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# Superheavy Lagrangian immersions in surfaces

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We show that the union of some circles in a closed Riemannian surface with positive genus is superheavy in the sense of Entov-Polterovich. By a result of Entov and Polterovich, this implies that the product of this union and the Clifford torus of $\mathbb{C}P^n$ with the Fubini-Study symplectic form cannot be displaced by any symplectomorphisms.

## 1. Introduction

A diffeomorphism $f$ of a symplectic manifold $(M, \omega)$ is called *symplectomorphism* if $f$ preserves the symplectic form $\omega$. A subset $U$ is said to be *strongly non-displaceable* if $f(U) \cap \bar{U} \neq \emptyset$ for any symplectomorphism $f$.

Entov and Polterovich [EP09] defined superheaviness for closed subsets in symplectic manifolds and showed that for a closed symplectic manifold $M$, $[M]$-superheavy subsets are strongly non-displaceable. Our main theorem is the following one.

**Theorem 1.1.** *Let $g$ be a positive integer. Let $(\Sigma_g, \omega)$ be a closed Riemannian surface with genus $g$ and a symplectic (area) form $\omega$ and $e^0 \cup e^1_1 \cup \cdots \cup e^1_{2g} \cup e^2$ its CW-decomposition. Then $e^1_1 \cup \cdots \cup e^1_{2g}$ is a $[\Sigma_g]$-superheavy subset of $(\Sigma_g, \omega)$,*

**Remark 1.2.** In the first draft of the present paper, Theorem 1.1 was written in the case only when $g = 1$ (for example, see Section 4 of [E]). After that work, Humiliére, Le Roux and Seyfaddini [HLS] and Ishikawa [Is] gave other proofs of the above result. They [HLS] also found its same generalization as Theorem 1.1 independently (see also [Is]).

**Remark 1.3.** As pointed out in (2) of Proposition 6 of [HLS], the removal of any curve $e^1_i$ destroy its superheaviness *i.e.* $e^1_1 \cup \cdots e^1_{i-1} \cup e^1_{i+1} \cup \cdots e^1_{2g}$ is not superheavy for any $i$.

The subset $e_1^1 \cup \cdots \cup e_{2g}^1$ of $\Sigma_g$ is not displaceable by any homeomorphisms of $\Sigma_g$ for topological reasons, however it gives rise to a non-trivial example of a strongly non-displaceable subset in $\mathbb{C}P^n \times \Sigma_g$. In fact, for the symplectic manifolds $M_1$, $M_2$, the product of superheavy subsets is superheavy in $M_1 \times M_2$. Thus we have the following corollary.

**Corollary 1.4.** *Let $(\mathbb{C}P^n, \omega_{FS})$ be the complex projective space with the Fubini-Study form $\omega_{FS}$ and $C$ the Clifford torus $\{[z_0 : \cdots : z_n] \in \mathbb{C}P^n; |z_0| = \cdots = |z_n|\}$ of $\mathbb{C}P^n$. Then $C \times (e_1^1 \cup \cdots \cup e_{2g}^1)$ is a strongly non-displaceable subset of $\mathbb{C}P^n \times \Sigma_g$.*

The present paper is organized as follows. We review the definitions in symplectic geometry and spectral invariants in Section 2 which are needed to prove Theorem 1.1. We introduce and prove the important proposition (Proposition 3.1) to prove Theorem 1.1. In Section 4, we prepare some definitions, proposition and theorem which are useful to prove Theorem 1.1. In Section 5, we prove Theorem 1.1 and Corollary 1.4.

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## 2. Preliminaries

### 2.1. Definitions

For a function $F\colon M \to \mathbb{R}$ with compact support, we define the *Hamiltonian vector field* $\operatorname{sgrad} F$ associated with $F$ by

$$\omega(\operatorname{sgrad} F, V) = -dF(V) \text{ for any } V \in \mathcal{X}(M),$$

where $\mathcal{X}(M)$ denotes the set of smooth vector fields on $M$.

