

An unoriented skein relation via bordered–sutured Floer homology

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We show that the bordered–sutured Floer invariant of the complement of a tangle in an arbitrary 3-manifold Y , with minimal conditions on the bordered–sutured structure, satisfies an unoriented skein exact triangle. This generalizes a theorem by Manolescu [Man07] for links in S^3 . We give a theoretical proof of this result by adapting holomorphic polygon counts to the bordered–sutured setting, and also give a combinatorial description of all maps involved and explicitly compute them. We then show that, for $Y = S^3$, our exact triangle coincides with Manolescu’s. Finally, we provide a graded version of our result, explaining in detail the grading reduction process involved.

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DSV was partially supported by NSF Grant DMS-1907654 and Simons Foundation Grant 524876. CMMW was partially supported by NSF Grant DMS-2039688 and an AMS–Simons Travel Grant.

1. Introduction

Knot Floer homology, defined by Ozsváth and Szabó [OSz04b], and independently by Rasmussen [Ras03], is an invariant of oriented, null-homologous knots K in an oriented, connected, closed 3-manifold Y . It is then extended by Ozsváth and Szabó [OSz08] to an invariant for oriented links L . The simplest flavor $\widehat{\text{HFK}}(Y, L)$ is a bigraded module over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ or \mathbb{Z} , which categorifies the Alexander polynomial of L when $Y = S^3$; for this reason, it is often compared to Khovanov homology [Kho00], a link invariant $\widehat{\text{Kh}}(L)$ that categorifies the Jones polynomial. Knot Floer homology is known to detect the fiberedness [Ghi08, Ni07] and the genus [OSz04a] of a knot, and has given rise to a number of concordance invariants [OSz03, Hom14, OSSz17]. Among other applications, it has also led to invariants of Legendrian and transverse links in a contact 3-manifold [NOT08, LOSSz09].

It is well known that the Alexander and Jones polynomials satisfy skein relations, which have played a very important role in our understanding of these invariants. In particular, both polynomials satisfy an oriented skein relation that relates two links that differ by a crossing and their common oriented resolution, while only the Jones polynomial satisfies an unoriented skein relation that relates a link with its two resolutions at a crossing. There are categorified statements of some of these skein relations, in the form of long exact sequences: Knot Floer homology satisfies an oriented skein exact triangle [OSz04b], while Khovanov homology satisfies an unoriented one [Kho00, Vir04, Ras05].

Curiously, while the Alexander polynomial does not satisfy an unoriented skein relation, Manolescu [Man07] shows that $\widehat{\text{HFK}}$ does satisfy an unoriented skein exact triangle over \mathbb{F}_2 , when $Y = S^3$. This is extended by the second author [Won17] to \mathbb{Z} -coefficients, using grid homology, a combinatorial version of link Floer homology.

This exact sequence has a number of consequences: First, quasi-alternating links are a large class of links in S^3 that include all alternating links. Manolescu's result immediately implies $\text{rk } \widehat{\text{HFK}}(S^3, L) = 2^{\ell-1} \det(L)$ for a quasi-alternating link L with ℓ components, explaining a rank equality between $\widehat{\text{HFK}}$ and $\widehat{\text{Kh}}$ for many small links in S^3 . By calculating the shifts in the δ -grading (defined as the difference of the two gradings) in the exact sequence, Manolescu and Ozsváth [MO08] show that a quasi-alternating link L is σ -thin over \mathbb{F}_2 , meaning that $\widehat{\text{HFK}}(S^3, L)$ is supported only in δ -grading $\sigma(L)/2$; this in turn implies that $\widehat{\text{HFK}}(S^3, L)$ is completely determined by

the signature and the Alexander polynomial of L . By the second author's work on grid homology, this fact is also true over \mathbb{Z} .

Second, the exact triangle is iterated by Baldwin and Levine [BL12] to obtain a cube-of-resolutions complex, giving a combinatorial description of link Floer homology that is distinct from grid homology.

Third, using the grid homology version of the exact triangle, Lambert-Cole [LC19] computes the rank of the maps involved, and uses this computation to prove that $\widehat{\text{HF}}\widehat{\text{K}}$ remains invariant under mutation by a large class of tangles.¹ This mutation invariance applies to the Kinoshita–Terasaka and Conway families, as well as all mutant knots with crossing number at most 12, giving a partial answer to the Mutation Invariance Conjecture [BL12].

The goal of the present paper is to generalize Manolescu's result to obtain an exact triangle for an unoriented skein triple of tangles in a 3-manifold with boundary, in an appropriate sense. There are currently several theories related to Heegaard Floer homology that provide a suitable gluing theorem; in this paper, we focus on one such theory, bordered–sutured Floer homology, defined by Zarev [Zar11]. Petkova and the second author [PW20] have proven a similar result for tangle Floer homology, a combinatorial tangle invariant defined by Petkova and Vértesi [PV16], similar to grid homology, for tangles in $S^2 \times [0, 1]$; meanwhile, Zibrowius [Zib17] has given an alternative proof of Manolescu's result for links in S^3 using peculiar modules, which are invariants of tangles in B^3 . The exact triangle in the present paper applies to tangles in any 3-manifold.

Sutured Floer homology, defined by Juhász [Juh06], is a variant of Heegaard Floer homology [OSz04c] for sutured manifolds. It recovers the hat-version of link Floer homology via the isomorphism

$$\text{SFH}(Y \setminus \nu(L), \Gamma_\mu) \cong \widehat{\text{HF}}\widehat{\text{K}}(Y, L),$$

where Γ_μ denotes a pair of oppositely oriented, meridional sutures for each link component, at the expense of breaking a uniform homological grading into a grading for each Spin^c -summand. (For this reason, this will be our view of the gradings on $\widehat{\text{HF}}\widehat{\text{K}}$ for the rest of the paper.) This isomorphism also gives a definition of $\widehat{\text{HF}}\widehat{\text{K}}$ for links that are not rationally null-homologous. Inspired by the bordered Heegaard Floer theory developed by Lipshitz, Ozsváth, and Thurston [LOT18, LOT15], Zarev [Zar11] defines an invariant of manifolds whose boundaries are partly sutured and partly

¹During the peer review process of the current article, Zibrowius [Zib19] announced a proof that $\widehat{\text{HF}}\widehat{\text{K}}$ is invariant under Conway mutation by all tangles.

bordered: One may then glue two manifolds together by identifying the bordered parts of their boundaries, and use a pairing theorem to calculate the invariant associated to the glued manifold. Since this theory is currently not defined over \mathbb{Z} , we shall work over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ throughout the paper.

For $k \in \{\infty, 0, 1\}$, let $\mathcal{T}_k^{\text{el}} = (\overline{B^3}, T_k^{\text{el}})$ be the tangles shown in Figure 1, which we shall refer to as *elementary tangles*. Here, $\partial T_k^{\text{el}} \subset \partial \overline{B^3}$ are identical 0-manifolds for $k \in \{\infty, 0, 1\}$.

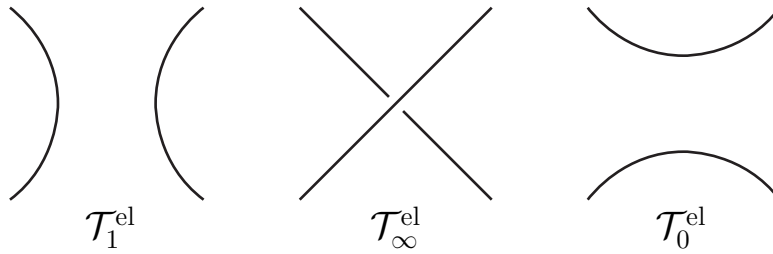


Figure 1. The elementary tangle $\mathcal{T}_\infty^{\text{el}}$ and its resolutions $\mathcal{T}_0^{\text{el}}$ and $\mathcal{T}_1^{\text{el}}$.

Consider now a triple of more general tangles $\mathcal{T}_k = (Y, T_k)$, where Y is a compact, oriented 3-manifold (possibly) with boundary and each $(T_k, \partial T_k) \subset (Y, \partial Y)$ is a smoothly embedded compact 1-dimensional submanifold, such that, for $k \in \{\infty, 0, 1\}$,

- There is an embedded $\overline{B^3}$ in the interior of Y in which T_k coincides with T_k^{el} ; and
- The tangles T_k are identical outside $\overline{B^3}$.

Suppose each tangle complement $Y_k = Y \setminus \nu(T_k)$ is equipped with a bordered–sutured structure $(\Gamma_k, \mathcal{Z}_1 \cup \mathcal{Z}_2, \phi_k)$, such that

- The sutures in Γ_k do not intersect $\overline{B^3}$, and are identical for $k \in \{\infty, 0, 1\}$;
- The intersection of ∂Y_k and $\overline{B^3}$ belongs to the positive region $R_+(\Gamma)$; and
- The bordered boundary, i.e. the image of the parametrization $\phi_k : G(\mathcal{Z}_1 \cup \mathcal{Z}_2) \hookrightarrow \partial Y_k$, does not intersect $\overline{B^3}$, and the parametrization maps are identical for $k \in \{\infty, 0, 1\}$.

Note that either of \mathcal{Z}_1 and \mathcal{Z}_2 may be empty or not connected. To each bordered–sutured manifold $\mathcal{Y}_k = (Y_k, \Gamma_k, \mathcal{Z}_1 \cup \mathcal{Z}_2, \phi_k)$, then, Zarev [Zar11]

associates a type DA bimodule

$$\mathcal{A}(-\mathcal{Z}_1)\widehat{\text{BSDA}}(\mathcal{Y}_k)_{\mathcal{A}(\mathcal{Z}_2)}.$$

over $\mathcal{A}(-\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$. A similar construction is used to obtain a bordered–sutured manifold $\mathcal{Y}_\Gamma(\mathcal{T})$ by Alishahi and Lipshitz [AL19], who prove that $\widehat{\text{BSD}}(\mathcal{Y}_\Gamma(\mathcal{T}))$ detects (partly) boundary parallel tangles. Our hypothesis differs from their construction in that we do not allow longitudinal sutures that intersect $\overline{B^3}$, but we do allow non–null-homologous \mathcal{T} and place no restrictions on the sutures or the bordered boundaries outside $\overline{B^3}$.

Theorem 1.1. *Let \mathcal{Y}_k be as described above. There exist type DA homomorphisms $F_k: \widehat{\text{BSDA}}(\mathcal{Y}_k) \rightarrow \widehat{\text{BSDA}}(\mathcal{Y}_{k+1})$ such that*

$$\widehat{\text{BSDA}}(\mathcal{Y}_k) \simeq \text{Cone}(F_{k+1}: \widehat{\text{BSDA}}(\mathcal{Y}_{k+1}) \rightarrow \widehat{\text{BSDA}}(\mathcal{Y}_{k+2}))$$

as type DA structures. Moreover, the homomorphisms F_k and the homotopy equivalence above respect the relative gradings on the bimodules in a sense to be made precise in Section 7; see Theorem 7.12.

Because of the more complicated nature of the gradings, we generally defer their discussions to Section 7.

The strategy to prove Theorem 1.1 is to consider bordered–sutured manifolds \mathcal{B}_k corresponding to the complements of the elementary tangles $\mathcal{T}_k^{\text{el}}$, first defining homomorphisms $f_k: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1})$, and then using the pairing theorem [Zar11, Theorem 8.5.1] to extend the result to our setting. In fact, it is evident from our strategy that the skein relation above holds for more general bordered–sutured manifolds \mathcal{Y}_k and not just tangle complements; for example, one may derive from Theorem 1.1 a skein relation for (a generalization of) graph Floer homology, defined by Harvey and O’Donnol [HO17], and independently by Bao [Bao18], for certain kinds of spatial graphs. The statement of Theorem 1.1 is chosen to emphasize our interest in the application of bordered–sutured Floer theory to the study of knots, links, and tangles. See Theorem 3.6 for the more general statement.

Our strategy illustrates and harnesses the power of a bordered theory, which allows one to use cut-and-paste techniques to prove general results by working locally. For example, in a similar spirit, Lipshitz, Ozsváth, and Thurston [LOT14, LOT16] have used bordered Floer homology to compute explicitly the spectral sequence from Khovanov homology of a link to the Heegaard Floer homology of its branched double cover [OSz05].

In particular, our strategy offers an advantage over the proofs in [Man07, Won17, PW20], as follows. In Manolescu’s original paper [Man07], the maps involved in the exact sequence are defined by counting holomorphic polygons in a high-dimensional manifold, which are of theoretical interest but not practically computable. In contrast, the maps in [Won17] are combinatorially defined, which is a feature specifically exploited in the application to mutation invariance [LC19]; the maps in [PW20] are likewise combinatorial. However, while we believe the maps in [Won17, PW20] coincide with holomorphic polygon counts, it is not immediately obvious that they do so; moreover, the complexes involved there are fairly large. The present paper unites the two approaches: We will first define f_k by counts of holomorphic triangles in Section 4, and then explicitly compute them combinatorially in Section 5, taking advantage of the fact that $\widehat{\text{BSD}}(\mathcal{B}_k)$ have at most three generators with our choice of bordered–sutured Heegaard diagrams \mathcal{H}_k . The reader is encouraged to take a cursory look at Figure 11 and Figure 13, which contain the entire computation.

A special case of Theorem 1.1 is when Y is a compact 3-manifold *without* boundary, with $\mathcal{Z}_1 = \mathcal{Z}_2 = \emptyset$. In this case, each T_k is an embedded 1-dimensional submanifold without boundary, and is hence a link in Y ; we thus denote it more conventionally by L_k . Let ℓ_k be the number of components of L_k , and set $m = \max\{\ell_\infty, \ell_0, \ell_1\}$.

Corollary 1.2. *There exists an exact triangle*

$$\dots \xrightarrow{(F_1)_*} \widehat{\text{HFK}}(Y, L_\infty) \otimes V^{m-\ell_\infty} \xrightarrow{(F_\infty)_*} \widehat{\text{HFK}}(Y, L_0) \otimes V^{m-\ell_0} \xrightarrow{(F_0)_*} \widehat{\text{HFK}}(Y, L_1) \otimes V^{m-\ell_1} \xrightarrow{(F_1)_*} \dots,$$

where V is a vector space of dimension 2. Moreover, the maps involved respect the relative gradings on the homology groups in a sense to be made precise in Section 7; see Theorem 7.12.

The reader familiar with gradings in sutured Floer homology may find it useful to examine Example 7.11 for an illustration of the graded version of Corollary 1.2.

The exact triangle in [Man07] is the special case $Y = S^3$. In [Man07], Manolescu constructs, from a given connected link projection of $L_\infty \subset S^3$ and a choice of edges in the projection, a *special Heegaard diagram* $\mathcal{H}_\infty^{\text{Man}}$ for L_∞ , modifies $\mathcal{H}_\infty^{\text{Man}}$ locally to obtain Heegaard diagrams $\mathcal{H}_0^{\text{Man}}$ and $\mathcal{H}_1^{\text{Man}}$ for L_0 and L_1 , and uses these Heegaard diagrams to prove the skein exact triangle. Here, we have added a superscript to distinguish these Heegaard

diagrams defined in [Man07] from those constructed in the present article; likewise, we denote by F_k^{Man} the maps involved in the skein exact triangle, which are denoted f_k in [Man07].

Note that one might be able to prove Corollary 1.2 by directly generalizing Manolescu's construction; however, that would require constructing Heegaard diagrams of a given form, for all (Y, L_∞) , which could be somewhat cumbersome before the advent of bordered–sutured Floer theory. (See, for contrast, the discussion after Theorem 6.5.)

Theorem 1.3. *For $Y = S^3$, the exact triangle in Corollary 1.2 agrees with the exact triangle in [Man07]. See Theorem 6.1 for the precise statement.*

In [MO08] (and in [Won17, PW20]), the exact triangles are equipped with an absolute δ -grading in \mathbb{Z} . While we expect the grading in Corollary 1.2 to be related to this δ -grading when applied to $Y = S^3$, we do not prove this in this paper. See, however, Remark 7.5.

Theorem 1.3 is inspired by [LOT16], in which the theory of bordered polygon counts is developed to prove that the exact triangle, and in fact the spectral sequence, constructed in [LOT14] using bordered Floer homology, agree with the original exact triangle and spectral sequence constructed directly using Heegaard Floer homology [OSz05].

Organization

The rest of the paper is organized as follows. In Section 2, we fix some notation and provide a lemma in homological algebra that will be useful in later sections. Then, in Section 3, we describe the bordered boundary and the bordered–sutured manifolds \mathcal{B}_k , which correspond to the complements of the elementary tangles T_k^{el} , and explicitly compute the algebra and modules associated to these objects. Stating and assuming the specialization of Theorem 1.1 to \mathcal{B}_k , we complete the general proof of Theorem 1.1 and Corollary 1.2. We provide two proofs of the specialization. We first give a theoretical proof by holomorphic polygon counts in Section 4, in the process adapting the technique of polygon counts from [LOT16] to the bordered–sutured setting. Then, in Section 5, we give a combinatorial proof, with explicit computations, and show that the two proofs are equivalent. We then compare our maps with Manolescu's maps [Man07] and establish Theorem 1.3 in Section 6. Finally, in Section 7, we discuss the gradings in bordered–sutured Floer theory in some detail and prove a graded version of Theorem 1.1.

Acknowledgements

The authors would like to thank Robert Lipshitz for helpful discussions. The authors are also very grateful to the anonymous referee for their many insightful and remarkably detailed comments on a previous draft.

Part of the research was conducted while CMMW was at Louisiana State University, and the authors thank the LSU Department of Mathematics for its support.

2. Algebraic preliminaries

We shall assume that the reader is familiar with the formal algebraic structures in the bordered Floer package [LOT18, LOT15] and the bordered–sutured Floer package [Zar11]. (For the reader familiar with the former but not the latter, the algebraic structures involved are the same.) These include type D and type A modules, bimodules of various types, morphisms of modules and their boundaries and compositions, and the box tensor operation of modules. In general, we follow the notation in [Zar11], which may differ from the notation in [LOT18]; see, however, Remark 4.4.

In the following, we will always use the calligraphic typeface (e.g. \mathcal{M} and \mathcal{N}) to denote an unspecified type D or type A module over some algebra \mathcal{A} , and reserve the regular italic typeface (e.g. M and N) for the underlying \mathbf{k} -module. However, we will denote both the type D bordered–sutured Floer module over the algebra $\mathcal{A}(\mathcal{Z})$, and the underlying \mathbf{k} -module, by $\widehat{\text{BSD}}(\mathcal{Y})$. We will also distinguish between the identity type D homomorphism $\mathbb{I}_{\mathcal{M}}$ of \mathcal{M} and the identity \mathbf{k} -module isomorphism id_M of M .

Given a type D morphism $f: \mathcal{M} \rightarrow \mathcal{N}$, let ∂f denote its boundary in $\text{Mor}^A(\mathcal{M}, \mathcal{N})$, given by

$$\partial f = (\mu \otimes \text{id}_N) \circ (\text{id}_A \otimes \delta_N) \circ f + (\mu \otimes \text{id}_N) \circ (\text{id}_A \otimes f) \circ \delta_M + (d \otimes \text{id}_N) \circ f.$$

For convenience, we sometimes denote by df the term $(d \otimes \text{id}_N) \circ f$.

The proof of Theorem 1.1 uses a lemma in homological algebra whose version for chain complexes first appeared in [OSz05]. Let $\mathcal{A} = (A, d, \mu)$ be a differential graded algebra, and let \mathcal{M} and \mathcal{N} be type D modules over \mathcal{A} .

Lemma 2.1. *Let $\mathcal{M}_k = \{(M_k, \delta_{M_k})\}_{k \in \{\infty, 0, 1\}}$ be a collection of type D modules over a unital differential graded algebra $\mathcal{A} = (A, d, \mu)$ over a base ring \mathbf{k} of characteristic 2, and let $f_k: \mathcal{M}_k \rightarrow \mathcal{M}_{k+1}$, $\varphi_k: \mathcal{M}_k \rightarrow \mathcal{M}_{k+2}$, and $\kappa_k: \mathcal{M}_k \rightarrow \mathcal{M}_k$ be morphisms satisfying the following conditions for each k :*

1) The morphism $f_k: \mathcal{M}_k \rightarrow \mathcal{M}_{k+1}$ is a type D homomorphism, i.e.

$$\partial f_k = 0;$$

2) The morphism $f_{k+1} \circ f_k$ is homotopic to zero via homotopy φ_k , i.e.

$$f_{k+1} \circ f_k + \partial \varphi_k = 0;$$

3) The morphism $f_{k+2} \circ \varphi_k + \varphi_{k+1} \circ f_k$ is homotopic to the identity \mathbb{I}_k via homotopy κ_k , i.e.

$$f_{k+2} \circ \varphi_k + \varphi_{k+1} \circ f_k + \partial \kappa_k = \mathbb{I}_k.$$

A graphical representation of the conditions above is given in Figure 2. Then for each k , the type D module \mathcal{M}_k is homotopy equivalent to the mapping cone $\text{Cone}(f_{k+1})$.

Proof. This is a special case of [PW20, Lemma 2.1], where the algebra $B = \mathbb{F}_2$ from [PW20, Lemma 2.1] is the trivial algebra. □

3. The bordered–sutured Floer package of elementary tangle complements

In this section, we explicitly compute the differential graded algebra and type D modules that we will be working with in the rest of the paper. For the sake of economy, we do not provide the definitions of the algebras and modules in bordered–sutured Floer theory, but direct the reader to [Zar11].

3.1. The algebra associated to a 4-punctured sphere

Let F be the 4-punctured sphere, and let \mathcal{F} be the sutured surface (F, Λ) , where Λ consists of 2 distinct points on each component of ∂F . In other words, each boundary circle of F is divided into a positive and a negative arc.

In the context of tangles, \mathcal{F} will be the bordered part of the boundary of a tangle complement, along which another bordered–sutured manifold can be glued. Thus, our first task is to parametrize \mathcal{F} by an arc diagram $-\mathcal{Z}$. (The orientation reversal is appropriate for type D structures.)

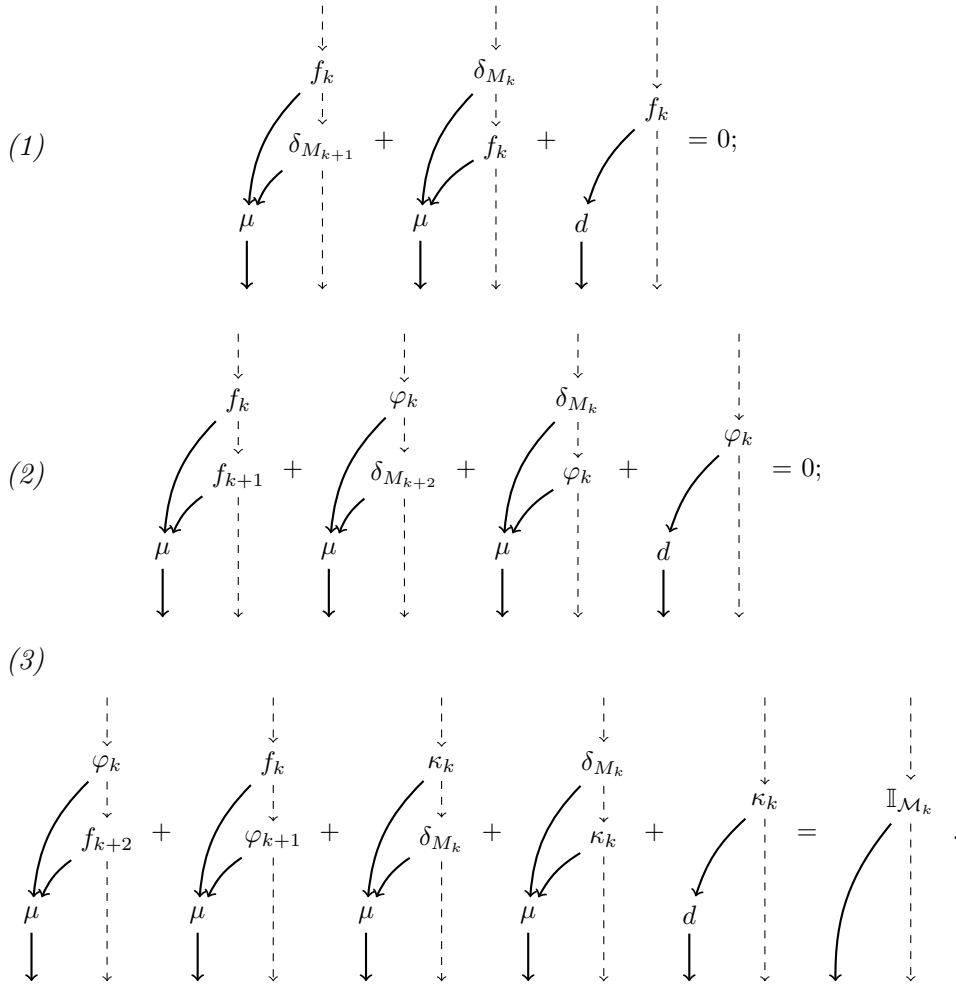


Figure 2. Graphical representations of the conditions in Lemma 2.1.

Let $\mathbf{Z} = \{Z_1, Z_2, Z_3, Z_4\}$ be a collection of oriented arcs, and let $\mathbf{a} = \{a_1, \dots, a_{12}\}$ be a collection of distinct points in \mathbf{Z} , such that

$$a_1 \in Z_1, \quad a_2, a_3, a_4, a_5 \in Z_2, \quad a_6, a_7, a_8, a_9 \in Z_3, \quad a_{10}, a_{11}, a_{12} \in Z_4,$$

in this order, if we traverse the arcs Z_i according to their orientations. Let $M: \mathbf{a} \rightarrow \{1, \dots, 6\}$ be the matching

$$\begin{aligned}
 M(a_1) = M(a_3) = 1, & \quad M(a_2) = M(a_5) = 2, & \quad M(a_4) = M(a_7) = 3, \\
 M(a_6) = M(a_9) = 4, & \quad M(a_8) = M(a_{11}) = 5, & \quad M(a_{10}) = M(a_{12}) = 6.
 \end{aligned}$$

If we define $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$, then $-\mathcal{Z}$ parametrizes $\mathcal{F}(-\mathcal{Z}) = \mathcal{F}$, as in Figure 3.

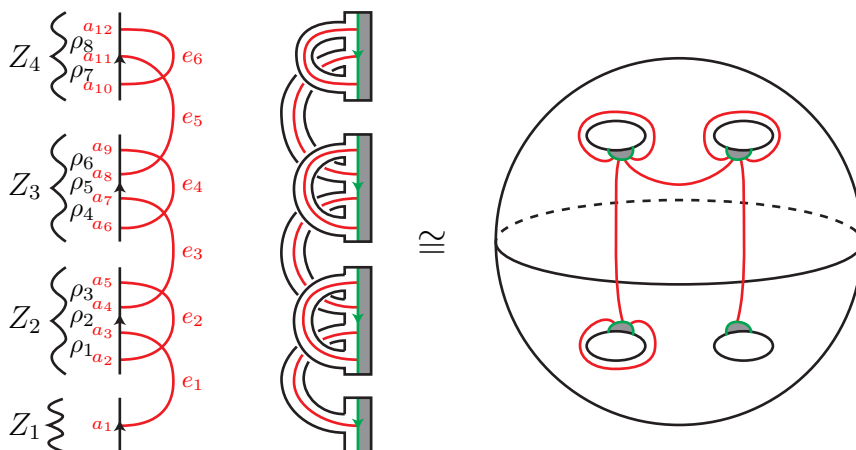


Figure 3. Left: The arc diagram \mathcal{Z} . Center and right: The sutured surface $\mathcal{F}(-\mathcal{Z}) = (F, \Lambda)$, where F is a 4-punctured sphere.

