. Then by taking a subsequence if necessary we have $z = \lim_{n\to\infty} z_n$, such that $y \in U^1_{\mathfrak{r}\mathfrak{q}}$ or $z \in U^1_{\mathfrak{q}\mathfrak{r}}$ and $\psi_{\mathfrak{q}}(y) = \psi_{\mathfrak{r}}(z)$.

Thus combining the above two, we have $\psi_{\mathfrak{p}}(x) = \psi_{\mathfrak{q}}(y) = \psi_{\mathfrak{r}}(z)$. Therefore either $\mathfrak{p} \le \mathfrak{r}$ or $\mathfrak{r} \le \mathfrak{p}$ holds. We may assume $\mathfrak{r} \le \mathfrak{p}$ without loss of generality. Then since $\psi_{\mathfrak{p}}(x) = \psi_{\mathfrak{r}}(z)$ we have $z \in U^1_{\mathfrak{p}\mathfrak{r}}$, $\varphi_{\mathfrak{p}\mathfrak{r}}(z) = x$ by Definition 2.2 3), 4). Therefore $z_n \in U^1_{\mathfrak{p}\mathfrak{r}}$ for sufficiently large n, since $U^1_{\mathfrak{p}\mathfrak{r}}$ is open in $U_{\mathfrak{r}}$. We use it to show:

Lemma 3.10. *We have $\varphi^1_{\mathfrak{p}\mathfrak{r}}(z_n) = x_n$ for sufficiently large n.*

Proof. Since $\psi_{\mathfrak{p}}(x) = \psi_{\mathfrak{q}}(y) = \psi_{\mathfrak{r}}(z)$ Definition 2.3 5) and $\mathfrak{r} \le \mathfrak{p}$ imply that one of the following holds.

(a) $\mathfrak{q} \le \mathfrak{r} \le \mathfrak{p}$. (b) $\mathfrak{r} \le \mathfrak{q} \le \mathfrak{p}$. (c) $\mathfrak{r} \le \mathfrak{p} \le \mathfrak{q}$.

In Case (a) we have $y \in U^1_{\mathfrak{r}\mathfrak{q}} \cap U^1_{\mathfrak{p}\mathfrak{q}} \cap (\varphi^1_{\mathfrak{r}\mathfrak{q}})^{-1}(U^1_{\mathfrak{p}\mathfrak{r}})$. Hence for all sufficiently large n, $y_n \in U^1_{\mathfrak{r}\mathfrak{q}} \cap U^1_{\mathfrak{p}\mathfrak{q}} \cap (\varphi^1_{\mathfrak{r}\mathfrak{q}})^{-1}(U^1_{\mathfrak{p}\mathfrak{r}})$ and $x_n = \varphi^1_{\mathfrak{p}\mathfrak{q}}(y_n) = \varphi^1_{\mathfrak{p}\mathfrak{r}} \circ \varphi^1_{\mathfrak{r}\mathfrak{q}}(y_n) = \varphi^1_{\mathfrak{p}\mathfrak{r}}(z_n)$, by Definition 2.3 4).

In Case (b), we have $z \in U^1_{\mathfrak{p}\mathfrak{r}} \cap U^1_{\mathfrak{q}\mathfrak{r}} \cap (\varphi^1_{\mathfrak{q}\mathfrak{r}})^{-1}(U^1_{\mathfrak{p}\mathfrak{q}})$. Hence, for all sufficiently large n, $z_n \in U^1_{\mathfrak{p}\mathfrak{r}} \cap U^1_{\mathfrak{q}\mathfrak{r}} \cap (\varphi^1_{\mathfrak{q}\mathfrak{r}})^{-1}(U^1_{\mathfrak{p}\mathfrak{q}})$ and $\varphi^1_{\mathfrak{p}\mathfrak{r}}(z_n) = \varphi^1_{\mathfrak{p}\mathfrak{q}} \circ \varphi^1_{\mathfrak{q}\mathfrak{r}}(z_n) = \varphi^1_{\mathfrak{p}\mathfrak{q}}(y_n) = x_n$.