For a function $F\colon M \times [0,1] \to \mathbb{R}$ and $t \in [0,1]$, we define $F_t\colon M \to \mathbb{R}$ by $F_t(x) = F(x,t)$. We denote by $\{f_t\}$ the isotopy which satisfies $f_0 = \mathrm{id}$

and $\frac{d}{dt}f_t(x) = (\operatorname{sgrad} F_t)_{f_t(x)}$. We call this *the Hamiltonian path generated by the Hamiltonian function* $F_t$. The time-1 map $f_1$ of $\{f_t\}$ is called *the Hamiltonian diffeomorphism generated by the Hamiltonian function* $F_t$. A diffeomorphism $f$ is called *a Hamiltonian diffeomorphism* if there exists a Hamiltonian function $F_t$ with compact support generating $f$. A Hamiltonian diffeomorphism is a symplectomorphism.

For a symplectic manifold $(M,\omega)$, let $\mathrm{Symp}(M,\omega)$, $\mathrm{Ham}(M,\omega)$ and $\widetilde{\mathrm{Ham}}(M,\omega)$ denote the group of symplectomorphisms, the group of Hamiltonian diffeomorphisms of $(M,\omega)$ and its universal cover, respectively.

Let $(M,\omega)$ be a symplectic manifold and $\{f_t\}_{t\in[0,1]}$ and $\{g_t\}_{t\in[0,1]}$ be the Hamiltonian paths generated by Hamiltonian functions $F_t$ and $G_t$, respectively. Then $\{f_tg_t\}_{t\in[0,1]}$ are generated by the Hamiltonian function $(F\sharp G)(x,t) = F(x,t) + G(f_t^{-1}(x),t)$.

A Hamiltonian function $H$ is called *normalized* if $\int_M H_t(x)\omega^n = 0$ for any $t \in [0,1]$.

## 2.2. Spectral invariants

For a closed connected symplectic manifold $(M,\omega)$, put

$$\Gamma = \frac{\pi_2(M)}{\mathrm{Ker}(c_1)\cap \mathrm{Ker}([\omega])},$$

where $c_1$ is the first Chern class of $TM$ with an almost complex structure compatible with $\omega$. The Novikov ring of the closed symplectic manifold $(M,\omega)$ is defined as follows:

$$\Lambda = \left\{ \sum_{A\in\Gamma} a_A A;\ a_A \in \mathbb{C},\ \#\left\{A; a_A \neq 0, \int_A \omega < R\right\} < \infty \right. \\ \left. \text{for any real number } R \right\}.$$

The quantum homology $QH_*(M,\omega)$ is a $\Lambda$-module isomorphic to $H_*(M;\mathbb{C})\otimes_{\mathbb{C}}\Lambda$ and $QH_*(M,\omega)$ has a ring structure with the multiplication called the *quantum product* ([O06]). For each element $a \in QH_*(M,\omega)$, a functional $c(a,\cdot)\colon C^\infty(M\times[0,1]) \to \mathbb{R}$ is defined in terms of the Hamiltonian Floer theory. The functional $c(a,\cdot)$ is called a *spectral invariant* ([O06]). To describe the properties of a spectral invariant, we define the spectrum of a Hamiltonian function as follows:

**Definition 2.1 ([O06]).** Let $H\colon M \times [0,1] \to \mathbb{R}$ be a Hamiltonian function on a closed symplectic manifold $M$. The *spectrum* $\mathrm{Spec}(H)$ of $H$ is defined as follows:

$$\mathrm{Spec}(H) = \left\{ \int_0^1 H(h_t(x),t)dt - \int_{\mathbb{D}^2} u^*\omega \right\} \subset \mathbb{R},$$

where $\{h_t\}_{t\in[0,1]}$ is the Hamiltonian path generated by $H$ and $x \in M$ is a fixed point of $h_1$ whose orbit defined by $\gamma^x(t) = h_t(x)$ ($t \in [0,1]$) is a contractible loop and $u\colon \mathbb{D}^2 \to M$ is a disc in $M$ such that $u|_{\partial\mathbb{D}^2} = \gamma^x$.

We define the *non-degeneracy* of Hamiltonian functions as follows:

**Definition 2.2.** A Hamiltonian function $H\colon M \times [0,1] \to \mathbb{R}$ is called *non-degenerate* if for any fixed point $x \in M$ of $h$ whose orbit $\gamma^x$ is a contractible loop, 1 is not an eigenvalue of the differential $(h_*)_x$.

