JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 15, Number 4, 1173–1208, 2017

The Hofer norm of a contactomorphism

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We show that the L^∞-norm of the contact Hamiltonian induces a non-degenerate right-invariant metric on the group of contactomorphisms of any closed contact manifold. This contact Hofer metric is not left-invariant, but rather depends naturally on the choice of a contact form α, whence its restriction to the subgroup of α-strict contactomorphisms is bi-invariant. The non-degeneracy of this metric follows from an analogue of the energy-capacity inequality. We show furthermore that this metric has infinite diameter in a number of cases by investigating its relations to previously defined metrics on the group of contact diffeomorphisms. We study its relation to Hofer's metric on the group of Hamiltonian diffeomorphisms, in the case of prequantization spaces. We further consider the distance in this metric to the Reeb one-parameter subgroup, which yields an intrinsic formulation of a small-energy case of Sandon's conjecture on the translated points of a contactomorphism. We prove this Chekanov-type statement for contact manifolds admitting a strong exact filling.

2010 Mathematics Subject Classification: 53D10, 37J55, 57R17, 57S05.
Key words and phrases: contact Hamiltonian, right-invariant metric, contactomorphism group, translated points, Hofer norm.

1. Introduction and main results

1.1. Introduction

Since the advent of the conjugation-invariant Hofer norm [16, 18] on the group of compactly supported Hamiltonian diffeomorphisms of any symplectic manifold, there has been a certain interest in investigating possible analogues of this norm in contact topology. The first fundamental difference between the Hamiltonian and the contact settings is the finding of Burago-Ivanov-Polterovich [9] that classical groups of diffeomorphisms that have a conformal freedom, the full group of diffeomorphisms and the contactomorphism group, do not to admit *fine* conjugation-invariant norms - that is norms whose image in $\mathbb{R}_{\geq 0}$ has 0 as an accumulation point (see [14] for the case of the contactomorphism group). Hence, all such norms must be *discrete*: there must exist a constant $c > 0$, such that the norm of every diffeomorphism other that the identity transformation exceeds c. And indeed a number of conjugation-invariant non-degenerate norms with values in $\mathbb{Z}$ on groups of contactomorphisms of certain contact manifolds were discovered in [12, 14, 26, 33] and certain other discrete norms are easily contructed from homogeneous quasi-morphisms on contactomorphism groups found in [7, 15].

In a different line of research, one notes that constructions similar to the Hofer metric using contact Hamiltonians tend to yield fine pseudo-norms on contactomorphism groups. In particular, a conjugation-invariant pseudo-norm was introduced on the group of *strict* contactomorphisms of any closed contact manifold equipped with a global contact form in [5] and its non-degeneracy was shown for a certain class of contact manifolds. This non-degeneracy result was improved in [21] to hold for all closed contact manifolds. By the observation of [9, 14], this norm cannot extend to a conjugation-invariant norm on the full contactomorphism group. While that is indeed so, the goal of this paper is to show that for all closed contact manifolds, the norm of [5, 21] is bounded from below by another conjugation-invariant norm on the group of strict contactomorphisms that extends to a non-degenerate fine norm on the full group of contactomorphisms, albeit without the property of conjugation-invariance.

The topology induced by this contact Hofer norm, and indeed the equivalence class of the metric it defines, turns out to be independent of the choice of a global contact form, and to admit a number of interesting closed subsets. We show, moreover, that the contact Hofer norm is related to a natural intersection problem for contactomorphisms - that of translated points of

S. Sandon [27, 28]. In particular we show how it gives an intrinsic Chekanov-type statement for the existence of translated points (cf. [1, 2, 4, 11]). Along the way we describe a geometric non-degeneracy condition for this problem.

We further show how the contact Hofer norm is related to certain previously introduced discrete bi-invariant norms on contactomorphism groups [7, 14, 15, 26, 33] and to the Hofer norm on the group of Hamiltonian diffeomorphisms.

1.2. Main results

Given a connected co-oriented contact manifold[1] (N,ξ), that we will assume to be closed unless stated otherwise, with a globally defined contact form α with $\xi = \ker\alpha$ we define a contactomorphism to be a diffeomorphism $\psi \in \mathrm{Diff}(N)$ satisfying $\psi_*\xi = \xi$ and preserving the co-orientation of ξ. This is equivalent to the existence of a positive function $\lambda_\psi = e^{g_\psi},\ g_\psi \in C^\infty(N,\mathbb{R})$ such that $\psi^*\alpha = \lambda_\psi \cdot \alpha$. We consider the group

$$\mathcal{G} = \mathrm{Cont}_0(N,\xi)$$

of contactomorphisms isotopic to the identity contactomorphism $\mathbb{1}$.

Recall that $\mathcal{G}$ is naturally isomorphic to the group of $\mathbb{R}_{>0}$-equivariant Hamiltonian diffeomorphisms of the (positive) symplectization SN of N, which is abstractly defined as the subspace

$$SN = \{(p,q)|\ p|_\xi = 0,\ p > 0_q\} \subset T^*N,$$

of the non-zero covectors vanishing on ξ, that are positive with respect to the co-orientation of ξ, endowed with the restriction of the canonical symplectic form $\omega_{can} = d\lambda_{can}$ on T^*N. A choice α of a global contact form for (N,ξ) determines an $\mathbb{R}_{>0}$-equivariant symplectomorphism from the symplectization SN to $N \times \mathbb{R}_{>0}$ with the symplectic form $\omega = d(r\widetilde{\alpha})$, where $\widetilde{\alpha}$ denotes the lift of α by the projection to the first co-ordinate and r denotes the coordinate function coinciding with the inclusion $\mathbb{R}_{>0} \hookrightarrow \mathbb{R}$. In this isomorphism, the graph of α, which is a subset of SN corresponds to the hypersurface $N_1 := N \times \{1\} \subset N \times \mathbb{R}_{>0}$. Similarly, the natural projection $\pi : SN \to N$ corresponds to the projection $N \times \mathbb{R}_{>0} \to N$ to the first coordinate. Denote by $r_\alpha : SN \to \mathbb{R}_{>0}$ the pullback of the coordinate function $r : N \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ under this symplectomorphism. Note that as the symplectomorphism depends on α, so does r_α. Given a contactomorphism

[1] This is what we call a contact manifold in this paper.

$\psi \in \mathcal{G}$, its natural lift $\overline{\psi}$ to SN is simply the restriction to SN of its canonical lift to T^*N. In the splitting $SN \cong N \times \mathbb{R}_{>0}$ given by a global contact form α, this lift takes the form $\overline{\psi}(x,r) = (\psi(x), \frac{r}{\lambda_\psi(x)})$. For later use we note that the differential $D(\overline{\psi})(x,1)$ of $\overline{\psi}$ at $(x,1)$ is

$$D(\overline{\psi})(x,1) = \begin{pmatrix} D(\psi)(x) & 0 \\ -\frac{1}{\lambda_\psi^2(x)} d\lambda_\psi(x) & \lambda_\psi(x)^{-1} \end{pmatrix}. \tag{1}$$

Note that with respect to the natural $\mathbb{R}_{>0}$ action on SN, for an interval $I = (a,b) \subset \mathbb{R}_{>0}$ we have the equality of subsets $I \cdot N_1 = \{r \cdot x \mid r \in I,\ x \in N_1\} = N \times I$ with respect to the splitting $SN \cong N \times \mathbb{R}_{>0}$. For a Hamiltonian $F \in C^\infty([0,1] \times SN)$ denote its Hamiltonian flow (generated by the vector field X_t given by $\iota_{X_t}\omega = -dF_t$) by $\{\phi_F^t\}_{t\in[0,1]}$.

For an isotopy $\{\psi_t\}_{t\in[0,1]}$, $\psi_0 = 1$, in $\mathcal{G}$, its time-dependent contact Hamiltonian $H \in C^\infty([0,1] \times N)$ is by definition the restriction of the corresponding $\mathbb{R}_{>0}$-homogeneous Hamiltonian of degree 1 on SN to N_1. In other words, for time $t \in [0,1]$ the contact Hamiltonian $H_t(-) = H(t,-)$ at time t satisfies

$$H_t = \alpha(Y_t)$$

where $Y_t = \frac{d}{dt'}|_{t'=t}\psi_{t'} \circ \psi_t^{-1}$ is the time-dependent vector field generating $\{\psi_t\}$. It is easy to see that the contact Hamiltonian determines the flow uniquely (the vector field Y_t is the unique solution of the system $\alpha(Y_t) = H_t$, $\iota_{Y_t} d\alpha = -dH_t + dH_t(R_\alpha)\alpha$). Denote by $\{\psi_H^t\}$ the flow determined by the contact Hamiltonian $H = H(t,x)$.

The one-parameter subgroup $\mathcal{R} = \mathcal{R}_\alpha$ of $\mathcal{G}$ defined by the flow of the autonomous degree-1-homogeneous Hamiltonian $H \equiv r$ on $SN \cong N \times \mathbb{R}_{>0}$ is called the *Reeb flow* of α, and the vector field $R = R_\alpha$ on N generating it is uniquely defined by the conditions $\alpha(R) = 1, \iota_R d\alpha = 0$. Hence the natural homomorphism $\mathbb{R} \to \mathcal{R}$ is given by $t \mapsto \phi_R^t$.

Following S. Sandon [27, 28], we call a point $x \in N$ a *translated point* of a contactomorphism ψ if $\psi(x) = \phi_R^\eta(x)$ for some $\eta \in \mathbb{R}$, and $(\psi^*\alpha)_x = \alpha_x$ (that is $\lambda_\psi(x) = e^{g_\psi(x)} = 1$), or alternatively (see [27, 28]) x is a leaf-wise intersection point of the lift $\overline{\psi}$ of ψ to SN, relative to the hypersurface $N_1 \subset SN$. In this instance we call (x,η) an *algebraic translated point* of ψ and note that for a given translated point x, there may be many algebraic translated points covering it (i.e. with first term x) [1, 4]. We remark that for any $\psi \in \mathcal{R}$, the set of translated points consists of the whole manifold N.

Definition 1. We call an algebraic translated point (x, η) of ψ *non-degenerate* if the differential

$$D(\phi_R^{-\eta}\psi)(x) : T_xN \to T_xN$$

of $\phi_R^{-\eta}\psi$ at x does not have an eigenvector with eigenvalue 1 that lies in the kernel of $d\lambda(x)$. In other words

$$\ker(D(\phi_R^{-\eta}\psi)(x) - 1) \cap \ker(d\lambda(x)) = 0. \tag{2}$$

We say that a contactomorphism $\psi \in \mathcal{G}$ is *non-degenerate* if all its algebraic translated points are non-degenerate.

We shall see that this condition is the restriction of the non-degeneracy condition of Albers-Frauenfelder to the class of degree 1 homogeneous Hamiltonians on SN.

Lemma 2. *The contactomorphism ψ is non-degenerate if and only if the corresponding Rabinowitz-Floer functional in the symplectization SN of N is Morse.*

Hence, whenever $\psi \in \mathcal{G}$ is non-degenerate, the algebraic translated points $(x, \eta) \in N \times \mathbb{R}$ are isolated, and hence are countably many. By an argument of Albers-Frauenfelder [1] that was brought to the contact setting by Albers-Merry [4], non-degenerate contactomorphisms are generic.

Lemma 3. *The set of contact Hamiltonians generating non-degenerate contactomorphisms is of the second Baire category inside $C^\infty([0,1] \times N, \mathbb{R})$.*

Given a global contact form α for ξ, we denote by $\rho(\alpha)$ the minimal period of a periodic Reeb orbit of α.

Recall that for a function $H \in C^\infty(N, \mathbb{R})$ its L^∞-norm is simply

$$|H|_{L^\infty(N)} = \max_N |H|.$$

Definition 4. Given a global contact form α, we define for a contactomorphism $\psi \in \mathcal{G}$ its Hofer energy $|\psi|_\alpha$ as

$$\inf \int_0^1 |H_t|_{L^\infty(N)} dt,$$

where the infimum runs over all isotopies $\{\psi_t^H\}$ with $\psi_1 = \psi$.

Similarly, for an element $\widetilde{\psi} \in \widetilde{\mathcal{G}}$ in the universal cover of $\mathcal{G}$, we define its Hofer energy $|\widetilde{\psi}|_\alpha$ by the same formula, where the infimum now runs over all isotopies $\{\psi_t^H\}$ in class $\widetilde{\psi}$.