In Case (c) we have $z \in U^1_{\mathfrak{p}\mathfrak{r}} \cap U^1_{\mathfrak{q}\mathfrak{r}} \cap (\varphi^1_{\mathfrak{p}\mathfrak{r}})^{-1}(U^1_{\mathfrak{q}\mathfrak{p}})$. Hence, for sufficiently large n, $z_n \in U^1_{\mathfrak{p}\mathfrak{r}} \cap U^1_{\mathfrak{q}\mathfrak{r}} \cap (\varphi^1_{\mathfrak{p}\mathfrak{r}})^{-1}(U^1_{\mathfrak{q}\mathfrak{p}})$. Moreover $y_n = \varphi^1_{\mathfrak{q}\mathfrak{p}}(x_n)$ and $y_n = \varphi^1_{\mathfrak{q}\mathfrak{r}}(z_n) = \varphi^1_{\mathfrak{q}\mathfrak{p}} \circ \varphi^1_{\mathfrak{p}\mathfrak{r}}(z_n)$. Since $\varphi^1_{\mathfrak{q}\mathfrak{p}}$ is injective, we find that $x_n = \varphi^1_{\mathfrak{p}\mathfrak{r}}(z_n)$. □

Lemma 3.10 implies $x_n \sim'_n z_n$ for sufficiently large n. This is a contradiction. □

We have thus proved that $\widehat{\mathcal{U}_n}$ satisfies Definition 2.5 7) for sufficiently large n. We now turn to the proof of Definition 2.5 8). Let $W_{\mathfrak{p}\mathfrak{q}} \Subset U^1_{\mathfrak{p}\mathfrak{q}}$ be an open neighborhood of $\psi_{\mathfrak{q}}^{-1}(K_{\mathfrak{p}} \cap K_{\mathfrak{q}})$.

Lemma 3.11. *For all sufficiently large n, we have*

$$(\varphi^1_{\mathfrak{p}\mathfrak{q}})^{-1}(U^n_{\mathfrak{p}}) \cap U^n_{\mathfrak{q}} \subseteq W_{\mathfrak{p}\mathfrak{q}}. \tag{3.10}$$

Proof. If (3.10) is false there exists a sequence $x_n \in (\varphi^1_{\mathfrak{p}\mathfrak{q}})^{-1}(U^n_{\mathfrak{p}}) \cap U^n_{\mathfrak{q}} \setminus W_{\mathfrak{p}\mathfrak{q}}$ with $n \to \infty$. We apply Lemma 3.6 and may assume 1), 2), 3), 4) of Lemma 3.6. Then $x \in U^1_{\mathfrak{p}\mathfrak{q}}$, $\psi_{\mathfrak{q}}(x) = \psi_{\mathfrak{p}}(y) \in K_{\mathfrak{q}} \cap K_{\mathfrak{p}}$. It implies $x \in W_{\mathfrak{p}\mathfrak{q}}$. Thus $x_n \in W_{\mathfrak{p}\mathfrak{q}}$ for large n. This is a contradiction. □

Lemma 3.12. *$C^n_{\mathfrak{p}\mathfrak{q}}$ is a compact subset of $C^n_{\mathfrak{q}}$ for all sufficiently large n.*

Proof. It suffices to show that $C^n_{\mathfrak{p}\mathfrak{q}}$ is a closed subset of $C^n_{\mathfrak{q}}$. Let $x_a \in C^n_{\mathfrak{p}\mathfrak{q}}$ be a sequence converging to $x \in C^n_{\mathfrak{q}}$. By definition

$$x_a \in \overline{U^{\delta_n}_{\mathfrak{q}}} \cap (\varphi^1_{\mathfrak{p}\mathfrak{q}})^{-1}(\overline{U^{\delta_n}_{\mathfrak{p}}}). \tag{3.11}$$

Now (3.11), (3.10) and $\overline{U^{\delta_n}_{\mathfrak{q}}} \subseteq U^{2\delta_n}_{\mathfrak{q}}$ imply that $x_a \in W_{\mathfrak{p}\mathfrak{q}} \Subset U^1_{\mathfrak{p}\mathfrak{q}}$ for sufficiently large n. Therefore $x \in U^1_{\mathfrak{p}\mathfrak{q}}$. Since $\varphi^1_{\mathfrak{p}\mathfrak{q}}$ is continuous on $U^1_{\mathfrak{p}\mathfrak{q}}$ we have $\lim_{a\to\infty} \varphi^1_{\mathfrak{p}\mathfrak{q}}(x_a) = \varphi^1_{\mathfrak{p}\mathfrak{q}}(x)$. Since $\varphi^1_{\mathfrak{p}\mathfrak{q}}(x_a) \in C^n_{\mathfrak{p}}$ and $C^n_{\mathfrak{p}}$ is compact, $x \in U^1_{\mathfrak{p}\mathfrak{q}}$ implies $\varphi^1_{\mathfrak{p}\mathfrak{q}}(x) \in C^n_{\mathfrak{p}}$. Thus $x \in C^n_{\mathfrak{p}\mathfrak{q}}$. This proves that $C^n_{\mathfrak{p}\mathfrak{q}}$ is closed in $C^n_{\mathfrak{q}}$ as required. □

We define $C^n = \hat{C}^n / \sim'_n$. The following is an immediate consequence of Lemma 3.12.

