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Asymptotic expansions of the coefficients in asymptotic series solutions of linear differential equations

F. W. J. Olver

ABSTRACT. In the neighborhood of an irregular singularity of rank one at infinity, a differential equation of the form

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)w = 0$$

has well-known asymptotic solutions of the form

$$e^{\lambda_1 z} z^{\mu_1} \sum_{n=0}^{\infty} \frac{a_{n,1}}{z^n}, \quad e^{\lambda_2 z} z^{\mu_2} \sum_{n=0}^{\infty} \frac{a_{n,2}}{z^n},$$

in which λ_1, λ_2, μ_1, and μ_2 are constants. It is proved that for large n, the coefficients $a_{n,1}$, and $a_{n,2}$ can be expanded in asymptotic series of inverse factorials with explicit coefficients.

1. Introduction

It is well known that the Gamma function has an asymptotic expansion of the form

$$\Gamma(n) \sim \sqrt{2\pi} n^{n-(1/2)} e^{-n} \left\{1 + \frac{1}{12n} + \frac{1}{288n^2} + \cdots \right\} \tag{1.1}$$

as $n \to \infty$ in the sector $|\operatorname{ph} n| \leq \pi - \delta$, where δ is an arbitrary positive constant. Now suppose that a and b are fixed (or bounded) real or complex parameters. Then from (1.1) it follows that

$$\frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+1)} \sim \sqrt{2\pi} n^{n+a+b-(3/2)} e^{-n} \sum_{j=0}^{\infty} \frac{u_j}{n^j} \tag{1.2}$$

in the same circumstances, where $u_0 = 1$ and higher coefficients are polynomials in a and b. It is not difficult to find explicit expressions for the first few, for example,

$$\begin{aligned} u_1 &= \frac{1}{2}(a^2 + b^2 - a - b) + \frac{1}{12}, \\ u_2 &= \frac{1}{24}\{3(a^2+b^2)^2 - 2(5a^3 + 3a^2b + 3ab^2 + 5b^3) \\ &\qquad + 2(5a^2 + 3ab + 5b^2) - 3(a+b)\} + \frac{1}{288}, \end{aligned}$$

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but a general formula for u_j is unavailable.

In consequence of Stirling's formula, the expansion (1.2) can be rearranged as a series of inverse factorials:

$$\frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+1)} \sim \sum_{j=0}^{\infty} v_j \Gamma(n+a+b-j-1), \tag{1.3}$$

again valid when $n \to \infty$ in $|\operatorname{ph} n| \leq \pi - \delta$. The interesting feature of this rearrangement is that in contrast to (1.2), the coefficients enjoy a simple general form given by

$$v_j = (-)^j \frac{(1-a)_j (1-b)_j}{j!}. \tag{1.4}$$

Here we have employed Pochhammer's notation for ascending factorials, given by $(z)_0 = 1$ and

$$(z)_j = z(z+1)(z+2)\cdots(z+j-1), \quad j \geq 1.$$

The results (1.3) and (1.4) are stated without proof by R. B. Dingle on p. 15 (and elsewhere) in [2].

An elegant proof of (1.4) has been supplied recently by R. B. Paris in the Appendix of [9]. Paris' proof is based on the identity

$$\frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+1)\Gamma(n+a+b-1)} = \{1 - \chi(n)\}{}_2F_1(1-a, 1-b; 2-a-b-n; 1), \tag{1.5}$$

in which

$$\chi(n) = \sin(\pi a)\sin(\pi b)\operatorname{cosec}(\pi n + \pi a)\operatorname{cosec}(\pi n + \pi b).$$

This result is valid when neither $n+a$ nor $n+b$ is a nonpositive integer. If we let $|\operatorname{Im} n| \to \infty$ with $\operatorname{Re} n$ fixed, then $\chi(n)$ becomes exponentially small and can be neglected. The coefficients in the required asymptotic expansion are then obtainable from the power-series expansion for the hypergeometric function in (1.5).[1]

Paris needed the expansion (1.3) in the construction of exponentially-improved asymptotic expansions for the confluent hypergeometric function $U(a, a-b+1, z)$, with large $|z|$, via Mellin-Barnes integral representations. It was also needed by the present writer [7] in solving the same problem via the confluent hypergeometric differential equation.

The first purpose of the present paper is to supply an alternative, and quite simple, proof of (1.4). This proof serves as a preliminary for the main problem considered in the paper: the asymptotic nature of the coefficients in asymptotic series solutions of second-order linear differential equations in the neighborhood of an irregular singularity of unit rank. It transpires that the same method of proof yields some elegant, and apparently new, results for this more general problem.