Clearly, if we denote by $\pi : \widetilde{\mathcal{G}} \to \mathcal{G}$ the canonical projection homomorphism, we have

$$|\psi|_\alpha = \inf_{\pi(\widetilde{\psi})=\psi} |\widetilde{\psi}|_\alpha.$$

The first main result of this paper is the non-degeneracy of this Hofer-type functional. We observe certain other properties of this functional, showing that it is a norm on the group $\mathcal{G}$. Note that this norm is not conjugation-invariant, but instead satisfies an equivariance property which is simply a statement of its naturality with respect to coordinate change. We remark that in the course of preparation of this paper it came to the attention of the author that this functional was previously considered by Rybicki [24], and Properties (ii), (iii), (iv), other than the non-degeneracy, were already observed by him.

Theorem A. *The Hofer energy functional $|\psi|_\alpha$ satisfies the following properties. Denote by $\phi, \psi \in \mathcal{G}$ two arbitrary elements.*

(i) (non-degeneracy) If $\psi \neq 1$, then $|\psi|_\alpha > 0$, and $|1|_\alpha = 0$.

(ii) (triangle inequality) $|\phi\psi|_\alpha \leq |\phi|_\alpha + |\psi|_\alpha$.

(iii) (symmetry) $|\psi^{-1}|_\alpha = |\psi|_\alpha$.

(iv) (naturality) $|\psi\phi\psi^{-1}|_\alpha = |\phi|_{\psi^*\alpha}$

Remark 5. The analogous functional on $\widetilde{\mathcal{G}}$ satisfies properties (ii), (iii), (iv). Property (i) may not hold, because it is not clear how to bound $|-|_\alpha$ from below on non-trivial elements of $\pi_1(\mathcal{G}) \subset \widetilde{\mathcal{G}}$.

Remark 6. We remark that these properties imply that $|\cdot|_\alpha$ descends to a conjugation-invariant norm on the subgroup $\mathcal{H}_\alpha = \mathrm{Cont}_0(N, \alpha)$ of strict contactomorphisms of the contact form α. Moreover for $\phi, \psi \in \mathcal{G}$ the distance function (metric)

$$d_\alpha(\phi, \psi) = |\psi\phi^{-1}|_\alpha$$

is invariant with respect to the action of $\mathcal{G}$ on the right, and the action of $\mathcal{H}_\alpha$ on the left.

Remark 7. Following Eliashberg-Polterovich [13], it is easily seen that for $1 \leq p < \infty$, the pseudo-norm $|-|_{p,\alpha}$ on $\mathcal{G}$ given by replacing $|H_t|_{L^\infty(N)}$ with $|H_t|_{L^p(N,\alpha(d\alpha)^n)}$ in Definition 4 is degenerate. Indeed, call a pseudo-norm ν on $\mathcal{G}$ *quasi-conjugation-invariant* if for each $\phi \in \mathcal{G}$ there exists $C(\phi) > 0$, such that for all $\psi \in \mathcal{G}$,

$$\nu(\phi\psi\phi^{-1}) \leq C(\phi) \cdot \nu(\psi).$$

Then for each such norm the non-degeneracy criterion of Eliashberg-Polterovich holds still: the pseudo-norm ν is non-degenerate if and only if the ν-displacement-energy

$$E_\nu(U) = \inf\{\nu(\phi) \mid \phi(U) \cap \overline{U} = \emptyset\}$$

of any open subset $U \subset N$ is positive. The proof of this fact is identical to the one in [13], with the sole difference that the lower estimate on the displacement energy via commutators takes the form

$$E_\nu(U) \geq \frac{1}{(C(\phi)+1)(C(\psi)+1)} \cdot \nu([\phi, \psi])$$

for all $\phi, \psi \in \mathcal{G}$ supported in U. One continues to show that $|-|_{p,\alpha}$ is indeed quasi-conjugation-invariant, and that, by introducing an appropriate time-dependent cutoff, the $|-|_{p,\alpha}$-displacement energy of each small Darboux ball vanishes.

Moreover, assuming the simplicity of $\mathcal{G}$ (see [25, 30]), one concludes that $|-|_{p,\alpha}$ vanishes identically. Indeed $\mathcal{K}_\nu = \{\phi \in \mathcal{G} \mid \nu(\phi) = 0\}$ is a normal subgroup of $\mathcal{G}$ for every quasi-conjugation-invariant norm ν, and in the case $\nu = |-|_{p,\alpha}$, it is non-empty by the argument above.

Remark 8. The conjugation-invariant norm $(|\cdot|_\alpha)|_{\mathcal{H}_\alpha}$ gives a lower bound for the conjugation-invariant norm $|\cdot|_{str,\alpha}$ on $\mathcal{H}_\alpha$ induced by the restriction of the Finsler metric defining $|\cdot|_\alpha$ to $\mathcal{H}_\alpha$, that is - we take only paths in $\mathcal{H}_\alpha$ in Definition 4. Hence $|\cdot|_{str,\alpha}$ is also a non-degenerate conjugation-invariant norm on $\mathcal{H}_\alpha$. The inequality

$$\begin{aligned}
|H_t|_{L^\infty(N)} &\leq \operatorname{osc}_N(H_t) + \frac{1}{\operatorname{vol}(N, \alpha(d\alpha)^n)} \left| \int_N H_t \alpha(d\alpha)^n \right| \\
&\leq \operatorname{osc}_N(H_t) + \frac{1}{\operatorname{vol}(N, \alpha(d\alpha)^n)} \int_N |H_t| \alpha(d\alpha)^n \leq 3|H_t|_{L^\infty(N)}
\end{aligned}$$

where $\mathrm{osc}_N(H_t) = \max_N(H_t) - \min_N(H_t)$, shows immediately an inequality for the corresponding norms

$$|\phi|_{str,\alpha} \leq |\phi|_{BD} \leq |\phi|_{MS} \leq 3|\phi|_{str,\alpha}, \tag{3}$$

where $|\cdot|_{BD}$ is the conjugation-invariant norm of Banyaga-Donato [5] on $\mathcal{H}_\alpha$, that was reinterpreted and shown to be non-degenerate for all closed contact manifolds by Müller-Spaeth in [21]. The inequality (3) appeared in [21], where it was also shown that $|\phi|_{BD} = |\phi|_{MS}$.

Remark 9. Note that $|\cdot|_{str,\alpha}$ on $\widetilde{\mathcal{H}}_\alpha$ is always unbounded. Indeed, the Calabi-Weinstein invariant $cw : \widetilde{\mathcal{H}}_\alpha \to \mathbb{R}$ (cf. [32]) defined as

$$cw([\{\phi_H^t\}]) = \frac{1}{\mathrm{vol}(N, \alpha(d\alpha)^n)} \int_0^1 dt \int_N H(t,x)\alpha(d\alpha)^n$$

for any path $\{\phi_H^t\}$ of strict contactormorphisms in a fixed class in $\widetilde{\mathcal{H}}_\alpha$, satisfies

$$cw([\{\phi_R^{\kappa \cdot t}\}_{t\in[0,1]}]) = \kappa,$$

for all $\kappa \in \mathbb{R}$, while $|cw(\widetilde{\phi})| \leq |\widetilde{\phi}|_{str,\alpha}$ for all $\widetilde{\phi} \in \widetilde{\mathcal{H}}_\alpha$. In fact it is easy to see that

$$|[\{\phi_R^{\kappa \cdot t}\}_{t\in[0,1]}]|_{str,\alpha} = \kappa.$$

As a metric, d_α defines a topology on $\mathcal{G}$. The following lemma states that the equivalence class of the metric d_α, and consequently this topology, does not depend on the choice α of a global contact form compatible with the co-orientation of ξ. Note that any two such global contact forms α, α' differ by a positive smooth function $f \in C^\infty(M, \mathbb{R}_{>0})$:

$$\alpha' = f \cdot \alpha.$$

Lemma 10.

$$\min_N(f) \cdot d_\alpha \leq d_{\alpha'} \leq \max_N(f) \cdot d_\alpha,$$

and therefore the metrics $d_\alpha, d_{\alpha'}$ are equivalent.

The main point of Theorem A is Property (i). It follows from the following statement. For a compact subset A of SN, let be the *Gromov radius* of A be

$$c(A) = \sup\{u| \text{ there exists a symplectic embedding } B(u) \hookrightarrow A\},$$

where $B(u) \subset (\mathbb{R}^{2n}, \omega_{std})$ the standard ball of capacity u (that is radius $\sqrt{\frac{u}{\pi}}$), and define the *height of* A to be

$$h_\alpha(A) := \sup_A r_\alpha$$

where $r_\alpha : SN \cong N \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is simply the projection to the second coordinate. Put

$$\widehat{c}_\alpha(A) := \frac{c(A)}{h_\alpha(A)}.$$

Note that for $\lambda \in \mathbb{R}_{>0}$ we have $\widehat{c}_\alpha(\lambda \cdot A) = \widehat{c}_\alpha(A)$.

Proposition 11. *If the lift $\overline{\psi}$ of ψ to SN displaces a compact subset $A \subset SN$, then*

$$|\psi|_\alpha \geq \frac{1}{4}\widehat{c}_\alpha(A).$$

Remark 12. Theorem A and Proposition 11 remain true in the non-compact setting, where $\mathcal{G}$ denotes the identity component $\mathrm{Cont}_{c,0}(N, \xi)$ of the group of compactly supported contactomorphisms of N.

Remark 13. Proposition 11 strengthens [21, Theorem 1.1] of Müller-Spaeth, combining the use of the energy-capacity inequality of Lalonde-McDuff [18] in the symplectization with techniques like Proposition 41 and Lemma 42. Consequently a few results in [21] can be strengthened. For example [21, Proposition 7.1] can be strengthened to the following Proposition 14. It is interesting to see which other applications to questions in C^0-contact topology the contact Hofer metric can have. This shall be investigated elsewhere.

Proposition 14. *Assume a sequence $\{\psi_i \in \mathcal{G}\}_{i \in \mathbb{Z}_{>0}}$ of contactomorphisms, a contactomorphism $\psi \in \mathcal{G}$ and a map $\phi : N \to N$ satisfy*

(i) $d_\alpha(\psi_i, \psi) \to 0$,

(ii) $\psi_i \to \phi$ *uniformly,*

as $i \to \infty$. Then $\phi = \psi$.

Recall that $\pi : SN \to N$ denotes the natural projection. For a subset $D \subset N$ with non-empty interior, define its *contact α-capacity* as

$$\widetilde{c}_\alpha(D) = \widehat{c}_\alpha(\pi^{-1}(D)) = \sup_{A \subset \pi^{-1}(D)} \widehat{c}_\alpha(A) > 0,$$

the supremum running over all compact subsets $A \subset \pi^{-1}(D)$. Moreover, we say that $\psi \in \mathcal{G}$ *displaces* D if $\psi(D) \cap \overline{D} = \emptyset$, and define the α-displacement energy of D to be

$$E_\alpha(D) = \inf\{\, |\psi|_\alpha \mid \psi \text{ displaces } D\}.$$

Then Proposition 11 immediately implies the following α-dependent *energy-capacity inequality* on N.

Corollary 15.

$$E_\alpha(D) \geq \frac{1}{4}\widetilde{c}_\alpha(D).$$

Remark 16. Note that in fact, at least for open $D \subset N$, $\widetilde{c}_\alpha(D) = c(\pi^{-1}(D) \cap \{r_\alpha < 1\})$, and therefore $\psi \mapsto \widetilde{c}_\alpha(\psi(D))$ is a C^0-continuous function on $\mathcal{G}$.

Remark 17. It is straightforward to show that in case when N is closed the contact α-capacity $\widehat{c}_\alpha$ (and hence $\widetilde{c}_\alpha$) is universally bounded. Indeed comparing the Gromov radius c with volume we have for any compact subset $A \subset SN$ the estimate

$$\widehat{c}_\alpha(A) \leq \frac{\pi}{(v_{2n+2})^{1/n+1}} \operatorname{vol}(N, \alpha(d\alpha)^n)^{1/n+1},$$

where v_{2n+2} is the volume of the unit ball $B \subset \mathbb{R}^{2n+2}$. In the non-compact setting this universal bound disappears, being replaced by

$$\widehat{c}_\alpha(A) \leq \frac{\pi}{(v_{2n+2})^{1/n+1}} \operatorname{vol}(\pi(A), \alpha(d\alpha)^n)^{1/n+1},$$

where $\pi : SN \to N$ is the natural projection.

Remark 17 brings us to the following interesting question.

Question 18. For which contact manifolds with a global contact form (N, ξ, α) is the norm $|\cdot|_\alpha$ unbounded, that is

$$\sup_{\psi \in \mathcal{G}} |\psi|_\alpha = \infty?$$

An example of such a manifold is a standard prequantization space (P_1, α_1) of the two-torus (T^2, ω_{std}) [17]. For more partial results on this question see Proposition 33. Put $\xi_1 = \operatorname{Ker} \alpha_1$ for the contact structure on P_1.