The following proposition summaries the properties of spectral invariants which we need.

**Proposition 2.3 ([O06], [U]).** *Spectral invariants has the following properties.*

(1) ***Non-degenerate spectrality*:** $c(a,H) \in \mathrm{Spec}(H)$ *for every non-degenerate* $H \in C^\infty(M\times[0,1])$.

(2) ***Hamiltonian shift property*:** $c(a, H + \lambda(t)) = c(a,H) + \int_0^1 \lambda(t)dt$.

(3) ***Lipschitz property*:** *The map* $H \mapsto c(a,H)$ *is Lipschitz on* $C^\infty(M \times [0,1])$ *with respect to the* $C^0$*-norm.*

(4) ***Homotopy invariance*:** $c(a,H_1) = c(a,H_2)$ *for any normalized* $H_1$ *and* $H_2$ *generating the same* $h \in \widetilde{\mathrm{Ham}}(M)$. *Thus one can define* $c(a,\cdot)\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ *by* $c(a,h) = c(a,H)$, *where* $H$ *is a normalized Hamiltonian function generating* $h$.

(5) ***Triangle inequality*:** $c(a*b, fg) \leq c(a,f) + c(b,g)$ *for elements* $f$ *and* $g \in \widetilde{\mathrm{Ham}}(M,\omega)$, *where* $*$ *denotes the quantum product.*

### 2.3. Superheaviness

Entov and Polterovich [EP09] defined *superheaviness* of closed subsets in closed symplectic manifolds and gave examples of strongly non-displaceable subsets.

For an idempotent $a$ of the quantum homology $QH_*(M,\omega)$, define the functional $\zeta_a\colon C^\infty(M)\to\mathbb{R}$ by

$$\zeta_a(H)=\lim_{l\to+\infty}\frac{c(a,lH)}{l},$$

where $c(a,H)$ is the spectral invariant.

**Definition 2.4 ([EP09]).** Let $(M,\omega)$ be a $2n$-dimensional closed symplectic manifold and $a$ be an idempotent of the quantum homology $QH_*(M,\omega)$. A closed subset $X$ of $M$ is said to be *$a$-superheavy* if

$$\zeta_a(H)\leq\sup_X H \text{ for any Hamiltonian function } H\colon M\to\mathbb{R}.$$

A closed subset $X$ of $M$ is called superheavy if $X$ is $a$-superheavy for some idempotent $a$ of $QH_*(M,\omega)$.

**Example 2.5.** Let $(\mathbb{C}P^n,\omega_{FS})$ be the complex projective space with the Fubini-Study form. The Clifford torus $C=\{[z_0:\cdots:z_n]\in\mathbb{C}P^n;|z_0|=\cdots=|z_n|\}\subset\mathbb{C}P^n$ is a $[\mathbb{C}P^n]$-superheavy subset of $(\mathbb{C}P^n,\omega_{FS})$, hence they are strongly non-displaceable ([BEP] Lemma 5.1, [EP09] Theorem 1.8).

For a closed oriented manifold $M$, we denote its fundamental class by $[M]$. It is known that $[M]$ is an idempotent of $QH_*(M,\omega)$.

**Theorem 2.6 (Theorem 1.4 of [EP09]).** *Every $[M]$-superheavy subset is strongly non-displaceable.*

**Definition 2.7.** Let $(M,\omega)$ be a $2n$-dimensional closed symplectic manifold. Let $a$ be an idempotent of the quantum homology $QH_*(M,\omega)$. An open subset $U$ of $M$ is said to satisfy *the bounded spectrum condition* (with respect to $a$) if there exists a constant $E>0$ such that

$$|c(a,F)|\leq E$$

for any Hamiltonian function $F\colon U\times[0,1]\to\mathbb{R}$ with compact support.

Open subsets satisfying the bounded spectrum condition play an essential role in the present paper.

**Example 2.8.** A stably displaceable open subset of a closed symplectic manifold satisfies the bounded spectrum condition with respect to any idempotent $a$ ([S] Lemma 4.1).