[1]Another proof of (1.3) and (1.4) will be found in [8]. This proof is based on contour integration.

2. The Gamma-function ratio

Let n be a nonnegative integer, and for $n > -\operatorname{Re} a$ and $n > -\operatorname{Re} b$ define

$$a_n = \frac{\Gamma(n+a)\Gamma(n+b)}{n!}. \tag{2.1}$$

Then a_n satisfies the recurrence relation

$$na_n = (n+a-1)(n+b-1)a_{n-1}. \tag{2.2}$$

Motivated by (1.3), we substitute the following generalized asymptotic expansions into this relation

$$\begin{aligned} a_n &\sim \sum_{j=0}^{\infty} v_j \Gamma(n+a+b-j-1), \\ a_{n-1} &\sim \sum_{j=0}^{\infty} v_j \Gamma(n+a+b-j-2). \end{aligned} \tag{2.3}$$

We seek to determine the unknown coefficients v_j. As a preliminary, we rearrange the coefficients in (2.2) as linear combinations of descending factorials; thus

$$n = (n+a+b-j-1) + (j+1-a-b),$$

and

$$\begin{aligned} (n+a-1)(n+b-1) =& (n+a+b-j-1)(n+a+b-j-2) \\ &+ (2j+1-a-b)(n+a+b-j-2) \\ &+ (j+1-a)(j+1-b). \end{aligned}$$

On substituting (2.3) into (2.2), we derive

$$\begin{aligned} &\sum_{j=0}^{\infty} v_j \Gamma(n+a+b-j) + \sum_{j=0}^{\infty} (j+1-a-b) v_j \Gamma(n+a+b-j-1) \\ &\quad \sim \sum_{j=0}^{\infty} v_j \Gamma(n+a+b-j) + \sum_{j=0}^{\infty} (2j+1-a-b) v_j \Gamma(n+a+b-j-1) \\ &\qquad + \sum_{j=0}^{\infty} (j+1-a)(j+1-b) v_j \Gamma(n+a+b-j-2). \end{aligned}$$

Upon reduction, this asymptotic identity becomes

$$\sum_{j=1}^{\infty} j v_j \Gamma(n+a+b-j-1) \sim -\sum_{j=1}^{\infty} (j-a)(j-b) v_{j-1} \Gamma(n+a+b-j-1),$$

and, equating coefficients, we find that

$$v_j = -\frac{(j-a)(j-b)}{j} v_{j-1}.$$

For the first coefficient we have, from (2.1) and (2.3),

$$v_0 = \lim_{n\to\infty} \frac{\Gamma(n+a)\Gamma(n+b)}{n!\,\Gamma(n+a+b-1)} = 1.$$

The required result (1.4) follows immediately.[2]

3. Linear differential equations

The asymptotic theory of the general homogeneous linear differential equation of the second order with an irregular singularity of rank one is given in [6], Chapter 7. Without loss of generality we may suppose the singularity is at infinity. The main features of the theory are as follows. The differential equation has the form

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)w = 0, \tag{3.1}$$

in which the functions $f(z)$ and $g(z)$ can be expanded in power series

$$f(z) = \sum_{s=0}^{\infty} \frac{f_s}{z^s}, \quad g(z) = \sum_{s=0}^{\infty} \frac{g_s}{z^s}, \tag{3.2}$$

that converge in a neighborhood of $z = \infty$. Furthermore, not all of the coefficients f_0, g_0, and g_1 vanish (otherwise, infinity would be a regular singularity). We may also suppose, without loss of generality, that

$$f_0^2 \neq 4g_0; \tag{3.3}$$

cf. [6], Chapter 7, §1.3. Let λ_1 and λ_2 be the roots of the characteristic equation

$$\lambda^2 + f_0\lambda + g_0 = 0. \tag{3.4}$$

In consequence of (3.3), we have $\lambda_1 \neq \lambda_2$. Also define

$$\mu_1 = \frac{f_1\lambda_1 + g_1}{\lambda_2 - \lambda_1}, \quad \mu_2 = \frac{f_1\lambda_2 + g_1}{\lambda_1 - \lambda_2}. \tag{3.5}$$