Proposition 19. *Let $H \in C^\infty(T^2, \mathbb{R})$ be an autonomous normalized Hamiltonian with the non-contractible closed curve $\{0\} \times S^1$ in T^2 as a component of a regular level set. Consider the contact flow $\{\psi^t = \psi^t_{\pi^* H}\}_{t>0}$ generated by the lift $\pi^* H$ to P_1 of the Hamiltonian H. This flow is a canonical lift to P_1 of the Hamiltonian flow of H. Then*

$$|\psi^t|_{\alpha_1} \geq \mathrm{const} \cdot t,$$

for a constant $\mathrm{const} > 0$.

Remark 20. The same argument works for prequantizations of a surface Σ of higher genus and a simple closed curve γ of infinite order in $\pi_1(\Sigma)$ instead of $(T^2, \{0\} \times S^1)$. Moreover, a very similar argument applies to show that $|[\{\phi_R^{s\cdot t}\}_{s\in[0,1]}]|_\alpha \geq \mathrm{const} \cdot t$, for $t > 0$.

The contact Hofer norm and its analogue for the universal cover can be explicitly computed for the contact manifolds $N = S^1 = \mathbb{R}/\mathbb{Z}$, and $N = \mathbb{R}$, endowed for example with the standard contact form $\alpha = dx$.

Proposition 21. *If under the natural isomorphism of $\widetilde{\mathrm{Cont}_0}(S^1)$ with the group of smooth $\mathbb{Z}$-equivariant maps $f : \mathbb{R} \to \mathbb{R}$ with $f' > 0$ everywhere, $\widetilde{\psi}$ corresponds to f, then*

$$|\widetilde{\psi}|_\alpha = \max_{x\in\mathbb{R}} |f(x) - x|.$$

Similarly, for $f \in \mathrm{Cont}_{c,0}(\mathbb{R})$,

$$|f|_\alpha = \max_{x\in\mathbb{R}} |f(x) - x|.$$

Finally, for $\phi \in \mathrm{Cont}_0(S^1)$,

$$|\phi|_\alpha = \max_{x\in S^1} d(\phi(x), x),$$

for the standard metric d on S^1. In particular, the (pseudo-)norm is unbounded in the first two cases and bounded in the third case.

Proof. The proofs of the three statements are essentially identical, so we prove only the second one. The contact isotopy

$$\{f^s(x) = sf(x) + (1-s)x\}_{s\in[0,1]}$$

has generating contact Hamiltonian h^s with $h^s(f^s(x)) = \alpha(\frac{d}{ds}f^s(x)) = f(x) - x$. Hence, as $|h^s|_{L^\infty} = |h^s \circ f^s|_{L^\infty}$ for all $s \in [0,1]$,

$$|f|_\alpha \leq \max_{x\in\mathbb{R}} |f(x) - x|.$$

In the other direction, let x_0 be such that $\max_{x\in\mathbb{R}} |f(x) - x| = |f(x_0) - x_0|$, and let f^s be any isotopy in $\mathrm{Cont}_{c,0}(\mathbb{R})$. Then, as

$$|f(x_0) - x_0| \leq \int_0^1 |h^s \circ f^s(x)|\, ds \leq \int_0^1 |h^s|_{L^\infty}\, ds,$$

we obtain

$$\max_{x\in\mathbb{R}} |f(x) - x| \leq |f|_\alpha.$$

□

Similarly to the non-degeneracy of the contact Hofer metric, we deduce from Proposition 11 the following basic topological property of $\mathcal{H}_\alpha$ in the topology defined by the distance function d_α.

Proposition 22. *If $\psi \in \mathcal{G} \setminus \mathcal{H}_\alpha$, then there exists $\varepsilon = \varepsilon_{\psi,\alpha} > 0$, such that $d_\alpha(\psi, \phi) \geq \varepsilon$ for all $\phi \in \mathcal{H}_\alpha$. In other words, $\mathcal{H}_\alpha$ is closed in the topology defined by d_α.*

This proposition establishes the lower bound

$$d_\alpha(\psi, \mathcal{H}_\alpha) = \inf_{\phi\in\mathcal{H}_\alpha} d_\alpha(\psi, \phi) \geq \varepsilon,$$

raising the following question.

Question 23. For which contact manifolds (N, ξ) with a global contact form α there are contactomorphisms $\psi \in \mathcal{G}$ that are arbitrarily d_α-far from $\mathcal{H}_\alpha$, or in other words, when

$$\sup_{\psi\in\mathcal{G}} d_\alpha(\psi, \mathcal{H}_\alpha) = \infty?$$

This question, for a different metric [33] on $\mathcal{G} = \mathrm{Cont}_{0,c}(T^*B \times S^1)$ where B is a closed connected manifold of positive dimension, was asked by F. Zapolsky.

Next we turn to a different functional on the group $\mathcal{G}$ and on its universal cover $\widetilde{\mathcal{G}}$, coming from a different way to express the Hofer norm in the usual Hamiltonian setting.

Definition 24. Given a global contact form α, we define for a contactomorphism $\psi \in \mathcal{G}$ and for an element $\widetilde{\psi} \in \widetilde{\mathcal{G}}$ in the universal cover of $\mathcal{G}$, their oscillation Hofer energy $|\psi|_\alpha^{\mathrm{osc}}$ and $|\widetilde{\psi}|_\alpha^{\mathrm{osc}}$ as

$$\inf \int_0^1 \mathrm{osc}_N(H_t)dt,$$

where

$$\mathrm{osc}_N(H_t) = \max_N(H_t) - \min_N(H_t),$$

and the infimum runs over all isotopies $\{\psi_t\}$ with $\psi_1 = \psi$ in the first case and all isotopies in class $\widetilde{\psi}$ in the second case.

It is curious that while this oscillation energy functional does not turn out to be equivalent to the L^∞ energy functional like in the Hamiltonian case, it does, in contrast to the Hamiltonian case, turn out to have a closed form expression in terms of the L^∞ energy.

Proposition 25. *For each $\psi \in \mathcal{G}$,*

$$\frac{1}{2}|\psi|_\alpha^{\mathrm{osc}} = d_\alpha(\psi, \mathcal{R}) = \inf_{t\in\mathbb{R}} d_\alpha(\psi, \phi_R^t).$$

This proposition has the following corollary.

Corollary 26. *For $\psi \in \mathcal{G}$, $|\psi|_\alpha^{\mathrm{osc}} = 0$ if and only if ψ belongs to the closure $\overline{\mathcal{R}}_\alpha$ of $\mathcal{R}_\alpha \subset \mathcal{G}$ in the topology defined by d_α.*

This corollary brings forth the following question.

Question 27. For which contact forms α, $\overline{\mathcal{R}}_\alpha = \mathcal{R}_\alpha$, that is $\mathcal{R}_\alpha$ is a closed subgroup of $\mathcal{G}$ in the topology defined by d_α?

Remark 28. It is immediate that those contact manifolds with a global contact form for which the Reeb flow is periodic, for example prequantization spaces, satisfy this property. Moreover, by Proposition 22, we see that $\overline{\mathcal{R}}_\alpha \subset \mathcal{H}_\alpha$, edging closer to an answer to this question. Indeed by an argument of Müller-Spaeth [20] and Casals-Spacil [10], whenever the Reeb flow $\{\phi^t_R\}$ of α has a dense orbit, we have $\mathcal{H}_\alpha = \mathcal{R}_\alpha$, implying $\overline{\mathcal{R}}_\alpha = \mathcal{R}_\alpha$. In the case of irrational ellipsoid with the standard contact form, it is easy to see that $T = \overline{\mathcal{R}}_\alpha$ is a torus and $\mathcal{R}_\alpha \subset T$ is an irrational one-parametric subgroup. It would be interesting to study the closure of $\mathcal{R}_\alpha$ in more general situations.

In [27, 28] Sandon introduced the notion of a translated point of a contactomorphism $\psi \in \mathcal{G}$ and observed that the number of translated points of a C^1-small (i.e. C^1-close to the identity transformation) contactomorphism ψ is at least the minimal number of critical points of a function on N and for generic such ψ, it is at least the minimal number of critical points of a Morse function on N. Consequently she conjectured the same conclusion to hold for all $\psi \in \mathcal{G}$. Here we study the following weak homological version of Sandon's conjecture.

Conjecture 29. Every contactomorphism $\psi \in \mathcal{G}$ has a translated point, and generic $\psi \in \mathcal{G}$ have at least $\dim H_*(N, \mathbb{Z}/(2))$ translated points.

The Sandon conjecture or its homological version has been shown to hold in a number of cases in [2, 4, 27, 28]. We remark that the genericity assumption, similarly to the case of the Arnol'd conjecture on Hamiltonian diffeomorphisms, should correspond to a non-degeneracy condition on the diffeomorphism ψ, for which our notion from Definition 1 is a candidate.

Proposition 25 suggests that if the oscillation energy of ψ is small enough, then it should have translated points. To this end we have the following definition and theorem.

Definition 30. We say that $\psi \in \mathcal{G}$ has *small oscillation energy* if $\frac{1}{2}|\psi|^{\mathrm{osc}}_\alpha < \rho(\alpha)$.

Theorem B. *Assume that (N, ξ) admits a strong exact symplectic filling $(M, \tau = d\overline{\beta})$, $N = \partial M$, with $\alpha = \overline{\beta}|_N$ a global contact form for (N, ξ). Then every contactomorphism ψ of small oscillation energy has at least one translated point, and if in addition ψ is non-degenerate, then it must have at least $\dim H_*(N, \mathbb{Z}/(2))$ translated points.*

Theorem B is an analogue of, and indeed follows from the Chekanov-type theorems [1, Theorem A, Theorem B] of Albers-Frauenfelder for the case of leafwise intersection points of Hamiltonian flows in an exact symplectic manifold $(W, d\beta)$ relative to bounding hypersurfaces Σ of restricted contact type ($\alpha = \beta|_\Sigma$ is a contact form). A point $x \in \Sigma$ is a leafwise intersection for $\phi \in \mathrm{Ham}_c(W, d\beta)$ relative to Σ if $\phi(x) \in \Sigma$ and moreover $\phi(x)$ lies in the same leaf as x of the characteristic foliation $\ker(d\beta|_\Sigma)$, that is - on the same orbit as x of the Reeb flow of α.

Theorem C (Albers-Frauenfelder [1]). *If $\phi \in \mathrm{Ham}_c(W, d\beta)$ has Hofer norm $|\phi|_{Hofer} < \rho(\alpha)$, then ϕ has a leafwise intersection point relative to Σ, and generic such ϕ have at least $\dim H_*(\Sigma, \mathbb{Z}/(2))$ leafwise intersection points.*

The paper [1] was used by Albers-Merry in [4] to prove results on translated points of contactomorphisms of contact manifolds with a strong exact filling. Techniques like Lemma 42 allow us to improve their estimates on the Hofer energy and prove a sharper Chekanov-type theorem in this setting. We also note that when $\rho(\alpha) = +\infty$, namely there are no closed Reeb orbits, it was shown in [2] that Conjecture 29 holds without the requirement on the existence of a symplectic filling. This result of Albers-Fuchs-Merry and Theorem B constitute reasonable evidence towards the following conjecture, removing the assumption on the existence of a strong exact symplectic filling.

Conjecture 31. Every contactomorphism ψ of small oscillation energy has at least one translated point, and if in addition ψ is non-degenerate, then it must have at least $\dim H_*(N, \mathbb{Z}/(2))$ translated points.

The methods of [2] seem to give a version of this conjecture (cf. Albers-Hein [3]) which is weaker in terms of assumptions. Additionally, following [3], one expects a cuplength estimate instead of the estimate ≥ 1 in the degenerate case, strengthening the conclusion.

