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A BIDIRECTIONAL LONG-WAVE MODEL

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ABSTRACT. For modeling weakly nonlinear and weakly dispersive long gravity waves of typical amplitude a and typical length λ, propagating in both directions in a straight, gradually varying channel of breath $b(x)$ and mean water depth $h(x)$, the Boussinesq equations provide a versatile model, with its validity based on the assumptions that $\alpha = a/h \ll 1$, $\epsilon = (h/\lambda)^2 = O(\alpha)$ and $(d/dx)\log(bh^{1/2}) = O(\epsilon^{3/2})$. On this same basis, a new bidirectional long-wave model is derived to evaluate the cross-sectional mean surface elevation $\zeta(x,t) = \zeta_+ + \zeta_- + \zeta_1$, where ζ_+ and ζ_- denote, respectively, right-going and left-going waves, both of $O(\epsilon)$, and ζ_1 is a term of $O(\epsilon^2)$ representing the interaction between ζ_+ and ζ_-. The evolution equations obtained for ζ_+ and ζ_- exhibit extensions of the Korteweg–de Vries equation to comprise the additional effects of slow variations in the admittance, $(bh^{1/2})$, of varying channels on evolving waves. Main features of this model include: (i) For wave-and-channel-wall interactions, this model accounts for reflection and transmission of long waves in varying channels while maintaining both mass and energy conserved adiabatically. (ii) For wave-wave interactions, head-on collisions between right- and left-going solitary waves are shown to gain a total phase shift which is an algebraic function of the amplitudes of the colliding waves. (iii) For forced generation of nonlinear waves in varying channels, this model admits weakly resonant disturbances with both right-going and left-going components.

1. Introduction

The general subject of nonlinear and dispersive waves evolving in non-uniform media has been of strong interest and active development, the thrust being motivated by attempts for the ultimate generality. For nonlinear long gravity waves, of typical amplitude a and length λ, propagating in a straight, gradually varying channel of breadth $b(x)$ and cross-sectional mean-water-depth $h(x)$, the various models of the Boussinesq family are based on the assumptions that

$$\alpha = a/h \ll 1, \quad \epsilon = (h/\lambda)^2 = O(\alpha), \quad (d/dx)\log(b\, h^{\frac{1}{2}}) = O(\epsilon^{3/2}), \tag{1}$$

(see, e.g. Peregrine [12]; Whitham [22]; Miles [9, 10, 11]; Wu [19, 20, 21]). The models in this class have been found by Teng and Wu [17, 18] to be in broad agreement with existing experiment.

Along a different approach, the additional assumption of unidirectional wave motion in varying channels has been applied to derive the Korteweg–de Vries (KdV) class of equations by Shuto [15], Miles [11], Teng and Wu [17, 18] and others. Similarly, variable-depth form of the Kadomtsev-Petviashvili (K-P) equation has been given by Liu et al. [7]. Whilst Shuto's equation and similar models are known to be adiabatic in conserving energy, they have, however, a crucial deficiency in not conserving mass, with an error which is first order and has a cumulative effect. This effect has been

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investigated by Chang et al. [3] with experiment for channels with gradually varying breadth and found numerically by Teng and Wu [18] for the channel KdV model to accumulate to a total loss (or gain) of considerable fraction (over 50%) of original mass for convergent (or divergent) channels, as used in the experiment. In sharp contrast, the channel Boussinesq model is found by Teng and Wu [18] to conserve very accurately both mass and energy. To overcome the flaw in the varying-channel KdV model on conserving mass, a remedy has been proposed by Kirby and Vengayil [4] who derived a set of coupled equations of the KdV type to model both transmitted and reflected waves due to channel variations. However, their model can be further improved in regards to describing more accurately local interactions between waves and between waves and varying channel walls, as will be seen later.

Another related problem of central interest is concerned with head-on collisions between right-going and left-going solitary waves, including, as a special case, the total reflection of solitary waves by a vertical end wall. For the latter case, the resonant interaction during reflection of solitary waves has been found (by Byatt-Smith [2], Maxworthy [8], Su and Mirie [16], Power and Chwang [14] among others) to yield a terminal phase shift for the waves. For channels with varying breadth and depth, such as those devised to model reflection of long waves in marine straits, analytical solutions to these similar nonlinear problems are of interest, especially as a valuable alternative to seeking numerical solutions to the Boussinesq equations.

The primary purpose of the present study is to derive a set of equations akin to the KdV family for evaluating generation, evolution, transmission and reflection of nonlinear dispersive waves propagating in *both* directions in gradually varying channels based on the same assumptions as listed in (1) and possessing solutions with the same accuracy as the Boussinesq model. The model equations will first be derived in §2 for channels with varying rectangular cross-section. This model will be shown to conserve both mass and energy adiabatically and to possess several conspicuous properties about leaving an impress of the nonlinear and dispersive effects on waves undergoing interactions with themselves and with varying channel boundaries. In §3, this new model is further extended for applications to varying channels of arbitrary shape. Applications of the model equations will be presented in subsequent papers addressing problems of wave reflection and transmission in nonhomogeneous media and resonant wave-wave interactions.