Then (3.1) has unique solutions $w_1(z)$ and $w_2(z)$ such that

$$w_1(z) \sim e^{\lambda_1 z}\{(\lambda_2 - \lambda_1)z\}^{\mu_1} \sum_{n=0}^{\infty} \frac{a_{n,1}}{z^n} \tag{3.6}$$

as $z \to \infty$ in the sector $|\,\mathrm{ph}(\lambda_2 - \lambda_1)z| \leq \frac{3}{2}\pi - \delta$, and

$$w_2(z) \sim e^{\lambda_2 z}\{(\lambda_2 - \lambda_1)z\}^{\mu_2} \sum_{n=0}^{\infty} \frac{a_{n,2}}{z^n} \tag{3.7}$$

as $z \to \infty$ in the sector $-\frac{1}{2}\pi + \delta \leq \mathrm{ph}(\lambda_2 - \lambda_1)z \leq \frac{5}{2}\pi - \delta$. Here δ again denotes an arbitrary positive constant, and the coefficients are given by $a_{0,1} = a_{0,2} = 1$ and, for $n \geq 1$,

$$\begin{aligned}(\lambda_1 - \lambda_2)na_{n,1} =& (n - \mu_1)(n - 1 - \mu_1)a_{n-1,1} \\ &+ \sum_{s=1}^{n}\{\lambda_1 f_{s+1} + g_{s+1} - (n - s - \mu_1)f_s\}a_{n-s,1},\end{aligned} \tag{3.8}$$

[2]It is only the formula for v_j that is at issue. By restricting n to be a nonnegative integer in the present section, we do not impair the overall region of validity $|\,\mathrm{ph}\, n| \leq \pi - \delta$ of (1.3).

$$(\lambda_2 - \lambda_1) n a_{n,2} = (n - \mu_2)(n - 1 - \mu_2) a_{n-1,2} + \sum_{s=1}^{n} \{\lambda_2 f_{s+1} + g_{s+1} - (n - s - \mu_2) f_s\} a_{n-s,2}. \quad (3.9)$$

The main objective of this paper is to derive asymptotic expansions for $a_{n,1}$ and $a_{n,2}$ as $n \to \infty$. This type of problem has been studied before. Indeed, the asymptotic forms of the coefficients in formal series solutions of more general differential equations are well known; see [1], [3], [4], [10] and [11]. In the case of (3.1) itself, it is known how to construct asymptotic expansions for $a_{n,1}$ and $a_{n,2}$ in descending powers of n. However, the determination of the coefficients of the successive powers is cumbersome and no general formula is available. We shall show that by employing inverse factorial series instead, the coefficients can be expressed in simple explicit forms. It should be noted that Dingle [2, Chapters 13 and 24] also used inverse factorial series for expanding the late coefficients of asymptotic series. However, he did not consider the present problem (except in the special case mentioned in §1); furthermore, his analyses are of a formal character.

As in §2, our derivation of the new expansions for $a_{n,1}$ and $a_{n,2}$ is based directly on the recurrence relations satisfied by these coefficients and we employ only elementary analysis. An alternative, and somewhat shorter, proof based on some new integral representations for $a_{n,1}$ and $a_{n,2}$ will be found in [5]. An advantage of the present approach is that it is applicable to more general recurrence relations for which an integral representation of the solution may be unavailable.

4. Main result

Theorem 4.1. *Let J be an arbitrary nonnegative fixed integer. Then as $n \to \infty$*

$$a_{n,1} = \frac{1}{(\lambda_1 - \lambda_2)^n} \left[\Lambda_1 \sum_{j=0}^{J-1} a_{j,2} (\lambda_1 - \lambda_2)^j \Gamma(n + \mu_2 - \mu_1 - j) + O\{\Gamma(n + \mu_2 - \mu_1 - J)\} \right], \quad (4.1)$$

$$a_{n,2} = \frac{1}{(\lambda_2 - \lambda_1)^n} \left[\Lambda_2 \sum_{j=0}^{J-1} a_{j,1} (\lambda_2 - \lambda_1)^j \Gamma(n + \mu_1 - \mu_2 - j) + O\{\Gamma(n + \mu_1 - \mu_2 - J)\} \right], \quad (4.2)$$

where Λ_1 and Λ_2 are constants.

This theorem is proved in the next two sections. At this stage we make the following observations:

(i) Actual values of the constants Λ_1 and Λ_2 are not derived in the present paper for the general case.[3] However, in [5] it is shown that

$$\Lambda_1 = -ie^{(\mu_2 - \mu_1)\pi i} C_1/(2\pi), \quad \Lambda_2 = iC_2/(2\pi),$$

[3] In [5] it is assumed that (3.1) has been normalized in such a way that $\lambda_2 - \lambda_1 = 1$. This normalization is not assumed in the formulas given here.

where C_1, C_2 are the coefficients in the connection formulas

$$w_1(z) = e^{2\pi i\mu_1} w_1(ze^{-2\pi i}) + C_1 w_2(z),$$
$$w_2(z) = e^{-2\pi i\mu_2} w_2(ze^{2\pi i}) + C_2 w_1(z).$$

See also [1], [3].