We proceed by studying the relationship of the contact Hofer norm with a number of conjugation invariant norms on the group $\mathcal{G}$ and $\widetilde{\mathcal{G}}$. The norms that were constructed on the group $\mathcal{G}$ are the norm ν_S of Sandon [26] on $\mathcal{G} = \mathrm{Cont}_{0,c}(\mathbb{R}^{2n} \times S^1)$ and ν_Z of Zapolsky [33] on $\mathcal{G} = \mathrm{Cont}_{0,c}(T^*B \times S^1)$ for a closed connected manifold B of positive dimension (remark, we take $\nu_Z = \rho_{\sup}$ in the notations of [33]). The norms in both of these cases are unbounded. The norms that were constructed on the group $\widetilde{\mathcal{G}}$ are ν_{CS} by Colin-Sandon [12] for any contact manifold and ν_{FPR} by Fraser-Polterovich-Rosen [14] for orderable contact manifolds whose Reeb flow generates a circle action (note: we only consider the subgroup $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_c$ of $\widetilde{\mathcal{G}}_e$ in the notation of [14]). The norm ν_{FPR} is unbounded for example if the contact manifold with contact form (P,λ) considered in [14] is closed. Moreover, the quasimorphisms μ_G of Givental [15] on $\widetilde{\mathcal{G}}$ for $\mathcal{G} = \mathrm{Cont}_0(\mathbb{R}P^{2n-1}, \xi_{std})$ and μ_{BZ} of Borman-Zapolsky [7] on $\widetilde{\mathcal{G}}$ for certain toric contact manifolds give norms ν_G, ν_{BZ} on $\widetilde{\mathcal{G}}$ defined by $\nu_G = |\mu_G| + D(\mu_G)$ and $\nu_{BZ} = |\mu_{BZ}| + D(\mu_{BZ})$ on $\widetilde{\mathcal{G}} \setminus \widetilde{1}$, where $D(\mu) \geq 0$ denotes the defect of the quasimorphism μ, and $\nu_G(\widetilde{1}) = 0, \nu_{BZ}(\widetilde{1}) = 0$. The two norms μ_G, μ_{BZ} are unbounded. It turns out that $|\cdot|_\alpha$ provides a sort of universal upper bound for most of these metrics, and is therefore unbounded whenever these metrics are unbounded. We list these estimates below, the proofs of which are either straightforward computations or are contained in existing work and hence are mostly omitted. However we note the following bound which facilitates these computations.

Lemma 32. *For $\phi \in \mathcal{G}$ and $\widetilde{\phi} \in \widetilde{\mathcal{G}}$ the numbers $\lceil |\phi|_\alpha \rceil$ and $\lceil |\widetilde{\phi}|_\alpha \rceil$ are bounded from below by*

$$\min \left(\max \left\{ \left| \left\lceil \int_0^1 \max_N H_t \, dt \right\rceil \right|, \left| \left\lfloor \int_0^1 \min_N H_t \, dt \right\rfloor \right| \right\} \right) - \epsilon$$

where the minimum is taken over all contact Hamiltonians H whose flow $\{\phi_H^t\}_{t\in[0,1]}$ satisfies $\phi = \phi_H^1$ in the first case, and $[\{\phi_H^t\}] = \widetilde{\phi} \in \widetilde{\mathcal{G}}$ in the second case, and $\epsilon \in \{0,1\}$ is 1 if and only if $|\phi|_\alpha$ (repsectively $|\widetilde{\phi}|_\alpha$) is a non-negative integer, and the infimum in the definition of the contact Hofer norm is not attained.

Lemma 32 is a consequence of the inequality

$$\left\lceil \int_0^1 |H_t|_{L^\infty(N)} \, dt \right\rceil \geq \max \left\{ \left| \left\lceil \int_0^1 \max_N H_t \, dt \right\rceil \right|, \left| \left\lfloor \int_0^1 \min_N H_t \, dt \right\rfloor \right| \right\}$$

which follows by straightforward case-by-case analysis, and the semicontinuity properties of the ceiling function $\lceil \cdot \rceil$.

Proposition 33. *For all $\phi \in \mathcal{G}, \widetilde{\phi} \in \widetilde{\mathcal{G}}$, we have the estimates*

(i) $\nu_S(\phi) \leq 2\lceil |\phi|_{\lambda_{std}} \rceil$, *where λ_{std} is the standard contact form on $\mathbb{R}^{2n} \times S^1$,*

(ii) $\nu_Z(\phi) \leq \lceil |\phi|_{\lambda_{std}} \rceil$, *where λ_{std} is the standard contact form on $T^*B \times S^1$,*

(iii) $\nu_{FPR}(\widetilde{\phi}) \leq \lceil |\widetilde{\phi}|_\lambda \rceil + 1$, *where λ is the contact form that appears in [14],*

(iv) $\nu_G(\widetilde{\phi}) \leq \mathrm{const} \cdot (\lceil |\widetilde{\phi}|_{\alpha_{std}} \rceil + 1) + D(\mu_G)$, *where* $\mathrm{const} = \mu_G([\{\phi_R^t\}_{t\in[0,1]}])$,

(v)

$$\nu_{BZ}(\widetilde{\phi}) \le \text{const} \cdot (\lceil |\widetilde{\phi}|_{\alpha_{std}} \rceil + 1) + D(\mu_{BZ}),$$

where const $= \mu_{BZ}([\{\phi_R^t\}_{t\in[0,1]}])$.

Hence $|\cdot|_\alpha$ *is unbounded whenever any one of the norms* ν_S*,* ν_Z*,* ν_{FPR}*,* ν_G*,* ν_{BZ} *is unbounded.*

Item (ii) is simply the upper bound on $\rho_{\text{sup}}(\phi)$ from [33, Proof of Theorem 1.3], and item (i) follows by tracing the identifications of contact manifolds in [26] from the same upper bound and a comparison between ρ_{osc} and ρ_{sup} in the notations of [33]. Item (iii) follows by the identities

$$\nu_+(\widetilde{\phi}) = \min_{[\{\phi_H^t\}]=\widetilde{\phi}} \left\lceil \int_0^1 \max H_t \, dt \right\rceil, \quad \nu_-(\widetilde{\phi}) = \max_{[\{\phi_H^t\}]=\widetilde{\phi}} \left\lfloor \int_0^1 \min H_t \, dt \right\rfloor,$$
$$\nu_{FPR} = \max\{|\nu_+|, |\nu_-|\},$$

in the notations of [14] and Lemma 32 by straightforward case-by-case analysis. Items (iv) and (v) follow from Item (iii) and [7, Lemma 1.33], the coefficient const being the one that appears in [7, Lemma 1.33].

Proposition 33 has the following consequence which is similar in spirit to [14, Proposition 3.11]. We call an open subset $U \subset N$ of N *downscalable* if for every compact subset $A \subset U$ and for every $K > 0$ there exists $\psi = \psi_K \in \text{Cont}(N)$, such that $\lambda_\psi = e^{g_\psi}$ satisfies $g_\psi|_A \le -K$. Note that the property of U being downscalable does not depend on the choice of the contact form α on N. A simple example of a downscalable U is the complement of a point in S^1 with the standard contact structure. A more general example is any contact Darboux ball, that is - the image of a contact embedding of a standard contact ball

$$B_{st} = \left\{ \sum_j (x_j^2 + y_j^2) + z^2 < 1 \right\} \tag{4}$$

in the standard contact $\mathbb{R}^{2n} \times \mathbb{R}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$.

Corollary 34. *Any element* $\widetilde{\phi} \in \widetilde{\mathcal{G}}$ *that lies in the image of the natural map* $\widetilde{\text{Cont}}_{0,c}(U) \to \widetilde{\mathcal{G}}$ *for a downscalable subset* U *has*

$$\inf_{\psi \in \mathcal{G}} |\psi \widetilde{\phi} \psi^{-1}|_\alpha = 0.$$

Hence $\nu_{FPR}(\widetilde{\phi}) \le 2$*, and similar statements hold for the other norms* ν_S*,* ν_Z*,* ν_G*,* ν_{BZ}*.*

Proof. Let $Y(t,x)$ be a contact vector field generating $\widetilde{\phi}$ such that $Y(t,x)=0$ outside a compact $A \subset U$. Choose $K > 0$, and let ψ be such that $\lambda_\psi \leq e^{-K}$ on A. For a choice of global contact form α, set $l(Y,\alpha) = \int_0^1 |H_t|_{L^\infty(N)}\, dt$, where $H = \alpha(Y)$ is the contact Hamiltonian of the vector field Y. Note that by Property (iv) of the contact Hofer metric, $|\psi\widetilde{\phi}\psi^{-1}|_\alpha = |\widetilde{\phi}|_{\psi^*\alpha}$. Moreover, since $\psi^*\alpha = \lambda_\psi \alpha$,

$$l(Y,\psi^*\alpha) = \int_0^1 |\lambda_\psi H_t|_{L^\infty(N)}\, dt, = \int_0^1 |\lambda_\psi H_t|_{L^\infty(U)}\, dt \leq e^{-K} l(Y,\alpha).$$

Taking infima now shows that

$$\inf_{\psi \in \mathcal{G}} |\widetilde{\phi}|_{\psi^*\alpha} \leq e^{-K} l(Y,\alpha)$$

for all $K > 0$. This finishes the proof of the first statement. The second statment is now an immediate corollary of Proposition 33 and the conjugation-invariance of the norm ν_{FPR}. □

Remark 35. Recall that the Lie algebra of $\mathcal{G}$ is identified with $C^\infty(N,\mathbb{R})$ by a choice of a contact form α, and under this identification, the adjoint action of $\mathcal{G}$ on its Lie algebra takes the form $\mathrm{Ad}_\psi(H) = (\lambda_\psi\, H) \circ \psi^{-1}$. It is curious to note that by the same argument as in the proof of Corollary 34, given any norm $|\cdot|_0$ that is invariant under pull-back by elements of $\mathcal{G}$ on the linear space $C^\infty(N,\mathbb{R})$ (for example the L^∞-norm), its Ad-invariantization (cf. [8])

$$|H|_0^{Ad} = \inf_{\substack{H_1+\ldots+H_l=H,\\ \psi_1,\ldots,\psi_l \in \mathcal{G}}} |\mathrm{Ad}_{\psi_1}(H_1)|_0 + \ldots + |\mathrm{Ad}_{\psi_l}(H_l)|_0,$$

yields an (Ad-invariant!) pseudo-norm that is identically zero. This confirms the intuition that conjugation-invariant norms on $\mathcal{G}$ are objects of global topological nature.

Considering conjugacy classes of elements in $\mathcal{G}$ leads one to the following question.

Question 36. For which contact manifolds (N,ξ) with contact form α the value

$$|\phi|_{conj,\alpha} = \sup_{\psi \in \mathcal{G}} |\psi\phi\psi^{-1}|_\alpha$$

is finite for all $\phi \in \mathcal{G}$?

Note that whenever this finiteness condition holds, $|\cdot|_{conj,\alpha}$ gives a conjugation-invariant non-degenerate norm on $\mathcal{G}$. Clearly there is an analogous definition for $\widetilde{\mathcal{G}}$. An example of ϕ with $|\phi|_{conj,\alpha} = +\infty$ would have to involve a sequence $\{\psi_j\}$ of contactomorphisms such that $\max \lambda_{\psi_j} \to \infty$ and $|\psi_j|_\alpha \to \infty$ as $j \to \infty$, and hence a partial answer to Question 18. In particular whenever $|\cdot|_\alpha$ is bounded, we obtain a bounded conjugation-invariant norm on $\mathcal{G}$. Moreover, according the theorem of [14], there must exist a constant $c > 0$, such that $|\phi|_{conj,\alpha} \geq c$ for all $\phi \neq 1$. The following is a numerical bound on $c > 0$.

Proposition 37. *For all $\phi \neq 1$,*

$$|\phi|_{conj,\alpha} \geq \frac{1}{4} c_0, \quad c_0 = \sup_B \widetilde{c}_\alpha(B) > 0,$$

where B runs over all contact embeddings of the standard contact ball B_{st} into N.