2. The model

Let us begin by adopting the generalized-channel-Boussinesq (gcB) equations derived, under assumption (1), by Teng and Wu [17] for gradually varying channels of arbitrary shape. For expediency and simplicity, we shall first consider channels of rectangular cross-section of gradually varying breadth $b(x)$ and depth $h(x)$; further extension to arbitrary channel shape will be made subsequently in §3. Then, for gradually varying rectangular channels, the gcB equations (see Teng and Wu [17], eqs. 43, 44, 73, 74 and Table 1) are

$$(b\tilde{\zeta})_t + [b(h+\tilde{\zeta})\bar{u}]_x = 0, \tag{2}$$

$$\bar{u}_t + \bar{u}\bar{u}_x + \tilde{\zeta}_x - \frac{1}{3}h^2\bar{u}_{xxt} = -(p_a)_x, \tag{3}$$

where the subscripts x and t denote differentiation, $\bar{u}(x,t)$ is the cross-sectional average of the longitudinal x-component velocity along the channel axis and $\tilde{\zeta}(x,t)$ is the sectional free-surface average of wave elevation, taken positive above the undisturbed free surface at the horizontal plane $z = 0$ and at time t,

$$\bar{u}(x,t) = \frac{1}{A}\iint_A u(x,y,z,t)\, dydz, \tag{4a}$$

$$\tilde{\zeta}(x,t) = \frac{1}{b}\int_{-b/2}^{b/2} \zeta(x,y,t)\, dy, \tag{4b}$$

$$A(x,t) = A_h(x) + A_\zeta(x,t), \quad A_h(x) = b(x)h(x), \tag{4c}$$

where A_h is the undisturbed channel cross-section of breadth $b(x)$ and uniform depth $h(x)$, $A_\zeta(x,t) = b(x)\tilde{\zeta}(x,t)$ is the variation of $A(x,t)$ due to wave elevation and $p_a(x,t)$ is an external pressure applied over the water surface. Here, the length is scaled by a typical depth h_0, time by $(h_0/g)^{1/2}$, g being the gravitational constant. The flow quantities and variations of $h(x)$ and $b(x)$ satisfy (1) as assumed, and $|p_a| = O(\epsilon^2)$. In addition, the flow at $x = \pm\infty$ will be assumed regular, i.e.

$$\bar{u}, \bar{u}_x, \bar{u}_{xx}, \tilde{\zeta}, \tilde{\zeta}_x \longrightarrow 0 \quad \text{sufficiently fast as} \quad |x| \longrightarrow \infty. \tag{5}$$

Under scaling (1), $\bar{u}$ and $\tilde{\zeta}$ are clearly of $O(\epsilon)$. We further note that (2) is exact in the sense that it describes the relationship for mass conservation exactly, and that (3) conserves momentum with an error of $O(\epsilon^2)$ relative to the leading terms $\bar{u}_t$ and $\tilde{\zeta}_x$, which are of $O(\epsilon^{3/2})$.

In terms of the "velocity potential" $\phi(x,t)$ and the critical speed $c(x)$ defined by

$$\bar{u}(x,t) = \phi_x, \quad c(x) = h^{1/2}, \tag{6}$$

(3) can be integrated once, yielding under conditions (5) the equation

$$\tilde{\zeta} + \phi_t + \frac{1}{2}(\phi_x)^2 - \frac{h^2}{3}\phi_{xxt} = -p_a. \tag{7}$$

Substituting (6) and (7) into (2) to eliminate $\tilde{\zeta}$, we obtain for ϕ the equation

$$\phi_{tt} - c\partial_x(c\partial_x\phi) = \frac{h^2}{3}\phi_{xxtt} - \Big[(\phi_x)^2 + \frac{1}{2c^2}(\phi_t)^2\Big]_t + c\phi_x c(\log bc)_x - (p_a)_t. \tag{8}$$

In deriving (8), the term $\partial_t(\phi_t)^2$ follows from using the lowest order approximation of (8), i.e. $\phi_{tt} = c^2\phi_{xx}$, to convert the quadratic term $(\phi_t\phi_x)_x$ into $\phi_x\phi_{xt} + \phi_t\phi_{tt}/c^2$, without altering the error estimate of (8), which remains at two orders above the lowest one. This approximation is to be noted to hold valid even when *both* right-going and left-going waves are present. In this respect, the preference of using ϕ over ζ as the primary variable is after Lin and Clark [6] and Benney and Luke [1].