(ii) As in the case of (1.3), the expansions (4.1) and (4.2) are generalized asymptotic expansions. Numerically their character resembles that of ordinary Poincaré expansions, since the successive Gamma functions decrease asymptotically by a factor n.

5. Proof of Theorem 4.1: The first approximation

Our objective in this section is to prove that

$$a_{n,1} = (\lambda_1 - \lambda_2)^{-n}\Gamma(n + \mu_2 - \mu_1)\{\Lambda_1 + O(n^{-1})\} \tag{5.1}$$

as $n \to \infty$. This result is a special case of known results; see, for example, [4]. However, for the sake of completeness and simplicity, we furnish a direct and elementary proof.

The underlying idea of the proof is as follows. In the recurrence relation (3.8), the first term on the right-hand side and the term in the sum for $s = 1$ can be combined. In this way we obtain

$$\begin{aligned}(\lambda_1 - \lambda_2)na_{n,1} = {} & (n-1-c_1)(n-1-c_2)a_{n-1,1} \\ & + \sum_{s=2}^{n}\{\lambda_1 f_{s+1} + g_{s+1} - (n-s-\mu_1)f_s\}a_{n-s,1},\end{aligned} \tag{5.2}$$

again valid when $n \geq 1$.[4] Here c_1, c_2 are constants that satisfy

$$c_1 + c_2 = 2\mu_1 + f_1 - 1, \quad c_1c_2 = \mu_1(\mu_1 + f_1 - 1) + \lambda_1 f_2 + g_2. \tag{5.3}$$

We note that, in passing,

$$\mu_1 + \mu_2 = -f_1 \tag{5.4}$$

(cf. (3.5)); in consequence, the first of (5.3) is equivalent to

$$c_1 + c_2 = \mu_1 - \mu_2 - 1. \tag{5.5}$$

If we neglect the sum on the right-hand side of (5.2), we obtain

$$(\lambda_1 - \lambda_2)na_{n,1} = (n-1-c_1)(n-1-c_2)a_{n-1,1}.$$

Since $a_{0,1} = 1$, this recurrence relation has the exact solution

$$a_{n,1} = \frac{1}{\Gamma(-c_1)\Gamma(-c_2)}\frac{\Gamma(n-c_1)\Gamma(n-c_2)}{(\lambda_1-\lambda_2)^n n!}. \tag{5.6}$$

In consequence of (5.5) and Stirling's formula, the right-hand sides of (5.1) and (5.6) share the same asymptotic form; more precisely

$$\frac{\Gamma(n-c_1)\Gamma(n-c_2)}{(\lambda_1-\lambda_2)^n n!} = \frac{\Gamma(n+\mu_2-\mu_1)}{(\lambda_1-\lambda_2)^n}\{1+O(n^{-1})\}, \quad n \to \infty. \tag{5.7}$$

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[4] As usual, empty sums are interpreted as zero.

Thus our essential object in this section is to show that the sum in (5.2) can be treated as a perturbation, whose effect is absorbable in the error term $O(n^{-1})$ in (5.1).

Since the series (3.2) are assumed to converge in a neighborhood of $z = \infty$, there exist constants F and ρ such that

$$|f_s| \le F\rho^s, \quad |g_s| \le F\rho^s, \tag{5.8}$$

for all s. Without loss of generality, we may suppose that $\rho \ge 1$. From these inequalities, it follows that for all positive values of the integer n and all s such that $1 \le s \le n$, we have

$$|\lambda_1 f_{s+1} + g_{s+1} - (n - s - \mu_1)f_s| \le G(n+\rho)\rho^s, \tag{5.9}$$

where G is another assignable constant.