## 3. Main proposition

Open subsets with volume greater than the half of that of $M$ are strongly non-displaceable but some of them satisfy the bounded spectrum condition for non simply connected symplectic manifold.

**Proposition 3.1.** *Let $(M,\omega)$ be a closed symplectic manifold. Let $\alpha$ be a nontrivial free homotopy class of free loops on $M$; $\alpha \in [\mathbb{S}^1, M]$, $\alpha \neq 0$. Let $U$ be an open subset of $M$. Assume that there exists a Hamiltonian function $H\colon M \times [0,1] \to \mathbb{R}$ which satisfies the following:*

(1) $h_1|_U = \mathrm{id}_U$,

(2) *for any $x \in U$, the free loop $\gamma^x\colon \mathbb{S}^1 \to M$ defined by $\gamma^x(t) = h_t(x)$ belongs to $\alpha$, and*

(3) $\alpha \notin i_*([\mathbb{S}^1, U])$.

*Here $i\colon U \to M$ is the inclusion map and $\{h_t\}_{t\in[0,1]}$ is the Hamiltonian path generated by $H$. Then $U$ satisfies the bounded spectrum condition with respect to any idempotent $a$ of $QH_*(M,\omega)$.*

The proof of Proposition 3.1 is based on the idea of K. Irie in the proof of Theorem 2.4 of [Ir]. We also use some ideas from [U]. For paths $\alpha, \beta\colon [0,1] \to M$ with $\alpha(1) = \beta(0)$, $\alpha\sharp\beta$ means the concatenation of $\alpha$ and $\beta$ . For free loops $\alpha', \beta'\colon \mathbb{S}^1 \to M$, $\alpha' \simeq \beta'$ means that $\alpha'$ and $\beta'$ are homotopic.

*Proof.* Fix a Hamiltonian function $F\colon U \times [0,1] \to \mathbb{R}$ with compact support. We denote by $\{f_t^u\}_{t\in[0,1]}$ the Hamiltonian path generated by $uF$ and denote by $\phi^{u,x}$ the path defined by $\phi^{u,x}(t) = f_t^u(x)$. To use the non-degenerate spectral property, we approximate $H$ by non-degenerate Hamiltonian functions. Take a sequence of non-degenerate Hamiltonian functions $H_n$ which converges to $H$ in the $C^2$-norm. We denote by $\{h_{n,t}\}_{t\in[0,1]}$ the Hamiltonian path generated by $H_n$ and denote by $\gamma_n^x$ the path defined by $\gamma_n^x(t) = h_{n,t}(x)$.

Choose a smooth function $\chi\colon [0,\frac{1}{2}] \to [0,1]$ and a positive real number $\epsilon \in (0,\frac{1}{4})$ such that

- $\chi'(t) \geq 0$ for any $t \in [0,\frac{1}{2}]$, and
- $\chi(t) = 0$ for any $t \in [0,\epsilon]$, and $\chi(t) = 1$ for any $t \in [\frac{1}{2}-\epsilon, \frac{1}{2}]$.

For $u \in [0,1]$, we define the new Hamiltonian function $L_n^u \colon M \times [0,1] \to \mathbb{R}$ as follows:

$$L_n^u(x,t) = \begin{cases} \chi'(t) H_n(x, \chi(t)) & \text{when } t \in [0, \frac{1}{2}] \\ u\chi'(t-\frac{1}{2}) F(x, \chi(t-\frac{1}{2})) & \text{when } t \in [\frac{1}{2}, 1]. \end{cases}$$

Since $\chi$ is constant on neighborhoods of 0 and $\frac{1}{2}$, $L_n^u$ is a smooth Hamiltonian function.