Lemma 3.13. *Let n be sufficiently large so that Lemma 3.11 holds. Then the space C^n is Hausdorff with respect to the quotient topology.*

Proof. We first note that $\hat{C}^n$ is Hausdorff since it is a disjoint union of Hausdorff spaces $C^n_{\mathfrak{q}}$ over $\mathfrak{q}$. On the other hand, Lemma 3.12 implies that the relation $\sim'_n$ is a closed relation defined on $\hat{C}^n$ for any such n. The lemma then is a standard consequence in general topology. □

We remark that $|\widehat{\mathcal{U}^n}| = \hat{U}^n / \sim_n$ by definition. The inclusion $\hat{U}^n \to \hat{C}^n$ induces a map $\hat{U}^n \to C^n$. Lemma 3.8 implies that it induces an *injective* map $|\widehat{\mathcal{U}^n}| \to C^n$. This map is continuous by the definition of the quotient topology. Therefore Lemma 3.13 implies that $|\widehat{\mathcal{U}^n}|$ is Hausdorff. The proof of Theorem 2.9 is now complete.

Remark 3.14. We would like to note that the domain $U^n_{\mathfrak{pq}}$ of the coordinate change of the shrinking $\widehat{\mathcal{U}_n}$ of $\widehat{\mathcal{U}}$ is *not* of the form

$$\varphi_{\mathfrak{pq}}^{-1}(U^n_{\mathfrak{p}}) \cap U^n_{\mathfrak{q}} \tag{3.12}$$

but is

$$U^n_{\mathfrak{pq}} = (\varphi^1_{\mathfrak{pq}})^{-1}(U^n_{\mathfrak{p}}) \cap U^n_{\mathfrak{q}} = \varphi_{\mathfrak{pq}}^{-1}(U^n_{\mathfrak{p}}) \cap U^n_{\mathfrak{q}} \cap U^1_{\mathfrak{pq}}.$$

In fact (3.12) is *not* relatively compact in $U_{\mathfrak{pq}}$ in general even when $U^n_{\mathfrak{p}}$ are relatively compact in $U_{\mathfrak{p}}$ for all $\mathfrak{p}$. We thank J. Solomon who found an example related to this point and informed it to us. Here we present a simpler and more directly related example.

Let $U_i = \{i\} \times \mathbb{R}$, $S_i = \{i\} \times \{0,1\} \subset U_i$ $(i = 1,2)$. We put $U_{21} = \{1\} \times (\mathbb{R} \setminus \{0\})$ and define $\varphi_{21} : U_{21} \to U_2$ by $\varphi_{21}(1,t) = (2,t)$. We put $X = \{(1,0), (2,0), 1\}$ and $\psi_1(1,0) = (1,0)$, $\psi_1(1,1) = 1$, $\psi_2(2,0) = (2,0), \psi_2(2,1) = 1$. They define an abstract good coordinate system of $Z = X$ in the weak sense.

Let U'_i be a small neighborhood of S_i in U_i. We may take $U'_i = \{i\} \times ((-\epsilon,\epsilon) \cup (1-\epsilon, 1+\epsilon))$, for example. We put

$$U'_{21} = \varphi_{21}^{-1}(U'_2) \cap U'_1 \cap U_{21} = \{1\} \times ((-\epsilon,0) \cup (0,\epsilon) \cup (1-\epsilon,1+\epsilon))$$

The space obtained by gluing U'_1 and U'_2 by $\varphi_{21}|_{U'_{21}}$ is not Hausdorff. In fact, $(1,0)$ and $(2,0)$ have no separating neighborhoods.

We shrink U'_{21} to

$$U''_{21} = \{1\} \times (1-\epsilon, 1+\epsilon).$$

Then the space obtained by gluing U'_1 and U'_2 by $\varphi_{21}|_{U''_{21}}$ is a disjoint union of three intervals and is Hausdorff.

We remark that U'_{21} is *not* relatively compact in U_{21} but U''_{21} is relatively compact in U_{21}.

4. Proof of metrizability

In this section we prove Proposition 2.11.