Let $\gamma_1 = \operatorname{Re} c_1$, $\gamma_2 = \operatorname{Re} c_2$, and take N to be a fixed positive integer that satisfies all of the following inequalities:

$$N > \gamma_1, \quad N > \gamma_2, \quad \frac{\rho|\lambda_1 - \lambda_2|(N+1)}{(N-\gamma_1)(N-\gamma_2)} \le \frac{1}{2}, \tag{5.10}$$

and

$$N \ge \sqrt{(1+\gamma_1)(1+\gamma_2)} - 1 \qquad \text{when} \qquad (1+\gamma_1)(1+\gamma_2) > 0. \tag{5.11}$$

(Obviously such an integer always exists.) When $n \ge N$, we substitute into (5.2) by means of the equation

$$a_{n,1} = \frac{\Gamma(n-c_1)\Gamma(n-c_2)}{(\lambda_1 - \lambda_2)^n\, n!}\alpha_n. \tag{5.12}$$

This leads to the following recurrence relation for the quantities α_n:

$$\alpha_n = \alpha_{n-1} + \sum_{s=2}^{n-N} h_s(n)\alpha_{n-s} + K(n), \quad n \ge N+1, \tag{5.13}$$

in which

$$\begin{aligned} h_s(n) = {}& \frac{\{\lambda_1 f_{s+1} + g_{s+1} - (n-s-\mu_1)f_s\}(\lambda_1 - \lambda_2)^{s-1}}{(n-1-c_1)(n-2-c_1)\cdots(n-s-c_1)} \\ & \times \frac{(n-1)(n-2)\cdots(n-s+1)}{(n-1-c_2)(n-2-c_2)\cdots(n-s-c_2)}, \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} K(n) = {}& \frac{(\lambda_1 - \lambda_2)^{n-1}(n-1)!}{\Gamma(n-c_1)\Gamma(n-c_2)} \\ & \times \sum_{s=n-N+1}^{n} \{\lambda_1 f_{s+1} + g_{s+1} - (n-s-\mu_1)f_s\} a_{n-s,1}. \end{aligned} \tag{5.15}$$

In consequence of (5.9) and (5.10), we have

$$|h_s(n)| \le H_s(n), \quad 2 \le s \le n-N, \tag{5.16}$$

where

$$H_s(n) = \frac{G(n+\rho)\rho^s|\lambda_1-\lambda_2|^{s-1}}{(n-1-\gamma_1)(n-2-\gamma_1)\cdots(n-s-\gamma_1)} \times \frac{(n-1)(n-2)\cdots(n-s+1)}{(n-1-\gamma_2)(n-2-\gamma_2)\cdots(n-s-\gamma_2)}. \tag{5.17}$$

We now introduce a majorizing sequence $\{\beta_n\}$ for $\{\alpha_n\}$, defined by

$$\beta_N = \max(|\alpha_N|, 1) \tag{5.18}$$

and

$$\beta_n = \beta_{n-1} + \sum_{s=2}^{n-N} H_s(n)\beta_{n-s} + |K(n)|, \quad n \geq N+1. \tag{5.19}$$

Obviously

$$|\alpha_n| \leq \beta_n, \quad n \geq N. \tag{5.20}$$

The next step is to prove that β_n tends to a limit as $n \to \infty$. Suppose now that $n \geq N+3$. From the definition (5.17), we have

$$\frac{H_{s+1}(n)}{H_s(n)} = \frac{\rho|\lambda_1-\lambda_2|(n-s)}{(n-s-1-\gamma_1)(n-s-1-\gamma_2)}, \quad 2 \leq s \leq n-N-1.$$

Let

$$\phi(\nu) = \frac{\nu}{(\nu-1-\gamma_1)(\nu-1-\gamma_2)}, \quad \nu \geq N+1.$$

Then

$$\frac{\phi'(\nu)}{\phi(\nu)} = \frac{(1+\gamma_1)(1+\gamma_2)-\nu^2}{\nu(\nu-1-\gamma_1)(\nu-1-\gamma_2)}.$$

In consequence of the assumed conditions (5.10), (5.11), we see that $\phi'(\nu) \leq 0$ when $\nu \geq N+1$. Thus as s increases the sequence of ratios $\{H_{s+1}(n)/H_s(n)\}$ is nondecreasing; accordingly

$$\frac{H_{s+1}(n)}{H_s(n)} \leq \frac{H_{n-N}(n)}{H_{n-N-1}(n)} = \frac{\rho|\lambda_1-\lambda_2|(N+1)}{(N-\gamma_1)(N-\gamma_2)} \leq \frac{1}{2}; \tag{5.21}$$

cf. again (5.10). Hence

$$H_s(n) \leq 2^{2-s}H_2(n), \quad 2 \leq s \leq n-N. \tag{5.22}$$

From (5.19), it is obvious that the sequence $\beta_N, \beta_{N+1}, \beta_{N+2}, \ldots$ is increasing. With the aid of (5.18), (5.19), and (5.22) we derive

$$\beta_n = \beta_{n-1}(1+\theta_n),$$

where

$$|\theta_n| \leq \sum_{s=2}^{n-N} H_s(n) + \frac{|K(n)|}{\beta_{n-1}} \leq 2H_2(n) + |K(n)|.$$