We claim that $\mathrm{Spec}(L_n^u) \subset \mathrm{Spec}(H_n)$ for a large enough integer $n$ and any $u \in [0,1]$. We denote by $\{l_{n,t}^u\}_{t\in[0,1]}$ the Hamiltonian path generated by $L_n^u$. Let $x \in M$ be a fixed point of $l_{n,1}^u$ whose orbit $\lambda_n^{u,x}$ defined by $\lambda_n^{u,x}(t) = l_{n,t}^u(x)$ is contractible. If $x \notin \bigcup_{t\in[0,1]} \mathrm{supp}(F_t)$, $x$ is also a fixed point of $h_1$ and $\lambda_n^{u,x}(t)$ coincides with $\gamma_n^x$ up to parameter change. Hence $\gamma_n^x$ is contractible. Since $\int_0^1 H_n(h_t(x),t)dt = \int_0^1 L_n^u(l_t^u(x),t)dt$, the element of $\mathrm{Spec}(L_n^u)$ given by the fixed point of $x$ belongs to $\mathrm{Spec}(H_n)$. If $x \in \bigcup_{t\in[0,1]} \mathrm{supp}(F_t)$, since $n$ is assumed to be large enough, there exists a path $\beta_n^x$ in $U$ such that $\beta_n^x(0) = h_{n,1}(x)$ and $\beta_n^x(1) = x$ and $\gamma_n^x \sharp \beta_n^x$ represents $\alpha \in [\mathbb{S}^1, M]$. Since

$$\bigcup_{t\in[0,1]} \mathrm{supp}(F_t) \subset U \quad \text{and} \quad (\bar\beta_n^x \sharp \bar\gamma_n^x) \sharp \lambda_n^{u,x} \simeq (\bar\beta_n^x \sharp \bar\gamma_n^x) \sharp (\gamma_n^x \sharp \phi^{u,x}) \simeq \bar\beta_n^x \sharp \phi^{u,x},$$

the free loop $(\bar\beta_n^x \sharp \bar\gamma_n^x) \sharp \lambda_n^{u,x}$ is homotopic to a free loop in $U$. Therefore, since $\lambda_n^{u,x}$ is contractible, $\bar\beta_n^x \sharp \bar\gamma_n^x \simeq ((\bar\beta_n^x \sharp \bar\gamma_n^x) \sharp \lambda_n^{u,x}) \sharp \bar\lambda_n^{u,x}$ is also homotopic to a free loop in $U$ and this contradicts $\alpha \notin i_*([\mathbb{S}^1, U])$. Hence we see $x \notin \bigcup_{t\in[0,1]} \mathrm{supp}(F_t)$ by contradiction and thus $\mathrm{Spec}(L_n^u) \subset \mathrm{Spec}(H_n)$ holds. By this argument, the non-degeneracy of $H_n$ implies the non-degeneracy of $L_n^u$. Since $L_n^0$ and $H_n$ generate the same element of $\widetilde{\mathrm{Ham}}(M,\omega)$, the homotopy invariance implies

$$c\left(a, L_n^0 - \int_M L_n^0 \omega^n\right) = c\left(a, H_n - \int_M H_n \omega^n\right).$$

By the Hamiltonian shift property and $\int_0^1 \int_M L_n^0 \omega^n dt = \int_0^1 \int_M H_n \omega^n dt$,

$$\begin{aligned} c(a, L_n^0) &= c\left(a, L_n^0 - \int_M L_n^0 \omega^n\right) + \int_0^1 \int_M L_n^0 \omega^n dt \\ &= c\left(a, H_n - \int_M H_n \omega^n\right) + \int_0^1 \int_M H_n \omega^n dt = c(a, H_n). \end{aligned}$$

The Lipschitz property asserts that $c(a, L_n^u)$ depends continuously on $u$. Since $L_n^u$ is non-degenerate and $\mathrm{Spec}(H_n)$ is a measure-zero set (Lemma 2.2

of [O02]), the non-degenerate spectrality implies that $c(a, L_n^u)$ is a constant function of $u$. Hence $c(a, L_n^u) = c(a, H_n)$ for any $u \in [0,1]$.