We recall the following well-known definition. A family of subsets $\{U_i \mid i \in I\}$ of a topological space Y containing $x \in Y$ is said to be a neighborhood basis of x if

(nbb 1) each U_i contains an open neighborhood of x,

(nbb 2) for each open set U containing x there exists i such that $U_i \subseteq U$.

A family of open subsets $\{U_i \mid i \in I\}$ of a topological space X is said to be a basis of the open sets if for each x the set $\{U_i \mid x \in U_i\}$ is a neighborhood basis of x. A topological space is said to satisfy the second axiom of countability if there exists a countable basis of open subsets $\{U_i \mid i \in I\}$. A classical result of Urysohn says a topological space is metrizable if it is regular and satisfies the second axiom of countability. (See a standard text book such as [Ke] for these facts.)

Proof of Proposition 2.11. We put $K_{\mathfrak{p}} = \overline{U'_{\mathfrak{p}}}$ and consider $K = \coprod_{\mathfrak{p}\in\mathfrak{P}} K_{\mathfrak{p}}/\sim_K$ in $|\widehat{\mathcal{U}}|$. (Here $\sim_K$ is the restriction of the equivalence relation $\sim_U$ obtained by applying Definition 2.4 to $\widehat{\mathcal{U}}$. ($\sim_U$ is an equivalence relation on $\coprod_{\mathfrak{p}\in\mathfrak{P}} U_{\mathfrak{p}} \supseteq \coprod_{\mathfrak{p}\in\mathfrak{P}} K_{\mathfrak{p}}$.) Let $\Pi_{\mathfrak{p}} : K_{\mathfrak{p}} \to K$ be the the natural inclusion followed by the projection. As a subset of $|\widehat{\mathcal{U}}|$, we can also write $K = \bigcup_{\mathfrak{p}\in\mathfrak{P}} K_{\mathfrak{p}} \subseteq |\widehat{\mathcal{U}}|$. Note the induced topology of the embedding $U' \to K$ coincides with the induced topology of the embedding $U' \to |\widehat{\mathcal{U}}|$. This is because the map $K \to |\widehat{\mathcal{U}}|$ is a topological embedding. (K is compact and $|\widehat{\mathcal{U}}|$ is Hausdorff.) Therefore, it suffices to show that K is metrizable with respect to the quotient topology of $\Pi_{\mathfrak{P},K} : \coprod_{\mathfrak{p}\in\mathfrak{P}} K_{\mathfrak{p}} \to K$. We remark that K is compact. K is Hausdorff since $|\widehat{\mathcal{U}}|$ is Hausdorff and $K \to |\widehat{\mathcal{U}}|$ is injective and continuous. Therefore K is regular. Now it remains to show that K satisfies the second axiom of countability. This is [FOOO2, Lemma 8.5]. We repeat its proof here for the convenience of the reader.

For each $\mathfrak{p}$, we take a countable basis $\mathfrak{U}_{\mathfrak{p}} = \{U_{\mathfrak{p},i_{\mathfrak{p}}} \subseteq K_{\mathfrak{p}} \mid i_{\mathfrak{p}} \in I_{\mathfrak{p}}\}$ of open sets of $K_{\mathfrak{p}}$. We may assume $\emptyset \in \mathfrak{U}_{\mathfrak{p}}$.

For each $\vec{i} = (i_{\mathfrak{p}})_{\mathfrak{p}\in\mathfrak{P}}$ ($i_{\mathfrak{p}} \in I_{\mathfrak{p}}$) we define $U(\vec{i})$ to be the interior of the set

$$U^+(\vec{i}) := \bigcup_{\mathfrak{p}\in\mathfrak{P}} \Pi_{\mathfrak{p}}(U_{\mathfrak{p},i_{\mathfrak{p}}}). \tag{4.1}$$

Then $\{U(\vec{i})\}$ is a countable family of open subsets of K. We will prove that this family is a basis of open sets of K.

Let $q \in K$, we put

$$\mathfrak{P}(q) = \{\mathfrak{p} \in \mathfrak{P} \mid \exists x,\ q = [x],\, x \in K_{\mathfrak{p}}\}. \tag{4.2}$$

Here and hereafter we identify $K_{\mathfrak{p}}$ to the image of $\Pi_{\mathfrak{P},K}(K_{\mathfrak{p}})$ in K. Note since K is Hausdorff and $K_{\mathfrak{p}}$ is compact, the natural inclusion map $K_{\mathfrak{p}} \to \coprod_{\mathfrak{p}\in\mathfrak{P}} K_{\mathfrak{p}}$ induces a topological embedding $K_{\mathfrak{p}} \to K$.