From (5.17) with $s = 2$, we see that $H_2(n) = O(n^{-2})$ as $n \to \infty$. Furthermore, since the only unknown coefficients that appear in the sum in (5.15) are $a_{0,1}, a_{1,1}, \ldots, a_{N-1,1}$, it follows from Stirling's approximation and (5.9) that

$$|K(n)| = \frac{(|\lambda_1 - \lambda_2|e\rho)^n}{n^{n-\gamma_1-\gamma_2-(3/2)}} O(1), \quad n \to \infty. \tag{5.23}$$

Thus $\theta_n = O(n^{-2})$. In consequence, the infinite product

$$\prod_{n=N+1}^{\infty} (1 + \theta_n)$$

converges, and so β_n tends to a finite limit β, say, as $n \to \infty$.

Finally, from (5.13), (5.16), (5.20), and (5.22), we see that

$$|\alpha_n - \alpha_{n-1}| \leq \beta \sum_{s=2}^{n-N} 2^{2-s} H_2(n) + |K(n)| = O\left(\frac{1}{n^2}\right), \tag{5.24}$$

as before. Hence α_n approaches a finite limit Λ_1, say; furthermore

$$\alpha_n = \Lambda_1 + O(n^{-1}), \quad n \to \infty. \tag{5.25}$$

In combination with (5.12) and (5.7), this establishes (5.1).

6. Proof of Theorem 4.1: Higher approximations

We now propose to extend the asymptotic approximation (5.1) into the desired asymptotic expansion (4.1). We first observe from (5.14) that in the relation (5.13), the functions $h_s(n)$ can be expanded in convergent series of the form

$$h_s(n) = \frac{1}{n^s} \sum_{j=0}^{\infty} \frac{h_{j,s}}{n^j} \tag{6.1}$$

when n is large; the coefficients $h_{j,s}$ depend on $f_s, f_{s+1}, g_{s+1}, \lambda_1, \lambda_2, \mu_1, c_1$, and c_2.

Assume that $n \geq N + 3$. Then we have

$$\left|\sum_{s=3}^{n-N} h_s(n)\alpha_{n-s} + K(n)\right| \leq \beta \sum_{s=3}^{n-N} 2^{3-s} H_3(n) + |K(n)| = O\left(\frac{1}{n^3}\right);$$

cf. (5.16), (5.17) (with $s = 3$), (5.21) and (5.23). Hence from (5.13) we have

$$\alpha_n - \alpha_{n-1} = h_2(n)\alpha_{n-2} + O\left(\frac{1}{n^3}\right) = \frac{h_{0,2}\Lambda_1}{n^2} + O\left(\frac{1}{n^3}\right);$$

cf. (5.25) and (6.1) (with $s = 2$). Accordingly, from the expansion

$$\alpha_n = \Lambda_1 + \sum_{s=n}^{\infty} (\alpha_s - \alpha_{s+1}),$$

we derive

$$\alpha_n = \Lambda_1 - \frac{h_{0,2}\Lambda_1}{n} + O\left(\frac{1}{n^2}\right), \quad n \to \infty.$$

Continuing the resubstitutions, we easily see that

$$\alpha_n \sim \sum_{j=0}^{\infty} \frac{u_j}{n^j},$$

where the coefficients u_j are constants that depend on Λ_1, the coefficients in the expansions (3.2) for $f(z)$ and $g(z)$, and the Bernoulli numbers.[5]

In consequence, from (5.12) we have

$$a_{n,1} \sim \frac{\Gamma(n-c_1)\Gamma(n-c_2)}{(\lambda_1-\lambda_2)^n\, n!} \sum_{j=0}^{\infty} \frac{u_j}{n^j}, \quad n \to \infty. \tag{6.2}$$

In turn, this expansion may be rearranged in the form

$$a_{n,1} \sim \frac{1}{(\lambda_1-\lambda_2)^n} \sum_{j=0}^{\infty} v_j \Gamma(n+\mu_2-\mu_1-j), \quad n \to \infty, \tag{6.3}$$

where $v_0, v_1, v_2, \ldots$ comprise another sequence of constants; cf. again (5.7).