Since $L_n^1$ and $F \sharp H_n$ generates the same element of $\widetilde{\mathrm{Ham}}(M, \omega)$, by a computation as above, $c(a, F \sharp H_n) = c(a, L_n^1)$. Thus $c(a, F \sharp H_n) = c(a, L_n^1) = c(a, H_n)$. Then the triangle inequality implies

$$\begin{aligned} c(a, F) &\leq c(a, F \sharp H_n) + c(a, \bar{H}_n) \\ &= c(a, L_n^1) + c(a, \bar{H}_n) \\ &= c(a, H_n) + c(a, \bar{H}_n), \quad \text{and} \end{aligned}$$

$$\begin{aligned} c(a, F) &\geq c(a, F \sharp H_n) - c(a, H_n) \\ &= c(a, H_n) - c(a, H_n) = 0. \end{aligned}$$

Since Lipschitz properties implies

$$\lim_{n \to \infty} c(a, H_n) = c(a, H) \quad \text{and} \quad \lim_{n \to \infty} c(a, \bar{H}_n) = c(a, \bar{H}),$$

we have

$$0 \leq c(a, F) \leq c(a, H) + c(a, \bar{H}).$$

□

## 4. the bounded spectrum condition and an *a*-stem

**Definition 4.1.** An open subset $U$ of $M$ is said to be $a$-null if for any Hamiltonian function $G \colon U \to \mathbb{R}$ with compact support,

$$\zeta_a(G) = 0.$$

An open subset $U$ of $M$ is said to be strongly $a$-null if for any Hamiltonian function $F \colon M \to \mathbb{R}$ and any Hamiltonian function $G \colon U \to \mathbb{R}$ with compact support with $\{F, G\} = 0$,

$$\zeta_a(F + G) = \zeta_a(F).$$

A subset $X$ of $M$ is said to be (strongly) $a$-null if there exists a (strongly) $a$-null open neighborhood $U$ of $X$.

$a$-nullity is defined in [MVZ]. If a subset $X$ of $M$ is strongly $a$-null, $X$ is $a$-null.

The arguments in [EP06] shows the following proposition.

**Proposition 4.2.** *Let $(M, \omega)$ be a $2n$-dimensional closed symplectic manifold. For an idempotent $a$ of $QH_*(M, \omega)$, if an open subset $U$ of $M$ satisfies the bounded spectrum condition with respect to $a$, then $U$ is strongly $a$-null.*

Entov and Polterovich defined stems to give examples of superheavy subsets ([EP09]). We define $a$-stems which generalizes a little the notion of stems and they exhibits $a$-superheaviness.

We generalize the argument of Entov and Polterovich as follows.

**Definition 4.3.** Let $\mathbb{A}$ be a finite-dimensional Poisson-commutative subspace of $C^\infty(M)$ and $\Phi\colon M \to \mathbb{A}^*$ the moment map defined by $\langle \Phi(x), F \rangle = F(x)$. Let $a$ be a non-trivial idempotent of $QH_*(M, \omega)$. A non-empty fiber $\Phi^{-1}(p)$, $p \in \mathbb{A}^*$ is called *an $a$-stem* of $\mathbb{A}$ if all non-empty fibers $\Phi^{-1}(q)$ with $q \neq p$ are strongly $a$-null. If a subset of $M$ is an $a$-stem of a finite-dimensional Poisson-commutative subspace of $C^\infty(M)$, it is called just *an $a$-stem*.

**Theorem 4.4.** *For every idempotent $a$ of $QH_*(M, \omega)$, every $a$-stem is a $a$-superheavy subset.*

The proof of Theorem 4.4 is same as the one of Theorem 1.8 of [EP09].

## 5. Proof of Theorem 1.1 and Corollary 1.4

*Proof of Theorem 1.1.* By cutting $\Sigma_g$ open along $e_1^1 \cup \cdots \cup e_{2g}^1$, we construct $4g$-gon $\tilde{\Sigma}_g$ and the natural quatient map $\pi\colon \tilde{\Sigma}_g \to \Sigma_g$. We mark all sides of $\tilde{\Sigma}_g$ with $e_1^u, \ldots, e_{2g}^u$ and $e_1^l, \ldots, e_{2g}^l$ such that $\pi(e_i^u) = e_i^1$ and $\pi(e_i^l) = e_i^1$.

