

Geometric quantization of almost toric manifolds

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Kostant gave a model for the geometric quantization via the cohomology associated to the sheaf of flat sections of a pre-quantum line bundle. This model is well-adapted for real polarizations given by integrable systems and toric manifolds. In the latter case, the cohomology can be computed by counting integral points inside the associated Delzant polytope. In this article we extend Kostant's geometric quantization to semitoric integrable systems and almost toric manifolds. In these cases the dimension of the acting torus is smaller than half of the dimension of the manifold. In particular, we compute the cohomology groups associated to the geometric quantization if the real polarization is the one induced by an integrable system with focus-focus type singularities in dimension four. As an application we determine a model for the geometric quantization of K3 surfaces under this scheme.

1. Introduction

An important contribution of Kostant has been the definition of geometric quantization via the cohomology associated to the sheaf of sections of a chosen pre-quantum line bundle that are flat along a given polarization. This construction using real polarizations is an abstraction of Kähler quantization and has been used in connection to representation theory (see for instance

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[12]). Generalizations of this scheme considering non-degenerate singularities have also been obtained by Hamilton [13], Hamilton and Miranda [14], and Solha [32].

A toric manifold is a symplectic manifold endowed with an effective Hamiltonian action of a torus whose rank is half of the dimension of the manifold. A theorem of Delzant [7] establishes a one-to-one correspondence between closed toric manifolds in dimension $2m$ and a class of polytopes (now called Delzant polytopes) on \mathbb{R}^m . The real geometric quantization of closed toric manifolds can be read from the Delzant polytope, as proved by Hamilton [13] (generalizing previous results by Śniatycki [31] to the singular context): given a toric manifold, its real geometric quantization is completely determined by the count of integral points inside (boundary points are excluded) its associated Delzant polytope.

Toric manifolds are central in the study of the geometry of symplectic manifolds and their symmetries, as are their generalizations, such as semitoric integrable systems [26] or almost toric manifolds [18], in which the rank of the torus is no longer half of the dimension of the manifold. Examples of almost toric manifolds are given by K3 surfaces, which are also of relevance in complex geometry. A semitoric integrable system (see for instance [26, 27]) is an integrable system admitting only non-degenerate singularities composed of elliptic and focus-focus components, but excluding any hyperbolic ones.

As observed in [32] the quantization of almost toric manifolds can be reduced to the computation of the contribution of a neighborhood of a Bohr–Sommerfeld focus-focus singular fiber by the use of factorization tools¹. This is because the geometric quantization of neighborhoods of Bohr–Sommerfeld fibers computes the geometric quantization of the whole manifold (by means of a standard Mayer–Vietoris sequence).

In this article we apply Kostant’s model to focus-focus singularities and conclude its computation, showing that the first cohomology group associated to the real geometric quantization of a small neighborhood of a focus-focus fiber of a 4-dimensional semitoric integrable system is trivial, but not the second cohomology group, which is infinite dimensional when the singular fiber is Bohr–Sommerfeld (Theorem 5.1). This determines completely the geometric quantization when the real polarization has focus-focus fibers (the cohomology group in degree zero is trivial and had already been computed in [32]) and, thus, brings to a close the problem of geometric quantization of integrable systems with non-degenerate singularities as initiated in [13] and [14] for 4-dimensional manifolds with no hyperbolic-hyperbolic fibers.

¹Which behave following a Künneth formula [21], as a simple sheaf cohomology.

As a motivation for these results, we present K3 surfaces as an example of almost toric manifolds and analyze the effect of nodal trades [18] in their real quantization. Other models of quantization for K3 surfaces have been recently obtained by Castejón [3] using the Berezin–Toeplitz operators approach [2]. For this direction see also [24].

Our main theorem (Theorem 6.1) also has applications to actual physical systems: A first example which can be considered as a toy model of the spin-spin system of [30] has the product of two spheres as starting point. As a toric manifold it can be represented by its Delzant polytope (which is a square), see Figure 1. A nodal trade can be performed in one of its vertices in order to obtain a semitoric system. The techniques of this article offer a clear method to compute the real quantization of this system. A direct application of Theorem 6.1 shows that its quantization is infinite dimensional.

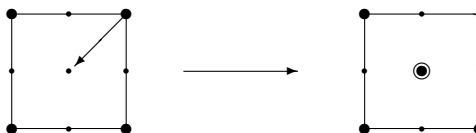


Figure 1: Nodal trade on $S^2 \times S^2$.

Particular choices of the parameters on the system of a spherical pendulum (mass and length of the pendulum, as well as the gravitational acceleration) imply that the focus-focus singular fiber is Bohr–Sommerfeld. As a consequence of Theorem 6.1, our representation space turns out to be infinite-dimensional. If the quantization program is compatible with Kostant’s model, our result yields the following interpretation: There can be infinite-dimensional eigenspaces associated to the eigenvalue which represents the energy level of the classical unstable equilibrium state of a focus-focus singular point.

Another physical example to which our method applies is the coupled classical spin and harmonic oscillator (see for instance [25]) which is a classical version of the Jaynes–Cummings model (see [16]). While the spherical pendulum is described by an integrable system on the cotangent bundle of the sphere, the spin-oscillator system is described by a semitoric system on the trivial bundle $\mathbb{R}^2 \times S^2$ over the sphere.

The common feature of these examples is that this sort of degeneracy at energy values associated with classical unstable equilibrium states does not represent the physics of the quantum systems (cf. [5, 6]), and it would be interesting to refine the definition of geometric quantization with singular

real polarizations to mod out these infinite dimensional contributions to get a finite dimensional Hilbert space instead.