In bordered–sutured Floer theory, an algebra $\mathcal{A}(\mathcal{Z})$ is associated to the arc diagram \mathcal{Z} . Each generator of $\mathcal{A}(\mathcal{Z})$ corresponds to a *strands diagram*. Multiplication is defined by concatenation of strands diagrams with the convention that the product is zero if they cannot be concatenated. The differential is given by the sum of all possible ways to resolve the crossings in a given strands diagram. In both operations, there is an additional condition that the resulting strands diagram must not have double crossings between any two strands; any offending strands diagram is set to be zero.

The algebra $\mathcal{A}(\mathcal{Z})$ decomposes into a direct sum

$$\mathcal{A}(\mathcal{Z}) = \bigoplus_{i=0}^6 \mathcal{A}(\mathcal{Z}, i),$$

corresponding to the number of *occupied* arcs (for type A modules) or *unoccupied* arcs (for type D modules). We will be working with type D modules

defined by bordered–sutured diagrams with no α -circles and exactly one β -circle, and so we will always have five unoccupied arcs. Thus, for our purposes, it is sufficient to consider the 5-summand $\mathcal{A}(\mathcal{Z}, 5)$.

The algebra $\mathcal{A}(\mathcal{Z}, 5)$ has six idempotents $I_{12345}, I_{12346}, I_{12356}, I_{12456}, I_{13456}$, and I_{23456} ; we denote the set of these idempotents by $\mathcal{I}(\mathcal{Z}, 5)$. To simplify notation, we will instead denote these by $I_{\bar{6}}, I_{\bar{5}}, I_{\bar{4}}, I_{\bar{3}}, I_{\bar{2}}$, and $I_{\bar{1}}$ respectively, with the subscripts indicating the *occupied* arc rather than the *unoccupied* arcs. We will denote by I the sum $I_{\bar{1}} + \cdots + I_{\bar{6}}$.

In \mathcal{Z} , there are 15 Reeb chords: ρ_1 from a_2 to a_3 , ρ_2 from a_3 to a_4 , ρ_3 from a_4 to a_5 , and their concatenations ρ_{12}, ρ_{23} , and ρ_{123} ; ρ_4 from a_6 to a_7 , ρ_5 from a_7 to a_8 , ρ_6 from a_8 to a_9 , and their concatenations ρ_{45}, ρ_{56} , and ρ_{456} ; ρ_7 from a_{10} to a_{11} , ρ_8 from a_{11} to a_{12} , and their concatenation ρ_{78} . Therefore, we define the index set

$$J' = \{1, 2, 3, 12, 23, 123, 4, 5, 6, 45, 56, 456, 7, 8, 78\}.$$

In fact, we will soon be working with a 12-element subset $J \subset J'$, defined by

$$J = \left\{ j \in J' \mid M(\rho_j^-) \neq M(\rho_j^+) \right\} = \{1, 2, 3, 12, 23, 4, 5, 6, 45, 56, 7, 8\}.$$

Here, ρ^- and ρ^+ denote the starting and ending points of ρ respectively. Similarly, given a collection $\boldsymbol{\rho}$ of Reeb chords, we use the notation $\boldsymbol{\rho}^- = \{\rho_j^- \mid \rho_j \in \boldsymbol{\rho}\}$ and $\boldsymbol{\rho}^+ = \{\rho_j^+ \mid \rho_j \in \boldsymbol{\rho}\}$.

Recall from [Zar11, Definition 2.3.1] the definition of a p -completion of a collection $\boldsymbol{\rho}$ of Reeb chords. Let $|\boldsymbol{\rho}| = n \leq 5$. Since we are working with $\mathcal{A}(\mathcal{Z}, 5)$, we are interested in the 5-completions of $\boldsymbol{\rho}$; in our context, such a 5-completion is a choice of a $(5 - n)$ -element subset of $\{1, \dots, 6\}$, corresponding to a choice of $5 - n$ unoccupied arcs not labeled by $M(\boldsymbol{\rho}^-)$ or $M(\boldsymbol{\rho}^+)$. As explained in [Zar11], every 5-completion s of $\boldsymbol{\rho}$ defines an element $a(\boldsymbol{\rho}, s) \in \mathcal{A}(\mathcal{Z}, 5)$, and the *associated element* of $\boldsymbol{\rho}$ in $\mathcal{A}(\mathcal{Z}, 5)$ is

$$a_5(\boldsymbol{\rho}) = \sum_{s \text{ a 5-completion of } \boldsymbol{\rho}} a(\boldsymbol{\rho}, s).$$

The algebra $\mathcal{A}(\mathcal{Z}, 5)$ is then generated over $\mathcal{I}(\mathcal{Z}, 5)$ by the elements $a_5(\boldsymbol{\rho})$ for all 5-completable $\boldsymbol{\rho}$.

Since we will be working with the type D invariant, we would like to draw our attention to the special case where $\boldsymbol{\rho} = \{\rho_j\}$ for some fixed $j \in J'$.

To simplify our notation, we denote the associated element of $\{\rho_j\}$ by

$$(j) = a_5(\{\rho_j\}).$$

More generally, for $j_1, \dots, j_\ell \in J'$, we let

$$(j_1, \dots, j_\ell) = a_5(\{\rho_{j_1}\}) \cdots a_5(\{\rho_{j_\ell}\}),$$

the product of these associated elements. We will be using this notation throughout the rest of this paper.

Remark 3.1. It will be helpful for us to observe that, if $(j_1, \dots, j_\ell) \neq 0$, then each of 1, 3, 4, 6, 7, 8 can appear in (j_1, \dots, j_ℓ) at most once, not only as indices in J' , but in fact as actual digits. For example, we have $(45, 4) = 0$; here, even though the index $4 \in J'$ only appears once, the digit 4 appears twice. To verify this claim, suppose the digit 4 appears more than once. Writing the product (j_1, \dots, j_ℓ) as $I_s \cdot a_5(\rho) \cdot I_e$, where I_s and I_e are idempotents, we see that the coefficient of $[\rho_4]$ in the homology class $[\rho] \in H_1(\mathbf{Z}, \mathbf{a})$ is greater than 1. This means that $\rho_j^- = \rho_{j'}^- = \rho_4^-$ for at least two Reeb chords ρ_j and $\rho_{j'}$ in ρ , and hence that ρ is not 5-completable. The same analysis works for the digits 1 and 7, while a similar argument involving ρ_j^+ works for 3, 6, and 8.

An easy generalization of the argument shows that the digits 2 and 5 can appear in (j_1, \dots, j_ℓ) at most twice.

Suppose further that $\rho = \{\rho_j\}$ for some $j \in J$. For such ρ , observe that since $M(\rho_j^-) \neq M(\rho_j^+)$, there is a unique 4-element subset of $\{1, \dots, 6\}$ disjoint from $\{M(\rho_j^-), M(\rho_j^+)\}$; in other words, $\{\rho_j\}$ has a unique 5-completion s_j . Thus, we see that

$$(j) = a_5(\{\rho_j\}) = a(\{\rho_j\}, s_j).$$

Unless otherwise stated, from now on, we will work only with the index set J rather than J' ; see Remark 3.3 (and Remark 3.1) for some justification.

It is evident that our notation gives a compact way to represent multiplication of many, but not all, algebra elements. (As we shall see, this will be sufficient.) Note that, however, for a given element $a \in \mathcal{A}(\mathcal{Z}, 5)$, there may be more than one way to represent a in this notation. For example, Figure 4

illustrates the fact that

$$\begin{aligned} (12, 3, 2) &= a_5(\{\rho_{12}\})a_5(\{\rho_3\})a_5(\{\rho_2\}) \\ &= a_5(\{\rho_{123}, \rho_2\}) = a_5(\{\rho_2\})a_5(\{\rho_1\})a_5(\{\rho_{23}\}) = (2, 1, 23). \end{aligned}$$

A similar calculation shows that $(45, 6, 5) = (5, 4, 56)$.

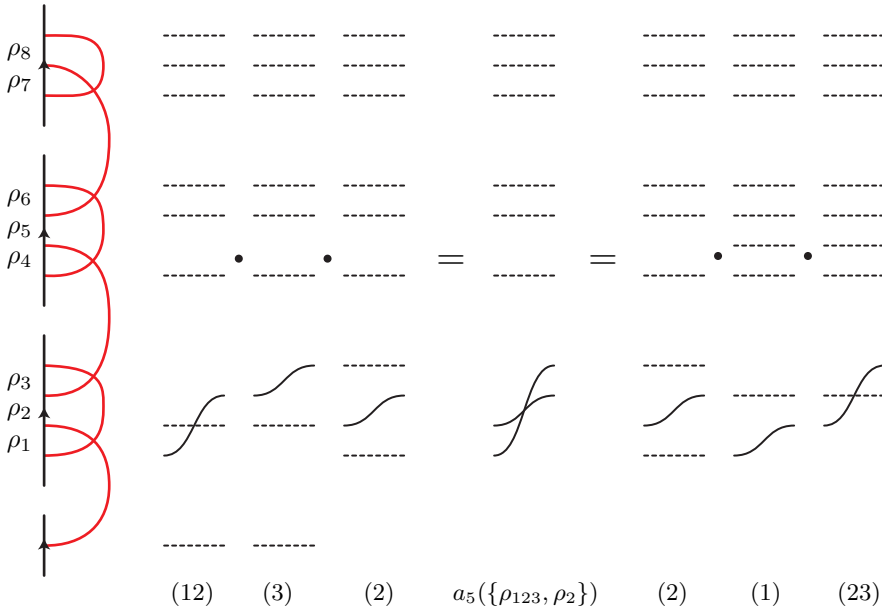


Figure 4. In our notation, $(12, 3, 2)$ and $(2, 1, 23)$ both represent the same algebra element, $a_5(\{\rho_{123}, \rho_2\})$.

Consider now an element $(j_1, \dots, j_\ell) \in \mathcal{A}(\mathcal{Z}, 5)$, with $j_1, \dots, j_\ell \in J$. For such an element, our notation gives a simple algorithm that computes its differential: It is the sum of all possible ways to insert a comma (such that the resulting notation makes sense) and exchange the indices adjacent to the new comma. For example, Figure 5 illustrates the fact that

$$d(12, 3, 4, 56, 2) = (2, 1, 3, 4, 56, 2) + (12, 3, 4, 6, 5, 2).$$

This observation follows from the Leibniz rule.

3.2. The modules associated to the three tangle complements

For $k \in \{\infty, 0, 1\}$, consider the tangle complement $B_k = B^3 \setminus \nu(T_k^{\text{el}})$. Let \mathcal{B}_k be the bordered–sutured manifold $(B_k, \Gamma_k^{\text{el}}, -\mathcal{Z}, \phi_k^{\text{el}})$, where Γ_k^{el} consists

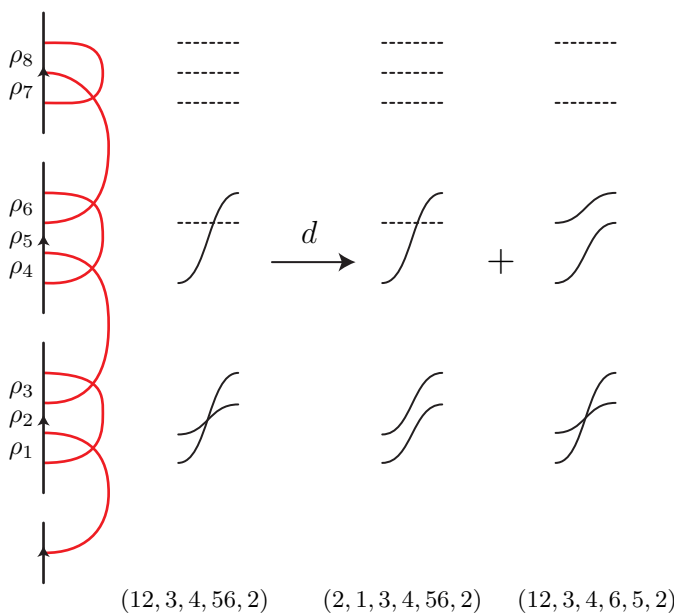


Figure 5. The differential of the algebra element $(12, 3, 4, 56, 2)$.

of two pairs of oppositely oriented meridional sutures, one for each arc of T_k^{el} , and $\phi_k^{\text{el}}: G(-\mathcal{Z}) \hookrightarrow \partial B_k$ gives the parametrization of the 4-punctured sphere $\partial B_k \cap \partial B^3$ shown in Figure 6. In other words, ϕ_k^{el} allows us to identify $\mathcal{F}(-\mathcal{Z})$ with the sutured surface $(\partial B_k \cap \partial B^3, \Gamma_k^{\text{el}} \cap \partial B^3 \cap \nu(T_k^{\text{el}}))$ that constitutes the bordered part of the boundary, and we will make this identification from now on. The “sutured” part of the boundary consists of the two inner cylinders, each of which is divided by Γ_k^{el} into one positive and two negative regions.

Below, we will compute the type D modules $\widehat{\text{BSD}}(\mathcal{B}_k)$. Recall that for a bordered–sutured manifold $\mathcal{Y} = (Y, \Gamma, \mathcal{Z}, \phi)$, $\widehat{\text{BSD}}(\mathcal{Y})$ is a type D module over $\mathcal{A}(-\mathcal{Z})$. Therefore, $\widehat{\text{BSD}}(\mathcal{B}_k)$ is a type D module over $\mathcal{A}(\mathcal{Z})$.

Bordered sutured diagrams $\mathcal{H}_k = (\Sigma, \alpha, \beta^k, -\mathcal{Z}, \psi)$ for \mathcal{B}_k are shown in Figure 7. Here, the three diagrams share the same surface with boundary Σ , collection $\alpha = \alpha^a$ of α -arcs, arc diagram \mathcal{Z} , and embedding $\psi: G(\mathcal{Z}) \hookrightarrow \Sigma$; the only difference is in the collections β^k , each of which consists of a single β -circle β^k . Note that \mathcal{H}_1 , \mathcal{H}_∞ , and \mathcal{H}_0 are provincially admissible but not admissible. (See [Zar11, Definition 4.4.1] for the definitions of admissibility.)

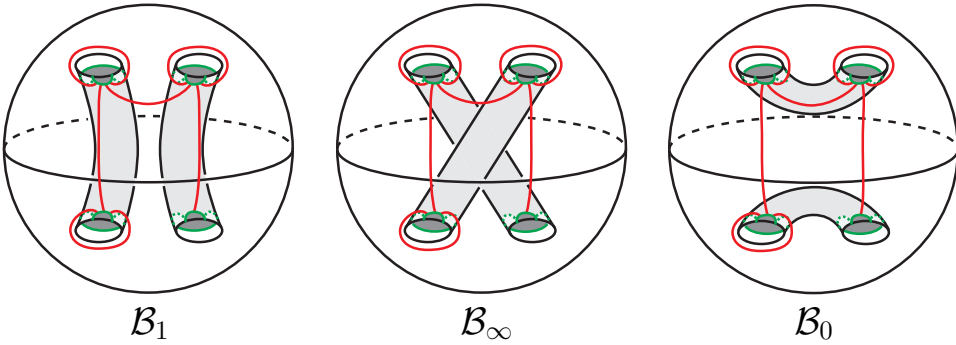


Figure 6. The bordered-sutured manifolds \mathcal{B}_k .

Since there is a single β -circle in each of these diagrams, there will always be five unoccupied arcs in \mathcal{Z} , and we may view $\widehat{\text{BSD}}(\mathcal{B}_k)$ as a type D module over $\mathcal{A} = \mathcal{A}(\mathcal{Z}, 5)$, confirming our claim in Subsection 3.1.

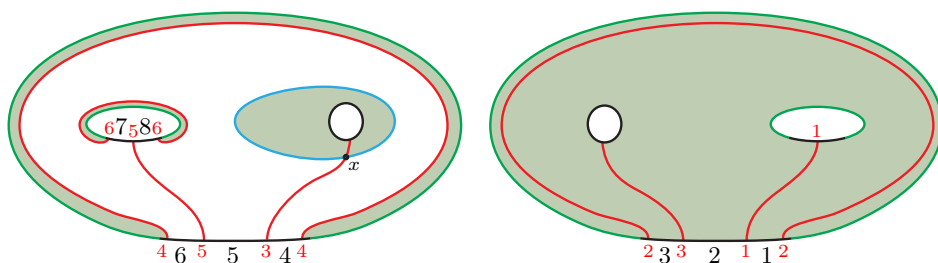
3.2.1. The module $\widehat{\text{BSD}}(\mathcal{B}_1)$. We begin by computing the type D module $\widehat{\text{BSD}}(\mathcal{B}_1)$. In the Heegaard diagram \mathcal{H}_1 , there is exactly one intersection point x between $\alpha = \alpha^a$ and $\beta^1 = \{\beta^1\}$, and so $\widehat{\text{BSD}}(\mathcal{B}_1)$ is generated by a single element \mathbf{x} , with idempotent

$$I_{\bar{3}} \cdot \mathbf{x} = \mathbf{x}.$$

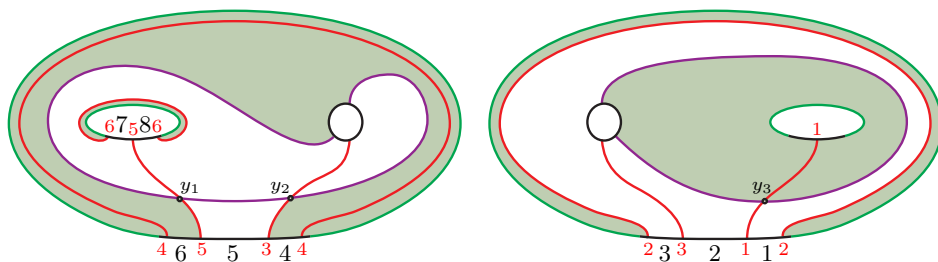
There is only one region R in \mathcal{H}_1 not adjacent to $\partial\Sigma \setminus \mathbf{Z}$, and so all domains are of the form nR . Note that $\partial^\partial R = [-\rho_4] + [-\rho_5] + [-\rho_6] + [-\rho_7] + [-\rho_8]$. If there exists a holomorphic curve u , with homology class $[u] = B \in \pi_2(\mathbf{x}, \mathbf{x})$, that contributes to $\delta(\mathbf{x})$, then $u \in \mathcal{M}_{\text{emb}}^B(\mathbf{x}, \mathbf{x}; S^\flat; \vec{P})$, where

- \vec{P} is a discrete ordered partition;
- The domain $[B]$ of B satisfies $\partial^\partial[B] = [\vec{\rho}(\vec{P})]$;
- $\text{ind}(B, \vec{\rho}(\vec{P})) = 1$; and
- $a(-\vec{\rho}(\vec{P})) \neq 0$.

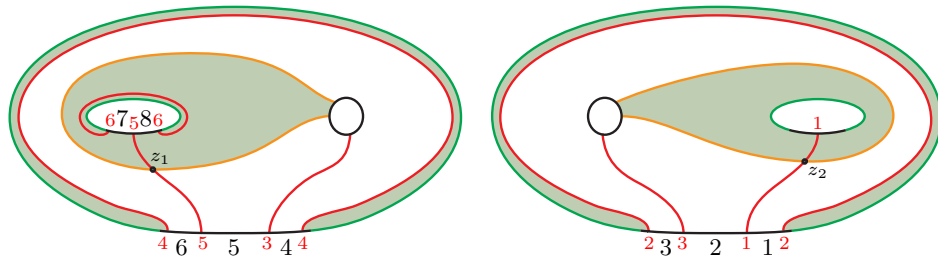
The second and the last of these conditions, together with Remark 3.1, imply that the coefficients of $[-\rho_4]$, $[-\rho_6]$, $[-\rho_7]$, and $[-\rho_8]$ in $\partial^\partial[B]$ cannot be greater than 1; therefore, the only domain that can possibly contribute to $\delta(\mathbf{x})$ is R , and we may consider only the homology class B with $[B] = R$. Now a simple calculation using [Zar11, Proposition 5.3.5] and the first and the third conditions above shows that the decorated source S^\flat must satisfy



(A) The bordered-sutured diagram \mathcal{H}_1 .



(B) The bordered-sutured diagram \mathcal{H}_∞ .



(C) The bordered-sutured diagram \mathcal{H}_0 .

Figure 7. The bordered-sutured diagrams \mathcal{H}_k for \mathcal{B}_k . Red curves represent $\alpha = \alpha^a$, green curves represent $\partial\Sigma \setminus \mathbf{Z}$, and the blue, purple, and gold curves represent β^1 , β^∞ , and β^0 respectively. A small red integer i indicates the arc $\alpha_i^a = \psi(e_i)$, and a larger black digit j indicates a Reeb chord ρ_j . (Only those Reeb chords where j consists of a single digit are shown.) The regions adjacent to $\partial\Sigma \setminus \mathbf{Z}$ are shaded in light green.

$\chi(S^\diamond) = 1$ and $\#\vec{\rho}(\vec{P})$ must be 5. Hence, S^\diamond must be a topological disk, with one (+)-puncture, one (-)-puncture, and five e -punctures labeled by $-\rho_4, -\rho_6, -\rho_7, -\rho_8,$ and $-\rho_5,$ in that order with the standard orientation. Here, one may deduce the order of the punctures by inspecting \mathcal{H}_1 ; in fact, this will also reveal the order of the elements of \vec{P} . With these choices of $B, S^\diamond,$ and $\vec{P},$ the Riemann mapping theorem implies that $\mathcal{M}_{\text{emb}}^B(\mathbf{x}, \mathbf{x}; S^\diamond; \vec{P})$ is indeed nonempty and in fact consists of exactly one holomorphic representative $u.$ This shows that

$$\delta(\mathbf{x}) = a_5(\rho_4)a_5(\rho_6)a_5(\rho_7)a_5(\rho_8)a_5(\rho_5) \otimes \mathbf{x} = (4, 6, 7, 8, 5) \otimes \mathbf{x}.$$

One may easily see from our notation that

$$d(4, 6, 7, 8, 5) = 0 = (4, 6, 7, 8, 5)^2,$$

where the second equality follows from Remark 3.1, with forbidden digits 4, 6, 7, 8 repeated; this confirms that $\widehat{\text{BSD}}(\mathcal{B}_1)$ does indeed satisfy the type D structure equation.

Remark 3.2. In the following discussion, we will generally not provide justification for the boundary operator computations, as each of them follows from a line of reasoning similar to the above: In general, Remark 3.1 provides an easy way to identify the domains that can possibly contribute, of which there are finitely many. The index formula [Zar11, Proposition 5.3.5] implies that S^\diamond must be a topological disk, and the Riemann mapping theorem then implies the existence of a unique holomorphic representative. See Remark 3.4 below for the only complication.

Remark 3.3. To aid in one’s computations, it is also worth noting that if a holomorphic curve with decorated source S^\diamond contributes to any map defined by counting holomorphic curves, none of the e -punctures of S^\diamond can be labeled by $-\rho_j$ with $j \in J' \setminus J = \{123, 456, 78\}.$ To see this, suppose q is an e -puncture labeled by, say, $-\rho_{123};$ then by examining $\mathcal{H}_k,$ the next puncture on ∂S^\diamond after $q,$ in the standard orientation, must be an e -puncture labeled by $-\rho_{j_0},$ where j_0 must contain the digit 3. This violates the restriction in Remark 3.1.

3.2.2. The module $\widehat{\text{BSD}}(\mathcal{B}_\infty).$ In $\mathcal{H}_\infty,$ there are three intersection points between α and $\beta^\infty,$ which we label by $y_1, y_2,$ and y_3 as in Figure 7.

The corresponding generators have idempotents

$$I_{\overline{5}} \cdot \mathbf{y}_1 = \mathbf{y}_1, \quad I_{\overline{3}} \cdot \mathbf{y}_2 = \mathbf{y}_2, \quad I_{\overline{1}} \cdot \mathbf{y}_3 = \mathbf{y}_3.$$

The boundary operation is given by

$$\begin{aligned} \delta(\mathbf{y}_1) &= (5) \otimes \mathbf{y}_2 + (7, 8, 5, 12, 3) \otimes \mathbf{y}_2 + (7, 8, 5, 2) \otimes \mathbf{y}_3, \\ \delta(\mathbf{y}_2) &= 0, \\ \delta(\mathbf{y}_3) &= (1, 3) \otimes \mathbf{y}_2. \end{aligned}$$

3.2.3. The module $\widehat{\text{BSD}}(\mathcal{B}_0)$. In \mathcal{H}_0 , there are two intersection points between α and β^0 , which we label by z_1 and z_2 as in Figure 7. The corresponding generators have idempotents

$$I_{\overline{5}} \cdot \mathbf{z}_1 = \mathbf{z}_1, \quad I_{\overline{1}} \cdot \mathbf{z}_2 = \mathbf{z}_2.$$

The boundary operation is given by

$$\begin{aligned} \delta(\mathbf{z}_1) &= (5, 12, 3, 4, 6) \otimes \mathbf{z}_1 + (5, 2) \otimes \mathbf{z}_2 + (5, 12, 3, 4, 56, 2) \otimes \mathbf{z}_2, \\ \delta(\mathbf{z}_2) &= (1, 3, 4, 6) \otimes \mathbf{z}_1 + (1, 3, 4, 56, 2) \otimes \mathbf{z}_2. \end{aligned}$$

Remark 3.4. We illustrate one consideration we must take in the calculation above, beyond the argument outlined in Remark 3.2. Let $B \in \pi_2(\mathbf{z}_2, \mathbf{z}_1)$ be the homology class from \mathbf{z}_1 to \mathbf{z}_2 that gives rise to the term $(5, 12, 3, 4, 56, 2) \otimes \mathbf{z}_2$ above; then $[B] = R_1 + 2R_2$, where R_2 is the unique region adjacent to $-\rho_2$ and $-\rho_5$, and R_1 is the only other non-forbidden region. Since $g = 1$ and $e(B) = -5/2$, applying [Zar11, Proposition 5.3.5] to B shows that we must have $\chi(S^\diamond) = \#\vec{P} - 5$. Because \vec{P} is discrete, inspecting \mathcal{H}_0 , we see that $\#\vec{P} \geq 4$. For all other domains involved in the calculations in this subsection, the analogous restriction on $\#\vec{P}$ and the fact that $\chi(S^\diamond) \leq 1$ (as S^\diamond is a connected surface with at least one boundary component) together imply that $\chi(S^\diamond) = 1$; here, however, we obtain two possibilities: $\chi(S^\diamond) = 1$ and $\#\vec{P} = 6$; and $\chi(S^\diamond) = -1$ and $\#\vec{P} = 4$. We must argue that we may discard the latter possibility, as follows. We note that $\#\vec{P} = 4$ implies that $\vec{\rho}(\vec{P})$ contains, in some order, either $-\rho_{123}$ and $-\rho_2$, or $-\rho_{12}$ and $-\rho_{23}$; and two other Reeb chords with index digits 4, 5, and 6. We know that $-\rho_{123} \notin \vec{\rho}(\vec{P})$ by Remark 3.3, and so $\vec{\rho}(\vec{P})$ must contain $-\rho_{12}$ and $-\rho_{23}$; but $(12, 23) = (23, 12) = 0$. (Alternatively, we may apply Remark 3.3 to $-\rho_{456}$ and observe that $(45, 56) = (56, 45) = 0$.) We may then proceed with the usual argument for the former possibility, where $\chi(S^\diamond) = 1$ and $\#\vec{P} = 6$, which gives the aforementioned term in $\delta(\mathbf{z}_1)$.