Proof. For a subset $A \subset N$, denote $\overline{c}_\alpha(A) = \sup_{\psi \in \mathcal{G}} \widetilde{c}_\alpha(\psi(A))$. By the energy-capacity inequality (Corollary 15) we have

$$|\phi|_{conj,\alpha} \geq \frac{1}{4} \overline{c}_\alpha(B_0)$$

where B_0 is a small contact ball displaced by ϕ. We claim that

$$\overline{c}_\alpha(B_0) \geq c_0.$$

Let B be the image of a contact embedding $\iota : B_{st} \to N$. Note that there exists a contactomorphism $\psi_1 \in \mathcal{G}$ mapping the center p_0 of B_0 to the center $p = \iota(0)$ of B. Then the image of B_0 under ψ_1 contains the image $\iota(B(\epsilon))$ of a small ellipsoid $B(\epsilon) = \{\sum_j \frac{(x_j^2+y_j^2)}{\epsilon^2} + \frac{z^2}{\epsilon^4} < 1\}$ around 0. Hence

$$\overline{c}_\alpha(B_0) \geq \overline{c}_\alpha(\iota(B(\epsilon))). \tag{5}$$

Moreover, given a number $0 < \rho < 1$, considering the autonomous contact flow $(x, y, z) \mapsto (e^\tau \cdot x, e^\tau \cdot y, e^{2\tau} \cdot z)$ on $\mathbb{R}^{2n} \times \mathbb{R}$, with contact Hamiltonian cut off sufficiently close to ∂B_{st}, we construct a contactomorphism $\psi_{2,\epsilon,\rho,st}$ supported in B_{st} and mapping $B(\epsilon)$ to $B(\rho)$. Clearly this contactomorphism extends to a contactomorphism $\psi_{2,\epsilon,\rho}$ of N. This, and the C^0-continuity of the contact capacity $\widetilde{c}_\alpha$ (Remark 16) gives the equality

$$\overline{c}_\alpha(\iota(B(\epsilon))) = \overline{c}_\alpha(B),$$

which combined with (5) finishes the proof. □

We conclude this section by studying the relation of the contact Hofer norm in the case of prequantization spaces and the usual Hofer norm on the group of Hamiltonian diffeomorphisms of a symplectic manifold. Given a symplectic manifold (M, ω), the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms of (M, ω) consists of the time-1-maps ϕ_H^1 of time-dependent Hamiltonian vector fields X_H obtained by the Hamiltonian construction $\iota_{X_H}\omega = -dH$ from compactly supported Hamiltonian functions $H \in C_c^\infty([0,1] \times M, \mathbb{R})$. In the case when M is closed it is enough to consider only $H \in C_0^\infty([0,1] \times M, \mathbb{R})$ normalized by the condition that $\int_M H(t,-)\,\omega^n = 0$ for all $t \in [0,1]$.

The Hofer norm on $\mathrm{Ham}(M)$ was introduced by Hofer [16] and shown to be always non-degenerate by Lalonde-McDuff [18]. It is defined as

$$|\phi|_{Hofer} = \inf \int_0^1 (\max_M H(t,\cdot) - \min_M H(t,\cdot))\,dt,$$

where the infimum runs over all $H \in C_0^\infty([0,1] \times M, \mathbb{R})$ with $\phi_H^1 = \phi$. Alternatively, one can take the infimum

$$|\phi|'_{Hofer} = \inf \int_0^1 |H(t,\cdot)|_{L^\infty(M)}\,dt$$

over the same set of H, which results in an equivalent norm:

$$\frac{1}{2}|\cdot|_{Hofer} \leq |\cdot|'_{Hofer} \leq |\cdot|_{Hofer}.$$

Let (P, α) be a prequantization space of the symplectic manifold (M, ω). That is $d\alpha = \widetilde{\omega}$, for the lift $\widetilde{\omega} = p^*\omega$ of ω to P by the fibration map $p : P \to M$. Such a prequantization exists if and only if the class $[\omega] \in H^2(M, \mathbb{R})$ is integral, that is lies in $\mathrm{Image}(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}))$. Then the group $\mathcal{H}_\alpha \subset \mathcal{G}$ of strict contactomorphisms (called quantomorphisms in this setting) and the group Ham $=$ Ham(M) enter the following central group extensions:

$$1 \to S^1 \to \mathcal{H}_\alpha \xrightarrow{pr} \mathrm{Ham} \to 1, \tag{6}$$

$$0 \to \mathbb{R} \to \widetilde{\mathcal{H}_\alpha} \xrightarrow{\widetilde{pr}} \widetilde{\mathrm{Ham}} \to 0. \tag{7}$$

Moreover the pullback map $p^* : C_0^\infty(M, \mathbb{R}) \to C^\infty(P, \mathbb{R})$, $H \mapsto H \circ p$ on functions induces a natural homomorphism

$$i : \widetilde{\mathrm{Ham}} \to \widetilde{\mathcal{H}_\alpha}$$

which is a section for $\widetilde{pr}$. Moreover $cw \circ i = 0$ (see Remark 9).

This allows us to define the following 8 conjugation-invariant norms on Ham (there are analogues for $\widetilde{\mathrm{Ham}}$, which we omit) from the (pseudo-)norms $|\cdot|_\alpha, |\cdot|_\alpha^{\mathrm{osc}}, |\cdot|_{str,\alpha}, |\cdot|_{str,\alpha}^{\mathrm{osc}}$, where the last one is the Finsler restriction from $\mathcal{G}$ to $\mathcal{H}_\alpha$ of the Finsler pseudo-metric giving $|\cdot|_\alpha^{\mathrm{osc}}$ on $\mathcal{G}$:

$$|\phi|_1 = \inf_{\pi(\widetilde{\phi})=\phi} |i(\widetilde{\phi})|_{str,\alpha}, \quad |\phi|_2 = \inf_{\pi(\widetilde{\phi})=\phi} |i(\widetilde{\phi})|_\alpha, \tag{8}$$

$$|\phi|_3 = \inf_{\pi(\widetilde{\phi})=\phi} |i(\widetilde{\phi})|_{str,\alpha}^{\mathrm{osc}}, \quad |\phi|_4 = \inf_{\pi(\widetilde{\phi})=\phi} |i(\widetilde{\phi})|_\alpha^{\mathrm{osc}}, \tag{9}$$

$$|\phi|_5 = \inf_{pr(\widehat{\phi})=\phi} |\widehat{\phi}|_{str,\alpha}, \quad |\phi|_6 = \inf_{pr(\widehat{\phi})=\phi} |\widehat{\phi}|_\alpha, \tag{10}$$

$$|\phi|_7 = \inf_{pr(\widehat{\phi})=\phi} |\widehat{\phi}|_{str,\alpha}^{\mathrm{osc}}, \quad |\phi|_8 = \inf_{pr(\widehat{\phi})=\phi} |\widehat{\phi}|_\alpha^{\mathrm{osc}}. \tag{11}$$

By Proposition 11 and Remark 28 it is seen that these norms are non-degenerate. Moreover, by arguments resembling the proof of Proposition 25, which we omit, one sees that

$$|\cdot|_3 = |\cdot|_5 = |\cdot|_7 = |\cdot|_{Hofer},$$

that

$$\frac{1}{2}|\cdot|_{Hofer} \leq |\cdot|_1 \leq |\cdot|_{Hofer},$$

that

$$\nu_1(\cdot) := |\cdot|_4 = |\cdot|_6 = |\cdot|_8 \leq |\cdot|_{Hofer}$$

and that

$$\nu_2(\cdot) := |\cdot|_2$$

satisfies

$$\nu_1(\cdot) \leq \nu_2(\cdot) \leq |\cdot|_1 \leq |\cdot|_{Hofer}.$$

This leads one to the following questions.

Question 38. Is ν_1 equivalent to the Hofer norm? Is ν_2 equivalent to the Hofer norm?

Question 39. Is ν_1 unbounded? Is ν_2 unbounded?

In this direction, using Givental's quasimorphism [15], we prove the following statement.

Proposition 40. *If $M = \mathbb{C}P^n$ with the symplectic form $2\,\omega_{FS}$, where ω_{FS} is the Fubini-Study form normalized so that $\langle[\omega_{FS}], [\mathbb{C}P^1]\rangle = 1$, and $P = \mathbb{R}P^{2n+1}$ with the standard contact form α_{st} is its prequantization, then ν_2 on $\mathrm{Ham}(\mathbb{C}P^n)$ is unbounded.*

2. Proofs

We open this section with the following general result from [31, Proof of Theorem 1.3], which we refer to as Usher's trick. We remark that while Usher's trick is originally stated for compactly supported Hamiltonians on general symplectic manifolds, it works equally well for degree 1 $\mathbb{R}_{>0}$-homogeneous Hamiltonians on the symplectization SN of N and their $\mathbb{R}_{>0}$-equivariant flows. Put $\mathcal{H}^1_{SN}$ for the space of degree 1 $\mathbb{R}_{>0}$-homogeneous Hamiltonians in $C^\infty([0,1] \times SN, \mathbb{R})$.

Proposition 41. *Given any $H \in \mathcal{H}^1_{SN}$, there exists $K \in \mathcal{H}^1_{SN}$ with flow $\{\phi^t_K\}_{t\in[0,1]}$, such that*

U1 $\{\phi^t_H\}$ and $\{\phi^t_K\}$ have the same endpoints (and are in fact homotopic through $\mathbb{R}_{>0}$-equivariant Hamiltonian flows with fixed endpoints), and

U2 for all $t \in [0,1]$, $x \in M$,

$$K(t, \phi^t_K(x)) = H(1-t, x).$$

For the reader's convenience, we present the proof of this statement. Given $H \in \mathcal{H}^1_{SN}$ generating the isotopy $\{\phi^t_H\}$ of $\mathbb{R}_{>0}$-equivariant Hamiltonian diffeomorphisms, consider the isotopy $\{\phi^1_H(\phi^{1-t}_H)^{-1}\}$ of $\mathbb{R}_{>0}$-equivariant Hamiltonian diffeomorphisms. We claim that the Hamiltonian $K \in \mathcal{H}^1_{SN}$ generating this isotopy is the required Hamiltonian. Indeed, Property U2 follows from the identity $(\phi^t_K)^{-1} = \phi^{1-t}_H \circ (\phi^1_H)^{-1}$ by comparing the generating Hamiltonians on both sides, and Property U1 is immediate.

We require the following cut-off technique. For an interval $I = (a,b) \subset \mathbb{R}_{>0}$ and $\kappa > 0$, denote $I^\kappa = (e^{-\kappa}a, e^\kappa b) \subset \mathbb{R}_{>0}$. Note that $I^\kappa \supset I$.

Lemma 42. *Let $\{\overline{\psi}_t\}$ be a path of $\mathbb{R}_{>0}$-equivariant Hamiltonian diffeomorphisms of SN. Let $\overline{H}$ be its degree 1 homogeneous Hamiltonian. Given an interval $I=(a,b)\subset\mathbb{R}_{>0}$ (so that $\sup I=b$) and $\delta>0$, fix a cutoff function $\lambda_0^{I,\delta}:\mathbb{R}_{>0}\to[0,1]$ that satisfies $\lambda_0^{I,\delta}\equiv 1$ on $I^{\delta/2}$, and $\lambda_0^{I,\delta}\equiv 0$ on $\mathbb{R}_{>0}\setminus I^{\delta}$. Denote by $\lambda^{I,\delta}:SN\cong N\times\mathbb{R}_{>0}\to[0,1]$ the lift $\lambda^{I,\delta}(x)=\lambda_0^{I,\delta}(r_\alpha(x))$ of $\lambda_0^{I,\delta}$ to SN by the projection r_α to the second coordinate. Then the new compactly supported Hamiltonian*

$$\overline{H}^{I,\delta}(t,x)=\overline{H}(t,x)\cdot\lambda^{I,\delta}((\overline{\psi}_t)^{-1}x)$$

with flow $\{\overline{\psi}_t^{I,\delta}\}$ satisfies the following properties.

C1 $\overline{\psi}_t^{I,\delta}|_{I\cdot N_1}\equiv\overline{\psi}_t|_{I\cdot N_1}$,

C2 $\int_0^1|\overline{H}_t^{I,\delta}|_{L^\infty(SN)}dt\le e^{\delta}(\sup I)\cdot\int_0^1|\overline{H}_t\circ\overline{\psi}_t|_{L^\infty(N_1)}dt.$

Proof of Lemma 42. The proof of Property C1 is simply the fact that for $x\in I\cdot N_1$ and $t\in[0,1]$, the Hamiltonian vector fields of $\overline{H}^{I,\delta}(t,x)$ and $\overline{H}(t,x)$ coincide on $\overline{\psi}_t(I\cdot N_1)$, and hence $\overline{\psi}_t(x)$ is a solution of the ODE defining the Hamiltonian flow of $\overline{H}^{I,\delta}$ with initial condition $x\in I\cdot N_1$. The conclusion follows by uniqueness of solutions to ODE.