2. Main definitions

2.1. Singular Lagrangian fibrations

The symplectic manifolds of interest to this article have a great deal of symmetry, and such symmetries are related to some particular classes of integrable systems: those admitting only non-degenerate singularities.

Definition 2.1. An integrable system on a symplectic manifold (M, ω) of dimension $2m$ is a set of m functions, $f_1, \dots, f_m \in C^\infty(M; \mathbb{R})$, satisfying

$$df_1 \wedge \dots \wedge df_m \neq 0 \text{ over an open dense subset of } M \text{ and}$$

$$\{f_j, f_k\}_\omega = 0 \text{ for all } j, k.$$

The *Poisson bracket* is defined by $\{f, \cdot\}_\omega = X_f(\cdot)$, where X_f is the unique vector field defined by the equation $\iota_{X_f}\omega + df = 0$, called the *Hamiltonian vector field* of f .

The next definition refers to the critical set of an integrable system, i.e. the set of points where $df_1 \wedge \dots \wedge df_m$ vanishes.

Definition 2.2. A critical point of rank $k_r = m - k_e - k_h - 2k_f$ of an integrable system $(f_1, \dots, f_m) : M \rightarrow \mathbb{R}^m$ is a non-degenerate singular point of Williamson type (k_e, k_h, k_f) if the quadratic parts of f_1, \dots, f_m can be written as:

$$\begin{array}{lll} h_j = x_j & \text{(regular)} & 1 \leq j \leq k_r \\ h_j = x_j^2 + y_j^2 & \text{(elliptic)} & k_r + 1 \leq j \leq k_r + k_e \\ h_j = x_j y_j & \text{(hyperbolic)} & k_r + k_e + 1 \leq j \leq k_r + k_e + k_h \\ \begin{cases} h_j = x_j y_j + x_{j+1} y_{j+1} \\ h_{j+1} = x_j y_{j+1} - x_{j+1} y_j \end{cases} & \text{(focus-focus)} & \begin{array}{l} j = k_r + k_e + k_h + 2l - 1, \\ 1 \leq l \leq k_f \end{array} \end{array}$$

in some Darboux local coordinates $(x_1, y_1, \dots, x_m, y_m)$.

The *fiber* of an integrable system is the preimage of a point in the image of $(f_1, \dots, f_m) : M \rightarrow \mathbb{R}^m$, and such a map will be referred to as *moment map*.

A singular fiber of an integrable system is said to be of *Williamson type* (k_e, k_h, k_f) if all of its singular points are non-degenerate singular points of

that same Williamson type. Another terminology is also used in this article: in dimension 2 an *elliptic fiber* and a *hyperbolic fiber* are singular fibers of Williamson type $(1, 0, 0)$ and $(0, 1, 0)$, in dimension 4 a *focus-focus fiber* is a singular fiber of Williamson type $(0, 0, 1)$.

When we refer to the *foliation associated to an integrable system* we refer to the foliation described by the orbits of the Hamiltonian vector fields. When we refer to the *fibration associated to an integrable system* we refer to the fibration defined by the moment map. The foliation associated to the last fibration and the first foliation do not necessarily coincide at the singular points.

As it was proved by Eliasson [8, 9] and Miranda [19, 20, 23], non-degenerate singularities are characterized by the fact that the foliation associated to an integrable system is equivalent to the foliation described by its quadratic part.

Let us define the notion of a *singular Lagrangian fibration*.

Definition 2.3. A singular Lagrangian fibration is a symplectic manifold (M, ω) of dimension $2m$ together with a surjective map $\mathcal{F} : M \rightarrow N$, where N is a topological space of dimension m , such that for every point in N there exist an open neighborhood $V \subset N$ and a homeomorphism $\chi : V \rightarrow U \subset \mathbb{R}^m$ satisfying that $\chi \circ \mathcal{F}|_{\mathcal{F}^{-1}(V)}$ is an integrable system on $(\mathcal{F}^{-1}(V), \omega|_{\mathcal{F}^{-1}(V)})$.

When the integrable systems in a singular Lagrangian fibration do not have singularities, one refers to it as a *regular Lagrangian fibration*. The real geometric quantization of such manifolds was computed in [31], whereas (closed) *locally toric manifolds* were considered in [13] (see [32] for the non-compact case); these symplectic manifolds are singular Lagrangian fibrations whose singularities are of Williamson type $(k_e, 0, 0)$ only.

Almost toric manifolds are singular Lagrangian fibrations admitting only singularities of Williamson type $(k_e, 0, k_f)$. In particular, regular Lagrangian fibrations and locally toric manifolds (which include toric manifolds), are examples of almost toric manifolds; as well as the *semitoric integrable systems* in dimension four, which are included in the almost toric manifolds whose bases are subsets of \mathbb{R}^2 . The semi-local and global classification of these symplectic manifolds has been the object of study of [4, 18, 26, 27, 33].

2.2. Real geometric quantization

Let (M, ω) be a symplectic manifold of dimension $2m$ whose de Rham class $[\omega]$ admits an integral lift. Such a symplectic manifold will be called *pre-quantizable*, and a complex line bundle over it with a connection ∇^ω satisfying $\text{curv}(\nabla^\omega) = -i\omega$ is said to be a *pre-quantum line bundle* for (M, ω) .

Definition 2.4. A real polarization \mathcal{P} is an integrable (in the Sussmann's sense) distribution of TM whose leaves are generically Lagrangian. The complexification of \mathcal{P} is denoted by P and will be called the polarization.