In Section 5 and in particular the proof of Proposition 5.5, this complication will again arise whenever the domain $[B]$ has multiplicity greater than 1 at some region.

3.3. Proof of the main theorem without gradings

The key step to proving Theorem 1.1 will be the following special case:

Proposition 3.5. *There exist type D homomorphisms $f_k: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1})$ such that*

$$\widehat{\text{BSD}}(\mathcal{B}_k) \simeq \text{Cone}(f_{k+1}: \widehat{\text{BSD}}(\mathcal{B}_{k+1}) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+2}))$$

as type D structures.

We will give two proofs of Proposition 3.5, using counts of bordered holomorphic polygons in Section 4 and by direct computation in Section 5. Using Proposition 3.5, we may prove the following generalization of Theorem 1.1:

Theorem 3.6. *Let $\mathcal{Y}' = (Y', \Gamma', \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}, \phi')$ be any bordered–sutured manifold, and let $\mathcal{Y}_k = \mathcal{Y}' \cup_{\mathcal{F}(\mathcal{Z})} \mathcal{B}_k$. There exist type DA homomorphisms $F_k: \widehat{\text{BSDA}}(\mathcal{Y}_k) \rightarrow \widehat{\text{BSDA}}(\mathcal{Y}_{k+1})$ such that*

$$\widehat{\text{BSDA}}(\mathcal{Y}_k) \simeq \text{Cone}(F_{k+1}: \widehat{\text{BSDA}}(\mathcal{Y}_{k+1}) \rightarrow \widehat{\text{BSDA}}(\mathcal{Y}_{k+2}))$$

as type DA structures. Moreover, the homomorphisms F_k and the homotopy equivalence above respect the relative gradings on the bimodules in a sense to be made precise in Section 7; see Theorem 7.12.

Proof of Theorem 3.6, without gradings. To the bordered–sutured manifold \mathcal{Y}' , Zarev [Zar11] associates (the homotopy type of) a type DA bimodule

$${}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{\text{BSDA}}(\mathcal{Y}')_{\mathcal{A}(\mathcal{Z}_2 \cup \mathcal{Z})} = {}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{\text{BSDA}}(\mathcal{Y}')_{\mathcal{A}(\mathcal{Z}_2) \otimes \mathcal{A}(\mathcal{Z})}$$

By choosing an admissible diagram \mathcal{H}' for \mathcal{Y}' , we may assume $\widehat{\text{BSDA}}(\mathcal{Y}')$ to be bounded.

Unlike in the familiar setting for dg modules, the dg category $\text{Mod}_{\mathcal{A}_1 \otimes \mathcal{A}_2}$ of (right) type A modules over $\mathcal{A}_1 \otimes \mathcal{A}_2$ is not the same as the dg category $\text{Mod}_{\mathcal{A}_1, \mathcal{A}_2}$ of (right-right) type AA bimodules over \mathcal{A}_1 and \mathcal{A}_2 . There is,

nonetheless, an equivalence

$$\mathcal{F}_{AA} : \mathbf{H}(\text{Mod}_{\mathcal{A}_1 \otimes \mathcal{A}_2}) \rightarrow \mathbf{H}(\text{Mod}_{\mathcal{A}_1, \mathcal{A}_2})$$

between the homotopy categories; see [LOT15, Proposition 2.4.11] and [Zar11, Section 8.3]. In our context, there is an analogous functor \mathcal{F}_{DAA} that allows us to view $\widehat{\text{BSDA}}(\mathcal{Y}')$ as a type DAA trimodule

$$\mathcal{A}^{(-\mathcal{Z}_1)} \widehat{\text{BSDAA}}(\mathcal{Y}')_{\mathcal{A}(\mathcal{Z}_2), \mathcal{A}(\mathcal{Z})} = \mathcal{F}_{DAA} \left(\mathcal{A}^{(-\mathcal{Z}_1)} \widehat{\text{BSDA}}(\mathcal{Y}')_{\mathcal{A}(\mathcal{Z}_2) \otimes \mathcal{A}(\mathcal{Z})} \right),$$

allowing us to write

$$\begin{aligned} \widehat{\text{BSDA}}(\mathcal{Y}_k) &\simeq \widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_k) \\ &\simeq \widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \text{Cone}(f_{k+1} : \widehat{\text{BSD}}(\mathcal{B}_{k+1}) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+2})) \\ &\simeq \text{Cone}(\mathbb{I}_{\widehat{\text{BSDAA}}(\mathcal{Y}')} \boxtimes f_{k+1} : \widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_{k+1}) \\ &\quad \rightarrow \widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_{k+2})) \\ &\simeq \text{Cone}(F_{k+1} : \widehat{\text{BSDA}}(\mathcal{Y}_{k+1}) \rightarrow \widehat{\text{BSDA}}(\mathcal{Y}_{k+2})) \end{aligned}$$

for some type DA homomorphism F_{k+1} , where we have respectively used the pairing theorem [Zar11, Theorem 8.5.1], Proposition 3.5, [LOT14, Lemma 2.9], and the pairing theorem again. \square

Proof of Theorem 1.1, without gradings. Define $Y' = Y_k \setminus B^3$, which is a compact 3-manifold with boundary. Considering Γ_k , note that by the definition of a bordered–sutured structure, there are sutures on each boundary component of Y_k ; thus, we may isotope each Γ_k such that it coincides with Γ_k^{el} in the interior of B^3 . Let $\Gamma' = (\Gamma_k \cap Y') \cup (-\Gamma_k^{\text{el}} \cap \partial \overline{B^3})$, and let $\phi' : G(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}) \hookrightarrow \partial Y'$ be the map obtained by setting $\phi'|_{G(\mathcal{Z}_1 \cup \mathcal{Z}_2)} = \phi_k$ and defining $\phi'|_{G(\mathcal{Z})}$ to be ϕ_k^{el} with the opposite orientation. (Recall that, with the codomain restricted to the image, neither ϕ_k nor ϕ_k^{el} depends on k .) Then $\mathcal{Y}' = (Y', \Gamma', \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}, \phi')$ is a bordered–sutured manifold, such that $\mathcal{Y}_k = \mathcal{Y}' \cup_{\mathcal{F}(\mathcal{Z})} \mathcal{B}_k$. The theorem now follows from Theorem 3.6. \square

Proof of Corollary 1.2, without gradings. For Y without boundary and $\mathcal{Z}_1 = \mathcal{Z}_2 = \emptyset$, the bordered–sutured manifold $\mathcal{Y}_k = (Y_k, \Gamma_k, \emptyset, \phi_k)$ is simply a sutured manifold (Y_k, Γ_k) (where $Y_k = Y \setminus \nu(L_k)$), and the type DA structure $\widehat{\text{BSDA}}(\mathcal{Y}_k)$ reduces to the sutured Floer chain group $\text{SFC}(Y_k, \Gamma_k)$; see, for example, [Zar11, Section 9.1]. Note that this is true for all $\{\Gamma_k\}_{k \in \{\infty, 0, 1\}}$ satisfying the conditions described in Section 1; below, we choose one such set to prove our corollary.

Observe that $T_k^{\text{el}} \subset L_k$ intersects exactly one component of L_k for two values k_1, k_2 of $k \in \{\infty, 0, 1\}$, and intersects two components of L_k for the other value k_3 ; we shall refer to these link components as the *special link components*. Noting that each component of ∂Y_k corresponds to a link component of L_k , choose $\{\Gamma_k\}$ such that

- For each point p in ∂T_k^{el} , there is a distinct meridional suture in Γ_k that is a push-off of the meridian of p into the interior of $\partial Y_k \cap Y'$;
- There are no other sutures on the components of ∂Y_k that correspond to the special link components; and
- There is exactly one pair of meridional sutures on each component of ∂Y_k that does not correspond to one of the special link components.

With this choice, (Y_k, Γ_k) is exactly the link complement with one pair of meridional sutures for each link component, and an additional pair of meridional sutures for the special link components when $k \in \{k_1, k_2\}$. By [Juh06, Proposition 9.2] and an easy extension of [MOS09, Proposition 2.3] to links in arbitrary 3-manifolds, we see that

$$\text{SFH}(Y_k, \Gamma_k) \cong \begin{cases} \widehat{\text{HFK}}(Y, L_k) \otimes V & \text{if } k = k_1, k_2, \\ \widehat{\text{HFK}}(Y, L_k) & \text{if } k = k_3. \end{cases}$$

Applying Theorem 1.1 to $\text{SFC}(Y_k, \Gamma_k)$, the mapping cone of chain complexes naturally gives rise to a short exact sequence, and hence a long exact triangle on homology. □

4. The skein relation via holomorphic polygon counts

In this section, we give a theoretical proof of Proposition 3.5 using a technique of counting holomorphic polygons in the bordered setting, developed by Lipshitz, Ozsváth, and Thurston [LOT16]. We shall use these polygon counts to define type D morphisms, which satisfy a type D version of some well-known \mathcal{A}_∞ -relations. That the morphisms satisfy the conditions in Lemma 2.1 will then be an easy consequence.

4.1. Polygon counting in bordered–sutured Floer theory

Lipshitz, Ozsváth, and Thurston [LOT16, Section 4] describe a theory of holomorphic polygon counting for bordered Floer homology. The same technique translates to the bordered–sutured Floer setting with small changes. Below, we outline the important definitions and results in this direction.

Definition 4.1 (cf. [LOT16, Definition 4.1]). Let Σ be a compact, oriented surface with boundary. Fix the data (\mathcal{Z}, ψ) , where \mathcal{Z} is an arc diagram and $\psi: G(\mathcal{Z}) \rightarrow \Sigma$ is an orientation-reversing embedding as in [Zar11, Definition 4.1]. (See [Zar11, Definition 2.4] for the definition of $G(\mathcal{Z})$, and of e_i below.) A *set of bordered attaching curves compatible with \mathcal{Z}* is a collection $\alpha = \alpha^a \cup \alpha^c$ of curves in Σ such that

- The collection $\alpha^a = \{\alpha_1^a, \dots, \alpha_q^a\}$ consists exactly of the arcs $\alpha_i^a = \psi(e_i)$;
- The collection $\alpha^c = \{\alpha_1^c, \dots, \alpha_r^c\}$ consists of simple closed curves in $\text{Int}(\Sigma)$;
- The curves in α are pairwise disjoint; and
- The map $\pi_0(\partial\Sigma \setminus \mathbf{Z}) \rightarrow \pi_0(\Sigma \setminus \alpha)$ is surjective.

Definition 4.2 (cf. [LOT16, Definition 3.1]). Let Σ be a compact, oriented surface with boundary. A *set of attaching circles* is a collection $\beta = \{\beta_1, \dots, \beta_t\}$ of simple closed curves in $\text{Int}(\Sigma)$ such that

- The curves in β are pairwise disjoint; and
- The map $\pi_0(\partial\Sigma) \rightarrow \pi_0(\Sigma \setminus \beta)$ is surjective.

Definition 4.3 (cf. [LOT16, Definition 4.2]). A *bordered–sutured Heegaard multi-diagram* is the data $(\Sigma, \alpha, \{\beta^k\}_{k=1}^m, \mathcal{Z}, \psi)$, where α is a set of bordered attaching curves compatible with \mathcal{Z} in Σ , and $\{\beta^k\}_{k=1}^m$ is some m -tuple of sets of t attaching circles, for some fixed t . Without loss of generality, we may assume that all curves involved are pairwise transverse, and that there are no triple intersection points. For the sake of economy, we often omit the word “Heegaard”.

A *generalized multi-periodic domain* is a relative homology class $B \in H_2(\Sigma, \alpha \cup \beta^1 \cup \dots \cup \beta^m \cup \partial\Sigma)$ whose boundary ∂B , viewed as an element of

$$H_1(\alpha \cup \beta^1 \cup \dots \cup \beta^m \cup \partial\Sigma, \partial\Sigma) \cong H_1(\alpha \cup \beta^1 \cup \dots \cup \beta^m, \mathbf{a}),$$

is contained in the image of the inclusion

$$H_1(\boldsymbol{\alpha}, \mathbf{a}) \oplus H_1(\boldsymbol{\beta}^1) \oplus \cdots \oplus H_1(\boldsymbol{\beta}^m) \rightarrow H_1(\boldsymbol{\alpha} \cup \boldsymbol{\beta}^1 \cup \cdots \cup \boldsymbol{\beta}^m, \mathbf{a}).$$

It is a *multi-periodic domain* if all of its local multiplicities at the regions adjacent to $\partial\Sigma \setminus \mathbf{Z}$ are zero. A multi-periodic domain is *provincial* if all of its local multiplicities at the regions adjacent to $\partial\Sigma$ are zero. The bordered–sutured multi-diagram $(\Sigma, \boldsymbol{\alpha}, \{\boldsymbol{\beta}^k\}_{k=1}^m, \mathcal{Z}, \psi)$ is *admissible* (resp. *provincially admissible*) if every non-zero multi-periodic domain (resp. non-zero provincial multi-periodic domain) has both positive and negative local multiplicities.

Given a bordered–sutured multi-diagram, one could always find an *admissible collection of almost-complex structures* $\{J_j\}_{j \in \text{Conf}(D_n)}$; see [LOT16, Definition 4.4]. ($\text{Conf}(D_n)$ denotes the moduli space of complex structures on the disk D_n with n labeled punctures on its boundary, i.e. an n -sided polygon.) We fix this data.

If $(\Sigma, \boldsymbol{\alpha}, \{\boldsymbol{\beta}^k\}_{k=1}^m, \mathcal{Z}, \psi)$ is a provincially admissible bordered–sutured multi-diagram, then for all k , the bordered–sutured diagram $\mathcal{H}_k = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^k, \mathcal{Z}, \psi)$ is provincially admissible in the sense of [Zar11, Definition 4.10]. Thus, we may consider the type D structure $\widehat{\text{BSD}}(\mathcal{H}_k)$; we denote this type D structure by $\widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^k)$. (For the expert, the implicit choice of a suitable almost-complex structure is specified by the data $\{J_j\}$ here.) Observe further that $\mathcal{H}_{k_1, k_2} = (\Sigma, \boldsymbol{\beta}^{k_1}, \boldsymbol{\beta}^{k_2})$ in fact is a sutured Heegaard diagram that is admissible in the sense of [Juh06, Definition 3.11]; we denote by $\text{SFC}(\boldsymbol{\beta}^{k_1}, \boldsymbol{\beta}^{k_2})$ its sutured Floer chain complex $\text{SFC}(\mathcal{H}_{k_1, k_2})$.

The discussion in [LOT16, Section 4] then carries over to the present context, allowing us to define maps by polygon counts. More precisely, if $(\Sigma, \boldsymbol{\alpha}, \{\boldsymbol{\beta}^k\}_{k=1}^m, -\mathcal{Z}, \psi)$ is a provincially admissible bordered–sutured multi-diagram, we obtain maps

$$\begin{aligned} \delta_n &: \text{SFC}(\boldsymbol{\beta}^{k_{n-1}}, \boldsymbol{\beta}^{k_n}) \otimes \cdots \otimes \text{SFC}(\boldsymbol{\beta}^{k_1}, \boldsymbol{\beta}^{k_2}) \otimes \widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_1}) \\ &\rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_n}), \end{aligned}$$

defined, for generators $\boldsymbol{\eta}^{k_i} \in \mathcal{G}(\boldsymbol{\beta}^{k_i}, \boldsymbol{\beta}^{k_{i+1}})$ and $\mathbf{x} \in \mathcal{G}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_1})$, by

$$\begin{aligned} &\delta_n(\boldsymbol{\eta}^{k_{n-1}} \otimes \cdots \otimes \boldsymbol{\eta}^{k_1} \otimes \mathbf{x}) \\ &= \sum_{\substack{\mathbf{y} \in \mathcal{G}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_n}) \\ S^\triangleright, \text{ discrete } \vec{P} \\ \text{ind}(B, \vec{\rho}(\vec{P}))=2-n}} \left(\# \mathcal{M}_{\text{emb}}^B(\mathbf{y}, \boldsymbol{\eta}^{k_{n-1}}, \dots, \boldsymbol{\eta}^{k_1}, \mathbf{x}; S^\triangleright; \vec{P}) \right) a(-\vec{\rho}(\vec{P})) \otimes \mathbf{y}. \end{aligned}$$

Here, $\mathcal{M}_{\text{emb}}^B(\mathbf{y}, \boldsymbol{\eta}^{k_{n-1}}, \dots, \boldsymbol{\eta}^{k_1}, \mathbf{x}; S^\triangleright; \vec{P})$ denotes the moduli space of pairs (j, u) , where j is a complex structure on D_{n+1} , and u is an embedded J_j -holomorphic representative of the homology class $B \in \pi_2(\mathbf{y}, \boldsymbol{\eta}^{k_{n-1}}, \dots, \boldsymbol{\eta}^{k_1}, \mathbf{x})$, with decorated source S^\triangleright , and compatible with a discrete ordered partition \vec{P} of the e -punctures of S^\triangleright . Note that δ_1 is the usual structure map on $\widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_1})$.

Remark 4.4. We have chosen to follow [LOT16] in presenting the arguments of δ_n in the order above. As a result, the generator arguments of the moduli spaces \mathcal{M} are presented in the order reverse to that in [Zar11]. See [LOT16, Convention 3.4 and Remark 4.22] for some justification.

Recall that in the non-bordered case, there are well-known maps

$$\begin{aligned} m_n &: \text{SFC}(\boldsymbol{\beta}^{k_{n-1}}, \boldsymbol{\beta}^{k_n}) \otimes \dots \otimes \text{SFC}(\boldsymbol{\beta}^{k_1}, \boldsymbol{\beta}^{k_2}) \otimes \text{SFC}(\boldsymbol{\beta}^{k_0}, \boldsymbol{\beta}^{k_1}) \\ &\rightarrow \text{SFC}(\boldsymbol{\beta}^{k_0}, \boldsymbol{\beta}^{k_n}), \end{aligned}$$

defined by holomorphic polygon counts, which are well known to satisfy some \mathcal{A}_∞ -relations; see, for example, [Sei08] and [FOOO09a, FOOO09b]. One may view the map m_n as a special case of δ_n , where $\boldsymbol{\alpha} = \boldsymbol{\beta}^{k_0}$ is a set of bordered attaching curves compatible with the empty arc diagram $\mathcal{Z} = \emptyset$, and $\mathcal{A}(\mathcal{Z}) = \mathcal{A}(\emptyset) = \mathbb{F}_2$. Conversely, δ_n can be viewed as the bordered version of m_n . Note also that m_1 is the usual differential on $\text{SFC}(\boldsymbol{\beta}^{k_0}, \boldsymbol{\beta}^{k_1})$.

The map δ_n can be naturally extended to a map

$$\begin{aligned} \tilde{\delta}_n &: \text{SFC}(\boldsymbol{\beta}^{k_{n-1}}, \boldsymbol{\beta}^{k_n}) \otimes \dots \otimes \text{SFC}(\boldsymbol{\beta}^{k_1}, \boldsymbol{\beta}^{k_2}) \otimes \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_1}) \\ &\rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_n}) \end{aligned}$$

defined by

$$\begin{aligned} \tilde{\delta}_n &(\boldsymbol{\eta}^{k_{n-1}} \otimes \dots \otimes \boldsymbol{\eta}^{k_1} \otimes a \otimes \mathbf{x}) \\ &= \left(\mu(a, -) \otimes \text{id}_{\widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_n})} \right) \circ \delta_n(\boldsymbol{\eta}^{k_{n-1}} \otimes \dots \otimes \boldsymbol{\eta}^{k_1} \otimes \mathbf{x}). \end{aligned}$$

Then the result we need, by adapting from [LOT16, Section 4], is the following proposition, which is the bordered version of the \mathcal{A}_∞ -relations for m_n .

Proposition 4.5 (cf. [LOT16, Proposition 4.23]). *The holomorphic polygon counts defined above satisfy the \mathcal{A}_∞ -relations:*

$$\begin{aligned}
 0 &= \sum_{1 \leq p \leq n} \tilde{\delta}_{n-p+1}(\boldsymbol{\eta}^{k_{n-1}} \otimes \cdots \otimes \boldsymbol{\eta}^{k_p} \otimes \delta_p(\boldsymbol{\eta}^{k_{p-1}} \otimes \cdots \otimes \boldsymbol{\eta}^{k_1} \otimes \mathbf{x})) \\
 &+ \sum_{1 \leq p < q \leq n} \delta_{n-q+p+1}(\boldsymbol{\eta}^{k_{n-1}} \otimes \cdots \otimes \boldsymbol{\eta}^{k_q} \otimes m_{q-p}(\boldsymbol{\eta}^{k_{q-1}} \otimes \cdots \otimes \boldsymbol{\eta}^{k_p}) \\
 &\quad \otimes \boldsymbol{\eta}^{k_{p-1}} \otimes \cdots \otimes \boldsymbol{\eta}^{k_1} \otimes \mathbf{x}) \\
 &+ \left(d \otimes \text{id}_{\widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_n})} \right) \circ \delta_n(\boldsymbol{\eta}^{k_{n-1}} \otimes \cdots \otimes \boldsymbol{\eta}^{k_1} \otimes \mathbf{x}).
 \end{aligned}$$

The maps δ_n have analogous versions for type A modules and for bi-modules of various types. For example, suppose $(\Sigma, \boldsymbol{\alpha}, \{\boldsymbol{\beta}^k\}_{k=1}^m, \mathcal{Z}_1 \cup \mathcal{Z}_2, \psi)$ is a provincially admissible bordered–sutured multi-diagram, and let $\mathcal{H}_k = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^k, \mathcal{Z}_1 \cup \mathcal{Z}_2, \psi)$; denoting by $\widehat{\text{BSDA}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^k)$ the type DA structure

$$\mathcal{A}(-\mathcal{Z}_1) \widehat{\text{BSDA}}(\mathcal{H}_k)_{\mathcal{A}(\mathcal{Z}_2)},$$

there are maps

$$\begin{aligned}
 {}_n \delta_\ell &: \text{SFC}(\boldsymbol{\beta}^{k_{n-1}}, \boldsymbol{\beta}^{k_n}) \otimes \cdots \otimes \text{SFC}(\boldsymbol{\beta}^{k_1}, \boldsymbol{\beta}^{k_2}) \otimes \widehat{\text{BSDA}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_1}) \\
 &\quad \otimes \underbrace{\mathcal{A}(\mathcal{Z}_2) \otimes \cdots \otimes \mathcal{A}(\mathcal{Z}_2)}_\ell \\
 &\rightarrow \mathcal{A}(-\mathcal{Z}_1) \otimes \widehat{\text{BSDA}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k_n}),
 \end{aligned}$$

defined by holomorphic polygon counts, which satisfy \mathcal{A}_∞ -relations similar to those in Proposition 4.5. We will not need these generalized maps in this section, but in the proof of Theorem 1.3 in Section 6, we will use the analogous maps for type DAA trimodules, which are similarly defined.

4.2. Counting provincial polygons

Before we define the type D morphisms needed in Lemma 2.1, we prepare ourselves by counting some provincial polygons and computing the associated maps m_n .

Consider the bordered–sutured diagrams $\mathcal{H}_k = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^k, -\mathcal{Z}, \psi)$ in Subsection 3.2, where $k \in \{\infty, 0, 1\}$. We shall view $\boldsymbol{\beta}^1$, $\boldsymbol{\beta}^\infty$, and $\boldsymbol{\beta}^0$ as sets of attaching circles on Σ , and combine them into one sutured multi-diagram as in Figure 8. Without loss of generality, we may assume, as in Figure 8, that the circles β^k intersect transversely, pairwise, at two points; we label

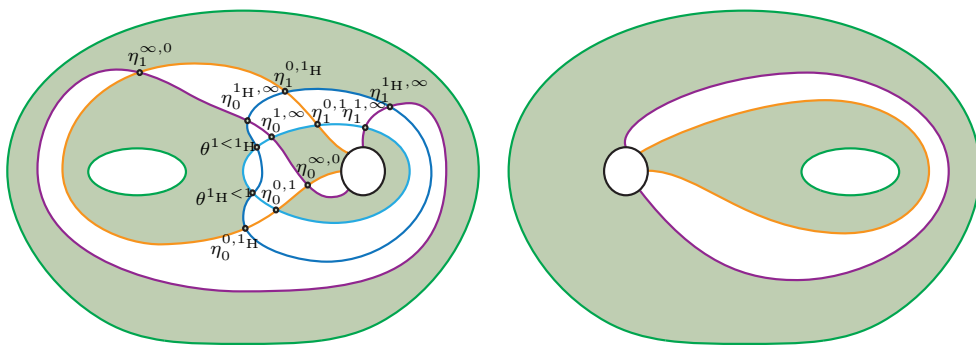


Figure 8. The sutured multi-diagram $\mathcal{H}_\beta^H = (\Sigma, \{\beta^k, \beta^{k_H}\}_{k \in \{\infty, 0, 1\}})$.

the two intersection points between β^k and β^{k+1} by $\eta_0^{k,k+1}$ and $\eta_1^{k,k+1}$. (The choice of which intersection point is $\eta_0^{k,k+1}$ and which is $\eta_1^{k,k+1}$, as indicated in Figure 8, is made to simplify the notation in Lemma 4.6 below, but is itself not material in the proof of Proposition 3.5.) Each $\eta_i^{k,k+1}$ is associated to *two* generators, one in $\mathcal{G}(\beta^k, \beta^{k+1})$ and one in $\mathcal{G}(\beta^{k+1}, \beta^k)$; we denote the former by $\eta_i^{k < k+1}$ and the latter by $\eta_i^{k+1 < k}$.

For each $k \in \{\infty, 0, 1\}$, let β^{k_H} be a small Hamiltonian perturbation of β^k , such that

- β^k and β^{k_H} intersect transversely at two points $\theta^{k < k_H}$ and $\theta^{k_H < k}$, both of which lie on the boundary ∂p of the embedded triangle p with vertices $\eta_0^{1,\infty}$, $\eta_0^{\infty,0}$, and $\eta_0^{0,1}$; and
- β^{k_H} intersects β^{k+1} (resp. β^{k+2}) transversely close to $\beta^k \cap \beta^{k+1}$ (resp. $\beta^k \cap \beta^{k+2}$).