The proof of Property C2 is that outside $I^\delta\cdot N_1$, where $I^\delta=(e^{-\delta}a,e^{\delta}b)$, the cut-off $\lambda^{I,\delta}$ vanishes. Therefore $\overline{H}_t^{I,\delta}$ vanishes outside $\overline{\psi}_t(I^\delta\cdot N_1)$. Hence

$$\begin{aligned}\int_0^1|\overline{H}_t^{I,\delta}|_{L^\infty(SN)}dt&\le\int_0^1|\overline{H}_t|_{L^\infty(\overline{\psi}_t(I^\delta\cdot N_1))}dt\\=\int_0^1|\overline{H}_t\circ\overline{\psi}_t|_{L^\infty(I^\delta\cdot N_1)}dt&\le e^{\delta}(\sup I)\cdot\int_0^1|\overline{H}_t\circ\overline{\psi}_t|_{L^\infty(N_1)}dt.\end{aligned}$$

In the last inequality we used that $\overline{\psi}_t$ is $\mathbb{R}_{>0}$-equivariant and $\overline{H}_t$ is $\mathbb{R}_{>0}$-homogeneous of degree 1, and noted that $\sup I^\delta=e^\delta\cdot\sup I$. □

Proof of Proposition 11. Assume that $\overline{\psi}$ displaces a compact subset A of SN. Let ψ be generated by the contact Hamiltonian $H(t,x)$ with flow $\{\psi_t\}$. Denote by $\overline{H}(t,x)$ its degree 1 homogeneous lift to SN with flow $\{\overline{\psi}_t\}$ the canonical lift of $\{\psi_t\}$ to SN. Apply Usher's trick (Propostion 41) to $\overline{H}(t,x)$ to obtain a degree 1 homogeneous Hamiltonian $\overline{K}(t,x)$, satisfying Properties U1 and U2. The time-one map $\overline{\psi}=\psi^1_{\overline{K}}$ of the flow of $\overline{K}$ displaces A. Hence, if we choose $I=(\min_A r_\alpha,\max_A r_\alpha)$, then for any $\delta>0$ by Property C1 so does the time-one map $\psi^1_{\overline{K}^{I,\delta}}$ of the cut-off $\overline{K}^{I,\delta}$ as performed in

Lemma 42. Therefore by [18, Theorem 1.1] of Lalonde-McDuff, we have

$$\frac{1}{4}c(A) \leq \int_0^1 |\overline{K}_t^{I,\delta}|_{L^\infty(SN)}\, dt.$$

Furthermore, by Properties C2 and U2 we estimate

$$\begin{aligned}\int_0^1 |\overline{K}_t^{I,\delta}|_{L^\infty(SN)}\, dt &\leq e^\delta h(A) \cdot \int_0^1 |\overline{K}_t \circ \psi^t_{\overline{K}}|_{L^\infty(N_1)}\, dt \\ &= e^\delta h(A) \cdot \int_0^1 |\overline{H}_t|_{L^\infty(N_1)}\, dt,\end{aligned}$$

recalling that $h(A) = \max_A r_\alpha$, where $r_\alpha : SN \to \mathbb{R}_{>0}$ is the composition of the symplectomorphism $SN \to N \times \mathbb{R}_{>0}$ determined by α, and the projection to the $\mathbb{R}_{>0}$ factor. As $i_1^* \overline{H}_t = H_t$ for the obvious isomorphism $i_1 : N \to N_1$, we conclude that

$$\frac{1}{4}c(A) \leq e^\delta h(A) \cdot \int_0^1 |H_t|_{L^\infty(N)}\, dt,$$

whence by taking infima over contact Hamiltonians $H(t,x)$ generating ψ and over $\delta > 0$, we obtain

$$\frac{1}{4}\widehat{c}_\alpha(A) \leq |\psi|_\alpha.$$

□

Proof of Theorem A. Property (i) follows from the fact that for $\psi \in \mathcal{G}$, the condition $\psi \neq 1_N$ is equivalent to $\overline{\psi} \neq 1_{SN}$, and the latter condition implies that $\overline{\psi}$ displaces a ball $B \subset SN$ of positive Gromov radius $c(B) > 0$ and height $h_\alpha(B) > 0$, and hence by Proposition 11 we have $|\psi|_\alpha \geq \frac{1}{4}\widehat{c}(B) > 0$.

For property (ii) assume that ϕ is generated by contact Hamiltonian F_t and ψ is generated by G_t. We claim that $\phi\psi$ is generated by contact Hamiltonian H_t with

$$\int_0^1 |H_t|_{L^\infty(N)}\, dt \leq \int_0^1 |F_t|_{L^\infty(N)}\, dt + \int_0^1 |G_t|_{L^\infty(N)}\, dt. \tag{12}$$

Then taking infima first on the left-hand side and then on the right hand side finishes the proof. Note that the functional $F \mapsto \int_0^1 |F_t|_{L^\infty(N)}\, dt$ is invariant under time-reparametrizations $F(t,x) \mapsto \tau'(t)F(\tau(t),x)$ acting as $\{\phi_t\}_{t\in[0,1]} \mapsto \{\phi_{\tau(t)}\}_{t\in[0,1]}$ on the flows, for $\tau : [0,1] \to [0,1]$ a smooth function with $\tau(0) = 0, \tau(1) = 1$ and $\tau' \geq 0$. Choose such reparametrizations

τ_1, τ_2 with $\mathrm{supp}(\tau_1') \subset [1/2, 1]$ and $\mathrm{supp}(\tau_2') \subset [0, 1/2]$. Then $\phi\psi$ is generated by the flow $\phi_{\tau_1(t)}\psi_{\tau_2(t)}$. The contact Hamiltonian H_t of this flow satisfies $H_t = \tau_2'(t)G_{\tau_2(t)}$ for $t \in [0, 1/2]$ and $H_t = \tau_1'(t)F_{\tau_1(t)}$ for $t \in [1/2, 1]$, whence (12) follows immediately.

Property (iii) is an immediate consequence of the fact that if $H(t, x)$ generates ψ_t with endpoint ψ then $-H(1-t, x)$ generates $\psi_{1-t}\psi^{-1}$ with endpoint ψ^{-1}.

Property (iv) follows from the fact that if X_t generates $\{\phi_t\}$ with $\phi_1 = \phi$, then $Y_t = \psi_*(X_t)$ that is $Y_t(x) = (D\psi)(\psi^{-1}x)X_t(\psi^{-1}x)$ generates $\{\psi\phi_t\psi^{-1}\}$ with $\psi\phi_1\psi^{-1} = \psi\phi\psi^{-1}$, and hence if $F_t = \alpha(X_t)$ is the contact Hamiltonian for X_t then $G_t = \alpha(\psi_*(X_t)) = (\psi^*\alpha)(X_t) \circ \psi^{-1}$ is the contact Hamiltonian for Y_t. □

Proof of Proposition 14. Since ϕ is a uniform limit of continuous maps, it is continuous. Assume that $\phi \neq \psi$. Then $\phi\psi^{-1}$ displaces a small closed ball $B \subset N$. By uniform convergence, this implies that $\psi_i\psi^{-1}$ displaces B for all i large enough. Therefore by Corollary 15

$$d_\alpha(\psi_i, \psi) \geq \frac{1}{4}\widetilde{c}_\alpha(B) > 0,$$

in contradiction to assumption (i). □

Proof of Proposition 19. Our approach is based on [22, Exercise 7.2.E] (cf. [19]) in the Hamiltonian case. This approach requires certain knowledge about the orbit class evaluation map

$$ev : \pi_1(\mathrm{Cont}_0(P_1, \xi)) \to \pi_1(P_1).$$

We first note that $\pi_1(P_1) \cong \mathrm{H}_{\mathbb{Z}}$, the integer Heisenberg group of unipotent upper triangular 3×3 matrices with integer coefficients. It is easy to see that $\mathrm{H}_{\mathbb{Z}} = \langle p, q, r \mid [p, q] = r,\ [r, p] = 1,\ [r, q] = 1\rangle$. Therefore the centre $\mathrm{Z}(\pi_1(P_1))$ of $\pi_1(P_1)$ satisfies $\mathrm{Z}(\pi_1(P_1)) = \langle r\rangle$. Moreover, r is the orbit class of the Reeb loop. Finally, as is true for all diffeomorphism groups (see e.g. [23, Section 3.7]), $\mathrm{image}(ev) \subset \mathrm{Z}(\pi_1(P_1))$. We conclude that

$$\mathrm{image}(ev) = \langle r\rangle. \tag{13}$$

Choosing appropriate action-angle coordinates around $\{0\} \times S^1$ in T^2 we can find a small open neighbourhood U of this curve symplectomorphic to $U = (-\epsilon, \epsilon) \times S^1$, all of whose slices $\{a\} \times S^1$, $|a| < \epsilon$, are invariant under the Hamiltonian flow $\{\phi_H^t\}_{t\in\mathbb{R}}$ of H, and moreover, there is a positive lower bound $\mathrm{const}(H|_U) > 0$ on the Riemannian length of the Hamiltonian vector field of $H|_U$ with respect to an auxiliary Riemannian metric on T^2 restricting to the standard metric in the action-angle coordinates on U.

In the neigbourhood U the prequantization can be trivialized as $U \times S^1$ with the contact form $\alpha_1 = d\theta + xdy$, where $x : (-\epsilon, \epsilon) \to \mathbb{R}$, $y : S^1 \to S^1$, and $\theta : S^1 \to S^1$ are the standard coordinates. Hence, over U the symplectization of P_1 trivializes as $U \times S^1 \times \mathbb{R}_{>0}$ with the symplectic form $\omega = d(r(d\theta + xdy)) = dr \wedge d\theta + d(rx) \wedge dy$. Putting $X = rx$ for a new coordinate, the symplectic form splits as $\omega = dr \wedge d\theta + dX \wedge dy$. Passing to universal covers we have $\widetilde{P_1|_U} = \widetilde{U} \times \mathbb{R}$, where $\widetilde{U} = (-\epsilon, \epsilon) \times \mathbb{R}$ and $\widetilde{SP_1|_U} = S(\widetilde{P_1|_U}) = \widetilde{U} \times \mathbb{R} \times \mathbb{R}_{>0}$ with the symplectic form $dr \wedge d\theta + d(rx) \wedge dy = dr \wedge d\theta + dX \wedge dy$, where now $\theta : \mathbb{R} \to \mathbb{R}$, and $y : \mathbb{R} \to \mathbb{R}$ are the standard coordinates. For a small $\delta > 0$, let $B'(t') = \{(r, \theta) \in \mathbb{R}_{>0} \times \mathbb{R} |\ \delta \le r \le 1 - \delta,\ 0 \le \theta \le \frac{t'}{1-2\delta}\} \subset \mathbb{R} \times \mathbb{R}_{>0}$ be a rectangle of capacity (area) t' in the (r, θ) half-plane. Note that in particular $r \ge \delta$ on $B'(t')$, and hence the image of $\widetilde{U} \times B'(t)$ in the new coordinates (X, y, θ, r) contains the strip $[-\epsilon \cdot \delta, \epsilon \cdot \delta] \times \mathbb{R} \times B'(t')$. Pick the rectangle $B(t') = \{(X, y) |\ X \in [-\epsilon \cdot \delta, \epsilon \cdot \delta],\ y \in [0, \frac{t'}{2\epsilon \cdot \delta}]\} \subset [-\epsilon \cdot \delta, \epsilon \cdot \delta] \times \mathbb{R}$ of capacity t'. Then the Gromov radius of $A(t') = B(t') \times B'(t')$ satisfies

$$c(A(t')) \ge t'.$$

Consider $A(t')$ as a subset of $\widetilde{SP_1}$.

Fixing $t > 0$, we claim that

$$|\psi_H^t|_{\alpha_1} \ge c \cdot t,$$

for $c = \frac{1}{2}\mathrm{const}(H|_U) \cdot \epsilon \cdot \delta$, where $\mathrm{const}(H|_U) > 0$ is (a lower bound on) the Riemannian length of the Hamiltonian vector field of $H|_U$ as above.

Fix any $t' < 4c \cdot t$, and let $F(s, x) \in C^\infty([0, 1] \times P_1, \mathbb{R})$ be a contact Hamiltonian generating a contact isotopy $\{\psi_F^s\}_{s \in [0,1]}$ with $\psi_F^1 = \psi_H^t$. Consider its deck-invariant lift $\widetilde{F}$ to the universal cover of P_1, generating $\widetilde{\psi}_F^s$. We first show that $\widetilde{\psi}_F^1$ displaces $A(t')$. Taking $F_0 = t\pi^* H$, it is immediate to see that $\widetilde{\psi}_{F_0}^1$ displaces $A(t')$ by considering the projection of $\widetilde{\psi}_{F_0}^1(A(t'))$ to the universal cover of T^2. Any other isotopy $\{\psi_F^s\}$ differs from $\{\psi_{F_0}^s\}$ by a loop $\gamma = \{\gamma_s\}_{s \in [0,1]}$ in $\mathrm{Cont}_0(P_1, \xi)$, based at the identity: that is $\psi_F^s = \gamma_s \psi_{F_0}^s$. Hence $\widetilde{\psi}_F^1 = D(ev([\gamma]))\widetilde{\psi}_{F_0}^1$, $D(g)$ denoting the Deck transformation corresponding to an element $g \in \pi_1(P_1)$. Hence by (13), the projections of $\widetilde{\psi}_F^1(A(t'))$ and $\widetilde{\psi}_{F_0}^1(A(t'))$ to the universal cover of T^2 coincide, and therefore $\widetilde{\psi}_F^1$ also displaces $A(t')$.