The most relevant real polarization for this work is $\langle X_{f_1}, \dots, X_{f_m} \rangle_{C^\infty(M; \mathbb{R})}$: the distribution of the Hamiltonian vector fields of an integrable system. The leaves of the associated (possibly singular) foliation are isotropic submanifolds and they are Lagrangian at points where the first integrals are functionally independent.

Definition 2.5. Let \mathcal{J} denote the sheaf of sections of a pre-quantum line bundle L such that for each open set $V \subset M$ the set $\mathcal{J}(V)$ is the module (over the ring of smooth leafwise constant complex-valued functions of V) of sections $s \in L$ defined over V satisfying $\nabla_X^\omega s = 0$ for all vector fields X in P defined over V .

Definition 2.6. The quantization of $(M, \omega, L, \nabla^\omega, P)$ is given by

$$\mathcal{Q}(M) = \bigoplus_{n \geq 0} H^n(M; \mathcal{J}),$$

where $H^n(M; \mathcal{J})$ are the sheaf cohomology groups associated to \mathcal{J} .

The following definition plays a very important role in the computation of the cohomology groups appearing in geometric quantization:

Definition 2.7. A leaf ℓ of \mathcal{P} is Bohr–Sommerfeld if there exists a non-vanishing section $s : \ell \rightarrow L$ such that $\nabla_X^\omega s = 0$ for any complex vector field X in the polarization P (restricted to ℓ). Fibers that are a union of Bohr–Sommerfeld leaves are called Bohr–Sommerfeld fibers.

3. A motivating example: K3 surfaces

A K3 surface is an example of a total space of an almost toric manifold: it admits an almost toric fibration over the sphere with 24 focus-focus fibers.

The base is a sphere with 24 marked points, and on the complement of these points one has a regular Lagrangian fibration with torus fibers.

A way to construct such an almost toric manifold, as done in [18] (cf. [10]), is to consider two copies of a (symplectic and toric) blowup of the complex projective plane at 9 different points as toric manifolds, apply nodal trades to all of their elliptic-elliptic singular fibers, and take their symplectic sum along the symplectic tori corresponding to the preimage of the boundary of their respective bases (as almost toric fibrations). Starting with a pre-quantizable K3 surface, this construction (together with a gluing result described in Section 7) allows one to obtain a K3 surface with up to 24 Bohr–Sommerfeld focus-focus fibers.

Here is how the construction works.

- Starting with a complex projective plane, understood as a toric manifold and described here by its Delzant polytope [7], one performs three blowups at different points, represented in their Delzant polytopes by cuts based on their three vertices (cf. [17]), followed by another six blowups at different points, represented in their Delzant polytopes by cuts based on their six new vertices formed after the first three blowups: see Figure 2.

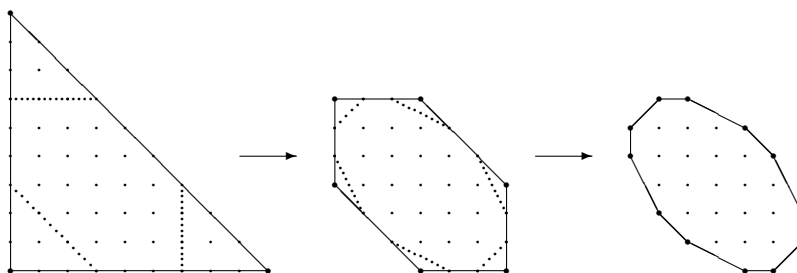


Figure 2: $\mathbb{C}P^2$, $\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2$, and $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$.

- Now, following [18], one can perform nodal trades to all the vertices of the resulting Delzant polytope. In Figure 3 each nodal trade is being represented by a vector based at a vertex, and the monodromy around each of the resulting focus-focus fibers can be read from those vectors.
- The resulting manifold, $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$, is endowed with an almost toric fibration, and the preimage of the boundary of its base is a symplectic torus. Thus, one can consider two copies of this symplectic manifold and perform a symplectic sum along these tori (cf. [10]), obtaining a K3 surface, $(\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2)\#_{T^2}(\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2)$; together with an almost toric

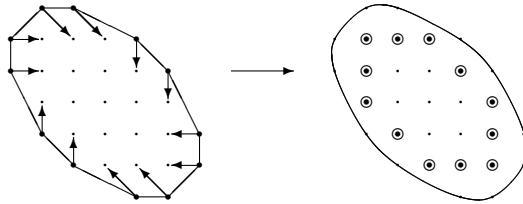


Figure 3: Nodal trades on $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$.

fibration whose base is the sphere formed by gluing two copies of the previously constructed disk (with its twelve marked points) along their boundary (see Figure 4).

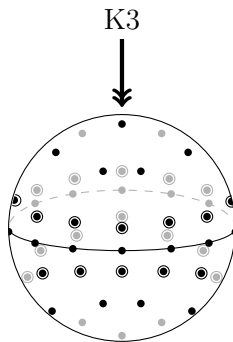


Figure 4: K3 surface as a singular fiber bundle over the sphere.

Before the nodal trades, the toric manifold $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ of Figure 2 admits a pre-quantum line bundle such that all the integer lattice points belonging to its Delzant polytope are images of Bohr–Sommerfeld fibers [12, 32]. Nodal trades produce a one parameter family of symplectomorphic manifolds (via nodal slides [18]), and the almost toric manifold constructed in Figure 3 is related by a symplectomorphism isotopic to the identity (the same is true for different nodal trades resulting in up to 12 focus-focus fibers outside the integer lattice). Therefore, it inherits a pre-quantum line bundle whose Bohr–Sommerfeld fibers are still given by the integer lattice points in the base; which now include 12 elliptic singular fibres (that corresponded to the vertices of the polytope before the nodal trades), up to 12 focus-focus fibers, and down to 7 regular fibres (depending on the size of the nodal trades).