Let $\beta^{k_H} = \{\beta^{k_H}\}$. In Figure 8, we have drawn β^{1_H} ; to prevent cluttering our diagrams, however, we will not draw β^{∞_H} or β^{0_H} here or in the sequel. As a consequence, whenever an argument below involves β^{k_H} (and its intersection points with other curves), the reader is invited to check the veracity of the argument for the case $k = 1$, and to observe that the cases $k = \infty$ and $k = 0$ are entirely analogous.

We now fix the notation for the intersection points on β^{k_H} . First, we label the intersection points between β^k and $\beta^{(k+1)_H}$ such that $\eta_i^{k,(k+1)_H}$ is close to $\eta_i^{k,k+1}$. Second, note that $(\Sigma, \beta^k, \beta^{k_H})$ is a sutured Heegaard diagram for the complement of a 2-component unlink in S^3 , with a pair of meridional

sutures on each unlink component. This means that

$$\text{SFC}(\beta^k, \beta^{k_H}) \cong \widehat{\text{CFK}}(S^3, \text{2-component unlink}),$$

and so is generated by two generators whose (relative) gradings differ by 1. Denote by $\theta^{k < k_H}$ the intersection point between β^k and β^{k_H} that corresponds to the “top-graded generator” $\theta^{k < k_H} \in \mathcal{G}(\beta^k, \beta^{k_H})$. The other intersection point, which we denote by $\theta^{k_H < k}$, will then correspond to the “top-graded generator” $\theta^{k_H < k} \in \mathcal{G}(\beta^{k_H}, \beta^k)$. (Since SFC is only relatively graded, what we really mean here is that we label the intersection points such that the moduli space $\mathcal{M}(\theta^{k_H < k}, \theta^{k < k_H})$ of holomorphic curves from $\theta^{k < k_H}$ to $\theta^{k_H < k}$ is not empty in the sutured Heegaard diagram $(\Sigma, \beta^k, \beta^{k_H})$.)

It is easy to check that the sutured multi-diagram

$$\mathcal{H}_\beta^H = (\Sigma, \{\beta^k, \beta^{k_H}\}_{k \in \{\infty, 0, 1\}})$$

is admissible. We now compute some \mathcal{A}_∞ -relations associated to \mathcal{H}_β^H .

Lemma 4.6. *For $k \in \{\infty, 0, 1\}$ and $i, j, \ell \in \mathbb{Z}/2\mathbb{Z}$, the maps m_n associated to \mathcal{H}_β^H satisfy:*

- (4.7) $m_1 \equiv 0;$
- (4.8) $m_2(\eta_j^{k+1 < k+2} \otimes \eta_i^{k < k+1}) = \eta_{i+j}^{k < k+2};$
- (4.9) $m_2(\eta_j^{k+1 < (k+2)_H} \otimes \eta_i^{k < k+1}) = \eta_{i+j}^{k < (k+2)_H};$
- (4.10) $m_2(\theta^{k_H < k} \otimes \eta_i^{k+2 < k_H}) = \eta_i^{k+2 < k};$
- (4.11) $m_3(\theta^{k_H < k} \otimes \eta_j^{k+2 < k_H} \otimes \eta_i^{k+1 < k+2}) = 0;$
- (4.12) $m_3(\eta_\ell^{k+2 < k_H} \otimes \eta_j^{k+1 < k+2} \otimes \eta_i^{k < k+1}) = \begin{cases} \theta^{k < k_H} & \text{if } i = j = \ell = 0, \\ 0 & \text{otherwise.} \end{cases}$

Proof. All calculations here follow from considerations similar to that in Subsubsection 3.2.1, summarized in Remark 3.2. (Remark 3.4 does not apply here.) Below, we mention some facts that may help the reader to verify the calculations.

For Equation (4.7), note that, for each pair of two sets of attaching circles (β, β') except (β^k, β^{k_H}) and (β^{k_H}, β^k) , it is clear that there are no domains in (Σ, β, β') that could possibly contribute to m_1 . For each of (β^k, β^{k_H}) and (β^{k_H}, β^k) , there are two bigons that cancel out. (All Heegaard diagrams involved are *nice* in the sense defined by Sarkar and Wang [SW10], and so the bigon counts are combinatorial.)

For Equation (4.8), note that if we completely ignore the three Hamiltonian perturbations β^{k_H} , then there are exactly four regions that are not adjacent to $\partial\Sigma \setminus \mathbf{Z}$, each a topological disk made up of three arcs (i.e. a triangle). Each of the twelve cases in Equation (4.8) arises from exactly one of these four regions, and each region gives rise to exactly three equations. Equation (4.9) is completely analogous.

There is exactly one region that gives rise to each case in Equation (4.10), and Equation (4.11) follows from the fact that there are no regions of the required index. Finally, for Equation (4.12), observe that there are no regions of the required index except when $i = j = \ell = 0$. In the case $i = j = \ell = 0$, there is exactly one quadrilateral formed by the four relevant intersection points, which reduces to one of the four triangles in the previous paragraph as the small Hamiltonian perturbation β^{k_H} is chosen to approach β^k ; this is the triangle p mentioned in our description of β^{k_H} , in the center of the left of Figure 8. □

4.3. Proof of Proposition 3.5 by polygon counting

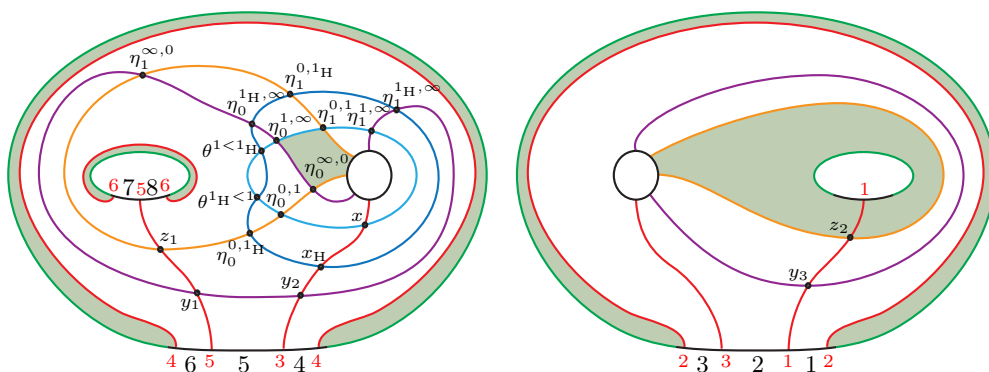


Figure 9. The bordered-sutured multi-diagram

$$\mathcal{H}_{\alpha, \beta}^H = (\Sigma, \alpha, \{\beta^k, \beta^{k_H}\}_{k \in \{\infty, 0, 1\}}, \mathcal{Z}, \psi).$$

We now define the type D morphisms needed in Lemma 2.1.

Consider the bordered-sutured multi-diagram

$$\mathcal{H}_{\alpha, \beta}^H = (\Sigma, \alpha, \{\beta^k, \beta^{k_H}\}_{k \in \{\infty, 0, 1\}}, \mathcal{Z}, \psi),$$

shown in Figure 9, which is provincially admissible but not admissible. Note that $\widehat{\text{BSD}}(\alpha, \beta^k) = \widehat{\text{BSD}}(\mathcal{B}_k)$ as in Subsection 3.2. For each $k \in \{\infty, 0, 1\}$,

let

$$\boldsymbol{\eta}^{k < k+1} = \boldsymbol{\eta}_0^{k < k+1} \oplus \boldsymbol{\eta}_1^{k < k+1},$$

and define

$$\begin{aligned} f_k : \widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^k) &\rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k+1}), \\ f_k(\mathbf{w}) &= \delta_2(\boldsymbol{\eta}^{k < k+1} \otimes \mathbf{w}), \\ \varphi_k : \widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^k) &\rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\boldsymbol{\alpha}, \boldsymbol{\beta}^{k+2}), \\ \varphi_k(\mathbf{w}) &= \delta_3(\boldsymbol{\eta}^{k+1 < k+2} \otimes \boldsymbol{\eta}^{k < k+1} \otimes \mathbf{w}). \end{aligned}$$

Then f_k and φ_k are type D morphisms.

Lemma 4.13. *The morphisms $f_k : \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1})$ are type D homomorphisms and thus satisfy Condition (1) of Lemma 2.1.*

Proof. This and following lemmas in this subsection are proven by combining the \mathcal{A}_∞ -relations in Proposition 4.5 and the computations in Lemma 4.6. We show the details explicitly in this proof and omit them in the sequel.

With inputs $\boldsymbol{\eta}^{k < k+1}$ and \mathbf{w} , the \mathcal{A}_∞ -relation in Proposition 4.5 is

$$\begin{aligned} 0 &= \tilde{\delta}_2(\boldsymbol{\eta}^{k < k+1} \otimes \delta_1(\mathbf{w})) + \tilde{\delta}_1(\delta_2(\boldsymbol{\eta}^{k < k+1} \otimes \mathbf{w})) \\ &\quad + \delta_2(m_1(\boldsymbol{\eta}^{k < k+1}) \otimes \mathbf{w}) + \left(d \otimes \text{id}_{\widehat{\text{BSD}}(\mathcal{B}_{k+1})}\right) \circ \delta_2(\boldsymbol{\eta}^{k < k+1} \otimes \mathbf{w}). \end{aligned}$$

Equation (4.7) now implies the third term is zero, and the above reduces to

$$0 = f_k \circ \delta(\mathbf{w}) + \delta \circ f_k(\mathbf{w}) + df_k(\mathbf{w}) = \partial f_k(\mathbf{w}),$$

which is the condition that f_k is a homomorphism. □

Lemma 4.14. *The morphisms $f_k : \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1})$ and $\varphi_k : \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+2})$ satisfy Condition (2) of Lemma 2.1.*

Proof. This is a combination of Proposition 4.5 with inputs $\boldsymbol{\eta}^{k+1 < k+2}$, $\boldsymbol{\eta}^{k < k+1}$, and \mathbf{w} , together with Equation (4.7) and the equality

$$(4.15) \quad m_2(\boldsymbol{\eta}^{k+1 < k+2} \otimes \boldsymbol{\eta}^{k < k+1}) = 0,$$

which is obtained by taking the sum of Equation (4.8) over all $i, j \in \{0, 1\}$. □

Lemma 4.16. *The morphisms $f_k: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1})$ and $\varphi_k: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+2})$ satisfy Condition (3) of Lemma 2.1.*

Proof. We begin by letting

$$\eta^{k < (k+1)_H} = \eta_0^{k < (k+1)_H} \oplus \eta_1^{k < (k+1)_H},$$

and defining the morphisms

$$\begin{aligned} f_{k,H}: \widehat{\text{BSD}}(\alpha, \beta^k) &\rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\alpha, \beta^{(k+1)_H}), \\ f_{k,H}(\mathbf{w}) &= \delta_2(\eta^{k < (k+1)_H} \otimes \mathbf{w}), \\ \varphi_{k,H}: \widehat{\text{BSD}}(\alpha, \beta^k) &\rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\alpha, \beta^{(k+2)_H}), \\ \varphi_{k,H}(\mathbf{w}) &= \delta_3(\eta^{k+1 < (k+2)_H} \otimes \eta^{k < k+1} \otimes \mathbf{w}), \end{aligned}$$

which are analogous to f_k and φ_k . We also define the morphisms

$$\begin{aligned} \Phi_{k < k_H}: \widehat{\text{BSD}}(\alpha, \beta^k) &\rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\alpha, \beta^{k_H}), \\ \Phi_{k < k_H}(\mathbf{w}) &= \delta_2(\theta^{k < k_H} \otimes \mathbf{w}), \\ \Phi_{k_H < k}: \widehat{\text{BSD}}(\alpha, \beta^{k_H}) &\rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\alpha, \beta^k), \\ \Phi_{k_H < k}(\mathbf{w}) &= \delta_2(\theta^{k_H < k} \otimes \mathbf{w}), \\ \Xi_k: \widehat{\text{BSD}}(\alpha, \beta^k) &\rightarrow \mathcal{A}(\mathcal{Z}) \otimes \widehat{\text{BSD}}(\alpha, \beta^{k+1}), \\ \Xi_k(\mathbf{w}) &= \delta_3(\theta^{(k+1)_H < k+1} \otimes \eta^{k < (k+1)_H} \otimes \mathbf{w}), \end{aligned}$$

We first claim that f_{k+2} is homotopic to $\Phi_{k_H < k} \circ f_{k+2,H}$. Indeed, this follows from Proposition 4.5 with inputs $\theta^{k_H < k}$, $\eta^{k+2 < k_H}$, and \mathbf{w} , together with Equation (4.7) and the equation

$$(4.17) \quad m_2(\theta^{k_H < k} \otimes \eta^{k+2 < k_H}) = \eta^{k+2 < k},$$

which is obtained by taking the sum of Equation (4.10) over $i \in \{0, 1\}$.

Similarly, we see that φ_{k+1} is homotopic to $\Phi_{k_H < k} \circ \varphi_{k+1,H} + \Xi_{k+2} \circ f_{k+1}$. (Note that φ_{k+1} is not a homomorphism.) This follows from Proposition 4.5 with inputs $\theta^{k_H < k}$, $\eta^{k+2 < k_H}$, $\eta^{k+1 < k+2}$, and \mathbf{w} , together with Equation (4.7), Equation (4.17), and the equations

$$(4.18) \quad m_2(\eta^{k+2 < k_H} \otimes \eta^{k+1 < k+2}) = 0,$$

$$(4.19) \quad m_3(\theta^{k_H < k} \otimes \eta^{k+2 < k_H} \otimes \eta^{k+1 < k+2}) = 0,$$

which are obtained respectively from Equation (4.9) and Equation (4.11).

Thus, we see that $f_{k+2} \circ \varphi_k + \varphi_{k+1} \circ f_k$ is homotopic to

$$\Phi_{k_{\mathbb{H}} < k} \circ (f_{k+2, \mathbb{H}} \circ \varphi_k + \varphi_{k+1, \mathbb{H}} \circ f_k) + \Xi_{k+2} \circ f_{k+1} \circ f_k.$$

By Lemma 4.14, the last term is nullhomotopic. Now we claim that $f_{k+2, \mathbb{H}} \circ \varphi_k + \varphi_{k+1, \mathbb{H}} \circ f_k$ is homotopic to $\Phi_{k < k_{\mathbb{H}}}$. Indeed, this follows from Proposition 4.5 with inputs $\eta^{k+2 < k_{\mathbb{H}}}$, $\eta^{k+1 < k+2}$, and $\eta^{k < k+1}$, together with Equation (4.7), Equation (4.15), Equation (4.18), and the equation

$$(4.20) \quad m_3(\eta^{k+2 < k_{\mathbb{H}}} \otimes \eta^{k+1 < k+2} \otimes \eta^{k < k+1}) = \theta^{k < k_{\mathbb{H}}},$$

which is obtained from Equation (4.12).

This means that $f_{k+2} \circ \varphi_k + \varphi_{k+1} \circ f_k$ is homotopic to $\Phi_{k_{\mathbb{H}} < k} \circ \Phi_{k < k_{\mathbb{H}}}$. Now recall that the Hamiltonian isotopy between β^k and $\beta^{k_{\mathbb{H}}}$ can be equivalently viewed as a deformation of the almost complex structure on $\Sigma \times D_2$, and [Lip06, Proposition 11.4], applied to the bordered–sutured Floer setting, implies that $\Phi_{k < k_{\mathbb{H}}}$ and $\Phi_{k_{\mathbb{H}} < k}$ are each homotopic to the homomorphism induced by the corresponding deformation. (Note that in [Lip06], a small Hamiltonian isotopy of β is denoted by β' , while β^H instead denotes the result of a handleslide.) By [Lip06, Lemma 9.4 and Lemma 9.5], $\Phi_{k_{\mathbb{H}} < k} \circ \Phi_{k < k_{\mathbb{H}}}$ is then homotopic to the homomorphism induced by the identity deformation, and hence homotopic to the identity. \square

Proof of Proposition 3.5, via holomorphic polygon counts. We apply Lemma 2.1, the conditions of which are satisfied according to Lemmas 4.13, 4.14, and 4.16. \square

5. The skein relation via direct computation

In this section, we give an alternative proof of Proposition 3.5 by direct computation.

Consider again the bordered–sutured multi-diagram $\mathcal{H}_{\alpha, \beta}$ in Figure 9, and let $i, j \in \{0, 1\}$.

Given generators $\mathbf{w}^k \in \mathcal{G}(\mathcal{H}_k)$, with corresponding intersection points w^k , and a sequence of Reeb chords $\vec{\rho} = (\rho^1, \dots, \rho^m)$, let $\text{Tri}_i(\mathbf{w}^{k+1}, \mathbf{w}^k; \vec{\rho})$ be the space of immersed disks p in Σ with the following properties:

- $\partial p \subset \alpha \cup \beta^k \cup \beta^{k+1} \cup \beta^{k+2} \cup \mathbf{Z}$;

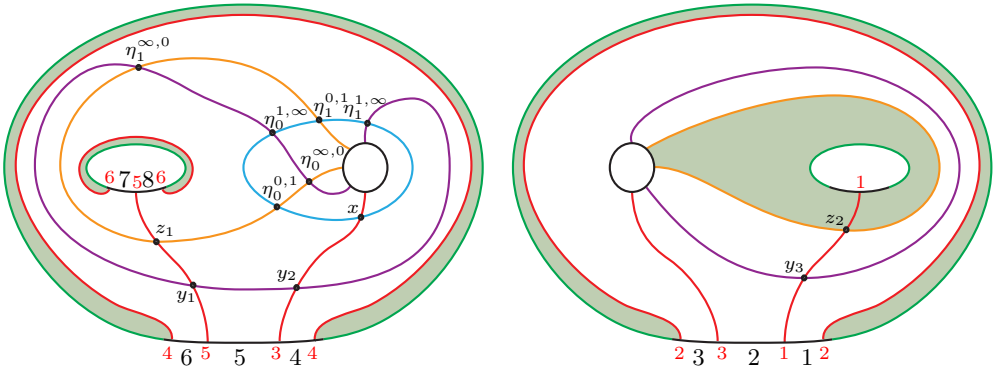


Figure 10. The bordered-sutured multi-diagram $\mathcal{H}_{\alpha,\beta} = (\Sigma, \alpha, \{\beta^k\}_{k \in \{\infty, 0, 1\}}, \mathcal{Z}, \psi)$.

- If we denote by α^a an α -arc in α^a , then, traversing ∂p in the standard orientation, we encounter

$$w^k, \alpha^a, \rho^1, \alpha^a, \rho^2, \alpha^a, \dots, \alpha^a, \rho^m, \alpha^a, w^{k+1}, \beta^{k+1}, \eta_i^{k,k+1}, \beta^k, w^k,$$

in this order; and

- At each of the turning points above, i.e. $w^k, w^{k+1}, \eta_i^{k,k+1}$, and $\{\partial \rho^\ell\} \subset \mathbf{a}$, the angle is acute.

Similarly, let $\text{Quad}_{ij}(\mathbf{w}^{k+2}, \mathbf{w}^k; \vec{\rho})$ be the space of immersed disks p in Σ with the following properties:

- $\partial p \subset \alpha \cup \beta^k \cup \beta^{k+1} \cup \beta^{k+2} \cup \mathbf{Z}$;
- If we denote by α^a an α -arc in α^a , then, traversing ∂p in the standard orientation, we encounter

$$w^k, \alpha^a, \rho^1, \alpha^a, \rho^2, \alpha^a, \dots, \alpha^a, \rho^m, \alpha^a, w^{k+2}, \beta^{k+2}, \eta_j^{k+1,k+2}, \beta^{k+1}, \eta_i^{k,k+1}, \beta^k, w^k,$$

in this order; and

- At each of the turning points above, i.e. $w^k, w^{k+2}, \eta_j^{k+1,k+2}, \eta_i^{k,k+1}$, and $\{\partial \rho^\ell\} \subset \mathbf{a}$, the angle is acute.

We use the spaces above to define the morphisms

$$\begin{aligned} \bar{f}_{k,i} &: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1}), \\ \bar{f}_{k,i}(\mathbf{w}^k) &= \sum_{\mathbf{w}^{k+1} \in \mathcal{G}(\mathcal{H}_{k+1})} \sum_{\text{Tri}_i(\mathbf{w}^{k+1}, \mathbf{w}^k; \bar{\rho})} a(-\bar{\rho}) \otimes \mathbf{w}^{k+1}, \\ \bar{\varphi}_{k,ij} &: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+2}), \\ \bar{\varphi}_{k,ij}(\mathbf{w}^k) &= \sum_{\mathbf{w}^{k+2} \in \mathcal{G}(\mathcal{H}_{k+2})} \sum_{\text{Quad}_{ij}(\mathbf{w}^{k+2}, \mathbf{w}^k; \bar{\rho})} a(-\bar{\rho}) \otimes \mathbf{w}^{k+2}, \end{aligned}$$

and the morphisms

$$\begin{aligned} \bar{f}_k &: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1}), & \bar{f}_k &= \bar{f}_{k,0} + \bar{f}_{k,1}, \\ \bar{\varphi}_k &: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+2}), & \bar{\varphi}_k &= \bar{\varphi}_{k,00} + \bar{\varphi}_{k,01} + \bar{\varphi}_{k,10} + \bar{\varphi}_{k,11}. \end{aligned}$$

Since the definitions of $\bar{f}_{k,i}$ and $\bar{\varphi}_{k,ij}$ refer to actual immersed disks rather than holomorphic representatives, by inspecting $\mathcal{H}_{\alpha,\beta}$, we may directly compute:

$$\begin{aligned} \bar{f}_{1,0}(\mathbf{x}) &= (4, 6) \otimes \mathbf{y}_1 + (12, 3) \otimes \mathbf{y}_2 + (4, 56) \otimes \mathbf{y}_2 + (2) \otimes \mathbf{y}_3, \\ \bar{f}_{1,1}(\mathbf{x}) &= (4, 6, 7, 8) \otimes \mathbf{y}_1 + I \otimes \mathbf{y}_2, \\ \bar{f}_{\infty,0}(\mathbf{y}_1) &= (45, 6) \otimes \mathbf{z}_1 + (7, 8) \otimes \mathbf{z}_1, \\ \bar{f}_{\infty,0}(\mathbf{y}_2) &= (4, 6) \otimes \mathbf{z}_1 + (4, 56, 2) \otimes \mathbf{z}_2, \\ \bar{f}_{\infty,0}(\mathbf{y}_3) &= I \otimes \mathbf{z}_2, \\ \bar{f}_{\infty,1}(\mathbf{y}_1) &= I \otimes \mathbf{z}_1, \\ \bar{f}_{\infty,1}(\mathbf{y}_2) &= (12, 3, 4, 6) \otimes \mathbf{z}_1 + (2) \otimes \mathbf{z}_2 + (12, 3, 4, 56, 2) \otimes \mathbf{z}_2, \\ \bar{f}_{\infty,1}(\mathbf{y}_3) &= (1, 23) \otimes \mathbf{z}_2, \\ \bar{f}_{0,0}(\mathbf{z}_1) &= (5) \otimes \mathbf{x} + (5, 12, 3, 4, 56) \otimes \mathbf{x} + (45, 6, 7, 8, 5) \otimes \mathbf{x}, \\ \bar{f}_{0,0}(\mathbf{z}_2) &= (1, 3, 4, 56) \otimes \mathbf{x}, \\ \bar{f}_{0,1}(\mathbf{z}_1) &= (5, 12, 3) \otimes \mathbf{x} + (7, 8, 5) \otimes \mathbf{x}, \\ \bar{f}_{0,1}(\mathbf{z}_2) &= (1, 3) \otimes \mathbf{x}. \end{aligned}$$

See Figure 11 for a graphical representation of $\bar{f}_{k,i}$.

Remark 5.1. Note that we have had to observe that $(12, 23) = (45, 56) = 0$ in this calculation; for instance, $\bar{f}_{\infty,0}(\mathbf{y}_1)$ in fact has a term $(45, 56, 2) \otimes \mathbf{z}_2 = 0$. To see the corresponding immersed disk, which has multiplicity 2 at two

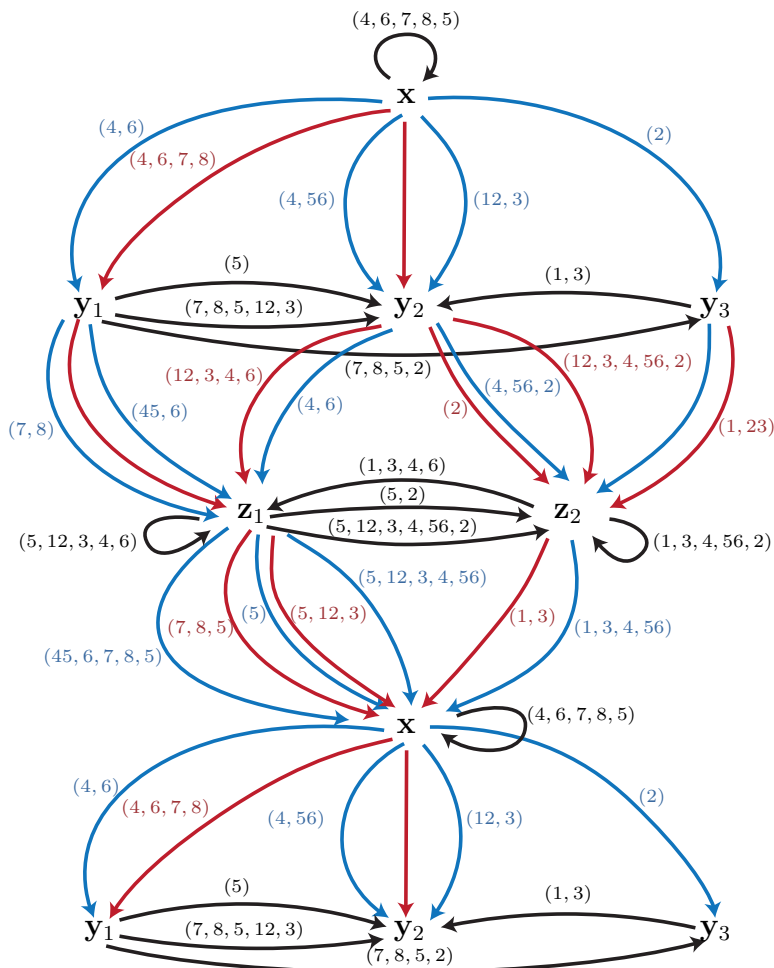


Figure 11. A graphical representation of the morphisms $\bar{f}_{k,i}$. Black arrows represent δ , blue arrows represent $\bar{f}_{k,0}$, and red arrows represent $\bar{f}_{k,1}$. The repeated level, from $\widehat{\text{BSD}}(\mathcal{B}_1)$ to $\widehat{\text{BSD}}(\mathcal{B}_\infty)$, is drawn here to facilitate checking Lemma 5.3.

non-adjacent regions, one may, for example, concatenate the domain for the term $(5) \otimes \mathbf{y}_2$ in $\delta(\mathbf{y}_1)$ with the domain for the term $(4, 56, 2) \otimes \mathbf{z}_2$ in $\bar{f}_{\infty,0}(\mathbf{y}_2)$; see Figure 12. In the holomorphic interpretation (see Proposition 5.5 below), these cases represent situations where we already know that $\chi(S^{\mathbb{P}^1}) = 1$ (cf. Remark 3.4, where we may have $\chi(S^{\mathbb{P}^1}) < 1$), but no partition \bar{P} supports a contribution to f_k .