For $\mathfrak{p} \in \mathfrak{P}(q)$, we have $x_{\mathfrak{p}} \in K_{\mathfrak{p}}$ with $[x_{\mathfrak{p}}] = q$. We put

$$I_{\mathfrak{p}}(q) = \{i_{\mathfrak{p}} \in I_{\mathfrak{p}} \mid x_{\mathfrak{p}} \in U_{\mathfrak{p},i_{\mathfrak{p}}}\}.$$

Then $\{U_{\mathfrak{p},i_{\mathfrak{p}}} \mid i_{\mathfrak{p}} \in I_{\mathfrak{p}}(q)\}$ is a countable neighborhood basis of $x_{\mathfrak{p}}$ in $K_{\mathfrak{p}}$. For each $\vec{i} = (i_{\mathfrak{p}}) \in \prod_{\mathfrak{p}\in\mathfrak{P}(q)} I_{\mathfrak{p}}(q)$, we set

$$U^{+}(\vec{i}) = \bigcup_{\mathfrak{p}\in\mathfrak{P}(q)} \Pi_{\mathfrak{p}}(U_{\mathfrak{p},i_{\mathfrak{p}}}) \subseteq K. \tag{4.3}$$

We claim that the collection $\{U^{+}(\vec{i}) \mid \vec{i} \in \prod_{\mathfrak{p}\in\mathfrak{P}(q)} I_{\mathfrak{p}}(q)\}$ is a neighborhood basis of q in K for any q. The claim follows from Lemmata 4.1, 4.2 below.

Lemma 4.1. *The subset $U^{+}(\vec{i})$ is a neighborhood of q in K.*

Proof. For $\mathfrak{p} \in \mathfrak{P}(q)$ the set $K_{\mathfrak{p}} \setminus U_{\mathfrak{p},i_{\mathfrak{p}}}$ is a closed subset of $K_{\mathfrak{p}}$ and so is compact. Therefore $\Pi_{\mathfrak{p}}(K_{\mathfrak{p}} \setminus U_{\mathfrak{p},i_{\mathfrak{p}}})$ is a compact subset in the Hausdorff space K and so is closed.

If $\mathfrak{p} \notin \mathfrak{P}(q)$ then we consider $\Pi_{\mathfrak{p}}(K_{\mathfrak{p}})$ which is closed.

Now we put

$$K_0 = \bigcup_{\mathfrak{p}\in\mathfrak{P}(q)} \Pi_{\mathfrak{p}}(K_{\mathfrak{p}} \setminus U_{\mathfrak{p},i_{\mathfrak{p}}}) \cup \bigcup_{\mathfrak{p}\notin\mathfrak{P}(q)} \Pi_{\mathfrak{p}}(K_{\mathfrak{p}}).$$

This is a finite union of closed sets and so is closed. It is easy to see that $q \in K \setminus K_0 \subseteq U^{+}(\vec{i})$. □

Lemma 4.2. *The collection $\{U^{+}(\vec{i})\}$ satisfies the property (nbb 2) of the neighborhood basis above.*

Proof. Let $U \subseteq K$ be an open subset containing q. Since the map $K_{\mathfrak{p}} \to K$ is a topological embedding, $U \cap K_{\mathfrak{p}}$ is an open set of $K_{\mathfrak{p}}$. Therefore for each

$\mathfrak{p} \in \mathfrak{P}(q)$, the set $U \cap K_{\mathfrak{p}}$ is a neighborhood of $x_{\mathfrak{p}}$ in $K_{\mathfrak{p}}$. By the definition of neighborhood basis in $K_{\mathfrak{p}}$, there exists $i_{\mathfrak{p}}$ such that $U_{\mathfrak{p},i_{\mathfrak{p}}} \subseteq U \cap K_{\mathfrak{p}}$. We put $\vec{i} = (i_{\mathfrak{p}})$. Then $U^+(\vec{i}) \subseteq U$ as required. □

We remark that $U^+(\vec{i})$ in (4.3) is a special case of $U^+(\vec{i})$ in (4.1). (We take $U_{\mathfrak{p},i_{\mathfrak{p}}} = \emptyset$ for $\mathfrak{p} \notin \mathfrak{P}(x)$.) The family $U(\vec{i})$ is a countable basis of open sets of K. Proposition 2.11 is now proved. □

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