When gluing two copies of those almost toric manifolds by a symplectic sum, one can glue the pre-quantum line bundles to obtain a pre-quantum line bundle on a K3 surface having any number (between 0 and 24) of focus-focus Bohr–Sommerfeld fibers. This last assertion is justified by applying Lemma 7.2 and Corollary 7.1 (which can be found, together with their proofs, in Section 7). Another important remark pointed out by Mayuko Yamashita is that, the symplectic sum takes each pair of elliptic Bohr–Sommerfeld fibers from $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ (obtained after nodal trades) and combines them to one regular Bohr–Sommerfeld fiber.

4. Poincaré lemma and Künneth formula

This section collects results from the literature needed for the proof of the main theorems of this article, and they are included here for the convenience of the reader.

Given a pre-quantizable symplectic manifold (M, ω) with polarization P and pre-quantum line bundle (L, ∇^ω) , it is possible to construct a fine resolution for the sheaf of flat sections \mathcal{J} , even when P has non-degenerate singularities [22, 29, 32].

Let $\Omega_P^n(M)$ denote the set of multi-linear maps

$$\mathrm{Hom}_{C^\infty(M; \mathbb{C})}(\wedge_{C^\infty(M; \mathbb{C})}^n P; C^\infty(M; \mathbb{C})),$$

usually called polarized n -forms, and

$$S_P^n(L) = \Omega_P^n(M) \otimes_{C^\infty(M; \mathbb{C})} \Gamma(L).$$

Then, the set of *line bundle valued polarized forms* is

$$S_P^\bullet(L) = \bigoplus_{n \geq 0} S_P^n(L).$$

Therefore, $\nabla = \nabla^\omega|_P : S_P^0(L) \rightarrow S_P^1(L)$, the restriction of the connection ∇^ω to the polarization, extends to a derivation of degree +1 on the space of line bundle valued polarized forms: if $\alpha \in \Omega_P^n(M)$ and $s \in \Gamma(L)$,

$$d^\nabla(\alpha \otimes s) = d_P \alpha \otimes s + (-1)^n \alpha \wedge \nabla s,$$

with the exterior derivative d_P being the restriction of the de Rham differential to the directions of the polarization.

Since $\omega = i \operatorname{curv}(\nabla^\omega)$ vanishes along P , d^∇ is a coboundary operator.

If \mathcal{S}_P^n denotes the associated sheaf of $S_P^n(L)$, one can extend d^∇ to a homomorphism of sheaves, $d^\nabla : \mathcal{S}_P^n \rightarrow \mathcal{S}_P^{n+1}$. The sheaf \mathcal{S} of sections of the line bundle L is isomorphic to \mathcal{S}_P^0 , and \mathcal{J} is isomorphic to the kernel of $\nabla : \mathcal{S} \rightarrow \mathcal{S}_P^1$, understood as a map between sheaves. The associated complex

$$0 \longrightarrow \mathcal{J} \hookrightarrow \mathcal{S} \xrightarrow{\nabla} \mathcal{S}_P^1 \xrightarrow{d^\nabla} \dots \xrightarrow{d^\nabla} \mathcal{S}_P^m \xrightarrow{d^\nabla} 0$$

is called the *Kostant complex*, and its cohomology is denoted by $H^\bullet(S_P^\bullet(L))$.

Theorem 4.1. *The Kostant complex is a fine resolution for \mathcal{J} when \mathcal{P} is a subbundle of TM , or when \mathcal{P} is induced by an integrable system whose moment map has only non-degenerate singularities. Therefore, each of its cohomology groups, $H^n(S_P^\bullet(L))$, is isomorphic to $H^n(M; \mathcal{J})$.*

This theorem is proved in [22, 29, 32] by showing that Poincaré lemmata exist for the Kostant complex.

Theorem 4.2 (Solha [32]). *The cohomology groups $H^n(M; \mathcal{J})$ vanish for all $n \geq 1$ in any sufficiently small contractible open neighborhood of a focus-focus singularity.*

Remark 4.1. The only property of L being used here is the existence of flat connections along P ; thus, the results here work if metaplectic correction is considered.

The classical Künneth formula also holds for the geometric quantization scheme [21]. Let (M_1, \mathcal{P}_1) and (M_2, \mathcal{P}_2) be a pair of pre-quantizable symplectic manifolds endowed with Lagrangian foliations. The natural Cartesian product for the foliations is Lagrangian with respect to the product symplectic structure. The induced sheaf of flat sections associated to the product foliation will be denoted \mathcal{J}_{12} . Note that we use the pre-quantum line bundle defined as pull-backs of the ones defined over M_1 and M_2 .

Theorem 4.3 (Miranda and Presas [21]). *There is an isomorphism*

$$H^n(M_1 \times M_2, \mathcal{J}_{12}) \cong \bigoplus_{p+q=n} H^p(M_1, \mathcal{J}_1) \otimes H^q(M_2, \mathcal{J}_2),$$

whenever M_1 admits a good cover, the geometric quantization associated to (M_2, \mathcal{J}_2) has finite dimension and M_2 is a submanifold of a compact manifold.

As an illustration, and to anticipate some needed results, let us mention what is the geometric quantization for $M = T^*I \times (I_s \times S^1)$ with $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, endowed with a trivial pre-quantum line bundle with connection $\nabla = d - i(x_1 dy_1 + x_2 dy_2)$, and \mathcal{P} generated by $\frac{\partial}{\partial y_1}$ and $\frac{\partial}{\partial y_2}$, where x_1 is the coordinate function along the fibers of T^*I , y_1 is the coordinate function along the open interval $I \subset \mathbb{R}$, x_2 is the coordinate function along the open interval $I_s \subset \mathbb{R}$, and y_2 is the periodic coordinate function along S^1 .