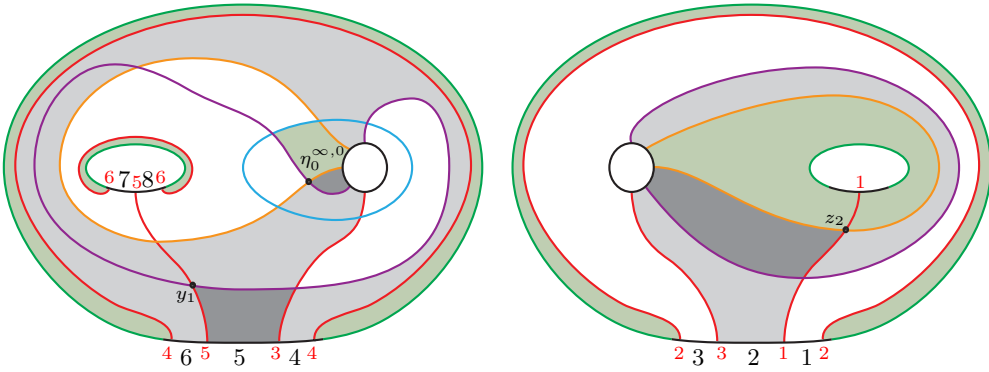


Figure 12. An immersed disk in $\text{Tri}_0(\mathbf{z}_2, \mathbf{y}_1; (-\rho_{45}, -\rho_{56}, -\rho_2))$ that does not contribute to $\bar{f}_{\infty,0}(\mathbf{y}_1)$ because $(45, 56, 2) = 0$. Regions with multiplicity 1 are shaded in light gray, and regions with multiplicity 2 in dark gray.

Lemma 5.2. *The morphisms $\bar{f}_k: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1})$ are type D homomorphisms and thus satisfy Condition (1) of Lemma 2.1.*

Proof. In fact, $\bar{f}_{k,i}$ is a homomorphism for $i \in \{0, 1\}$. The proof is a direct computation, and the reader may find Figure 11 helpful in checking this computation. We provide a sample calculation here:

$$\begin{aligned} \delta \circ \bar{f}_{\infty,0}(\mathbf{y}_1) &= \tilde{\delta}((45, 6) \otimes \mathbf{z}_1 + (7, 8) \otimes \mathbf{z}_1) \\ &= (45, 6, 5, 12, 3, 4, 6) \otimes \mathbf{z}_1 + (45, 6, 5, 2) \otimes \mathbf{z}_2 \\ &\quad + (45, 6, 5, 12, 3, 4, 56, 2) \otimes \mathbf{z}_2 + (7, 8, 5, 12, 3, 4, 6) \otimes \mathbf{z}_1 \\ &\quad + (7, 8, 5, 2) \otimes \mathbf{z}_2 + (7, 8, 5, 12, 3, 4, 56, 2) \otimes \mathbf{z}_2 \\ &= (45, 6, 5, 2) \otimes \mathbf{z}_2 + (7, 8, 5, 12, 3, 4, 6) \otimes \mathbf{z}_1 + (7, 8, 5, 2) \otimes \mathbf{z}_2 \\ &\quad + (7, 8, 5, 12, 3, 4, 56, 2) \otimes \mathbf{z}_2, \\ \bar{f}_{\infty,0} \circ \delta(\mathbf{y}_1) &= \left(\mu \otimes \text{id}_{\widehat{\text{BSD}}(\mathcal{B}_1)} \right) \circ (\text{id}_A \otimes \bar{f}_{\infty,1})((5) \otimes \mathbf{y}_2 \\ &\quad + (7, 8, 5, 12, 3) \otimes \mathbf{y}_2 + (7, 8, 5, 2) \otimes \mathbf{y}_3) \\ &= (5, 4, 6) \otimes \mathbf{z}_1 + (5, 4, 56, 2) \otimes \mathbf{z}_2 + (7, 8, 5, 12, 3, 4, 6) \otimes \mathbf{z}_1 \\ &\quad + (7, 8, 5, 12, 3, 4, 56, 2) \otimes \mathbf{z}_2 + (7, 8, 5, 2) \otimes \mathbf{z}_2, \\ d\bar{f}_{\infty,0}(\mathbf{y}_1) &= \left(d \otimes \text{id}_{\widehat{\text{BSD}}(\mathcal{B}_1)} \right) (\mathbf{y}_1) = d(45, 6) \otimes \mathbf{z}_1 + d(7, 8) \otimes \mathbf{z}_1 \\ &= (5, 4, 6) \otimes \mathbf{z}_1. \end{aligned}$$

Observing that $(45, 6, 5, 2) = (5, 4, 56, 2)$ (see Figure 4 and the surrounding text), we see that the sum of the three terms above is zero. \square

We may also directly compute:

$$\begin{aligned} \bar{\varphi}_{\infty,00}(\mathbf{y}_2) &= (4, 56) \otimes \mathbf{x}, \\ \bar{\varphi}_{\infty,10}(\mathbf{y}_2) &= I \otimes \mathbf{x}, \\ \bar{\varphi}_{\infty,11}(\mathbf{y}_2) &= (12, 3) \otimes \mathbf{x}, \\ \bar{\varphi}_{0,00}(\mathbf{z}_1) &= (45, 6) \otimes \mathbf{y}_1, \\ \bar{\varphi}_{0,01}(\mathbf{z}_1) &= I \otimes \mathbf{y}_1, \\ \bar{\varphi}_{0,11}(\mathbf{z}_1) &= (7, 8) \otimes \mathbf{y}_1, \\ \bar{\varphi}_{0,10}(\mathbf{z}_2) &= (1, 23) \otimes \mathbf{y}_3, \\ \bar{\varphi}_{0,11}(\mathbf{z}_2) &= I \otimes \mathbf{y}_3. \end{aligned}$$

All other combinations of i, j, k, \mathbf{w} yield $\bar{\varphi}_{k,ij}(\mathbf{w}) = 0$. See Figure 13 for a graphical representation of the sums $\bar{\varphi}_k$.

Lemma 5.3. *The morphisms $\bar{f}_k: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1})$ and $\bar{\varphi}_k: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+2})$ satisfy Condition (2) of Lemma 2.1.*

Proof. The proof is a direct computation analogous to that of Lemma 5.2. The reader may find it helpful to peruse both Figure 11 and Figure 13. \square

Lemma 5.4. *The morphisms $\bar{f}_k: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1})$ and $\bar{\varphi}_k: \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+2})$ satisfy Condition (3) of Lemma 2.1, with $\kappa_k \equiv 0$.*

Proof. The proof is a direct computation analogous to that of Lemma 5.2. Again, it may be helpful to peruse both Figure 11 and Figure 13. \square

Proof of Proposition 3.5, via direction computation. We apply to \bar{f}_k and $\bar{\varphi}_k$ Lemma 2.1, the conditions of which are satisfied according to Lemma 5.2, Lemma 5.3, and Lemma 5.4. \square

We end this section by relating this proof with the one in Section 4.

Proposition 5.5. *The morphisms \bar{f}_k and $\bar{\varphi}_k$ coincide with f_k and φ_k respectively.*

Proof. The morphisms f_k are defined using δ_2 , which in turn is defined by counting holomorphic polygons, i.e. by considering the moduli space $\mathcal{M}_{\text{emb}}^B(\mathbf{w}^{k+1}, \boldsymbol{\eta}^{k < k+1}, \mathbf{w}^k; S^{\mathbb{D}}, \bar{P})$; the morphisms φ_k are defined similarly. As

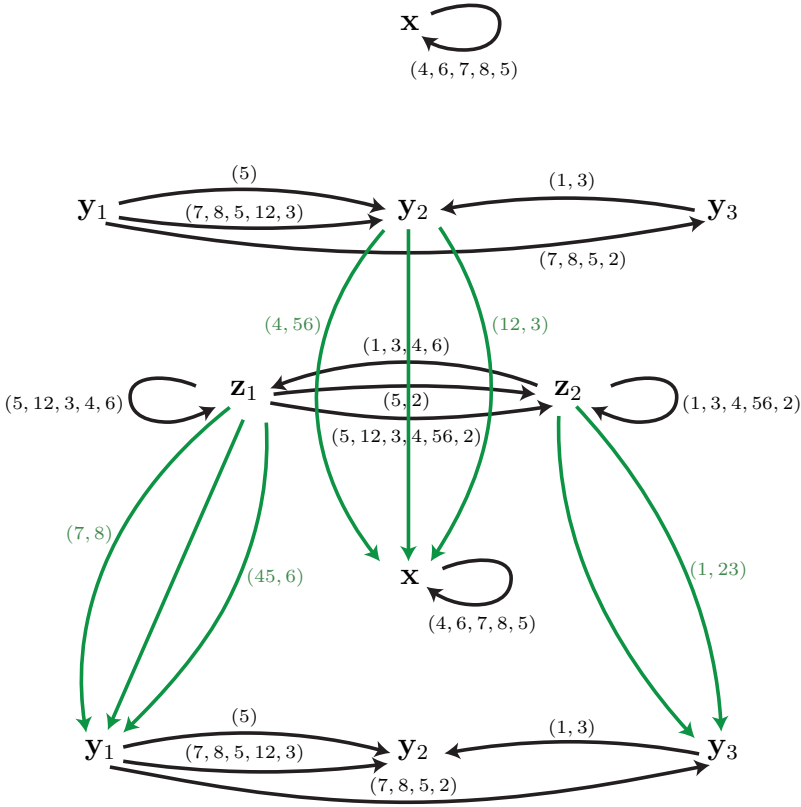


Figure 13. A graphical representation of the morphisms $\bar{\varphi}_k$. Black arrows represent δ and green arrows represent $\bar{\varphi}_k$.

in the proof of Lemma 4.6, these maps may be computed from considerations similar to that in Subsubsection 3.2.1, summarized in Remark 3.2, with the index formula [Zar11, Proposition 5.3.5] replaced by the one for polygons [LOT16, Proposition 4.9]. Remark 3.3 and Remark 3.4 both apply. This analysis yields the fact that the holomorphic polygon counts are in fact counts of immersed disks, and consideration of the term $\sum \partial_j(B) \cdot \partial_\ell(B)$ in [LOT16, Proposition 4.9] implies that the angles at the intersection points are acute. \square

6. Comparison with the skein relation for knot Floer homology

In this section, we prove that the skein exact triangle in Corollary 1.2 agrees with the one in [Man07], using a pairing theorem for triangles adapted from [LOT16]. Precisely, we prove the following, elaborated version of Theorem 1.3.

Theorem 6.1. *There exists a connected link projection of L_∞ and a choice of edges, such that the special Heegaard diagram $\mathcal{H}_\infty^{\text{Man}}$, when once quasi-stabilized, can be obtained by gluing together two bordered–sutured Heegaard diagrams \mathcal{H}' and \mathcal{H}_∞ , where \mathcal{H}' is admissible, and \mathcal{H}_∞ is as described in Subsection 3.2 and used in the proof of Theorem 1.1; moreover, for each $k \in \{\infty, 0, 1\}$, the diagram*

$$\begin{array}{ccc}
 \widehat{\text{BSA}}(\mathcal{H}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_k) & \xrightarrow{\mathbb{I}_{\widehat{\text{BSA}}(\mathcal{H}')} \boxtimes f_k} & \widehat{\text{BSA}}(\mathcal{H}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_{k+1}) \\
 \simeq \downarrow & & \downarrow \simeq \\
 \widetilde{\text{CFL}}(\mathcal{H}_k^{\text{Man}}) \otimes V & \xrightarrow{F_k^{\text{Man}} \otimes \text{id}_V} & \widetilde{\text{CFL}}(\mathcal{H}_{k+1}^{\text{Man}}) \otimes V,
 \end{array}$$

where V is a 2-dimensional vector space, and where the vertical arrows are induced by the pairing theorem [Zar11, Theorem 7.6.1], commutes up to homotopy.

Remark 6.2. Due to a difference in the orientation convention, the arrows in Theorem 6.1 point in the direction opposite to those in [Man07, MO08]. As in [Won17, PW20], we follow the convention in [OSz04b, Zar11], where the Heegaard surface is the oriented boundary of the α -handlebody.

6.1. Pairing theorem for triangles

In Subsection 4.1, we briefly summarized a theory of holomorphic polygon counting for bordered Floer homology developed by Lipshitz, Ozsváth, and Thurston [LOT16, Section 4]. In fact, they also provide a pairing theorem for these bordered polygons in [LOT16, Section 5], which, roughly speaking, establishes that polygon counting in the bordered setting commutes with the (usual) operation of pairing with another bordered diagram, up to homotopy. Below, we will translate a special case, a pairing theorem for triangles, to the bordered–sutured Floer setting. To avoid confusion, we shall refer to the

usual pairing theorems (e.g. [Zar11, Theorem 7.6.1 and Theorem 8.5.1]) as *pairing theorems for bigons*, to distinguish them from the pairing theorem for triangles.

Let $(\Sigma, \alpha, \{\beta^k\}_{k=1}^m, -\mathcal{Z}, \psi)$ be a provincially admissible bordered–sutured multi-diagram. As in Subsection 4.1, the bordered–sutured diagram $\mathcal{H}_k = (\Sigma, \alpha, \beta^k, -\mathcal{Z}, \psi)$ is provincially admissible, and the sutured diagram $\mathcal{H}_{k_1, k_2} = (\Sigma, \beta^{k_1}, \beta^{k_2})$ is admissible. Here, we are concerned with the type D modules

$$\mathcal{A}(\mathcal{Z})\widehat{\text{BSD}}(\mathcal{H}_k).$$

Let $\eta^{k_1} \in \text{SFC}(\mathcal{H}_{k_1, k_2})$ be a cycle; then Proposition 4.5 implies that the morphism

$$\delta_2(\eta^{k_1} \otimes -): \widehat{\text{BSD}}(\mathcal{H}_{k_1}) \rightarrow \widehat{\text{BSD}}(\mathcal{H}_{k_2})$$

is a type D homomorphism.

Now let $\mathcal{H}' = (\Sigma', \alpha', \beta', \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}, \psi')$ be an admissible bordered–sutured diagram. Let β'^{H} be a small Hamiltonian perturbation of β' ; then $\mathcal{H}'^{\text{H}} = (\Sigma', \alpha', \beta'^{\text{H}}, \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}, \psi')$ is also an admissible bordered–sutured diagram. We will be concerned with the type DAA trimodules

$$\mathcal{A}(-\mathcal{Z}_1)\widehat{\text{BSDAA}}(\mathcal{H}')_{\mathcal{A}(\mathcal{Z}_2), \mathcal{A}(\mathcal{Z})}, \mathcal{A}(-\mathcal{Z}_1)\widehat{\text{BSDAA}}(\mathcal{H}'^{\text{H}})_{\mathcal{A}(\mathcal{Z}_2), \mathcal{A}(\mathcal{Z})}.$$

Noting that there is an obvious relative \mathbb{Z} -grading on $\text{SFC}(\beta', \beta'^{\text{H}})$, let $\theta \in \mathcal{G}(\beta', \beta'^{\text{H}})$ be the top-graded generator. As mentioned at the end of Subsection 4.1, there are maps ${}_n\delta_{\ell_1, \ell_2}$ for type DAA trimodules analogous to δ_n . Observing that the differential vanishes on $\text{SFC}(\beta', \beta'^{\text{H}})$, we see that θ is also a cycle, and so by a type DAA version of Proposition 4.5, the morphism given by the collection

$$\left\{ \begin{array}{l} {}_{2\delta_{\ell_1, \ell_2}}(\theta \otimes - \otimes a'_1 \otimes \cdots \otimes a'_{\ell_1} \otimes a_1 \otimes \cdots \otimes a_{\ell_2}): \text{SFC}(\beta', \beta'^{\text{H}}) \otimes \widehat{\text{BSDAA}}(\alpha', \beta') \\ \otimes \underbrace{\mathcal{A}(\mathcal{Z}_2) \otimes \cdots \otimes \mathcal{A}(\mathcal{Z}_2)}_{\ell_1} \otimes \underbrace{\mathcal{A}(\mathcal{Z}) \otimes \cdots \otimes \mathcal{A}(\mathcal{Z})}_{\ell_2} \rightarrow \mathcal{A}(-\mathcal{Z}_1) \otimes \widehat{\text{BSDAA}}(\alpha', \beta'^{\text{H}}) \end{array} \right\}$$

is also a type DAA homomorphism; we denote this homomorphism by

$${}_{2\delta}(\theta \otimes -): \widehat{\text{BSDAA}}(\mathcal{H}') \rightarrow \widehat{\text{BSDAA}}(\mathcal{H}'^{\text{H}}).$$

Finally, we may glue the Heegaard diagrams \mathcal{H}' (resp. \mathcal{H}'^{H}) and \mathcal{H}_k along \mathcal{Z} to obtain a bordered–sutured diagram $\mathcal{H}' \cup \mathcal{H}_k$ (resp. $\mathcal{H}'^{\text{H}} \cup \mathcal{H}_k$).

This gives rise to type DA bimodules

$$\mathcal{A}(-\mathcal{Z}_1)\widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}_k)_{\mathcal{A}(\mathcal{Z}_2)}, \mathcal{A}(-\mathcal{Z}_1)\widehat{\text{BSDA}}(\mathcal{H}'^{\text{H}} \cup \mathcal{H}_k)_{\mathcal{A}(\mathcal{Z}_2)}.$$

Noting that

$$\boldsymbol{\theta} \otimes \boldsymbol{\eta}^{k_1} \in \text{SFC}(\boldsymbol{\beta}' \cup \boldsymbol{\beta}^{k_1}, \boldsymbol{\beta}'^{\text{H}} \cup \boldsymbol{\beta}^{k_2})$$

is a cycle, considerations similar to the above yield a type DA homomorphism

$${}_2\delta((\boldsymbol{\theta} \otimes \boldsymbol{\eta}^{k_1}) \otimes -): \widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}_{k_1}) \rightarrow \widehat{\text{BSDA}}(\mathcal{H}'^{\text{H}} \cup \mathcal{H}_{k_2}).$$

The discussion in [LOT16, Section 5] now carries over to our context to give us the following proposition, a pairing theorem for triangles.

Proposition 6.3 (cf. [LOT16, Proposition 5.35]). *The diagram*

$$\begin{array}{ccc} \widehat{\text{BSDAA}}(\mathcal{H}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_{k_1}) & \xrightarrow{{}_2\delta(\boldsymbol{\theta} \otimes -) \boxtimes \delta_2(\boldsymbol{\eta}^{k_1} \otimes -)} & \widehat{\text{BSDAA}}(\mathcal{H}'^{\text{H}}) \boxtimes \widehat{\text{BSD}}(\mathcal{H}_{k_2}) \\ \simeq \downarrow & & \downarrow \simeq \\ \widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}_{k_1}) & \xrightarrow{{}_2\delta((\boldsymbol{\theta} \otimes \boldsymbol{\eta}^{k_1}) \otimes -)} & \widehat{\text{BSDA}}(\mathcal{H}'^{\text{H}} \cup \mathcal{H}_{k_2}), \end{array}$$

where the vertical arrows are induced by the pairing theorem for bigons [Zar11, Theorem 8.5.1], commutes up to homotopy.

While this proposition may be generalized by replacing $\widehat{\text{BSD}}$ with tri-modules $\widehat{\text{BSDDA}}$, or by slightly relaxing the admissibility of the diagrams, the version stated suffices for our purposes.

The discussion in [LOT16, Section 5] in fact gives a pairing theorem for general polygons, which would be a necessary ingredient if we wished to identify the cube-of-resolutions spectral sequence [BL12, Section 5] (see also [Won17, Section 5]) arising from the iteration of the exact triangle in [Man07] with one arising from Theorem 1.1 in the bordered context. However, we will not be concerned with this identification.

For clarity in the sequel, we provide a variant of Proposition 6.3 here.

Proposition 6.4. *The diagram*

$$\begin{array}{ccc} \widehat{\text{BSDAA}}(\mathcal{H}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_{k_1}) & \xrightarrow{\mathbb{I}_{\widehat{\text{BSDAA}}(\mathcal{H}')} \boxtimes \delta_2(\boldsymbol{\eta}^{k_1} \otimes -)} & \widehat{\text{BSDAA}}(\mathcal{H}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_{k_2}) \\ \simeq \downarrow & & \downarrow \simeq \\ \widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}_{k_1}) & \xrightarrow{{}_2\delta((\boldsymbol{\theta} \otimes \boldsymbol{\eta}^{k_1}) \otimes -)} & \widehat{\text{BSDA}}(\mathcal{H}'^{\text{H}} \cup \mathcal{H}_{k_2}), \end{array}$$

where the left vertical arrow is induced by the pairing theorem for bigons [Zar11, Theorem 8.5.1], and the right vertical arrow is induced by the deformation of almost complex structure relating \mathcal{H}' and \mathcal{H}'^{H} and the pairing theorem, commutes up to homotopy.

Proof. This follows directly from Proposition 6.3 by precomposing with the commutative diagram that shows the definition of the box tensor morphism ${}_2\delta(\boldsymbol{\theta} \otimes -) \boxtimes \delta_2(\boldsymbol{\eta}^{k_1} \otimes -)$ (see [LOT15, Section 2.3.2]),

$$\begin{array}{ccc}
 & & \widehat{\text{BSDAA}}(\mathcal{H}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_{k_2}) \\
 & \nearrow \mathbb{I}_{\widehat{\text{BSDAA}}(\mathcal{H}')} \boxtimes \delta_2(\boldsymbol{\eta}^{k_1} \otimes -) & \downarrow {}_2\delta(\boldsymbol{\theta} \otimes -) \boxtimes \mathbb{I}_{\widehat{\text{BSD}}(\mathcal{H}_{k_2})} \\
 \widehat{\text{BSDAA}}(\mathcal{H}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_{k_1}) & \xrightarrow{{}_2\delta(\boldsymbol{\theta} \otimes -) \boxtimes \delta_2(\boldsymbol{\eta}^{k_1} \otimes -)} & \widehat{\text{BSDAA}}(\mathcal{H}'^{\text{H}}) \boxtimes \widehat{\text{BSD}}(\mathcal{H}_{k_2}),
 \end{array}$$

and adapting [Lip06, Proposition 11.4] to identify ${}_2\delta(\boldsymbol{\theta} \otimes -)$ with the homotopy equivalence induced by the deformation of almost complex structure, as in the last paragraph of the proof of Lemma 4.16.

Alternatively, one could directly unpack the definitions in [LOT16, Theorem 7], which was used to prove [LOT16, Proposition 5.35] and hence Proposition 6.3. \square

We are now ready to prove that the maps F_k in Theorem 1.1 could be obtained by counting holomorphic polygons in a bordered–sutured multi-diagram that encodes \mathcal{Y}_k , instead of doing so for \mathcal{B}_k and invoking the pairing theorem.

Theorem 6.5. *Let \mathcal{H}' be the admissible diagram for \mathcal{Y}' in the proof of Theorem 1.1 in Section 3, and define β'^{H} and \mathcal{H}'^{H} as in this section. For $k \in \{\infty, 0, 1\}$, the diagram*

$$\begin{array}{ccc}
 \widehat{\text{BSDAA}}(\mathcal{H}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_k) & \xrightarrow{\mathbb{I}_{\widehat{\text{BSDAA}}(\mathcal{H}')} \boxtimes f_k} & \widehat{\text{BSDAA}}(\mathcal{H}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_{k+1}) \\
 \downarrow \simeq & & \downarrow \simeq \\
 \widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}_k) & \xrightarrow{{}_2\delta((\boldsymbol{\theta} \otimes \boldsymbol{\eta}^{k < k+1}) \otimes -)} & \widehat{\text{BSDA}}(\mathcal{H}'^{\text{H}} \cup \mathcal{H}_{k+1}),
 \end{array}$$

where the left vertical arrow is induced by the pairing theorem for bigons [Zar11, Theorem 8.5.1], and the right vertical arrow is induced by the deformation of almost complex structure relating \mathcal{H}' and \mathcal{H}'^{H} and the pairing theorem, commutes up to homotopy.

Proof. This is a direct application of Proposition 6.4. \square

The significance of Theorem 6.5 is the following. By combining the diagrams $\mathcal{H}' \cup \mathcal{H}_k$, $\mathcal{H}'^H \cup \mathcal{H}_{k+1}$, and the analogous diagram for $k + 2$, one obtains an admissible bordered–sutured multi-diagram encoding $\{\mathcal{J}_k\}_{k \in \{\infty, 0, 1\}}$. By counting holomorphic polygons in this multi-diagram, one could directly prove Theorem 1.1. (Of course, in this approach, one would not in general be able to obtain an explicit description of the maps involved as in Section 5.) The resulting type DA homomorphisms are given exactly by ${}_2\delta((\theta \otimes \eta^{k < k+1}) \otimes -)$. Theorem 6.5 shows that these homomorphisms in fact agree with the ones we used in our proof of Theorem 1.1.

6.2. The exact triangles agree

As mentioned in Section 1, Manolescu [Man07] constructs from a connected link projection a special Heegaard diagram for (S^3, L) . We do not repeat the complete definitions here, and instead direct the reader to [Man07, Section 2]; we provide a quick summary in this paragraph. In this construction, the Heegaard surface is the boundary of a regular neighborhood of the singularization of the link projection, with an α -circle for each bounded region of the link diagram, and a β -circle β_v for each crossing. (The unbounded region is denoted A_0 .) We denote by \mathcal{P}_v the piece of the Heegaard diagram associated to the crossing v , as shown in [Man07, Figure 2]. It also involves a choice of a distinguished edge e , as well as a collection of edges $\{s_i\}$, in the link projection. At each of these edges, the local diagram is modified as in [Man07, Figure 3]: A meridional β -curve is added, with a puncture on each side; we further add an α -curve encircling the two punctures for each edge s_i .

In accordance with Remark 6.2, we must slightly modify the construction in [Man07] to fit with our convention, as follows. Since the difference in the conventions switches the roles of the α - and β -handlebodies, the special Heegaard diagram for (S^3, L) in [Man07] in fact encodes $(-S^3, L) \cong (S^3, m(L))$ in our convention, where $m(L)$ denotes the mirror of a link L . To compensate for the difference, we replace \mathcal{P}_v with the piece \mathcal{P}'_v of Heegaard surface constructed for v with the opposite crossing information; in other words, we replace the β -circle β_v with the circle β'_v , unique up to homotopy, that is not homotopic to β_v in the 4-punctured sphere, and that separates the four punctures as β_v does. The rest of the construction is unchanged.