In (6.3) it is clear that $v_0 = u_0 = \Lambda_1$. We shall find $v_1, v_2, \ldots$ by substituting into (3.8) by means of (6.3) and equating coefficients. We first need to overcome a difficulty stemming from the fact that the number of terms in the sum in (3.8) increases with n. This is achieved by truncating the sum at a fixed number of terms and estimating the remainder, as follows:

Lemma 6.1. *If M is any fixed nonnegative integer, then as $n \to \infty$*

$$\sum_{s=M+1}^{n} \{\lambda_1 f_{s+1} + g_{s+1} - (n-s-\mu_1) f_s\} a_{n-s,1} = O\left\{\frac{\Gamma(n+\mu_2-\mu_1-M)}{(\lambda_1-\lambda_2)^n}\right\}. \tag{6.4}$$

Proof. Define M_0 to be the least nonnegative integer such that

$$\mathrm{Re}(M_0+\mu_2-\mu_1) > 0 \text{ and } |M_0+\mu_2-\mu_1| \geq 2\rho|\lambda_1-\lambda_2|, \tag{6.5}$$

where ρ is defined as in (5.8). From (5.1) it follows that

$$|a_{s,1}| \leq A|(\lambda_1-\lambda_2)^{-s}\Gamma(s+\mu_2-\mu_1)|, \quad s \geq M_0, \tag{6.6}$$

where A is assignable independently of s.

From (5.9) and (6.6), we derive

$$|\{\lambda_1 f_{s+1} + g_{s+1} - (n-s-\mu_1) f_s\} a_{n-s,1}| \leq K_s(n), \quad 1 \leq s \leq n - M_0,$$

where

$$K_s(n) = AG(n+\rho)\rho^s |(\lambda_1-\lambda_2)^{s-n}\Gamma(n-s+\mu_2-\mu_1)|.$$

When $1 \leq s \leq n - M_0 - 1$, we have

$$\frac{K_{s+1}(n)}{K_s(n)} = \frac{\rho|\lambda_1-\lambda_2|}{|n-s+\mu_2-\mu_1-1|} \leq \frac{\rho|\lambda_1-\lambda_2|}{|M_0+\mu_2-\mu_1|} \leq \frac{1}{2};$$

[5]Compare [6], p. 292, Example 3.2.

cf. (6.5). Hence

$$\sum_{s=M+1}^{n-M_0} \left|\{\lambda_1 f_{s+1} + g_{s+1} - (n-s-\mu_1)f_s\}a_{n-s,1}\right| \le \sum_{s=M+1}^{n-M_0} 2^{M+1-s}K_{M+1}(n)$$
$$\le 2K_{M+1}(n)$$
$$= O\left\{\frac{\Gamma(n+\mu_2-\mu_1-M)}{(\lambda_1-\lambda_2)^n}\right\}. \tag{6.7}$$

In the remaining part of the sum in (6.4), the only a's that appear are $a_{0,1}$, $a_{1,1}$, ..., $a_{M_0-1,1}$. With the aid, again, of (5.9), we see that

$$\sum_{s=n-M_0+1}^{n} |\{\lambda_1 f_{s+1}+g_{s+1} - (n-s-\mu_1)f_s\}a_{n-s,1}|$$
$$= O(n\rho^n) = O\left\{\frac{\Gamma(n+\mu_2-\mu_1-M)}{(\lambda_1-\lambda_2)^n}\right\} \tag{6.8}$$

(cf. also (5.15) and (5.23).) Combination of (6.7) and (6.8) completes the proof of the lemma.

To continue the preparation of (3.8) for substitution by means of (6.3), we rearrange the coefficients as descending factorials (as in §2). Thus we set

$$\begin{aligned} n &= (n+\mu_2-\mu_1-j)+(\mu_1-\mu_2+j), \\ (n-\mu_1)(n-1-\mu_1) &= (n+\mu_2-\mu_1-j)(n+\mu_2-\mu_1-j-1) \\ &\quad +2(j-\mu_2)(n+\mu_2-\mu_1-j-1) \\ &\quad +\mu_2(\mu_2-1)+(1-2\mu_2)j+j^2, \end{aligned}$$

and

$$\lambda_1 f_{s+1}+g_{s+1}-(n-s-\mu_1)f_s = -f_s(n+\mu_2-\mu_1-j-s)+(\mu_2-j)f_s+\lambda_1 f_{s+1}+g_{s+1}.$$