Proposition 4.1. *The geometric quantization of $M = T^*I \times (I_s \times S^1)$, with the extra structures described above, is given by*

$$H^1(T^*I \times (I_s \times S^1); \mathcal{J}_{12}) \cong (H^0(T^*I; \mathcal{J}_1))^n \cong (C^\infty(\mathbb{R}; \mathbb{C}))^n,$$

where n is the number of integers inside I_s .

Remark 4.2. For some (possibly infinite dimensional) vector space \mathcal{H} and positive integer k , \mathcal{H}^k stands for the direct sum of k copies of \mathcal{H} .

The techniques from [21, 31, 32] provide isomorphisms from $H^1(T^*I \times (I_s \times S^1); \mathcal{J}_{12})$ to flat sections of trivial pre-quantum line bundles L over T^*I , for a given open interval $I \subset \mathbb{R}$. These flat sections are all of the form

$$T^*I \ni (x, y) \mapsto h(x)e^{ixy}s(x, y) \in L|_{(x,y)} \cong \mathbb{C}$$

where $h \in C^\infty(\mathbb{R}; \mathbb{C})$ and $s \in \Gamma(L)$ is a unitary section of L with potential 1-form $-x dy$. For example, Theorem 4.3 gives

$$\begin{aligned} H^1(T^*I \times (I_s \times S^1); \mathcal{J}_{12}) &\cong H^0(T^*I; \mathcal{J}_1) \otimes H^1(I_s \times S^1; \mathcal{J}_2) \\ &\cong C^\infty(\mathbb{R}; \mathbb{C}) \otimes \mathbb{C}^n. \end{aligned}$$

5. Contribution from focus-focus singularities

Let $V \subset M$ be an open neighborhood of a non-degenerate focus-focus fiber ℓ_f (compact or not) over which a Hamiltonian S^1 -action is defined [34]. Note that a focus-focus fiber might have more than one singular point (also called a *node*, or *nodal point* [18]).

Lemma 5.1 (Solha [32]). *In the neighborhood of ℓ_f over which a Hamiltonian S^1 -action is defined, there exists a neighborhood V containing only ℓ_f as a Bohr–Sommerfeld fiber such that $H^0(V; \mathcal{J}|_V) = \{0\}$.*

Therefore, without loss of generality, one can assume that V contains no Bohr–Sommerfeld fiber, or only one if ℓ_f is itself Bohr–Sommerfeld. Such a neighborhood V of a focus-focus fiber ℓ_f is called a *saturated neighborhood* (since it is saturated by the orbits of the S^1 -action).

Theorem 5.1. *The geometric quantization of a saturated neighborhood of a focus-focus fiber with n nodes is:*

- 0 if the singular fiber is not Bohr–Sommerfeld.
- isomorphic to

$$(C^\infty(\mathbb{R}; \mathbb{C}))^{n_f},$$

if the singular fiber is Bohr–Sommerfeld, where $n_f = n$ (for compact fibers) and $n_f = n - 1$ otherwise.

Proof. Let $p_1, \dots, p_n \in \ell_f$ be n singular points on the focus-focus fiber. Take $W_1, \dots, W_n \subset V$ contractible open neighborhoods of the singular points such that $W_j \cap W_k = \emptyset$ for $j \neq k$, and $V_0 \subset V$ an open (not connected) neighborhood satisfying $p_1, \dots, p_n \notin V_0$, as well as, $V = V_0 \cup W_1 \cup \dots \cup W_n$, and $V_0 \cap W_j = W_j^- \sqcup W_j^+$ for each W_j .

The neighborhood V is the total space of a singular Lagrangian fibration over an open disk $D^2 \cong \mathbb{R} \times I_s$ (with $I_s \subset \mathbb{R}$ an open interval representing the circle action direction), as well as $V_n = W_1 \cup \dots \cup W_n$ (which is diffeomorphic to a disjoint union of open 4-balls centered in the nodal points), while W_j^- , W_j^+ , and V_0 are regular trivial Lagrangian fibrations. Indeed,

$$V_0 \cong (I_0 \times S^1) \sqcup \dots \sqcup (I_j \times S^1) \sqcup \dots \sqcup (I_{2\pi} \times S^1) \times D^2,$$

with $I_0 = (0, b_1^-)$, $I_j = (a_j^+, b_{j+1}^-)$, $I_{2\pi} = (a_n^+, 2\pi)$, and $a_j^+ > b_{j-1}^-$,

$$W_j^- \cong (I_j^- \times S^1) \times D^2$$

with $I_j^- = (a_j^-, b_j^-)$ and $a_j^- \in (b_{j-1}^+, b_j^-)$, and

$$W_j^+ \cong (I_j^+ \times S^1) \times D^2$$

with $I_j^+ = (a_j^+, b_j^+)$ and $b_j^+ \in (a_j^+, a_{j+1}^-)$ (see Figure 5). For a compact fiber ℓ_f , one connects I_0 and $I_{2\pi}$ via $0 \sim 2\pi$.

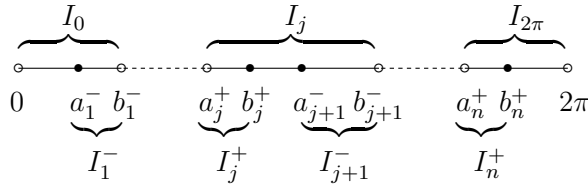


Figure 5: Intervals along a focus-focus fiber.