We proceed by cutting $\widetilde{F}$ off by a function $\lambda_c : \widetilde{P_1} \to [0,1]$ outside a compact subset containing $\bigcup_{s\in[0,1]} \widetilde{\psi}^s_F(\pi(A(t'))$ in its interior, where $\pi : S(\widetilde{P_1}) \to \widetilde{P_1}$ is the natural projection. We obtain a compactly supported contactomorphism of $\widetilde{P_1}$ with compactly supported Hamiltonian $\widetilde{F}^{\lambda_c}$ satisfying

$$\int_0^1 |\widetilde{F}^{\lambda_c}_s|_{L^\infty(\widetilde{P_1})} ds \leq \int_0^1 |F_s|_{L^\infty(P_1)} ds.$$

Moreover the lift $\overline{\psi}^s_{\widetilde{F}^{\lambda_c}}$ of the flow of $\widetilde{F}^{\lambda_c}$ to $S(\widetilde{P_1})$ displaces $A(t')$. Hence, noting that $h(A(t')) < 1$, by Proposition 11 we have

$$\int_0^1 |\widetilde{F}^{\lambda_c}_s|_{L^\infty(\widetilde{P_1})} ds \geq \frac{t'}{4}.$$

Therefore

$$\int_0^1 |F_s|_{L^\infty(P_1)} ds \geq \frac{t'}{4},$$

and hence first taking the infimum over all contact Hamiltonians F_s generating ψ^t_H and then taking supremum over all $t' < 4c \cdot t$, we finish the proof. □

Proof of Proposition 22. Assume $\psi \in \mathcal{G} \setminus \mathcal{H}_\alpha$, and $\phi \in \mathcal{H}_\alpha$. Let $\beta = \psi^{-1} \in \mathcal{G} \setminus \mathcal{H}_\alpha$. Then there exists a point $x \in N$ with $\mu := \max_N \lambda_\beta^{-1} = \lambda_\beta^{-1}(x) > 1$. There is a ball $B(x) \subset N$ such that $(\lambda_\beta^{-1})|_{B(x)} \geq \mu^{3/4}$. Consider the interval $I = (\mu^{-1/4}, \mu^{1/4})$. We claim that $\overline{\phi}\,(\overline{\psi})^{-1} = \overline{\phi}\,\overline{\beta}$ displaces $B(x) \times I$, and therefore denoting $\varepsilon = \frac{1}{4}\widehat{c}_\alpha(B(x) \times I)$ we have $d_\alpha(\psi, \phi) \geq \varepsilon > 0$ by Proposition 11. Indeed, for $(y, r) \in SN$, $\overline{\beta}(y, r) = (\beta(y), (\lambda_\beta)^{-1}(y)r)$, hence while $\max_{B(x)\times I} r = \mu^{1/4}$,

$$\min_{(\overline{\phi\beta})(B(x)\times I)} r = \min_{\overline{\beta}(B(x)\times I)} r = \mu^{-1/4} \min_{B(x)} (\lambda_\beta)^{-1} \geq \mu^{3/4-1/4} = \mu^{1/2} > \mu^{1/4}.$$

This finishes the proof. □

Proof of Propostion 25. Note that

$$d_\alpha(\psi, \mathcal{R}) = \inf_{\eta\in\mathbb{R}} |\phi^\eta_R \psi|_\alpha,$$

where we use Properties (iii) and (iv) of Theorem A. Now

$$\inf_{\eta\in\mathbb{R}} |\phi^\eta_R \psi|_\alpha = \inf_{H, b(t)} \int_0^1 | - b(t) + H_t|_{L^\infty(N)}\, dt,$$

where H is a contact Hamiltonian with $\phi_H^1 = \psi$ and $b : [0,1] \to \mathbb{R}$ is a smooth function. Indeed, for a given η and b with $B(t) = -\int_0^t b(s)\,ds$ satisfying $B(1) = \eta$ and any path $\{\chi_t\}$ generating $\phi_R^\eta \psi$, we can write $\chi_t = \phi_R^{B(t)} \phi_H^t$, for a contact path $\{\phi_H^t\}$ with contact Hamiltonian H and time-one map $\phi_H^1 = \psi$. For any given time t,

$$\inf_{b \in \mathbb{R}} | - b + H_t|_{L^\infty(N)} = \frac{1}{2} \operatorname{osc}_N H_t,$$

and in fact the infimum is achieved for $b_*(t) = \frac{1}{2}(\max H_t + \min H_t)$ which is a continuous function of t, and hence

$$\inf_H \inf_{b(t)} \int_0^1 | - b(t) + H_t|_{L^\infty(N)}\, dt = \frac{1}{2} \inf_H \int_0^1 \operatorname{osc}_N H_t\, dt = \frac{1}{2} |\psi|_\alpha^{\mathrm{osc}}.$$

This finishes the proof. □

Before we continue with the proof of Theorem B we recall some preliminary notions on the Rabinowitz-Floer functional [1, 2, 4] and prove several useful auxiliary results.

Consider the exact strong symplectic filling $(M, d\overline{\beta})$, $\overline{\beta}|_N = \alpha$, $N = \partial M$, of N. Denote by

$$(W, \omega = d\beta)$$

the completion of $(M, d\overline{\beta})$ with a cylindrical end along $N = \partial M$. Note that SN embeds symplectically into W by the flow of the Liouville vector field L defined by $\iota_L d\beta = \beta$, and that N is a hypersurface of restricted contact type in $(W, d\beta)$.

Consider a compactly supported Hamiltonian $G \in C_c^\infty([0,1] \times W, \mathbb{R})$, and a defining function $F = F^{\kappa',\kappa}$ with $\kappa' > \kappa > 0$ for N given by $F|_{W \setminus SN} \equiv c_-$, and $F|_{SN} \equiv c_-$ for $r < e^{-\kappa'}$, $F|_{SN} \equiv c_+$ for $r > e^{\kappa'}$, where $c_- < 0 < c_+$ are appropriately chosen constants, while $F|_{SN} \equiv r - 1$ on $N \times (e^{-\kappa}, e^{\kappa})$.

Choose reparametrization functions $\tau_1, \tau_2 : [0,1] \to [0,1]$ as in the proof of Theorem A, that is for $j \in 1, 2$, $\tau_j(t) \equiv 0$ near $t = 0$, $\tau_j(t) \equiv 1$ near $t = 1$, $\tau_j'(t) \geq 0$ for all $t \in [0,1]$ and $\operatorname{supp}(\tau_1') \subset [1/2, 1]$, $\operatorname{supp}(\tau_2') \subset [0, 1/2]$.

Let $\mathcal{L}W$ denote the loop space of W. Following [1, 2, 4] define the (perturbed) Rabinowitz-Floer functional[2]

$$\mathcal{A}_G^F : \mathcal{L}W \times \mathbb{R} \to \mathbb{R}$$

[2]The sign differences between the functional here and in [1] are explained by differing conventions for the Hamiltonian vector field X_G of a Hamiltonian G and by our using $(-\eta)$ as the Lagrange multiplier. The convention we use is $\iota_{X_G} d\beta = -dG$.

by

$$\mathcal{A}_G^F(x,\eta) = -\int_0^1 x^*\beta + \int_0^1 \tau_1'(t)\cdot G(\tau_1(t),x(t))\,dt - \eta\int_0^1 \tau_2'(t)F(x(t))\,dt.$$

The compactly supported Hamiltonians we use are of the form $\overline{H}^{I,\delta}$ extended by 0 to $W\setminus SN$, for H a contact Hamiltonian on N. Given such a Hamiltonian we choose κ so that $\operatorname{supp}(\overline{H}^{I,\delta})\subset[0,1]\times N\times(e^{-\kappa},e^{\kappa})$. Then we have the following.

Lemma 43. *Let $H\in C^\infty([0,1]\times N,\mathbb{R})$ be a contact Hamiltonian. Given any interval I containing 1 and any $\delta>0$, the critical points of the perturbed Rabinowitz functional $\mathcal{A}^F_{\overline{H}^{I,\delta}}$ on $\mathcal{L}W\times\mathbb{R}$ (and consequently on $\mathcal{L}SN\times\mathbb{R}$) coincide with the critical points of $\mathcal{A}^F_{\overline{H}}$ on $\mathcal{L}SN\times\mathbb{R}$. Moreover a critical point (x_0,η_0) is Morse for $\mathcal{A}^F_{\overline{H}^{I,\delta}}$ if and only if it is Morse for $\mathcal{A}^F_{\overline{H}}$.*

Proof of Lemma 43. The first statement is a direct consequence of Lemma 42 property C1, because each leafwise-intersection point lies on N_1. Moreover, the Hessians of the two functionals at such a common critical point agree, whence the second statement is immediate. □

Moreover, a computation that is by now standard (compare [4, Lemma 2.2]) shows that there is a bijection between critical points (x_0,η_0) of $\mathcal{A}^F_{\overline{H}}$ and algebraic translated points (w_0,η_0), $w_0\in N_1\cong N$ of the contact flow on N generated by H, given by $(x_0,\eta_0)\mapsto(w_0,\eta_0)=(x_0(0),\eta_0)$. Moreover, $\mathcal{A}^F_{\overline{H}}(x_0,\eta_0)=\eta_0$.

Proof of Lemma 2. We begin by recalling useful formulas from [1]. Let (x_0,η_0) be a critical point of $\mathcal{A}^F_{\overline{H}}$. For a Hamiltonian $P\in C^\infty_c([0,1]\times SN,\mathbb{R})$ put

$$\mathcal{L}_P SN = \{w\in W^{1,2}([0,1],SN)\,|\,w(0)=\phi^1_P(w(1))\}$$

for the twisted loop space, and consider the diffeomorphism

$$\Phi_P:\mathcal{L}_P SN\to\mathcal{L}SN,$$

given by

$$x(t)\mapsto\phi^t_P(x(t)).$$

Use $\Phi_{-\eta_0 F+\overline{H}}$ to pull back $\mathcal{A}^F_{\overline{H}}$ to

$$\widetilde{\mathcal{A}}^F_{\eta_0,\overline{H}} = (\Phi_{-\eta_0F+\overline{H}}\times 1_{\mathbb{R}})^*\mathcal{A}^F_{\overline{H}}:\mathcal{L}_{-\eta_0F+\overline{H}}SN\times\mathbb{R}\to\mathbb{R}.$$

Set $w_0 := \Phi^{-1}_{-\eta_0 F+\overline{H}}(x_0) = \text{const}$. By [1, Equation A.11] we see that the Hessian of $\widetilde{\mathcal{A}}^F_{\eta_0,\overline{H}}$ at (w_0,η_0) is given by

$$\mathcal{H}_{\widetilde{\mathcal{A}}^F_{\eta_0,\overline{H}}}(w_0,\eta_0)[(\widehat{w}_1,\widehat{\eta}_1),(\widehat{w}_2,\widehat{\eta}_2)] = \int_0^1 \omega(\partial_t \widehat{w}_1,\widehat{w}_2) - \widehat{\eta}_1 \int_0^1 \tau_2'(t)dF(w_0)[\widehat{w}_2] - \widehat{\eta}_2 \int_0^1 \tau_2'(t)dF(w_0)[\widehat{w}_1].$$

Choosing an inner product on

$$T_{w_0}(\mathcal{L}_{-\eta_0 F+\overline{H}}SN)\times T_{\eta_0}\mathbb{R} \cong T_{w_0}(\mathcal{L}_{-\eta_0 F+\overline{H}}SN)\times \mathbb{R}$$

one turns $\mathcal{H}_{\widetilde{\mathcal{A}}^F_{\eta_0,\overline{H}}}(w_0,\eta_0)$ to an operator $L_{(w_0,\eta_0)} : T_{w_0}(\mathcal{L}_{-\eta_0 F+\overline{H}}SN)\times\mathbb{R} \to (\mathcal{E}_{-\eta_0 F+\overline{H}})^*_{w_0}\times\mathbb{R}$, where

$$(\mathcal{E}_{-\eta_0 F+\overline{H}})_{w_0} = L^2([0,1], {w_0}^*T(SN)),$$

that is a Fredholm operator of index 0, since its extension to L^2 is self-adjoint. Therefore, to show that $\widetilde{\mathcal{A}}^F_{\eta_0,\overline{H}}$ is Morse at (w_0,η_0), or equivalently $\mathcal{A}^F_{\overline{H}}$ is Morse at (x_0,η_0), it is necessary and sufficent that the kernel of $\mathcal{H}_{\widetilde{\mathcal{A}}^F_{\eta_0,\overline{H}}}(w_0,\eta_0)$ be trivial.