Theorem 1.3 now follows from Theorem 6.5:

Proof of Theorem 1.3, without gradings. In [Man07], the maps F_k^{Man} are constructed as follows. First, a special Heegaard diagram $\mathcal{H}_\infty^{\text{Man}}$ for L_∞ is constructed from a connected link projection with a choice of e and $\{s_i\}$, with $|\{s_i\}| = m - 1$. Then $\mathcal{H}_0^{\text{Man}}$ and $\mathcal{H}_1^{\text{Man}}$ are constructed by changing the β -circle β'_v in $\mathcal{H}_\infty^{\text{Man}}$ near the crossing v in question, and modifying all other β -circles by a small Hamiltonian perturbation. The maps F_k^{Man} are then defined by holomorphic triangle counts between $\mathcal{H}_k^{\text{Man}}$ and $\mathcal{H}_{k+1}^{\text{Man}}$, similar to the morphisms f_k in Subsection 4.3.

In light of Theorem 6.5, we will find a link projection for L_∞ and a choice of e and $\{s_i\}$ with $|\{s_i\}| = m$, such that the associated special Heegaard diagram $\mathcal{H}_\infty^{\text{Man, st}}$ can be decomposed into two bordered–sutured Heegaard diagrams \mathcal{H}' and \mathcal{H}_∞ , where \mathcal{H}' is admissible and \mathcal{H}_∞ is the diagram in Section 3. We illustrate how to do so.

Fix a connected link projection D for L_∞ , with a crossing v at which the resolutions of D are projections for L_0 and L_1 . Using Reidemeister moves, find another connected link projection D' for L_∞ such that exactly one of the four regions adjacent to v is the unbounded region A_0 in \mathbb{R}^2 . In fact, we may require that, after rotating the diagram to match v with the left of Figure 14, the top region is A_0 and the other three regions are bounded.

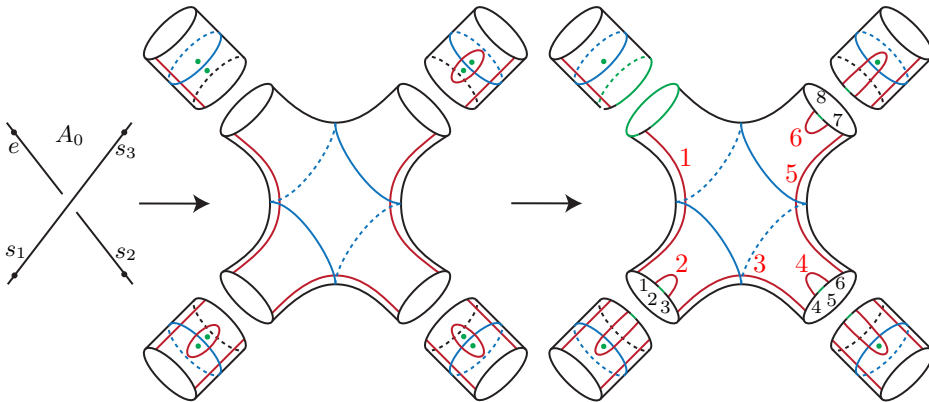


Figure 14. Left: The crossing v in a connected projection of L_∞ , with all adjacent edges marked. Center: The piece \mathcal{P}'_v associated with v in the special Heegaard diagram $\mathcal{H}_\infty^{\text{Man, st}}$, together with the local modifications according to the choice of e and $\{s_i\}$. Right: After a small isotopy of $\mathcal{H}_\infty^{\text{Man, st}}$, we may identify \mathcal{P}'_v with the bordered–sutured diagram \mathcal{H}_∞ in Section 3.

Viewing D' as a 4-valent graph, we choose the distinguished edge e to be that to the top left of v , and choose s_1 , s_2 , and s_3 to be the other

three edges adjacent to v ; see the left of Figure 14. Complete this to a minimal collection $\{s_i\}$; since the two strands at v belong to at most two components, the cardinality of $\{s_i\}$ will be one larger than the minimum, meaning $|\{s_i\}| = m$. Thus, $\mathcal{H}_\infty^{\text{Man, st}}$, and hence $\mathcal{H}_0^{\text{Man, st}}$ and $\mathcal{H}_1^{\text{Man, st}}$, will be once quasi-stabilized when compared to the Heegaard diagrams $\mathcal{H}_k^{\text{Man}}$ used in [Man07].

Now using D' , construct a special Heegaard diagram $\mathcal{H}_\infty^{\text{Man, st}}$ according to [Man07, Section 2]. Under this construction, the piece \mathcal{P}'_v of the Heegaard diagram associated with v is as in the center of Figure 14. In the same figure, the local diagrams associated to the edges e , s_1 , s_2 , and s_3 are also shown. $\mathcal{H}_\infty^{\text{Man, st}}$ is formed by attaching these pieces, as well as other pieces not shown, to each other.

After a small isotopy of $\mathcal{H}_\infty^{\text{Man, st}}$ (or equivalently, by choosing a different place to cut it apart), we obtain the local diagram as in the right of Figure 14. In this diagram, the 4-punctured sphere in the center, which we also denote by \mathcal{P}'_v by a slight abuse of notation, intersects with three more α -curves than before. Moreover, four of the punctures in the previous picture are now each manifested as an arc on $\partial\mathcal{P}'_v$ together with a corresponding arc on the boundary of another piece. Now it is easy to check that \mathcal{P}'_v is a provincially admissible bordered–sutured diagram, with the arcs mentioned above serving as $\partial\Sigma \setminus \mathcal{Z}$. In fact, \mathcal{P}'_v now coincides with \mathcal{H}_∞ (see Figure 7B); in the right of Figure 14, we have included the numbering of the α -arcs and some of the Reeb chords to help the reader verify this identification. We may now declare the rest of $\mathcal{H}_\infty^{\text{Man, st}}$ to be \mathcal{H}' ; then the admissibility of \mathcal{H}' follows from that of $\mathcal{H}_\infty^{\text{Man, st}}$. Note that, gluing \mathcal{H}' with $\mathcal{P}'_v \cong \mathcal{H}_\infty$ to obtain $\mathcal{H}_\infty^{\text{Man, st}}$, each arc in $\partial\Sigma \setminus \mathcal{Z}$ combines with the corresponding arc to form the boundary of the corresponding puncture in $\mathcal{H}_\infty^{\text{Man, st}}$.

Writing $\mathcal{H}_k^{\text{Man, st}}$ as $\mathcal{H}' \cup \mathcal{H}_k$, we see that $\mathcal{H}_{k+1}^{\text{Man, st}}$ is $\mathcal{H}'^{\text{H}} \cup \mathcal{H}_{k+1}$. Thus, we may directly apply Theorem 6.5 to conclude the proof. \square

7. Gradings

We have avoided any discussion of gradings so far, primarily because they are somewhat more complicated than usual in (bordered Floer theory and) bordered–sutured Floer theory.

In the following, we first give a brief overview of these gradings in Subsection 7.1, before proceeding to prove graded versions of Theorem 1.1 and Corollary 1.2. The strategy we employ reflects that of our proof of the ungraded versions in Section 3; in particular, we will first endow Proposition 3.5 with gradings, and then use the pairing theorem to extend these gradings to

the general case. Thus, we will compute the gradings on the algebra $\mathcal{A}(\mathcal{Z})$ in Subsection 7.2 and the gradings on the modules $\widehat{\text{BSD}}(\mathcal{B}_k)$ in Subsection 7.3, for \mathcal{Z} and \mathcal{B}_k as defined in Section 3. Finally, we state the graded statements and complete the proofs in Subsection 7.4.

7.1. Gradings in bordered–sutured Floer theory

Since Proposition 3.5 is stated for type D modules, we focus on the gradings on type D structures here. Again, we omit many details, and direct the interested reader to [Zar11, LOT18]. Note that we mostly follow the notation in [Zar11], which differs from that in [LOT18]. (See, however, Remark 7.1.)

Let $\mathcal{Y} = (Y, \Gamma, -\mathcal{Z}, \phi)$ be a bordered–sutured manifold, and $\mathcal{H} = (\Sigma, \alpha, \beta, -\mathcal{Z}, \psi)$ be a bordered–sutured diagram for \mathcal{Y} . First of all, the type D module $\widehat{\text{BSD}}(\mathcal{Y}) \simeq \widehat{\text{BSD}}(\mathcal{H})$ over $\mathcal{A}(\mathcal{Z})$ decomposes into direct-sum components corresponding to relative Spin^c -structures:

$$\widehat{\text{BSD}}(\mathcal{H}) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y, \partial Y \setminus F(-\mathcal{Z}))} \widehat{\text{BSD}}(\mathcal{H}, \mathfrak{s})$$

Here, the space $\text{Spin}^c(Y, \partial Y \setminus F(-\mathcal{Z}))$ of relative Spin^c -structures is an affine copy of $H^2(Y, \partial Y \setminus F(-\mathcal{Z})) \cong H_1(Y, F(-\mathcal{Z}))$. This decomposition arises from an assignment $\mathbf{x} \mapsto \mathfrak{s}(\mathbf{x})$ of an Euler structure to each generator (by the standard construction of a vector field), and the observation that the differential $\delta_{\widehat{\text{BSD}}(\mathcal{H})}$ respects this assignment. The set of all generators \mathbf{x} with $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}$ is denoted $\mathcal{G}(\mathcal{H}, \mathfrak{s})$. This direct sum decomposition is invariant for different Heegaard diagrams and choices of complex structures.

The structure of gradings in bordered–sutured Floer theory is as follows. The differential algebra $\mathcal{A}(\mathcal{Z})$ is graded by a possibly non-Abelian group G with a fixed central element $\lambda \in G$, via a map $g: \mathcal{A}(\mathcal{Z}) \rightarrow G$, such that for homogeneous elements a and b , we have

- $g(ab) = g(a)g(b)$; and
- $g(d(a)) = \lambda^{-1}g(a)$.

Then, each $\widehat{\text{BSD}}(\mathcal{H}, \mathfrak{s})$ has a (separate) *relative grading in a G -set*, by which we mean a left action by G on some set S , and a map $g: \mathcal{G}(\mathcal{H}, \mathfrak{s}) \rightarrow S$, such that, for a homogeneous element $a \in \mathcal{A}(\mathcal{Z})$ and a generator $\mathbf{x} \in \mathcal{G}(\mathcal{H}, \mathfrak{s})$, we have

- $g(a \otimes \mathbf{x}) = g(a) \cdot g(\mathbf{x})$; and

- $g(\delta(\mathbf{x})) = \lambda^{-1} \cdot g(\mathbf{x})$.

Two relative gradings g and g' , respectively in G -sets S and S' , are equivalent if there exists a bijective G -set map $\phi: S \rightarrow S'$ that commutes with g and g' .

Here, the story gets one more twist. There are in fact two related but distinct gradings on $\mathcal{A}(\mathcal{Z})$, and two corresponding ways to grade each $\widehat{\text{BSD}}(\mathcal{H}, \mathfrak{s})$. More concretely, choose a generator $\mathbf{w}_0 \in \mathcal{G}(\mathcal{H}, \mathfrak{s})$. One defines the (possibly non-Abelian) group $\text{Gr}(\mathcal{Z})$ and subgroup $\underline{\text{Gr}}(\mathcal{Z})$, and further defines (not necessarily normal) subgroups $\mathcal{P}(\mathbf{w}_0) \subset \text{Gr}(\mathcal{Z})$ and $\underline{\mathcal{P}}_r(\mathbf{w}_0) \subset \underline{\text{Gr}}(\mathcal{Z})$. Denoting the left cosets by

$$\begin{aligned} \text{Gr}(\mathcal{H}, \mathfrak{s}) &= \text{Gr}(\mathcal{Z})/\mathcal{P}(\mathbf{w}_0), \\ \underline{\text{Gr}}_r(\mathcal{H}, \mathfrak{s}) &= \underline{\text{Gr}}(\mathcal{Z})/\underline{\mathcal{P}}_r(\mathbf{w}_0), \end{aligned}$$

there is a natural left action by $\text{Gr}(\mathcal{Z})$ on $\text{Gr}(\mathcal{H}, \mathfrak{s})$, and by $\underline{\text{Gr}}(\mathcal{Z})$ on $\underline{\text{Gr}}_r(\mathcal{H}, \mathfrak{s})$. One then defines

- A map $\text{gr}: \mathcal{A}(\mathcal{Z}) \rightarrow \text{Gr}(\mathcal{Z})$;
- A map $\underline{\text{gr}}_r: \mathcal{A}(\mathcal{Z}) \rightarrow \underline{\text{Gr}}(\mathcal{Z})$;
- and
- and
- A map $\text{gr}: \mathcal{G}(\mathcal{H}, \mathfrak{s}) \rightarrow \text{Gr}(\mathcal{H}, \mathfrak{s})$,
- A map $\underline{\text{gr}}_r: \mathcal{G}(\mathcal{H}, \mathfrak{s}) \rightarrow \underline{\text{Gr}}_r(\mathcal{H}, \mathfrak{s})$,

such that gr (resp. $\underline{\text{gr}}_r$) provides a grading on $\mathcal{A}(\mathcal{Z})$ and a relative grading on $\widehat{\text{BSD}}(\mathcal{H}, \mathfrak{s})$. Unfortunately, while gr is independent of all choices made in its construction, the grading group $\text{Gr}(\mathcal{Z})$ is too large to support a graded version of the pairing theorem; on the other hand, while $\underline{\text{gr}}_r$ allows for a graded pairing theorem, it depends on some *refinement data* r on $\mathcal{A}(\mathcal{Z})$ (known as a *grading reduction* in [Zar11]), written as a subscript above. Since we will use the pairing theorem, we will focus on the $(\underline{\text{gr}}_r)$ -graded versions when we compute the gradings below.

To end this subsection, we mention here the structure of the graded pairing theorem. Let $\mathcal{Y}' = (Y', \Gamma', \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}, \phi')$ be another bordered–sutured manifold, and let \mathcal{Y}'' be the bordered–sutured manifold obtained by gluing \mathcal{Y}' and \mathcal{Y} along $\mathcal{F}(\mathcal{Z})$, with underlying manifold Y'' . Now let \mathcal{H}' be a bordered sutured diagram for \mathcal{Y}' , and let \mathfrak{s}' be a relative Spin^c -structure on $(Y', \partial Y' \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}))$. Choosing a generator $\mathbf{w}'_0 \in \mathcal{G}(\mathcal{H}', \mathfrak{s}')$, then, $\widehat{\text{BSDAA}}(\mathcal{H}', \mathfrak{s}')$ has a relative grading in

$$\underline{\text{Gr}}_{r_1, r_2, r}(\mathcal{H}', \mathfrak{s}') = \underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}),$$

which has a left action by $\underline{\text{Gr}}(-\mathcal{Z}_1)$ (equivalent to a right action by $\underline{\text{Gr}}(\mathcal{Z}_1)$) and a right action by $\underline{\text{Gr}}(\mathcal{Z}_2)$, and most pertinently, a right action by $\underline{\text{Gr}}(\mathcal{Z})$.

Suppose that \mathfrak{s}' is compatible with $\mathfrak{s} \in \text{Spin}^c(Y, \partial Y \setminus F(-\mathcal{Z}))$, i.e. that there exists some $\mathfrak{s}'' \in \text{Spin}^c(Y'', \partial Y'' \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$ whose restrictions to Y' and Y are \mathfrak{s}' and \mathfrak{s} respectively. The set of all such Spin^c -structures \mathfrak{s}'' , which we denote by $\text{Spin}^c(Y'', \partial Y'' \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2), \mathfrak{s}', \mathfrak{s})$, is an affine copy of

$$\text{Im}(\iota_*: H_1(F(\mathcal{Z})) \rightarrow H_1(Y'', F(\mathcal{Z}_1 \cup \mathcal{Z}_2))).$$

(Here, we have implicitly used Poincaré duality to regard the set of Spin^c -structures as an affine space of first homology, rather than the usual second cohomology.) Now Let $\mathbf{w}'_0 \in \mathcal{G}(\mathcal{H}', \mathfrak{s}')$ and $\mathbf{w}_0 \in \mathcal{G}(\mathcal{H}, \mathfrak{s})$. Then, viewing $\underline{\text{Gr}}(\mathcal{Z})$ as a subgroup of $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})$, the box tensor product $\widehat{\text{BSDAA}}(\mathcal{H}', \mathfrak{s}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}, \mathfrak{s})$ has a relative grading $\underline{\text{gr}}_{r_1, r_2, r}^\boxtimes$ with values in

$$\underline{\text{Gr}}_{r_1, r_2, r}(\mathcal{H}', \mathcal{H}, \mathfrak{s}', \mathfrak{s}) = \underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}) / \underline{\mathcal{P}}_r(\mathbf{w}_0).$$

The key feature of this grading, as shown in [LOT18, Zar11], is the following. Suppose the idempotents of \mathbf{w}'_0 and \mathbf{w}_0 match along \mathcal{Z} ; then there is a projection $\pi: \underline{\text{Gr}}_{r_1, r_2, r}(\mathcal{H}', \mathcal{H}, \mathfrak{s}', \mathfrak{s}) \rightarrow \text{Spin}^c(Y'', \partial Y'' \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$, such that

- π distinguishes different Spin^c -structures \mathfrak{s}'' in $(Y'', \partial Y'' \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$ that restrict to \mathfrak{s}' in Y' and \mathfrak{s} in Y ;
- Each fiber $\pi^{-1}(\mathfrak{s}'')$ can be identified with $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}, \mathfrak{s}'')$; and
- The homotopy equivalence

$$\begin{aligned} \Phi: \widehat{\text{BSDAA}}(\mathcal{H}', \mathfrak{s}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}, \mathfrak{s}) \\ \rightarrow \bigoplus_{\substack{\mathfrak{s}'' \in \text{Spin}^c(Y'', \partial Y'' \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2)) \\ \mathfrak{s}''|_{Y'} = \mathfrak{s}', \mathfrak{s}''|_Y = \mathfrak{s}}} \widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}, \mathfrak{s}'') \end{aligned}$$

respects the identification above, in the sense that, if $\Phi(\mathbf{x}) \in \widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}, \mathfrak{s}'')$, then $\underline{\text{gr}}_{r_1, r_2, r}^\boxtimes(\mathbf{x})$ is valued in $\pi^{-1}(\mathfrak{s}'')$, and the grading in this fiber is equivalent to the grading $\underline{\text{gr}}_{r_1, r_2}(\Phi(\mathbf{x}))$ valued in $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}, \mathfrak{s}'')$.

7.2. Grading on the algebra

We now turn to computing the gradings on the algebra $\mathcal{A}(\mathcal{Z}) = (\mathbf{Z}, \mathbf{a}, M)$. In fact, recall that in Subsection 3.1, we defined the index sets J' and J . Since we work with a type D module, it is sufficient for our purposes to

compute the gradings of $(j) = a_5(\{\rho_j\}) \in \mathcal{A}(\mathcal{Z}, 5)$ for $j \in J'$. (We may then compute the gradings of (j_1, \dots, j_ℓ) by group multiplication.)

7.2.1. The unrefined grading on the algebra. The unrefined $\text{Gr}(\mathcal{Z})$ -grading on $\mathcal{A}(\mathcal{Z})$ is defined as follows. For $a_i \in \mathbf{a}$ and $[\rho] \in H_1(\mathbf{Z}, \mathbf{a})$, let $m(a_i, [\rho])$ be the average multiplicity with which $[\rho]$ covers the regions on either side of a_i , and extend this linearly to a map $m: H_0(\mathbf{a}) \times H_1(\mathbf{Z}, \mathbf{a}) \rightarrow (1/2)\mathbb{Z}$. Define now $L: H_1(\mathbf{Z}, \mathbf{a}) \times H_1(\mathbf{Z}, \mathbf{a}) \rightarrow (1/2)\mathbb{Z}$ by

$$L([\rho], [\rho']) = m(\partial[\rho], [\rho']),$$

where ∂ is the (signed) connecting homomorphism in the long exact sequence for a pair. For $[\rho] \in H_1(\mathbf{Z}, \mathbf{a})$, we also define $\epsilon([\rho]) \in ((1/2)\mathbb{Z})/\mathbb{Z}$ by

$$\epsilon([\rho]) = \left(\frac{1}{4} \# \left\{ a_i \in \mathbf{a} \mid m(a_i, [\rho]) \equiv \frac{1}{2} \pmod{1} \right\} \right) \pmod{1}.$$

We now let

$$\text{Gr}(\mathcal{Z}) = \left\{ (n, [\rho]) \in \frac{1}{2}\mathbb{Z} \times H_1(\mathbf{Z}, \mathbf{a}) \mid n \equiv \epsilon([\rho]) \pmod{1} \right\}.$$

For $(n, [\rho]) \in \text{Gr}(\mathcal{Z})$, n is called the *Maslov component*, and $[\rho]$ the *homological component*.

Remark 7.1. The definitions of $\text{Gr}(\mathcal{Z})$ here, and consequently of $\underline{\text{Gr}}(\mathcal{Z})$ below, conform to the more recent definitions in [LOT18]. In [Zar11], $\text{Gr}(\mathcal{Z})$ is simply defined to be $(1/2)\mathbb{Z} \times H_1(\mathbf{Z}, \mathbf{a})$, which is twice as large as presented above and in [LOT18]. By [LOT18, Proposition 3.39], $\text{gr}(a)$ defined below is guaranteed to be in the smaller $\text{Gr}(\mathcal{Z})$ as defined here.

To define the grading of an element $a = I_s \cdot a_p(\rho) \cdot I_e \in \mathcal{A}(\mathcal{Z}, p)$, begin by choosing a value of $m_i \in M^{-1}(i) \subset \mathbf{a}$ for each unoccupied arc e_i that is in both I_s and I_e . (Informally, this is equivalent to making one of each matched pair of horizontal dotted lines solid in the strands diagram associated to a .) Let ϕ be the corresponding strands diagram, and let $\text{inv}(\phi)$ be the minimum number of crossings in ϕ . Further, let

$$S = \{m_i \mid e_i \text{ is in both } I_s \text{ and } I_e\} \cup \rho^- \subset \mathbf{a}$$

be the set of all starting points of ϕ (consisting of exactly p elements), and let $[S]$ denote the corresponding element of $H_0(\mathbf{a})$. The grading of a is then

defined to be

$$\text{gr}(a) = (\text{inv}(\phi) - m([S], [\rho]), [\rho]) \in \text{Gr}(\mathcal{Z}).$$

According to [Zar11, Proposition 2.4.5], $\text{gr}(a)$ is independent of the choice of values of m_i , and gr gives a well-defined grading on $\mathcal{A}(\mathcal{Z})$, with central element $\lambda = (1, 0)$. Furthermore, $\text{gr}(a)$ turns out to be independent of p , I_s , and I_e .

Focusing on our context, consider now some $(j) = a_5(\{\rho_j\}) \in \mathcal{A}(\mathcal{Z}, 5)$ with $j \in J'$. By the last sentence of the previous paragraph, we may easily compute

$$\text{gr}((j)) = \text{gr}(a_5(\{\rho_j\})) = \text{gr}(a_1(\{\rho_j\})),$$

where the strands diagram ϕ for $a_1(\{\rho_j\})$ has no horizontal dotted lines at all. This means that $\text{inv}(\phi) = 0$ and $S = \{\rho^-\}$, allowing us to compute

$$\text{gr}((j)) = \text{gr}(a_1(\{\rho_j\})) = \left(-\frac{1}{2}, [\rho_j]\right).$$

For economy and clarity, from now on we will write $\text{gr}(j)$ instead of $\text{gr}((j))$.

7.2.2. The refined grading on the algebra. For $\mathcal{A}(\mathcal{Z}) = (\mathbf{Z}, \mathbf{a}, M)$, the matching M induces a map $M_*: H_0(\mathbf{a}) \rightarrow H_0(\{1, \dots, 6\})$. Again denoting by ∂ the connecting homomorphism, let $\partial' = M_* \circ \partial$. The refined grading group is defined to be the subgroup

$$\underline{\text{Gr}}(\mathcal{Z}) = \{(n, [\rho]) \in \text{Gr}(\mathcal{Z}) \mid \partial'[\rho] = 0\} \subset \text{Gr}(\mathcal{Z}),$$

consisting of elements with homological component in $\ker \partial' \cong H_1(F(\mathcal{Z}))$. In fact, this allows us to view $\underline{\text{Gr}}(\mathcal{Z})$ as a central extension of $H_1(F(\mathcal{Z}))$.

In our context, $\ker \partial'$ is generated by $[\rho_{123}]$, $[\rho_{456}]$, and $[\rho_{78}]$. Moreover, it is easy to see that L is identically zero on $\ker \partial' \times \ker \partial'$, and so $\underline{\text{Gr}}(\mathcal{Z})$ is Abelian and in fact isomorphic to $\mathbb{Z} \oplus \mathbb{Z}^3$, with generators

$$\lambda = (1, 0), \quad A_1 = \left(-\frac{1}{2}, [\rho_{123}]\right), \quad A_2 = \left(-\frac{1}{2}, [\rho_{456}]\right), \quad A_3 = \left(-\frac{3}{2}, [\rho_{78}]\right).$$

The generators here are chosen to facilitate our discussion below. (Also, observe that, for example, $(0, [\rho_{123}]) \notin \text{Gr}(\mathcal{Z})$; see Remark 7.1.)

In [Zar11], in order to define the refinement data necessary to define $\underline{\text{gr}}_r$, the notion of *connected components* of idempotents is introduced: Two idempotents I and I' are connected if $I - I'$ is in the image of ∂' . Now let

$\pi_{\text{hom}} : \text{Gr}(\mathcal{Z}) \rightarrow H_1(\mathbf{Z}, \mathbf{a})$ be the projection homomorphism; the refinement data then consists of a *base idempotent* I_0 in each connected component, and an element $r(I) \in (\partial' \circ \pi_{\text{hom}})^{-1}(I - I_0) \subset \text{Gr}(\mathcal{Z})$ for each I in the same component as I_0 . Then $\underline{\text{gr}}_r : \mathcal{A}(\mathcal{Z}) \rightarrow \underline{\text{Gr}}(\mathcal{Z})$ is defined by

$$\underline{\text{gr}}_r(a) = r(I_s) \text{gr}(a) r(I_e)^{-1},$$

for a generator $a = I_s \cdot a \cdot I_e \in \mathcal{A}(\mathcal{Z})$.

For our \mathcal{Z} , it is obvious that there is exactly one connected component of idempotents. Thus, we may arbitrarily choose any idempotent in $\mathcal{I}(\mathcal{Z}, 5)$ to be I_0 . We choose $I_0 = I_{\bar{6}}$ for convenience of computation, and choose

$$\begin{aligned} r(I_{\bar{1}}) &= \left(-\frac{1}{2}, [\rho_2] + [\rho_5] + [\rho_8] \right), \\ r(I_{\bar{2}}) &= \left(-\frac{1}{2}, [\rho_1] + [\rho_2] + [\rho_5] + [\rho_8] \right), \\ r(I_{\bar{3}}) &= (0, [\rho_5] + [\rho_8]), \\ r(I_{\bar{4}}) &= (0, [\rho_4] + [\rho_5] + [\rho_8]), \\ r(I_{\bar{5}}) &= \left(-\frac{1}{2}, [\rho_8] \right), \\ r(I_{\bar{6}}) &= (0, 0). \end{aligned}$$

(Again, see Remark 7.1.) Then, as an example, we may compute $\underline{\text{gr}}_r(7, 8)$ as follows. Since $(7, 8) = I_{\bar{5}} \cdot (78) \cdot I_{\bar{5}}$, we have

$$\begin{aligned} \underline{\text{gr}}_r(7, 8) &= r(I_{\bar{5}}) \text{gr}(78) r(I_{\bar{5}})^{-1} \\ &= \left(-\frac{1}{2}, [\rho_8] \right) \left(-\frac{1}{2}, [\rho_{78}] \right) \left(\frac{1}{2}, -[\rho_8] \right) = \left(-\frac{3}{2}, [\rho_{78}] \right). \end{aligned}$$

Similarly, we have $\underline{\text{gr}}_r(1, 23) = \underline{\text{gr}}_r(12, 3) = (-1/2, [\rho_{123}])$ and $\underline{\text{gr}}_r(4, 56) = \underline{\text{gr}}_r(45, 6) = (-1/2, [\rho_{456}])$.