Let us represent the trivial regular Lagrangian fibrations as products of two cotangent bundles

$$\begin{aligned}
 V_0 &\cong (T^*(I_0 \sqcup \dots \sqcup I_j \sqcup \dots \sqcup I_{2\pi})) \times (I_s \times S^1), \\
 W_j^- &\cong T^*I_j^- \times (I_s \times S^1),
 \end{aligned}$$

and

$$W_j^+ \cong T^*I_j^+ \times (I_s \times S^1).$$

We do so in order to use Lemma 5.1, Theorem 4.2, and Proposition 4.1, which give:

$$H^0(V; \mathcal{J}|_V) = \{0\},$$

$$H^0(V_n; \mathcal{J}|_{V_n}) = H^1(V_n; \mathcal{J}|_{V_n}) = H^2(V_n; \mathcal{J}|_{V_n}) = \{0\},$$

$$H^0(V_0; \mathcal{J}|_{V_0}) = H^0(W_j^-; \mathcal{J}|_{W_j^-}) = H^0(W_j^+; \mathcal{J}|_{W_j^+}) = \{0\},$$

$$H^2(V_0; \mathcal{J}|_{V_0}) = H^2(W_j^-; \mathcal{J}|_{W_j^-}) = H^2(W_j^+; \mathcal{J}|_{W_j^+}) = \{0\},$$

$$H^1(V_0; \mathcal{J}|_{V_0}) \cong \begin{cases} \{0\}, & \text{if } \ell_f \text{ is not Bohr-Sommerfeld} \\ (C^\infty(\mathbb{R}; \mathbb{C}))^{n+1}, & \text{if } \ell_f \text{ is non-compact} \\ (C^\infty(\mathbb{R}; \mathbb{C}))^n, & \text{if } \ell_f \text{ is compact,} \end{cases}$$

and

$$\begin{aligned}
 H^1(V_0 \cap V_n; \mathcal{J}|_{V_0 \cap V_n}) &\cong (H^1(W_j^-; \mathcal{J}|_{W_j^-}) \oplus H^1(W_j^+; \mathcal{J}|_{W_j^+}))^n \\
 &\cong \begin{cases} (C^\infty(\mathbb{R}; \mathbb{C}))^{2n}, & \text{if } \ell_f \text{ is Bohr-Sommerfeld} \\ \{0\}, & \text{otherwise.} \end{cases}
 \end{aligned}$$

Considering the open covering $\{V_0, V_n\}$ of V , one has the following Mayer–Vietoris sequence (see [21] for a proof of its existence):

$$\begin{aligned} 0 \rightarrow H^0(V_0 \cup V_n; \mathcal{J}|_{V_0 \cup V_n}) &\rightarrow H^0(V_0; \mathcal{J}|_{V_0}) \oplus H^0(V_n; \mathcal{J}|_{V_n}) \\ &\rightarrow H^0(V_0 \cap V_n; \mathcal{J}|_{V_0 \cap V_n}) \rightarrow H^1(V_0 \cup V_n; \mathcal{J}|_{V_0 \cup V_n}) \\ &\rightarrow H^1(V_0; \mathcal{J}|_{V_0}) \oplus H^1(V_n; \mathcal{J}|_{V_n}) \rightarrow H^1(V_0 \cap V_n; \mathcal{J}|_{V_0 \cap V_n}) \\ &\rightarrow H^2(V_0 \cup V_n; \mathcal{J}|_{V_0 \cup V_n}) \rightarrow H^2(V_0; \mathcal{J}|_{V_0}) \oplus H^2(V_n; \mathcal{J}|_{V_n}) \\ &\rightarrow H^2(V_0 \cap V_n; \mathcal{J}|_{V_0 \cap V_n}) \rightarrow H^3(V_0 \cup V_n; \mathcal{J}|_{V_0 \cup V_n}) \rightarrow \dots \end{aligned}$$

Exploiting the dimension of V (cohomology groups in degree higher than two vanish) and the fact that the various cohomology groups in degree zero vanish, as well as in degree two for V_0, V_n , and $V_0 \cap V_n$, one has

$$\begin{aligned} 0 \rightarrow H^1(V_0 \cup V_n; \mathcal{J}|_{V_0 \cup V_n}) &\hookrightarrow H^1(V_0; \mathcal{J}|_{V_0}) \oplus H^1(V_n; \mathcal{J}|_{V_n}) \\ &\rightarrow H^1(V_0 \cap V_n; \mathcal{J}|_{V_0 \cap V_n}) \twoheadrightarrow H^2(V_0 \cup V_n; \mathcal{J}|_{V_0 \cup V_n}) \rightarrow 0. \end{aligned}$$