Put $A = \int_{\Sigma_g} \omega$ and let $S_A$ be a square in $\mathbb{R}^2$ defined by $S_A = [0,1] \times [0, A]$. Let $s^u$ and $s^l$ denote the sides $[0,1] \times \{A\}$ and $[0,1] \times \{0\}$ of $S_A$, respectively. Then we can take an area-preserving diffeomorphism $f\colon S_A \to \tilde{\Sigma}_g$ such that $f(s^u) = e_1^u$, $f(s^l) = e_1^l$ and $\pi(f(t,0)) = \pi(f(t,A))$ for any $t \in [0,1]$.

Consider a function $\Phi\colon \Sigma_g \to \mathbb{R}$ such that $\Phi(x) = 0$ if $x \in e_1^1 \cup \cdots \cup e_{2g}^1$ and $\Phi(x) > 0$ if $x \notin e_1^1 \cup \cdots \cup e_{2g}^1$. Take a real number $\epsilon \neq 0$. We view $\Phi$ as the moment map of the 1-dimensional Poisson-commutative algebra spanned by $\Phi$ itself. Take a positive number $\epsilon$ so that $\Phi^{-1}(\epsilon)$ is not empty. Let us prove that $\Phi^{-1}(\epsilon)$ is $a$-null. There exists a positive number $\delta$ and an open neighborhood $U$ of $(\Phi \circ \pi \circ f)^{-1}(\epsilon)$ such that $U \subset (\delta, 1-\delta) \times (\delta, A-\delta) \subset S_A$.

Consider a function $\hat{H}\colon S_A \to \mathbb{R}$ such that $\hat{H}((p,q)) = Ap$ for any $p \in [\delta, 1-\delta]$ and $\hat{H}((p,q)) = 0$ for any $p \in [0, \frac{\delta}{2}] \cup [1-\frac{\delta}{2}, 1]$. Since $\pi(f(t,0)) =$

$\pi(f(t,A))$ for any $t \in [0,1]$ and $\hat{H}((p,q)) = 0$ for any $p \in [0, \frac{\delta}{2}] \cup [1 - \frac{\delta}{2}, 1]$, there exists a Hamiltonian function $H\colon \Sigma_g \to \mathbb{R}$ such that $\hat{H} = H \circ \pi \circ f$.

Define the path $\hat{\gamma}\colon [0,1] \to S_A$ by $\hat{\gamma}(t) = (0, At)$ and the free loop $\gamma\colon \mathbb{S}^1 \to \Sigma_g$ by $\gamma = \pi \circ f \circ \hat{\gamma}$. Let $\alpha \in [\mathbb{S}^1, \Sigma_g]$ be the homotopy class of free loops represented by $\gamma$. Then $\alpha$, $U$ and $H$ satisfy the assumptions of Proposition 3.1, hence $U$ satisfies the bounded spectrum condition with respect to any idempotent $a \in QH_*(\Sigma_g, \omega)$. Thus, by Proposition 4.2, $\Phi^{-1}(\epsilon)$ is strongly $a$-null for all $\epsilon > 0$ such that $\Phi^{-1}(\epsilon)$ is non-empty and therefore $e_1^1 \cup \cdots \cup e_{2g}^1$ is an $a$-stem, hence it is $a$-superheavy by Theorem 4.4. □

Though the above example cannot be displaced by homeomorphisms, it gives rise to a nontrivial strongly non-displaceable example by using the following theorem.

**Theorem 5.1 ([EP09] Theorem 1.7).** *Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be closed symplectic manifolds. Take non-zero idempotents $a_1$, $a_2$ of $QH_*(M_1)$, $QH_*(M_1)$, respectively. Assume that for $i = 1, 2$, $X_i$ be a $a_i$-superheavy subset. Then the product $X_1 \times X_2$ is $a_1 \otimes a_2$-superheavy subset of $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ with respect to the idempotent of $QH_*(M_1 \times M_2)$.*

*Proof of Corollary 1.4.* By Example 2.5, Theorem 1.1 and Theorem 5.1, $C \times (e_1^1 \cup \cdots \cup e_{2g}^1)$ is a $[\mathbb{C}P^n \times \Sigma_g]$-superheavy subset of $(\mathbb{C}P^n \times \Sigma_g, \omega_{FS} \oplus \omega)$ and thus, by Theorem 2.6, strongly non-displaceable. □

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