7.2.3. A skein reduction. We now make a further reduction in the grading set, which is necessary for stating a graded version of Theorem 1.1: We let

$$\begin{aligned} \mathcal{P}_r^{\text{sk}} &= \langle A_1, A_2, A_3 \rangle, \\ \underline{\text{Gr}}_r^{\text{sk}} &= \underline{\text{Gr}}(\mathcal{Z}) / \mathcal{P}_r^{\text{sk}} \cong \mathbb{Z}. \end{aligned}$$

We will denote an element in $\underline{\mathbf{G}}_r^{\text{sk}}$ by an integer, given by sending the class $\lambda \cdot \underline{\mathcal{P}}_r^{\text{sk}}$ to 1. (The reason we chose the symbol $\underline{\mathcal{P}}_r^{\text{sk}}$ will become clear in the next subsection.)

Composing $\underline{\text{gr}}_r$ with the canonical projection, we thus obtain a map $\underline{\text{gr}}_r^{\text{sk}}: \mathcal{A}(\mathcal{Z}) \rightarrow \underline{\mathbf{G}}_r^{\text{sk}}$, which we call the *skein-reduced grading*. We may now compute all colored algebra elements appearing in Figure 11:

$$\begin{array}{ll}
 \underline{\text{gr}}_r(I) = (0, 0), & \underline{\text{gr}}_r^{\text{sk}}(I) = 0, \\
 \underline{\text{gr}}_r(4, 6) = (-1/2, [\rho_{456}]), & \underline{\text{gr}}_r^{\text{sk}}(4, 6) = 0, \\
 \underline{\text{gr}}_r(4, 6, 7, 8) = (-2, [\rho_{456}] + [\rho_{78}]), & \underline{\text{gr}}_r^{\text{sk}}(4, 6, 7, 8) = 0, \\
 \underline{\text{gr}}_r(4, 56) = (-1/2, [\rho_{456}]), & \underline{\text{gr}}_r^{\text{sk}}(4, 56) = 0, \\
 \underline{\text{gr}}_r(12, 3) = (-1/2, [\rho_{123}]), & \underline{\text{gr}}_r^{\text{sk}}(12, 3) = 0, \\
 \underline{\text{gr}}_r(2) = (0, 0), & \underline{\text{gr}}_r^{\text{sk}}(2) = 0, \\
 \underline{\text{gr}}_r(45, 6) = (-1/2, [\rho_{456}]), & \underline{\text{gr}}_r^{\text{sk}}(45, 6) = 0, \\
 \underline{\text{gr}}_r(7, 8) = (-3/2, [\rho_{78}]), & \underline{\text{gr}}_r^{\text{sk}}(7, 8) = 0, \\
 \underline{\text{gr}}_r(12, 3, 4, 6) = (-1, [\rho_{123}] + [\rho_{456}]), & \underline{\text{gr}}_r^{\text{sk}}(12, 3, 4, 6) = 0, \\
 \underline{\text{gr}}_r(4, 56, 2) = (-1/2, [\rho_{456}]), & \underline{\text{gr}}_r^{\text{sk}}(4, 56, 2) = 0, \\
 \underline{\text{gr}}_r(12, 3, 4, 56, 2) = (-1, [\rho_{123}] + [\rho_{456}]), & \underline{\text{gr}}_r^{\text{sk}}(12, 3, 4, 56, 2) = 0, \\
 \underline{\text{gr}}_r(1, 23) = (-1/2, [\rho_{123}]), & \underline{\text{gr}}_r^{\text{sk}}(1, 23) = 0, \\
 \underline{\text{gr}}_r(45, 6, 7, 8, 5) = (-3, [\rho_{456}] + [\rho_{78}]), & \underline{\text{gr}}_r^{\text{sk}}(45, 6, 7, 8, 5) = -1, \\
 \underline{\text{gr}}_r(7, 8, 5) = (-5/2, [\rho_{78}]), & \underline{\text{gr}}_r^{\text{sk}}(7, 8, 5) = -1, \\
 \underline{\text{gr}}_r(5) = (-1, 0), & \underline{\text{gr}}_r^{\text{sk}}(5) = -1, \\
 \underline{\text{gr}}_r(5, 12, 3) = (-3/2, [\rho_{123}]), & \underline{\text{gr}}_r^{\text{sk}}(5, 12, 3) = -1, \\
 \underline{\text{gr}}_r(5, 12, 3, 4, 56) = (-2, [\rho_{123}] + [\rho_{456}]), & \underline{\text{gr}}_r^{\text{sk}}(5, 12, 3, 4, 56) = -1, \\
 \underline{\text{gr}}_r(1, 3) = (-3/2, [\rho_{123}]), & \underline{\text{gr}}_r^{\text{sk}}(1, 3) = -1, \\
 \underline{\text{gr}}_r(1, 3, 4, 56) = (-2, [\rho_{123}] + [\rho_{456}]), & \underline{\text{gr}}_r^{\text{sk}}(1, 3, 4, 56) = -1.
 \end{array}$$

7.3. Gradings on the modules

We now compute the gradings on $\widehat{\text{BSD}}(\mathcal{B}_k) \simeq \widehat{\text{BSD}}(\mathcal{H}_k)$.

7.3.1. Unrefined and refined gradings on the modules. The gradings on the modules are defined in [Zar11] as follows. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{G}(\mathcal{H}, \mathfrak{s})$. First, for each homology class $B \in \pi_2(\mathbf{w}_2, \mathbf{w}_1)$ with domain $[B]$, its unrefined

and refined gradings are defined by

$$\begin{aligned} \text{gr}(B) &= (-e([B]) - n_{\mathbf{w}_1}([B]) - n_{\mathbf{w}_2}([B]), \partial^\theta[B]) \in \text{Gr}(\mathcal{Z}), \\ \underline{\text{gr}}_r(B) &= r(I_D(\mathbf{w}_2)) \text{gr}(B) r(I_D(\mathbf{w}_1))^{-1} \in \underline{\text{Gr}}_r(\mathcal{Z}), \end{aligned}$$

where $e([B])$ denotes the Euler measure of $[B]$, $n_{\mathbf{w}}([B])$ the sum of the average local multiplicities of $[B]$ in the four regions adjacent to the intersection points of \mathbf{w} , and $I_D(\mathbf{w})$ the type D idempotent of \mathbf{w} . (For the reader that is comparing the above with [Zar11, Definition 6.1.1], the order in which $r(I_D(\mathbf{w}_1))$ and $r(I_D(\mathbf{w}_2))$ appear in the second equation arises from the fact that we are working with $\underline{\text{Gr}}_r(\mathcal{Z})$ instead of $\underline{\text{Gr}}(-\mathcal{Z})$; see [LOT18, Section 10.4 and Section 10.5].)

Then, for each $\mathfrak{s} \in \text{Spin}^c(Y, \partial Y \setminus F(-\mathcal{Z}))$, one picks a generator $\mathbf{w}_0 \in \mathcal{G}(\mathcal{H}, \mathfrak{s})$, called a *base generator*, and defines the *stabilizer subgroups*

$$\begin{aligned} \mathcal{P}(\mathbf{w}_0) &= \{ \text{gr}(B) \mid B \in \pi_2(\mathbf{w}_0, \mathbf{w}_0) \} \subset \text{Gr}(\mathcal{Z}), \\ \underline{\mathcal{P}}_r(\mathbf{w}_0) &= \left\{ \underline{\text{gr}}_r(B) \mid B \in \pi_2(\mathbf{w}_0, \mathbf{w}_0) \right\} \subset \underline{\text{Gr}}_r(\mathcal{Z}). \end{aligned}$$

As mentioned in Subsection 7.1, the grading sets $\text{Gr}(\mathcal{H}, \mathfrak{s})$ and $\underline{\text{Gr}}_r(\mathcal{H}, \mathfrak{s})$ are respectively defined as the set of left cosets $\text{Gr}(\mathcal{Z})/\mathcal{P}(\mathbf{w}_0)$ and $\underline{\text{Gr}}_r(\mathcal{Z})/\underline{\mathcal{P}}_r(\mathbf{w}_0)$. For a generator $\mathbf{w} \in \mathcal{G}(\mathcal{H}, \mathfrak{s})$, then, one picks a homology class $B_0 \in \pi_2(\mathbf{w}, \mathbf{w}_0)$, and the gradings of \mathbf{w} are defined by

$$\begin{aligned} \text{gr}(\mathbf{w}) &= \text{gr}(B_0) \cdot \mathcal{P}(\mathbf{w}_0) \in \text{Gr}(\mathcal{H}, \mathfrak{s}), \\ \underline{\text{gr}}_r(\mathbf{w}) &= \underline{\text{gr}}_r(B_0) \cdot \underline{\mathcal{P}}_r(\mathbf{w}_0) \in \underline{\text{Gr}}_r(\mathcal{H}, \mathfrak{s}). \end{aligned}$$

The map $\underline{\text{gr}}_r : \mathcal{G}(\mathcal{H}, \mathfrak{s}) \rightarrow \underline{\text{Gr}}_r(\mathcal{H}, \mathfrak{s})$ gives a relative grading by a $\underline{\text{Gr}}_r(\mathcal{Z})$ -set, which is independent of the choice of \mathbf{w}_0 and B_0 (but not r in general).

In our context, first note that, for each $k \in \{\infty, 0, 1\}$, we have $H_1(B_k, F(-\mathcal{Z})) = 0$, and so there is a unique $\widehat{\text{Spin}}^c$ -structure $\mathfrak{s}_k \in \text{Spin}^c(B_k, \partial B_k \setminus F(-\mathcal{Z}))$. This means that $\widehat{\text{BSD}}(\mathcal{H}_k) = \widehat{\text{BSD}}(\mathcal{H}_k, \mathfrak{s}_k)$; in other words, $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in \mathcal{G}(\mathcal{H}_\infty, \mathfrak{s}_\infty)$ share the same grading set, and so do $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{G}(\mathcal{H}_0, \mathfrak{s}_0)$. In the following, we focus on the refined gradings.

Arbitrarily, we pick $\mathbf{x} \in \mathcal{G}(\mathcal{H}_1)$, $\mathbf{y}_3 \in \mathcal{G}(\mathcal{H}_\infty)$, and $\mathbf{z}_1 \in \mathcal{G}(\mathcal{H}_0)$ to define our grading sets. Since there is only one region R in \mathcal{H}_1 not adjacent to $\partial\Sigma \setminus \mathbf{Z}$, we see that $\pi_2(\mathbf{x}, \mathbf{x})$ is generated by the homology class B with domain $[B] = R$. Seeing $I_D(\mathbf{x}) = I_{\bar{3}}$, we may thus compute $e([B]) = -2$ and

$n_{\mathbf{x}}([B]) = 1/2$ to obtain

$$\begin{aligned} \underline{\text{gr}}_r(B) &= (0, [\rho_5] + [\rho_8])(1, -[\rho_{456}] - [\rho_{78}]) (0, -[\rho_5] - [\rho_8]) \\ &= (2, -[\rho_{456}] - [\rho_{78}]) = -A_2 - A_3. \end{aligned}$$

This means that $\underline{\mathcal{P}}_r(\mathbf{x}) = \langle -A_2 - A_3 \rangle \subset \underline{\text{Gr}}(\mathcal{Z})$, and so $\underline{\text{Gr}}_r(\mathcal{H}_1, \mathfrak{s}_1) = \langle \lambda, A_1, A_2, A_3 \rangle / \langle -A_2 - A_3 \rangle$. Similarly, we obtain that $\underline{\mathcal{P}}_r(\mathbf{y}_3) = \langle -A_1 - A_3 \rangle$, and $\underline{\mathcal{P}}_r(\mathbf{z}_1) = \langle -A_1 - A_2 \rangle$.

Clearly, $\underline{\text{gr}}_r(\mathbf{x}) = \underline{\mathcal{P}}_r(\mathbf{x}) \in \underline{\text{Gr}}_r(\mathcal{H}_1, \mathfrak{s}_1)$, and similarly for \mathbf{y}_3 and \mathbf{z}_1 . To compute $\underline{\text{gr}}_r(\mathbf{y}_2)$, for example, one might choose $B_0 \in \pi_2(\mathbf{y}_2, \mathbf{y}_3)$ to be the homology class that corresponds to the term $(1, 3) \otimes \mathbf{y}_2$ in $\delta(\mathbf{y}_3)$, to see that

$$\underline{\text{gr}}_r(\mathbf{y}_2) = \underline{\text{gr}}_r(B_0) \cdot \underline{\mathcal{P}}_r(\mathbf{y}_3) = \left(\frac{1}{2}, -[\rho_{123}] \right) \cdot \underline{\mathcal{P}}_r(\mathbf{y}_3) = -A_1 \cdot \langle -A_1 - A_3 \rangle.$$

Alternatively, one could use the fact that $(1, 3) \otimes \mathbf{y}_2$ appears in $\delta(\mathbf{y}_3)$ and that the grading shift of δ is $-\lambda$ to arrive at the same conclusion.

7.3.2. Skein-reduced gradings. Our goal is to show that the maps f_k , φ_k , and κ_k respect gradings in a suitable sense. Towards this goal, we now recall the definitions of $\underline{\mathcal{P}}_r^{\text{sk}}$ and $\underline{\text{Gr}}_r^{\text{sk}}$ in Subsubsection 7.2.3:

$$\begin{aligned} \underline{\mathcal{P}}_r^{\text{sk}} &= \langle A_1, A_2, A_3 \rangle, \\ \underline{\text{Gr}}_r^{\text{sk}} &= \underline{\text{Gr}}(\mathcal{Z}) / \underline{\mathcal{P}}_r^{\text{sk}} \cong \mathbb{Z}. \end{aligned}$$

Since $\underline{\mathcal{P}}_r(\mathbf{x})$, $\underline{\mathcal{P}}_r(\mathbf{y}_3)$, and $\underline{\mathcal{P}}_r(\mathbf{z}_1)$ are all subgroups of $\underline{\mathcal{P}}_r^{\text{sk}}$, we obtain maps $\underline{\text{gr}}_r^{\text{sk}} : \mathcal{G}(\mathcal{H}_k, \mathfrak{s}_k) \rightarrow \underline{\text{Gr}}_r^{\text{sk}}$ for each $k \in \{\infty, 0, 1\}$, which are relative gradings in the $\underline{\text{Gr}}(\mathcal{Z})$ -set $\underline{\text{Gr}}_r^{\text{sk}}$. In other words, we have united the three gradings on \mathcal{H}_k . We also call these gradings the *skein-reduced gradings*.

We record the gradings of all generators below. Again, we denote elements of $\underline{\text{Gr}}_r^{\text{sk}}$ by the integer obtained by sending $\lambda \cdot \underline{\mathcal{P}}_r^{\text{sk}}$ to 1.

$$\begin{array}{ll} \underline{\text{gr}}_r(\mathbf{x}) = \langle -A_2 - A_3 \rangle, & \underline{\text{gr}}_r^{\text{sk}}(\mathbf{x}) = 0, \\ \underline{\text{gr}}_r(\mathbf{y}_1) = A_3 \cdot \langle -A_1 - A_3 \rangle, & \underline{\text{gr}}_r^{\text{sk}}(\mathbf{y}_1) = 0, \\ \underline{\text{gr}}_r(\mathbf{y}_2) = -A_1 \cdot \langle -A_1 - A_3 \rangle, & \underline{\text{gr}}_r^{\text{sk}}(\mathbf{y}_2) = 0, \\ \underline{\text{gr}}_r(\mathbf{y}_3) = \langle -A_1 - A_3 \rangle, & \underline{\text{gr}}_r^{\text{sk}}(\mathbf{y}_3) = 0, \\ \underline{\text{gr}}_r(\mathbf{z}_1) = \langle -A_1 - A_2 \rangle, & \underline{\text{gr}}_r^{\text{sk}}(\mathbf{z}_1) = 0, \\ \underline{\text{gr}}_r(\mathbf{z}_2) = \langle -A_1 - A_2 \rangle, & \underline{\text{gr}}_r^{\text{sk}}(\mathbf{z}_2) = 0. \end{array}$$

Finally, we mention that, for $a \in \mathcal{A}(\mathcal{Z})$ and $\mathbf{w} \in \mathcal{G}(\mathcal{H}_k)$, we have

$$\underline{\text{gr}}_r^{\text{sk}}(a \otimes \mathbf{w}) = \underline{\text{gr}}_r(a) \cdot \underline{\text{gr}}_r^{\text{sk}}(\mathbf{w}) = \underline{\text{gr}}_r^{\text{sk}}(a) + \underline{\text{gr}}_r^{\text{sk}}(\mathbf{w}),$$

since the action by $\underline{\text{gr}}_r(a) \in \underline{\text{Gr}}(\mathcal{Z})$ on $\underline{\text{Gr}}_r^{\text{sk}}$ is exactly given by its value $\underline{\text{gr}}_r^{\text{sk}}(a)$ under the canonical projection.

7.4. Graded versions of results and their proofs

We are now ready to state and prove a graded version of Proposition 3.5.

Proposition 7.2. *With our choice of refinement data r , the morphisms $f_k : \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+1})$ in Proposition 3.5 preserve the relative gradings $\underline{\text{gr}}_r^{\text{sk}}$ in the $\underline{\text{Gr}}(\mathcal{Z})$ -set $\underline{\text{Gr}}_r^{\text{sk}} \cong \mathbb{Z}$, with grading shifts given by the following diagram:*

$$\begin{array}{ccc} \widehat{\text{BSD}}(\mathcal{B}_0) & \xrightarrow{-1} & \widehat{\text{BSD}}(\mathcal{B}_1) \\ & \searrow 0 & \swarrow 0 \\ & \widehat{\text{BSD}}(\mathcal{B}_\infty) & \end{array}$$

(The reader is reminded that there is a $\underline{\text{Gr}}(\mathcal{Z})$ -action on the grading set $\underline{\text{Gr}}_r^{\text{sk}}$ for each of the three modules $\widehat{\text{BSD}}(\mathcal{B}_k)$, shifting by any integer; accordingly, the diagram above should be interpreted up to simultaneous and opposite integer shifts on any two consecutive arrows.) The homotopy equivalence

$$\widehat{\text{BSD}}(\mathcal{B}_k) \simeq \text{Cone}(f_{k+1} : \widehat{\text{BSD}}(\mathcal{B}_{k+1}) \rightarrow \widehat{\text{BSD}}(\mathcal{B}_{k+2}))$$

also respects the relative gradings $\underline{\text{gr}}_r^{\text{sk}}$.

Proof. Combining Figure 11, Figure 13, and the grading computations in Subsubsection 7.2.3 and Subsubsection 7.3.2, we see that the maps \bar{f}_k in Section 5 preserve the relative gradings $\underline{\text{gr}}_r^{\text{sk}}$, and have grading shifts indicated by the diagram. Now $\bar{\varphi}_k$ can also be easily checked to be homogeneous, and have the appropriate grading shifts, using the same grading computations. Recalling that $\kappa_k \equiv 0$ in Lemma 5.4, this means that all morphisms in Lemma 2.1 are graded, implying that the homotopy equivalence also respects the gradings. (Note that Proposition 5.5 implies that the gradings are the same for f_k and φ_k as defined in Section 4.) □

Remark 7.3. A version of the statements in the proposition above should be true for every choice of refinement data r . Here, one would possibly have

to change the basis of $\underline{\text{Gr}}(\mathcal{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}^3$, i.e. to pick different generators A_1 , A_2 , and A_3 , in order to define $\underline{\mathcal{P}}_r^{\text{sk}}$ and $\underline{\text{Gr}}_r^{\text{sk}}$. In fact, one could first work with the unrefined gradings gr to obtain an unrefined version of the proposition, and only then, refine the gradings. For economy and clarity, we elected not to present this approach; this saved us, for example, from computations in the quotient of $\text{Gr}(\mathcal{Z})$ by the normal closure of $\{A_1, A_2, A_3\}$.

Remark 7.4. One could also remove the indeterminacy of simultaneous and opposite integer shifts as follows. Instead of considering domain gradings for bigons in the three Heegaard diagrams \mathcal{H}_1 , \mathcal{H}_∞ , and \mathcal{H}_0 in Figure 7 separately, one may introduce domain gradings for polygons in the bordered–sutured multi-diagram $\mathcal{H}_{\alpha,\beta}$ in Figure 10. Again, for simplicity, we opted not to take this approach.

Remark 7.5. The astute reader might have noticed in Subsubsection 7.3.2 that $\underline{\mathcal{P}}_r^{\text{sk}}$ is not the smallest subgroup $\underline{\mathcal{P}}_r^{\text{min}}$ containing $\underline{\mathcal{P}}_r(\mathbf{x})$, $\underline{\mathcal{P}}_r(\mathbf{y}_3)$, and $\underline{\mathcal{P}}_r(\mathbf{z}_1)$. Indeed, while $\underline{\text{Gr}}(\mathcal{Z})/\underline{\mathcal{P}}_r^{\text{sk}} \cong \mathbb{Z}$, we have $\underline{\text{Gr}}(\mathcal{Z})/\underline{\mathcal{P}}_r^{\text{min}} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In fact, it is easy to check that, for each $k \in \{\infty, 0, 1\}$, the morphisms $f_{k,0}$ and $\bar{f}_{k,1}$ are both homogeneous with respect to the gradings in $\underline{\text{Gr}}(\mathcal{Z})/\underline{\mathcal{P}}_r^{\text{min}}$, but their grading shifts differ by the generator of the $\mathbb{Z}/2\mathbb{Z}$ factor. If $\mathcal{Y}_k = \mathcal{Y}' \cup_{\mathcal{F}(\mathcal{Z})} \mathcal{B}_k$ is the sutured manifold associated to a link $L \subset S^3$, then the $\mathbb{Z}/2\mathbb{Z}$ factor may be seen as the Alexander grading on $\widehat{\text{HFK}}(S^3, L)$ modulo 2. Assuming that Proposition 6.3 holds with gradings, one could use this fact, as well as the fact that $\eta_0^{k < k+1}$ and $\eta_1^{k < k+1}$ differ in mod-2 Maslov gradings, to obtain the mod-2 grading reduction of the δ -graded skein relation in [MO08] (with gradings suitably shifted).

Next, we turn to a graded version of Theorem 3.6. Let $\mathcal{Y}' = (Y', \Gamma', \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}, \phi')$ be the bordered–sutured manifold defined in Subsection 3.3, with $\mathcal{Y}' \cup_{F(\mathcal{Z})} \mathcal{B}_k = \mathcal{Y}_k$, and let \mathcal{H}' be a bordered–sutured diagram for \mathcal{Y}' ; then

$$\widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_k) \simeq \widehat{\text{BSDA}}(\mathcal{Y}_k).$$

For $\mathfrak{s}' \in \text{Spin}^c(Y', \partial Y' \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}))$, pick a generator $\mathbf{w}'_0 \in \mathcal{G}(\mathcal{H}', \mathfrak{s}')$. By the graded pairing theorem mentioned in Subsection 7.1, $\widehat{\text{BSDAA}}(\mathcal{H}', \mathfrak{s}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_k)$ has a relative grading $\underline{\text{gr}}_{r_1, r_2, r}^{\boxtimes}$ with values in

$$\underline{\text{Gr}}_{r_1, r_2, r}(\mathcal{H}', \mathcal{H}_k, \mathfrak{s}', \mathfrak{s}_k) = \underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}) / \underline{\mathcal{P}}_r(\mathbf{w}_{0,k}),$$

where $\mathbf{w}_{0,k}$ is \mathbf{x} , \mathbf{y}_3 , and \mathbf{z}_1 , for $k = 1, 0$, and ∞ , respectively. We may, as before, take a skein reduction to get a relative grading $\underline{\text{gr}}_{r_1, r_2, r}^{\boxtimes, \text{sk}}$ with values

in the set

$$\underline{\text{Gr}}_{r_1, r_2, r}^{\text{sk}}(\mathcal{H}', \mathcal{H}_k, \mathfrak{s}', \mathfrak{s}_k) = \mathcal{P}_{r_1, r_2, r}(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}) / \mathcal{P}_r^{\text{sk}}.$$

Proposition 7.6. *For each $\mathfrak{s}' \in \text{Spin}^c(Y', \partial Y' \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}))$, and with our choice of refinement data r , the morphisms*

$$\mathbb{I}_{\widehat{\text{BSDAA}}(\mathcal{Y}')} \boxtimes f_k : \widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_{k+1})$$

preserve the relative gradings $\underline{\text{gr}}_{r_1, r_2, r}^{\boxtimes, \text{sk}}$ in the $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ -set

$$\underline{\text{Gr}}_{r_1, r_2, r}^{\text{sk}}(\mathcal{H}', \mathcal{H}_k, \mathfrak{s}', \mathfrak{s}_k),$$

with grading shifts given by the following diagram:

$$\begin{array}{ccc} \widehat{\text{BSDAA}}(\mathcal{H}', \mathfrak{s}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_0) & \xrightarrow{\lambda^{-1} = (-1, 0)} & \widehat{\text{BSDAA}}(\mathcal{H}', \mathfrak{s}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_1) \\ & \swarrow (0, 0) & \searrow (0, 0) \\ & \widehat{\text{BSDAA}}(\mathcal{H}', \mathfrak{s}') \boxtimes \widehat{\text{BSD}}(\mathcal{H}_\infty) & \end{array}$$

(Again, the diagram above should be interpreted up to simultaneous and inverse (right) actions by elements of $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ on any two consecutive arrows.) The homotopy equivalence

$$\widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_k) \simeq \text{Cone}(\mathbb{I}_{\widehat{\text{BSDAA}}(\mathcal{Y}')} \boxtimes f_{k+1})$$

also respects the relative gradings $\underline{\text{gr}}_{r_1, r_2, r}^{\boxtimes, \text{sk}}$.

Proof. Given a G -set-graded type D morphism $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$, with grading shifts in G/\mathcal{P} , it is straightforward to check that, for any G -set-graded type A module \mathcal{N} with gradings in $\mathcal{P}' \backslash G$, we have that $\mathbb{I}_{\mathcal{N}} \boxtimes f : \mathcal{N} \boxtimes \mathcal{M}_1 \rightarrow \mathcal{N} \boxtimes \mathcal{M}_2$ is a \mathbb{Z} -set graded chain map with grading shifts given by the obvious reduction to $\mathcal{P}' \backslash G/\mathcal{P}$. (More generally, the functor $(\mathcal{N} \boxtimes -, \mathbb{I}_{\mathcal{N}} \boxtimes -)$ can be upgraded to a dg functor in an appropriate sense; see [LOT15, Example 2.5.35].)