Let $(\widehat{w}_1,\widehat{\eta}_1)$ be such that $\mathcal{H}_{\widetilde{\mathcal{A}}^F_{\eta_0,\overline{H}}}(w_0,\eta_0)[(\widehat{w}_1,\widehat{\eta}_1),(\widehat{w}_2,\widehat{\eta}_2)] = 0$ for all $(\widehat{w}_2,\widehat{\eta}_2)$. Note that

$$\widehat{w}_1,\widehat{w}_2 \in W^{1,2}([0,1],T_{w_0}SN)$$

with the condition that

$$\widehat{w}_1(0) = D\overline{\phi}_{-\eta_0 F+\overline{H}}(\widehat{w}_1(1)), \qquad \widehat{w}_2(0) = D\overline{\phi}_{-\eta_0 F+\overline{H}}(\widehat{w}_2(1)). \tag{14}$$

Noting that $w_0 \in N_1$, we consider the splitting

$$T_{w_0}(SN) \cong \mathbb{R}\langle R\rangle \oplus \mathbb{R}\langle\partial_r\rangle \oplus \xi_{w_0},$$

and write

$$\widehat{w}_1(t) = a_1(t)R + b_1(t)\partial_r + \widehat{w}_1^{\xi}(t),$$
$$\widehat{w}_2(t) = a_2(t)R + b_2(t)\partial_r + \widehat{w}_2^{\xi}(t)$$

with respect to this splitting. Plugging in different $(\widehat{w}_2,\widehat{\eta}_2)$ - of the form $(0,\widehat{\eta}_2)$, $(b_2\partial_r,0)$, $(a_2R,0)$, and $(\widehat{w}_2^{\xi},0)$, we obtain the identities

1) $\int_0^1 \tau_2'(t) b_1(t)\, dt = 0$,

2) $a_1' + \widehat{\eta}_1 \tau_1' \equiv 0$,

3) $b_1' \equiv 0$,

4) $\partial_t \widehat{w}_1^\xi \equiv 0$.

From identities 1 and 3, we conclude that $b_1 \equiv 0$, as $\int_0^1 \tau_2'(t)\, dt = 1$. Identity 4 means that $\widehat{w}_1^\xi$ is a constant vector. Identity 2 implies that

$$a_1(1) - a_1(0) = -\widehat{\eta}_1. \tag{15}$$

Now Equation 14 implies that

$$a_1(0)R + \widehat{w}_1^\xi(0) = D(\overline{\phi}_{-\eta_0 F + \overline{H}})(a_1(1)R + \widehat{w}_1^\xi(1))$$

which, noting that $\phi_R^{-\eta_0} w_0 = x_0$, is equivalent by Equation 1 to the conditions

$$\begin{aligned} &d\lambda_{\psi\phi_R^{-\eta_0}}(w_0)(a_1(1)R + \widehat{w}_1^\xi(1)) \\ &= d\lambda_\psi(x_0) \circ D(\phi_R^{-\eta_0})(w_0)(a_1(1)R + \widehat{w}_1^\xi(1)) = 0, \end{aligned}$$

and

$$a_1(0)R + \widehat{w}_1^\xi(0) = D(\phi_{-\eta_0 F + \overline{H}})(a_1(1)R + \widehat{w}_1^\xi(1)).$$

Evaluating α_{w_0} on both sides of the equality, and noting that $\lambda_\psi(x_0) = 1$, we obtain

$$\begin{aligned} a_1(0) &= \alpha_{w_0}(D(\phi_{-\eta_0 F + \overline{H}})(a_1(1)R + \widehat{w}_1^\xi(1))) \\ &= \lambda_\psi(x_0) \cdot \alpha_{w_0}(a_1(1)R + \widehat{w}_1^\xi(1))) = a_1(1). \end{aligned} \tag{16}$$

Equations (15) and (16) imply that $\widehat{\eta}_1 = 0$, and therefore $a_1' \equiv 0$. This means that a_1 and therefore $\widehat{w}_1 = a_1 R + \widehat{w}_1^\xi$ is a constant vector, which we denote by θ_{w_0}. We conclude that $\theta_{w_0} \mapsto D(\phi_R^{-\eta_0})(w_0)(\theta_{w_0})$ induces an injection $\iota(w_0, \eta_0)$ of $\ker \mathcal{H}_{\widetilde{\mathcal{A}}^F_{\eta_0,\overline{H}}}(w_0, \eta_0)$ into $\ker(D(\phi_R^{-\eta_0}\psi)(w_0) - 1_{TN_{w_0}}) \cap \ker(d\lambda_\psi(w_0))$. The Lemma follows immediately. □

Remark 44. Retracing the steps in the above argument we easily see that the injection $\iota(w_0, \eta_0)$ is in fact surjective, and hence an isomorphism of vector spaces

$$\ker \mathcal{H}_{\widetilde{\mathcal{A}}^F_{\eta_0,\overline{H}}}(w_0, \eta_0) \cong \ker(D(\phi_R^{-\eta_0}\psi)(w_0) - 1_{TN_{w_0}}) \cap \ker(d\lambda_\psi(w_0)).$$

Proof of Theorem B. Given a contactomorphism ψ we note that for each $\eta \in \mathbb{R}$, x is a translated point of ψ if and only if x is a translated point of $\phi_R^\eta \psi$. Moreover by the definition of non-degeneracy (Definition 1) (x, η_0) is a non-degenerate algebraic translated point of ψ if and only if $(x, \eta_0 + \eta)$ is a non-degenerate algebraic translated point of $\phi_R^\eta \psi$. If ψ is of small oscillation energy there exists $\eta_* \in \mathbb{R}$ such that $|\phi_R^{\eta_*}\psi|_\alpha < \rho(\alpha)$. Therefore there exists a Hamiltonian H' generating $\{\phi_{H'}^t\}$ with $\phi_{H'}^1 = \phi_R^{\eta_*}\psi$ with $\int_0^1 |H'_t|_{L^\infty(N)}\, dt < \rho(\alpha)$. Then by Usher's trick there exists a cutoff $\overline{H}^{I,\delta}$ with $I = (e^{-\epsilon}, e^{\epsilon})$ for $\epsilon > 0$ of the degree 1 homogeneous lift $\overline{H}$ to SN of another contact Hamiltonian H with $\phi_H^1 = \phi_R^{\eta_*}\psi$ such that

$$\int_0^1 |\overline{H}^{I,\delta}|_{L^\infty(SN)}\, dt < \rho(\alpha).$$

Since M is a strong exact symplectic filling of (N, α) we can consider $\overline{H}^{I,\delta}$ as a function on W and apply [1, Theorem A] to see that the Hamiltonian flow of $\overline{H}^{I,\delta}$ possesses a leafwise-intersection point relative to the hypersurface $N_1 \subset W$. By Lemma 43 this gives us a translated point of $\phi_R^{\eta_*}\psi$ and therefore a translated point of ψ. If ψ is non-degenerate, then so is $\phi_R^{\eta_*}\psi$, and therefore $\overline{H}^{I,\delta}$ is non-degenerate relative to $N_1 \subset W$. Therefore [1, Proposition 2.20] and Lemma 43 finish the proof (see also [1, Theorem B]). □

Proof of Proposition 40. Let $c_1, c_2, c > 0$ denote generic positive constants, and $c_0 \neq 0$ denote a generic non-zero constant. Consider the Givental quasimorphism [15]

$$\mu_G : \widetilde{\mathcal{G}} \to \mathbb{R},$$

where $\mathcal{G} = \mathrm{Cont}_0(\mathbb{R}P^{2n+1}, \xi_{st})$, for $\xi_{st} = \ker \alpha_{st}$. The main result of [6] by Ben Simon applied to this setting states that for any open ball B in $\mathbb{C}P^n$ that is displaceable by a Hamiltonian diffeomorphism, the restriction of the Givental quasimorphism to $\widetilde{\mathrm{Ham}}_c(B)$ by the chain of natural maps $\widetilde{\mathrm{Ham}}_c(B) \to \widetilde{\mathrm{Ham}}(\mathbb{C}P^n) \xrightarrow{i} \widetilde{\mathcal{H}}_\alpha \to \widetilde{\mathcal{G}}$ is equal to $c_0 \cdot Cal_B$, where $Cal_B(\widetilde{\phi})$ is defined as

$$\int_0^1 dt \int_B H_t\, \omega_B^n$$

for $\omega_B = 2\,\omega_{FS}|_B$ and $H \in C_c^\infty([0,1] \times B, \mathbb{R})$ any Hamiltonian normalized to vanish near ∂B generating a path $\{\phi_t\}_{t=0}^1$ whose class in $\widetilde{\mathrm{Ham}}_c(B)$ is $\widetilde{\phi}$. In particular $|\mu_G|$ restricted to the image of the composition $\widetilde{\mathrm{Ham}}(\mathbb{C}P^n) \xrightarrow{i} \widetilde{\mathcal{H}}_\alpha \to \widetilde{\mathcal{G}}$ is unbounded. We recall that the image of the map $\widetilde{\mathrm{Ham}}(\mathbb{C}P^n) \xrightarrow{i} \widetilde{\mathcal{H}}_\alpha$ lies in the kernel of the Calabi-Weinstein homomorphism $\widetilde{\mathcal{H}}_\alpha \to \mathbb{R}$ (see

Remark 9). Moreover, by [29, Section 2.2] for all $\gamma \in \pi_1(\mathrm{Ham}(\mathbb{C}P^n))$, the element $i(\gamma^{n+1}) = i(\gamma)^{n+1} \in \widetilde{\mathcal{H}}_\alpha$ in fact lies in $\pi_1(\mathcal{H}_\alpha)$. Moreover, as μ_G is homogeneous, by [29, Theorem 1] we have

$$\mu_G(i(\gamma)) = \frac{1}{n+1}\mu_G(i(\gamma)^{n+1}) = 2\, cw(i(\gamma)^{n+1}) = 0.$$

Hence, for $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(\mathbb{C}P^n)$ and for any $\gamma \in \pi_1(\mathrm{Ham}(\mathbb{C}P^n))$, as μ_G is homogeneous and $i(\gamma) \in Z(\widetilde{\mathcal{H}}_\alpha)$ is a central element, we have $\mu_G(i(\widetilde{\phi}\gamma)) = \mu_G(i(\widetilde{\phi})) + \mu_G(i(\gamma)) = \mu_G(i(\widetilde{\phi}))$, and hence

$$\overline{\mu}_G(\phi) := \mu_G(i(\widetilde{\phi}))$$

depends only on $\phi = \pi(\widetilde{\phi}) \in \mathrm{Ham}(\mathbb{C}P^n)$. Moreover $|\overline{\mu}_G|$ is unbounded on $\mathrm{Ham}(\mathbb{C}P^n)$.

Now Proposition 33 (iv) implies that for $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(\mathbb{C}P^n)$ with $\pi(\widetilde{\phi}) = \phi$,

$$|\overline{\mu}_G(\phi)| = |\mu_G(i(\widetilde{\phi}))| \leq c_1\, |i(\widetilde{\phi})|_\alpha + c_2,$$

and hence

$$|\overline{\mu}_G(\phi)| \leq c_1\, \nu_2(\phi) + c_2,$$

as $\nu_2(\phi) = \inf_{\pi(\widetilde{\phi})=\phi} |i(\widetilde{\phi})|_\alpha$. This implies that ν_2 is unbounded, as required. □

Acknowledgements

I thank Sheila Margherita Sandon and Leonid Polterovich for introducing me to the main subjects of this paper over the course of numerous conversations. I thank Strom Borman, Viktor Ginzburg, Michael Khanevsky, Boris Khesin, François Lalonde, Dmitry Tonkonog, Michael Usher and Frol Zapolsky for useful conversations. I thank Strom Borman, Leonid Polterovich, Sheila Margherita Sandon and Frol Zapolsky for their comments on an earlier version of this manuscript. The work on this paper has started right after the workshop "Rigidity and Flexibility in Symplectic Topology and Dynamics" at the Lorentz Center (Leiden, 2014). I thank its organizers and participants for a very enjoyable and stimulating event. This work was carried out at CRM, University of Montreal, and I thank this institution for its warm hospitality. I thank the referees for their very careful reading and useful suggestions.

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Received September 20, 2015
Accepted September 15, 2016