Because $H^1(V_n; \mathcal{J}|_{V_n}) = \{0\}$, the middle map

$$H^1(V_0; \mathcal{J}|_{V_0}) \oplus H^1(V_n; \mathcal{J}|_{V_n}) \longrightarrow H^1(V_0 \cap V_n; \mathcal{J}|_{V_0 \cap V_n})$$

is injective. This can be seen by identifying the pertinent cohomology groups with $C^\infty(\mathbb{R}; \mathbb{C})$ (see Proposition 4.1 and comments below it); thus, the map can be identified with

$$(C^\infty(\mathbb{R}; \mathbb{C}))^n \oplus \{0\} \ni h \oplus 0 \mapsto h \oplus h \in (C^\infty(\mathbb{R}; \mathbb{C}))^{2n},$$

when the fiber is compact and Bohr–Sommerfeld, and

$$(h_1, \dots, h_{n+1}, 0) \in (C^\infty(\mathbb{R}; \mathbb{C}))^{n+1} \oplus \{0\}$$

is mapped to

$$(h_1, h_2, h_2, \dots, h_j, h_j, \dots, h_n, h_n, h_{n+1}) \in (C^\infty(\mathbb{R}; \mathbb{C}))^{2n},$$

when the fiber is Bohr–Sommerfeld but not compact; otherwise every cohomology group vanish and the map is trivial. From the exactness of the

sequence and using the first isomorphism theorem, this implies the following:

$$H^1(V_0 \cup V_n; \mathcal{J}|_{V_0 \cup V_n}) = \{0\}$$

and

$$H^2(V_0 \cup V_n; \mathcal{J}|_{V_0 \cup V_n}) \cong \frac{H^1(V_0 \cap V_n; \mathcal{J}|_{V_0 \cap V_n})}{H^1(V_0; \mathcal{J}|_{V_0})}.$$

Thus, identifying $V_0 \cup V_n = V$, a nodal point on a compact Bohr–Sommerfeld focus-focus fiber provides an infinite dimensional contribution,

$$H^1(V; \mathcal{J}|_V) = \{0\}$$

and

$$H^2(V; \mathcal{J}|_V) \cong \begin{cases} \{0\}, & \text{if } \ell_f \text{ is not Bohr–Sommerfeld} \\ (C^\infty(\mathbb{R}; \mathbb{C}))^{n-1}, & \text{if } \ell_f \text{ is non-compact} \\ (C^\infty(\mathbb{R}; \mathbb{C}))^n, & \text{if } \ell_f \text{ is compact.} \end{cases}$$

□

6. Semitoric systems and almost toric manifolds

Reference [32] states a formula (its very last equation) for the geometric quantization of a 4-dimensional closed almost toric manifold. Theorem 5.1 together with that formula proves the next theorem; whose proof is provided for the convenience of the reader.

Theorem 6.1. *For a 4-dimensional closed almost toric manifold M , with n_r regular Bohr–Sommerfeld fibers and n_f focus-focus Bohr–Sommerfeld fibers:*

$$\mathcal{Q}(M) \cong \mathbb{C}^{n_r} \oplus \left(\bigoplus_{j \in \{1, \dots, n_f\}} (C^\infty(\mathbb{R}; \mathbb{C}))^{n(j)} \right),$$

with $n(j)$ the number of nodes on the j -th focus-focus Bohr–Sommerfeld fiber.

Proof. A 4-dimensional closed almost toric manifold is a particular example satisfying Definition 2.3; therefore, there exist a symplectic structure ω , a topological space N , a surjective map $\mathcal{F} : M \rightarrow N$, and an open neighborhood V_b , for every $b \in N$, together with a homeomorphism $\chi_b : V_b \rightarrow U_b \subset \mathbb{R}^2$ such

that $\chi_b \circ \mathcal{F}|_{\mathcal{F}^{-1}(V_b)}$ is an integrable system admitting only singularities of Williamson type $(k_e, 0, k_f)$ of $(\mathcal{F}^{-1}(V_b), \omega|_{\mathcal{F}^{-1}(V_b)})$.

Because M is compact and Bohr–Sommerfeld fibers are isolated, one can choose a finite open cover for N in such a way that no Bohr–Sommerfeld fiber is contained in more than one of the open sets $\mathcal{F}^{-1}(V_b)$. For each of those open sets their cohomology groups $H^k(\mathcal{F}^{-1}(V_b); \mathcal{J}|_{\mathcal{F}^{-1}(V_b)})$ are computed using Proposition 4.1, if $\mathcal{F}^{-1}(V_b)$ contains a regular Bohr–Sommerfeld fiber, Theorem 5.1, if $\mathcal{F}^{-1}(V_b)$ contains a focus-focus Bohr–Sommerfeld fiber, or the results from [13, 32] which state that $H^k(\mathcal{F}^{-1}(V_b); \mathcal{J}|_{\mathcal{F}^{-1}(V_b)}) = \{0\}$ if $\mathcal{F}^{-1}(V_b)$ is a neighborhood of an elliptic Bohr–Sommerfeld fiber. If V_{b_1} and V_{b_2} are two open sets such that $V_{b_1} \cap V_{b_2} \neq \emptyset$, then $W := \mathcal{F}^{-1}(V_{b_1}) \cap \mathcal{F}^{-1}(V_{b_2})$ has no Bohr–Sommerfeld fiber and all cohomology groups $H^k(W; \mathcal{J}|_W)$ vanish.

The Mayer–Vietoris sequences associated to the finite open cover of M formed by the subsets $\mathcal{F}^{-1}(V_b)$ reduces the computation of $\mathcal{Q}(M)$ to a finite direct sum of neighborhoods containing regular and focus-focus Bohr–Sommerfeld fibers; hence, proving the desired formula. □

Remark 6.1. The previous Theorem fits in line with previous results that state that real Geometric Quantization is a very rough invariant when used to classify almost toric manifolds (see [21, 31, 32]). In particular, the symplectic type of a local neighborhood of a focus–focus singularity was classified in [33] and it has been extended to multiple focus–focus singularities on the same fiber quite recently [1, 28]. There is a complete invariant that is a formal series in two variables. However, Theorem 6.1 clearly shows that the Geometric Quantization is the same for any element of that family, i.e. any neighborhood of a focus–focus fiber has the same Geometric Quantization. This implies that the associated real quantization does not distinguish the different elements of the moduli.