The statement above generalizes in an obvious manner to the case where \mathcal{N} is a type DAA trimodule. Specializing this generalization to our context would mean the following: Given the $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})$ -set-graded morphism $f_k : \widehat{\text{BSD}}(\mathcal{H}_k) \rightarrow \widehat{\text{BSD}}(\mathcal{H}_{k+1})$, with gradings in $\underline{\text{Gr}}(\mathcal{Z})/\mathcal{P}_r^{\text{sk}} \subset \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})/\mathcal{P}_r^{\text{sk}}$, the morphism $\mathbb{I}_{\widehat{\text{BSDAA}}(\mathcal{H}', \mathfrak{s}')} \boxtimes f_k$ is a $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ -set-graded morphism, with gradings given by the reduction to $\mathcal{P}_{r_1, r_2, r}(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})/\mathcal{P}_r^{\text{sk}}$.

Recalling that the grading shift -1 is a shorthand for $\lambda^{-1} \in \underline{\text{Gr}}(\mathcal{Z})/\underline{\mathcal{P}}_r^{\text{sk}} \subset \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})/\underline{\mathcal{P}}_r^{\text{sk}}$, the proposition now follows directly from Proposition 7.2. \square

We now work towards interpreting the result above in terms of the Spin^c -structures and intrinsic gradings on $\mathcal{Y}_k = \mathcal{Y}' \cup_{\mathcal{F}(\mathcal{Z})} \mathcal{B}_k$. To do so, we must first understand the grading set $\underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})/\underline{\mathcal{P}}_r^{\text{sk}}$, or in fact

$$\underline{\text{Gr}}_{r_1, r_2, r}(\mathcal{H}', \mathcal{H}_k, \mathfrak{s}', \mathfrak{s}_k) = \underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})/\underline{\mathcal{P}}_r(\mathbf{w}_{0,k}),$$

in terms of the discussion in Subsection 7.1.

First, assume for the moment that there exist generators $\mathbf{w}'_1, \mathbf{w}'_3, \mathbf{w}'_5 \in \mathcal{G}(\mathcal{H}', \mathfrak{s}')$ in idempotents I_1, I_3 , and I_5 respectively; then

- \mathbf{w}'_1 is complementary to $\mathbf{y}_3 = \mathbf{w}_{0,\infty}$ and \mathbf{z}_2 ;
- \mathbf{w}'_3 is complementary to $\mathbf{x} = \mathbf{w}_{0,1}$ and \mathbf{y}_2 ; and
- \mathbf{w}'_5 is complementary to \mathbf{y}_1 and $\mathbf{z}_1 = \mathbf{w}_{0,0}$.

Let $\mathbf{w}'_{0,k}$ be $\mathbf{w}'_3, \mathbf{w}'_1$, and \mathbf{w}'_5 , for $k = 1, 0$, and ∞ , respectively, so that $\mathbf{w}'_{0,k}$ is complementary to $\mathbf{w}_{0,k}$. Let $B'_{0,k} \in \pi_2(\mathbf{w}'_0, \mathbf{w}'_{0,k})$ be any homology class from \mathbf{w}'_0 to $\mathbf{w}'_{0,k}$. Adapting [LOT18, Corollary 10.16], one could choose to change the base generator in \mathcal{H}' from \mathbf{w}'_0 to $\mathbf{w}'_{0,k}$, and identify the grading in $\underline{\text{Gr}}_{r_1, r_2, r}(\mathcal{H}', \mathfrak{s}') = \underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})$ with the grading in $\underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})$, such that the grading of a generator $\mathbf{w}' \in \mathcal{G}(\mathcal{H}', \mathfrak{s}')$ with respect to $\mathbf{w}'_{0,k}$ is

$$\underline{\text{gr}}_{r_1, r_2, r}(B'_{0,k}) \cdot \underline{\text{gr}}_{r_1, r_2, r}(\mathbf{w}').$$

This correspondence carries directly to the double-coset space, allowing one to view gradings in $\underline{\text{Gr}}_{r_1, r_2, r}(\mathcal{H}', \mathcal{H}_k, \mathfrak{s}', \mathfrak{s}_k)$ as gradings in

$$\underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})/\underline{\mathcal{P}}_r(\mathbf{w}_{0,k}).$$

Similarly, one may view gradings in $\underline{\text{Gr}}_{r_1, r_2, r}^{\text{sk}}(\mathcal{H}', \mathcal{H}_k, \mathfrak{s}', \mathfrak{s}_k)$ as gradings in

$$\underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})/\underline{\mathcal{P}}_r^{\text{sk}}.$$

Denote by π_k the projection

$$\begin{aligned} \pi : \underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})/\underline{\mathcal{P}}_r(\mathbf{w}_{0,k}) \\ \rightarrow \text{Spin}^c(Y_k, \partial Y_k \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2), \mathfrak{s}', \mathfrak{s}_k) \end{aligned}$$

as mentioned in Subsection 7.1. According to [LOT18], this projection to the set of Spin^c -structures on $(Y_k, \partial Y_k \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$ that restrict to \mathfrak{s}' and \mathfrak{s}_k , is defined as follows. The quotient of the double-coset space by the action by \mathbb{Z} on the Maslov component is naturally identified with the image of $H_1(F(\mathcal{Z}))$ in $H_1(Y_k, F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$; let $[-]$ denote this quotient map. More concretely, for $g \in \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})$, let $[\underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \cdot g \cdot \underline{\mathcal{P}}_r(\mathbf{w}_{0,k})]$ denote the image of the $H_1(F(\mathcal{Z}))$ -component of g in $H_1(Y_k, F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$. Then π_k is defined by

$$\begin{aligned} \pi_k(\underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \cdot g \cdot \underline{\mathcal{P}}_r(\mathbf{w}_{0,k})) \\ = \mathfrak{s}(\mathbf{w}'_{0,k} \times \mathbf{w}_{0,k}) + [\underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \cdot g \cdot \underline{\mathcal{P}}_r(\mathbf{w}_{0,k})], \end{aligned}$$

where $\mathbf{w}'_{0,k} \times \mathbf{w}_{0,k}$ is viewed as a generator in $\mathcal{G}(\mathcal{H}' \cup \mathcal{H}_k)$. (Recall that $\text{Spin}^c(Y_k, \partial Y_k \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2), \mathfrak{s}', \mathfrak{s}_k)$ is an affine copy of the image of $H_1(F(\mathcal{Z}))$ in $H_1(Y_k, F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$.) Each fiber $\pi_k^{-1}(\mathfrak{s}'')$ is then isomorphic as a $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ -set to $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, \mathfrak{s}'')$.

Since Proposition 7.6 pertains to the skein-reduced gradings in $\underline{\text{Gr}}_{r_1, r_2, r}^{\text{sk}}(\mathcal{H}', \mathcal{H}_k, \mathfrak{s}', \mathfrak{s}_k)$, which we identify with $\underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \setminus \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}) / \underline{\mathcal{P}}_r^{\text{sk}}$, we must find an analogous correspondence for these grading sets too. To do so, recall that $\underline{\mathcal{P}}_r^{\text{sk}}$ is generated by A_1, A_2 , and A_3 ; the homological components of these elements are $[\rho_{123}]$, $[\rho_{456}]$, and $[\rho_{78}]$ respectively. Direct inspection of Figure 3 shows that these elements, viewed in $H_1(F(\mathcal{Z}))$, are represented by meridian loops of three of the four punctures of $F(\mathcal{Z})$; we denote the images of these meridian loops in $H_1(Y_k, F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$ by μ_1, μ_2 , and μ_3 respectively. Letting $M_k^{\text{sk}} \subset H_1(Y_k, F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$ be the subgroup generated by μ_1, μ_2 , and μ_3 , we may thus define a projection

$$\begin{aligned} \pi_k^{\text{sk}} : \underline{\mathcal{P}}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \setminus \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}) / \underline{\mathcal{P}}_r^{\text{sk}} \\ \rightarrow \text{Spin}^c(Y_k, \partial Y_k \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2), \mathfrak{s}', \mathfrak{s}_k) / \sim_{\text{sk}}, \end{aligned}$$

where the equivalence relation \sim_{sk} is defined by

$$\mathfrak{s}_1'' \sim_{\text{sk}} \mathfrak{s}_2'' \quad \text{if } \text{PD}(\mathfrak{s}_1'' - \mathfrak{s}_2'') \in M_k^{\text{sk}}.$$

Here, PD denotes the isomorphism under Poincaré duality. We denote an equivalence class of Spin^c -structures by $[\mathfrak{s}'']$, and the set of equivalence classes in $\text{Spin}^c(Y_k, \partial Y_k \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2), \mathfrak{s}', \mathfrak{s}_k)$ by E_k . Recalling that the grading sets $\underline{\text{Gr}}_{r_1, r_2, r}^{\text{sk}}(\mathcal{H}', \mathcal{H}_k, \mathfrak{s}', \mathfrak{s}_k)$ are in fact the same for $k \in \{\infty, 0, 1\}$, the projections π_k^{sk} provide a family of bijections $e_k : E_k \rightarrow E_{k+1}$ such that $e_{k+2} \circ e_{k+1} \circ e_k$ is the identity. Furthermore, $(\pi_k^{\text{sk}})^{-1}([\mathfrak{s}''])$ is isomorphic to $(\pi_{k+1}^{\text{sk}})^{-1}(e_k([\mathfrak{s}'']))$.

In order to understand the fibers of π_k^{sk} , we first note the following.

Lemma 7.7. *For $k \in \{\infty, 0, 1\}$ and $i \in \{1, 2, 3\}$, the map*

$$\mathcal{P}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \cdot g \cdot \mathcal{P}_r(\mathbf{w}_{0,k}) \mapsto \mathcal{P}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \cdot gA_i \cdot \mathcal{P}_r(\mathbf{w}_{0,k})$$

defines an isomorphism $\pi_k^{-1}(\mathfrak{s}'') \cong \pi_k^{-1}(\mathfrak{s}'' + \text{PD}(\mu_i))$ as $\text{Gr}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ -sets. This extends to multiplication by $A_1^{n_1} A_2^{n_2} A_3^{n_3}$, giving isomorphisms $\pi_k^{-1}(\mathfrak{s}'') \cong \pi_k^{-1}(\mathfrak{s}'' + \text{PD}(h))$ for some $h \in M_k^{\text{sk}}$.

Proof. First, note that $\text{Gr}(\mathcal{Z})$ is contained in the center of $\text{Gr}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z})$, since $\text{Gr}(\mathcal{Z})$ is Abelian, and \mathcal{Z}_1 and \mathcal{Z}_2 are disjoint from \mathcal{Z} . The fact that the given map is well defined and bijective, and respects the action of $\text{Gr}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$, follows from the observation that A_i is central for each i . □

Remark 7.8. It is possible that $\mu_i = 0$ (or, more generally, $h = 0$), in which case Lemma 7.7 provides an isomorphism of $\pi_k^{-1}(\mathfrak{s}'')$ with itself, which may not be the identity isomorphism.

Remark 7.9. When $\mathcal{Z}_1 = \mathcal{Z}_2 = \emptyset$, i.e. when \mathcal{Y}_k is a sutured manifold for each k , the isomorphism in Lemma 7.7 takes on a particularly simple form. By [Zar11, Proposition 6.3.2], $\text{Gr}(\mathcal{H}' \cup \mathcal{H}_k, \mathfrak{s}'')$, and hence the fiber $\pi_k^{-1}(\mathfrak{s}'')$, is (non-canonically) isomorphic to $\mathbb{Z}/\text{div}(\mathfrak{s}'')\mathbb{Z}$. (Recall that the divisibility $\text{div}(\mathfrak{s}'')$ of \mathfrak{s}'' is the integer $\text{gcd}_{\alpha \in H_2(Y_k)} \langle c_1(\mathfrak{s}''), \alpha \rangle$, where $c_1: \text{Spin}^c(Y_k, \partial Y_k \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$ sends \mathfrak{s}'' to the Euler class of the orthogonal bundle of a vector field representative of \mathfrak{s}'' .) Thus, the lemma states that the fibers $\pi_k^{-1}(\mathfrak{s}'')$ and $\pi_k^{-1}(\mathfrak{s}'' + \text{PD}(\mu_i))$ are isomorphic as \mathbb{Z} -sets, or, equivalently, that $\text{div}(\mathfrak{s}'') = \text{div}(\mathfrak{s}'' + \text{PD}(\mu_i))$. Indeed, under the map $H_1(Y_k, F(\mathcal{Z}_1 \cup \mathcal{Z}_2)) = H_1(Y_k) \rightarrow H_1(Y_k, \partial Y_k)$ from the long exact sequence for a pair, the images of μ_i are trivial. Under Poincaré duality, this is equivalent to saying that the connecting homomorphism $\delta: H^2(Y_k, \partial Y_k \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2)) = H^2(Y_k, \partial Y_k) \rightarrow H^2(Y_k)$ sends $\text{PD}(\mu_i)$ to zero. This implies that

$$c_1(\mathfrak{s}'' + \text{PD}(\mu_i)) = c_1(\mathfrak{s}'') + 2\delta(\text{PD}(\mu_i)) = c_1(\mathfrak{s}'')$$

which in turn implies that $\text{div}(\mathfrak{s}'') = \text{div}(\mathfrak{s}'' + \text{PD}(\mu_i))$.

Lemma 7.7 allows us to understand the fibers of π_k^{sk} as follows. Viewing $\mathcal{P}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \backslash \text{Gr}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}) / \mathcal{P}_r(\mathbf{w}_{0,k})$ as the disjoint union of fibers $\pi_k^{-1}(\mathfrak{s}'')$, each isomorphic to $\text{Gr}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, \mathfrak{s}'')$, the reduction to $\mathcal{P}_{r_1, r_2, r}(\mathbf{w}'_{0,k}) \backslash \text{Gr}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}) / \mathcal{P}_r^{\text{sk}}$ may be seen as obtained by identifying

the fibers $\pi_k^{-1}(\mathfrak{s}'')$ for all \mathfrak{s}'' in the same equivalence class under \sim_{sk} , using the relators $A_1 = A_2 = A_3 = e$ (where e is the identity element). We denote the resulting $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ -set by $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, [\mathfrak{s}''])$. Note that, for each k , at least one of the three relators is redundant, and so we may consider only two relators; for example, setting $A_1 = e$ in $\underline{\mathcal{P}}_r(\mathbf{w}'_{0,0}) \backslash \underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}) / \underline{\mathcal{P}}_r(\mathbf{w}_{0,0})$ immediately implies $A_2 = e$ (since $A_1 A_2 = e$). We illustrate this identification process with two examples; as we shall see, $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, [\mathfrak{s}''])$ may or may not be isomorphic to $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, \mathfrak{s}'')$.

Example 7.10. Let $\mathcal{Z}_1 = \mathcal{Z}_2 = \emptyset$, so that \mathcal{Y}_k is a sutured manifold for each k . Suppose $\underline{\mathcal{P}}_r(\mathbf{w}'_0) = \langle A_1 + A_3 \rangle$. Since $\underline{\text{Gr}}(\mathcal{Z})$ is Abelian, the conjugate $\underline{\mathcal{P}}_r(\mathbf{w}'_{0,k})$ is simply $\underline{\mathcal{P}}_r(\mathbf{w}'_0)$. Then

$$\begin{aligned} \underline{\mathcal{P}}_r(\mathbf{w}'_{0,1}) \backslash \underline{\text{Gr}}(\mathcal{Z}) / \underline{\mathcal{P}}_r(\mathbf{w}_{0,1}) &= \langle A_1 + A_3 \rangle \backslash \langle \lambda, A_1, A_2, A_3 \rangle / \langle A_2 + A_3 \rangle \\ &\cong \mathbb{Z} \oplus \mathbb{Z}, \\ \underline{\mathcal{P}}_r(\mathbf{w}'_{0,\infty}) \backslash \underline{\text{Gr}}(\mathcal{Z}) / \underline{\mathcal{P}}_r(\mathbf{w}_{0,\infty}) &= \langle A_1 + A_3 \rangle \backslash \langle \lambda, A_1, A_2, A_3 \rangle / \langle A_1 + A_3 \rangle \\ &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \\ \underline{\mathcal{P}}_r(\mathbf{w}'_{0,0}) \backslash \underline{\text{Gr}}(\mathcal{Z}) / \underline{\mathcal{P}}_r(\mathbf{w}_{0,0}) &= \langle A_1 + A_3 \rangle \backslash \langle \lambda, A_1, A_2, A_3 \rangle / \langle A_1 + A_2 \rangle \\ &\cong \mathbb{Z} \oplus \mathbb{Z}, \\ \underline{\mathcal{P}}_r(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}) / \underline{\mathcal{P}}_r^{\text{sk}} &= \langle A_1 + A_3 \rangle \backslash \langle \lambda, A_1, A_2, A_3 \rangle / \langle A_1, A_2, A_3 \rangle \\ &\cong \mathbb{Z}, \end{aligned}$$

where the first summand on the right hand side is the Maslov component. This situation may arise, for example, in the familiar setting in [MO08], when \mathcal{Y}_k is the complement of $L_k \subset S^3$ with meridional sutures, where L_0 and L_1 are knots and L_∞ is a 2-component link. The \mathbb{Z} -summands in the homological components are then in correspondence with the link components, and may be interpreted as the relative Alexander gradings. On each fixed Spin^c -structure (i.e. in each Alexander grading), there is a relative Maslov \mathbb{Z} -grading. For $k = \infty$, the two relators $A_1 = 0$ (or equivalently, $A_3 = 0$) and $A_2 = 0$ identify the $(\mathbb{Z} \oplus \mathbb{Z})$ -worth of fibers $\pi_\infty^{-1}(\mathfrak{s}'')$, each isomorphic to \mathbb{Z} , with each other uniquely. For $k = 0, 1$, one of the two relators identifies the \mathbb{Z} -worth of fibers uniquely, and the other relator is redundant. Thus, in all three cases, the skein reduction is the projection onto the Maslov component. Although we do not prove it here, we expect the skein reduction to be related to the relative δ -grading in [MO08], and the graded exact sequence in Theorem 7.12 below to be related to that in [MO08].

Example 7.11. Let $\mathcal{Z}_1 = \mathcal{Z}_2 = \emptyset$. Suppose $\underline{\mathcal{P}}_r(\mathbf{w}'_0) = \langle 12\lambda + A_1 + A_2, 18\lambda + A_1 + A_3 \rangle$. Then

$$\begin{aligned} \underline{\mathcal{P}}_r(\mathbf{w}'_{0,1}) \backslash \underline{\text{Gr}}(\mathcal{Z}) / \underline{\mathcal{P}}_r(\mathbf{w}_{0,1}) &\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \\ \underline{\mathcal{P}}_r(\mathbf{w}'_{0,\infty}) \backslash \underline{\text{Gr}}(\mathcal{Z}) / \underline{\mathcal{P}}_r(\mathbf{w}_{0,\infty}) &\cong \mathbb{Z}/18\mathbb{Z} \oplus \mathbb{Z}, \\ \underline{\mathcal{P}}_r(\mathbf{w}'_{0,0}) \backslash \underline{\text{Gr}}(\mathcal{Z}) / \underline{\mathcal{P}}_r(\mathbf{w}_{0,0}) &\cong \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}, \\ \underline{\mathcal{P}}_r(\mathbf{w}'_0) \backslash \underline{\text{Gr}}(\mathcal{Z}) / \underline{\mathcal{P}}_r^{\text{sk}} &\cong \mathbb{Z}/6\mathbb{Z}, \end{aligned}$$

where again the first summand on the right is the Maslov component. For $k = 1$, there are exactly two Spin^c -structures in $\text{Spin}^c(Y_1, \partial Y_1, \mathfrak{s}', \mathfrak{s}_1)$, with a relative Maslov \mathbb{Z} -grading on each. As the $(\mathbb{Z}/2\mathbb{Z})$ -summand is generated by $15\lambda + A_1 = 3\lambda - A_2$, we may identify A_1 with $(-15, 1)$, and A_2 with $(3, 1)$. The relator $A_1 = 0$ does more than identify the two fibers $\pi_1^{-1}(\mathfrak{s}'')$; since $2A_1 = (-30, 0) = 0$, it identifies each fiber with itself by a degree-30 shift, reducing each fiber to $\mathbb{Z}/30\mathbb{Z}$. Similarly, $A_2 = 0$ further reduces each fiber to $\mathbb{Z}/6\mathbb{Z}$. This describes the skein reduction for $k = 1$. In contrast, for $k = \infty$, the element A_1 has infinite order, and so the relator $A_1 = 0$ only identifies distinct fibers uniquely and does not reduce the fibers; $A_2 = 0$ also identifies the fibers uniquely. However, since the isomorphisms given by A_1 and A_2 are different, the two relators together force a reduction of the fibers from $\mathbb{Z}/18\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$. The skein reduction for $k = 0$ is similar. While we do not provide a diagram \mathcal{H}' where $\underline{\mathcal{P}}_r(\mathbf{w}'_0)$ arises, the phenomena this example illustrates could occur in general.

Note that the discussion above remains valid if we choose some other pair of complementary base generators $\mathbf{w}'_{0,k}$ and $\mathbf{w}_{0,k}$ in \mathcal{H}' and \mathcal{H}_k . For example, we could have chosen to work with $\mathbf{w}'_{\frac{1}{5}} \times \mathbf{y}_1$ instead of $\mathbf{w}'_{\frac{1}{1}} \times \mathbf{y}_3$ for $k = \infty$. Indeed, the correspondence between gradings with respect to different base generators respects both the projection π_k and the $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ -action on the fibers, and commutes with the isomorphism in Lemma 7.7 (since A_i is central). This means that the identification process in the skein reduction is in fact independent of the pair of complementary base generators. This allows us to deal with the case when one or more of $\mathbf{w}'_{\frac{1}{1}}$, $\mathbf{w}'_{\frac{1}{3}}$, and $\mathbf{w}'_{\frac{1}{5}}$ does not exist: For example, if $\mathbf{w}'_{\frac{1}{5}}$ does not exist (i.e. if there does not exist a generator $\mathcal{G}(\mathcal{H}', \mathfrak{s}')$ in idempotent $I_{\frac{1}{5}}$), it would serve us well to let $\mathbf{w}'_{0,0} \times \mathbf{w}_{0,0}$ be $\mathbf{w}'_{\frac{1}{1}} \times \mathbf{z}_2$ instead of $\mathbf{w}'_{\frac{1}{5}} \times \mathbf{z}_1$ as defined above. (In the case that no generator in $\mathcal{G}(\mathcal{H}', \mathfrak{s}')$ is compatible with one of the generators of $\mathcal{G}(\mathcal{H}_k)$, then the situation is in fact simpler: The summand $\overline{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}_k, \mathfrak{s}'')$ is trivial for all $\mathfrak{s}'' \in \text{Spin}^c(Y_k, \partial Y_k \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2), \mathfrak{s}', \mathfrak{s}_k)$, and the skein relation,

when restricted to the summand associated to \mathfrak{s}' , is an isomorphism between modules associated to \mathcal{Y}_{k+1} and \mathcal{Y}_{k+2} .)

We are now ready to state the interpretation of Proposition 7.6 in terms of the identification above.

Theorem 7.12. *Let $\mathcal{Y}_k = \mathcal{Y}' \cup_{\mathcal{F}(\mathcal{Z})} \mathcal{B}_k$ be as in Theorem 3.6, and fix $\mathfrak{s}' \in \text{Spin}^c(Y', \partial Y' \setminus F(\mathcal{Z}_1 \cup \mathcal{Z}_2))$. Viewing $\underline{\text{Gr}}_{r_1, r_2, r}(\mathcal{H}', \mathcal{H}_k, \mathfrak{s}', \mathfrak{s}_k)$ as the disjoint union of fibers $\pi_k^{-1}(\mathfrak{s}'') \cong \underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, \mathfrak{s}'')$, the isomorphisms in Lemma 7.7 provide a way to identify $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, \mathfrak{s}'')$ for Spin^c -structures \mathfrak{s}'' that differ by a linear combination of the Poincaré duals of the meridians of T_k^{el} , in the process possibly reducing the grading set, such that the resulting grading set $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, [\mathfrak{s}''])$ satisfies*

$$\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, [\mathfrak{s}'']) \cong \underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_{k+1}, e_k([\mathfrak{s}'']))$$

as $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ -sets. Then, interpreting $\underline{\text{gr}}_{r_1, r_2, r}^{\boxtimes, \text{sk}}$ as assigning to each generator in $\widehat{\text{BSDA}}(\mathcal{Y}_k)$ an equivalence class $[\mathfrak{s}'']$ and a relative grading in $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, [\mathfrak{s}''])$, the morphisms

$$F_k : \widehat{\text{BSDA}}(\mathcal{Y}_k) \rightarrow \widehat{\text{BSDA}}(\mathcal{Y}_{k+1})$$

in Theorem 3.6, and hence Theorem 1.1, send the summand associated to $[\mathfrak{s}'']$ to $e_k([\mathfrak{s}''])$, and preserve the relative gradings in $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, [\mathfrak{s}''])$, with grading shifts given by the following diagram:

$$\begin{array}{ccc} \widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}_0, [\mathfrak{s}'']) & \xrightarrow{\lambda^{-1}=(-1,0)} & \widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}_1, e_0([\mathfrak{s}''])) \\ & \swarrow (0,0) & \searrow (0,0) \\ & \widehat{\text{BSDA}}(\mathcal{H}' \cup \mathcal{H}_\infty, e_\infty^{-1}([\mathfrak{s}''])) & \end{array}$$

(Again, the diagram above should be interpreted up to simultaneous and inverse (right) actions by elements of $\underline{\text{Gr}}(\mathcal{Z}_1 \cup \mathcal{Z}_2)$ on any two consecutive arrows.) The homotopy equivalence

$$\widehat{\text{BSDA}}(\mathcal{Y}_k) \simeq \text{Cone}(F_{k+1} : \widehat{\text{BSDA}}(\mathcal{Y}_{k+1}) \rightarrow \widehat{\text{BSDA}}(\mathcal{Y}_{k+2}))$$

also respects the relative gradings in $\underline{\text{Gr}}_{r_1, r_2}(\mathcal{H}' \cup \mathcal{H}_k, [\mathfrak{s}''])$.

Proof. Proposition 7.6 and the discussion above implies that the morphisms

$$\mathbb{I}_{\widehat{\text{BSDAA}}(\mathcal{Y}')} \boxtimes f_k : \widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_k) \rightarrow \widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_{k+1})$$

satisfy the description in the theorem. Since the homotopy equivalence

$$\widehat{\text{BSDAA}}(\mathcal{Y}') \boxtimes \widehat{\text{BSD}}(\mathcal{B}_k) \simeq \widehat{\text{BSDA}}(\mathcal{Y}_k)$$

is graded, the theorem follows. \square

Remark 7.13. For the result of applying Theorem 7.12 to Corollary 1.2, the reader is invited to consider Example 7.11 and other similar cases.

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RECEIVED MARCH 22, 2019

ACCEPTED DECEMBER 10, 2020