We then find that

$$\begin{aligned} &(\lambda_1-\lambda_2)^{n-1}\{(\lambda_1-\lambda_2)na_{n,1}-(n-\mu_1)(n-1-\mu_1)a_{n-1,1}\} \\ &\sim \sum_{j=0}^{\infty}(\mu_1+\mu_2-j)v_j\Gamma(n+\mu_2-\mu_1-j) \\ &\quad -\sum_{j=0}^{\infty}\{\mu_2(\mu_2-1)+(1-2\mu_2)j+j^2\}v_j\Gamma(n+\mu_2-\mu_1-j-1), \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} &(\lambda_1-\lambda_2)^{n-1}\sum_{s=1}^{M}\{\lambda_1 f_{s+1}+g_{s+1}-(n-s-\mu_1)f_s\}a_{n-s,1} \\ &\sim -\sum_{s=1}^{M} f_s(\lambda_1-\lambda_2)^{s-1}\sum_{j=0}^{\infty} v_j\Gamma(n+\mu_2-\mu_1-j-s+1) \\ &\quad +\sum_{s=1}^{M}(\lambda_1-\lambda_2)^{s-1}\sum_{j=0}^{\infty}\{(\mu_2-j)f_s+\lambda_1 f_{s+1}+g_{s+1}\}v_j\Gamma(n+\mu_2-\mu_1-j-s). \end{aligned} \tag{6.10}$$

Bearing in mind Lemma 6.1, we may equate coefficients in the right members in (6.9) and (6.10) up to (but excluding) the terms in $\Gamma(n+\mu_2-\mu_1-M)$. From the leading terms, we derive

$$(\mu_1+\mu_2)v_0=-f_1v_0,$$

in agreement with (5.4).

Next, on equating coefficients of $\Gamma(n+\mu_2-\mu_1-m)$, with $1\le m\le M-1$, we have

$$\begin{aligned}(\mu_1+\mu_2-m)v_m&-\left\{\mu_2(\mu_2-1)+(1-2\mu_2)(m-1)+(m-1)^2\right\}v_{m-1}\\&=-\sum_{s=1}^{m+1}f_s(\lambda_1-\lambda_2)^{s-1}v_{m-s+1}\\&\quad+\sum_{s=1}^{m}\left\{(\mu_2-m+s)f_s+\lambda_1f_{s+1}+g_{s+1}\right\}(\lambda_1-\lambda_2)^{s-1}v_{m-s}.\end{aligned}$$

Upon rearrangement and use once again of (5.4), the last equation reduces to

$$\begin{aligned}-mv_m=&(m-\mu_2)(m-1-\mu_2)v_{m-1}\\&+\sum_{s=1}^{m}\left\{\lambda_2f_{s+1}+g_{s+1}-(m-s-\mu_2)f_s\right\}(\lambda_1-\lambda_2)^{s-1}v_{m-s}.\end{aligned}$$

From (3.9) it is easily verified that this is exactly the same as the recurrence relation satisfied by $(\lambda_1-\lambda_2)^m a_{m,2}$. And since $a_{0,2}=1$, we conclude that

$$v_m=(\lambda_1-\lambda_2)^m a_{m,2}v_0.$$

This result has been derived on the assumption that $m\le M-1$, but since M is arbitrary it becomes valid for all m. Substituting this result into (6.3), and recalling that $v_0=\Lambda_1$, we derive the required result (4.1).

The companion result (4.2) is proved in a similar manner, or we can appeal to symmetry. This completes the proof of Theorem 4.1.

7. Summary

In §§1 and 2, we discussed the problem of constructing asymptotic expansions for a ratio of Gamma functions, and gave a new proof of the expansion

$$\frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+1)}\sim\sum_{j=0}^{\infty}(-)^j\frac{(1-a)_j(1-b)_j}{j!}\Gamma(n+a+b-j-1)\tag{7.1}$$

as $n\to\infty$ in $|\operatorname{ph} n|\le\pi-\delta\ (<\pi)$.

In §§3 to 6, we used a similar method of proof to obtain new asymptotic expansions, again in series of inverse factorials, for the coefficients in asymptotic series solutions of homogeneous linear differential equations of the second order in the neighborhood of an irregular singularity of rank one. The results are summarized by Theorem 4.1.

It is easily verified that if Theorem 4.1 is applied to the following form of the confluent hypergeometric differential equation:

$$\frac{d^2w}{dz^2}+\left(\frac{a-b+1}{z}-1\right)\frac{dw}{dz}-\frac{a}{z}w=0,$$

then the expansion (7.1) results. Thus (7.1) may be regarded as a special case of Theorem 4.1. Apart from obvious analytical and computational interests, the new expansions are of importance in the general theory of exponentially-improved asymptotic solutions of ordinary linear differential equations [5], just as the expansion (7.1) arises in the corresponding theory for the confluent hypergeometric equation [7], [9].

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INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742, U. S. A.