Closed almost toric manifolds in dimension four were classified, up to diffeomorphism, in [18], and in order to obtain their real geometric quantization it is enough to identify the image of the Bohr–Sommerfeld fibers at each of the seven possible base spaces of such fibrations, and then apply Theorem 6.1. The total number of regular Bohr–Sommerfeld fibers is determined by the symplectic volume of the almost toric manifold, and the number of focus-focus fibers can be read from table 1 in [18]. Via nodal slides is always possible to modify the real polarization to change the number of focus-focus fibers that are actually Bohr–Sommerfeld fibers; this is exemplified in Section 7 for the K3 surface.

As mentioned in the last paragraph of Subsection 2.1, semitoric systems [26] are a particular example of almost toric manifolds. Their total spaces, however, need not to be closed symplectic manifolds resulting in the possibility of no upper bound on the number of Bohr–Sommerfeld fibers. The formula of Theorem 6.1 still holds, except that their set of regular and focus-focus Bohr–Sommerfeld fibers can be a countably infinite set; thus, $\mathcal{Q}(M)$ may be a direct sum of infinitely many copies of \mathbb{C} and $C^\infty(\mathbb{R}; \mathbb{C})$.

7. Quantization of K3 surfaces

As mentioned in the construction of a pre-quantizable K3 surface (Section 3), one can obtain a K3 surface with up to 24 Bohr–Sommerfeld focus-focus fibers. In the particular example constructed in Section 3, an application of Theorem 6.1 yields

$$\mathcal{Q}(K3) \cong \mathbb{C}^{26} \oplus (C^\infty(\mathbb{R}; \mathbb{C}))^{24}.$$

The real geometric quantization of the K3 surface can be, then, drastically different from the Kähler case, which is always finite dimensional.

On the K3 surface, the dimension of the vector space of holomorphic sections for a given ample holomorphic line bundle L equals $\frac{1}{2}c_1(L)^2 + 2$ (cf. [15]), and the dimension of its Kähler quantization is exactly this number. Since the first Chern class of a pre-quantum line bundle L is represented by the symplectic form ω , it holds that

$$c_1(L)^2 = \int_{K3} \omega \wedge \omega.$$

In the particular example above $K3 = (\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}) \#_{T^2} (\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2})$, and the symplectic volume of a symplectic sum is the sum of the symplectic volumes [10]. Thus, the symplectic volume can be computed from the symplectic volume of each toric manifold (as nodal trades produce symplectomorphic manifolds [18]), which is simply two times the Euclidean volume of each Delzant polytope [11] (up to a $(2\pi)^2$ factor due to different conventions); and the volume of the Delzant polytopes are 24 in this case. Therefore, for the particular example computed above, the dimension of its Kähler quantization is

$$\frac{1}{2}c_1(L)^2 + 2 = \frac{1}{2}(2 \cdot 24 + 2 \cdot 24) + 2 = 50,$$

coinciding with the real geometric quantization when none of the 24 focus-focus fibers are Bohr–Sommerfeld².

What is missing is to actually show how to glue pre-quantum line bundles when performing a symplectic sum. We begin by reviewing the symplectic sum construction, and we, then, keep track of this construction when considering pre-quantum line bundles in the picture.

Lemma 7.1 (Gompf [10]). *Let (M_j, ω_j) , $j = 1, 2$, be two symplectic manifolds. Assume that there are two codimension 2 symplectic submanifolds $\Sigma_j \subset M_j$, a symplectomorphism $\Psi : \Sigma_1 \rightarrow \Sigma_2$ and a complex isomorphism identifying the symplectic normal bundle ν_1 of Σ_1 and the dual symplectic normal bundle ν_2^* of Σ_2 . Then, there is a symplectic structure on the fiber connected sum $M_1 \#_{\Psi} M_2$ of M_1 and M_2 along $\Sigma_2 \simeq \Psi(\Sigma_1)$.*

Denote by U_j a small tubular neighborhood of Σ_j and $U_j^* = U_j \setminus \Sigma_j$. Recall that Gompf’s construction provides a symplectomorphism $\Phi : U_1^* \rightarrow U_2^*$ that takes the outer boundary of one domain to the inner boundary of the other one and vice versa. This is used as a gluing morphism. It is, then, simple to obtain the next lemma and its corollary.

Lemma 7.2. *Let $(M_j, \omega_j, L_j, \nabla_j)$, $j = 1, 2$, be two symplectic manifolds equipped with pre-quantum line bundles. Under the hypotheses of Lemma 7.1, assume moreover that $\Psi^* L_2 \simeq L_1$ (as topological complex line bundles), then the fiber connected sum $M_1 \#_{\Psi} M_2$ admits a pre-quantum line bundle (L, ∇) whose restriction to $M_j \setminus U_j$ coincides with (L_j, ∇_j)*

Corollary 7.1. *With the same hypothesis, further assume that there exists a symplectomorphism $\psi : (M, \omega_1, L_1, \nabla_1) \rightarrow (M, \omega_2, L_2, \nabla_2)$ with ψ isotopic to the identity ($M = M_1 = M_2$), then for any Σ we have that $L_1|_{\Sigma}$ and $L_2|_{\Sigma}$ are isomorphic as topological complex line bundles*

Do note that $\psi(\Sigma) \neq \Sigma$ in general. If this were not the case, the statement and the gluing would be trivial.

²We want to thank Mayuko Yamashita for observing that the preimage of the equator of the sphere of Figure 4 does contain 12 regular Bohr–Sommerfeld fibers, obtained from the 12 elliptic singular Bohr–Sommerfeld fibers from $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ (Figure 3) which become regular after the symplectic sum.

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