We now apply the multiple scale expansion in terms of the new variables:

$$\xi_\pm = \epsilon^{1/2}\Big[t \mp \int \frac{dx}{c(x)}\Big], \quad \tau = \epsilon^{3/2}t, \tag{9a}$$

with

$$\phi(x,t) = \epsilon^{1/2}\left[\phi_0(\xi_+,\xi_-;\tau) + \epsilon\phi_1(\xi_+,\xi_-;\tau) + \cdots\right], \tag{9b}$$

$$b(x) = b_0 + \epsilon b_1(x) + \cdots, \quad c(x) = c_0 + \epsilon c_1(x) + \cdots, \quad (b_0 = 1,\ c_0 = 1) \tag{9c}$$

$$p_a = \epsilon^2[P_+(\xi_+) + P_-(\xi_-)]. \tag{9d}$$

The corresponding differential operators are related by

$$\partial_t = \epsilon^{1/2}[(\partial_- + \partial_+) + \epsilon\partial_\tau], \quad c\partial_x = \epsilon^{1/2}(\partial_- - \partial_+), \tag{10a}$$

where

$$\partial_+ = \partial/\partial\xi_+, \quad \partial_- = \partial/\partial\xi_-, \quad \partial_\tau = \partial/\partial\tau. \tag{10b}$$

Here, the external forcing is taken to consist of a right-going and a left-going component. With this asymptotic expansion, (8) yields, to the leading $O(\epsilon^{3/2})$, the equation

$$4\partial_+\partial_-\phi_0 = 0, \tag{11}$$

which has the general solution

$$\phi_0 = \phi_+(\xi_+;\tau) + \phi_-(\xi_-;\tau), \tag{12}$$

comprising a right-going and a left-going wave, or R-wave and L-wave for brevity, they being so far indeterminate until the next order equation is considered.

The next order terms of $O(\epsilon^{5/2})$ in (8) are found to give for ϕ_1 the equation

$$\begin{aligned} -4\partial_-\partial_+\phi_1 = {} & \\ & 2\partial_\tau\partial_+\phi_+ + \frac{3}{2}\partial_+(\partial_+\phi_+)^2 - \frac{1}{3}\partial_+^4\phi_+ + (\partial_+\phi_+)(\partial_- - \partial_+)(\log bc) + \partial_+P_+ \\ & +2\partial_\tau\partial_-\phi_- + \frac{3}{2}\partial_-(\partial_-\phi_-)^2 - \frac{1}{3}\partial_-^4\phi_- - (\partial_-\phi_-)(\partial_- - \partial_+)(\log bc) + \partial_-P_- \\ & -(\partial_+ + \partial_-)(\partial_+\phi_+)(\partial_-\phi_-). \end{aligned} \tag{13}$$

The solvability condition to prevent ϕ_1 from growing linearly with increasing ξ_+ or ξ_- requires that the secular terms on the right-hand side of (13) vanish separately with respect to ξ_+ and ξ_-. For nonhomogeneous channels, the variable quantity $(bc) = bh^{1/2}$ represents the effect of channel variation called the (dimensionless) *channel admittance* (see Lighthill [5]) which is, in physical dimensions,

$$Y = \text{admittance} = \frac{\text{volumetric mass flux}}{\text{pressure excess}} = \frac{\rho A\bar{u}}{\rho g\tilde{\zeta}} = \frac{bc}{g},$$

on noting that $\bar{u}/c = \tilde{\zeta}/h$ to leading order. For uniform channels, or, more generally, for varying channels with uniform admittance, so that $\partial_\pm(\log bc) = 0$, the secular equations result simply from setting the first and second lines on the right-hand side of (13) separately to zero. For channels of variable admittance, however, the solvability condition may not be totally separable since the term with (bc) in (13), being a function of x, is generally not separable into additive functions of ξ_+ and ξ_-.

For the general case with variable channel admittance, we propose to adopt an optimum solvability condition based on the following principles:

(i) *that there be no preferred positive direction for the longitudinal axis of any given channel;*
(ii) *that the basic principles of conservation of mass and energy be satisfied, at least adiabatically.*

The property (i) implies that if $x \to -x'$, then $\xi_\pm \to \xi'_\mp$, $\partial_\pm \to \partial'_\mp$, $\phi_\pm \to \phi'_\mp$, and $P_\pm \to P'_\mp$, so the two secular equations will interchange and the system is therefore invariant. Following these principles, we obtain from (13) the following set of equations:

$$\begin{aligned}\partial_\tau\partial_\pm\phi_\pm &+ \frac{3}{4}\partial_\pm(\partial_\pm\phi_\pm)^2 - \frac{1}{6}\partial_\pm^4\phi_\pm \\ &+ \frac{1}{2}(\partial_+\phi_+ - \partial_-\phi_-)(\partial_- - \partial_+)(\log bc) = -\frac{1}{2}\partial_\pm P_\pm, \qquad \text{(14a,b)}\end{aligned}$$

$$\begin{aligned}4\partial_+\partial_-\phi_1 = &(\partial_+ + \partial_-)(\partial_+\phi_+)(\partial_-\phi_-) \\ &+ (\partial_+\phi_+ - \partial_-\phi_-)(\partial_- - \partial_+)(\log bc), \qquad \text{(14c)}\end{aligned}$$

where the upper and lower subscripts in (14a,b) are vertically ordered. Here, (14a,b) results from setting the first and second lines on the right-hand side of (13) separately to zero, but only after each having admitted one additional term, and having these terms subtracted from the right-hand side of (14c) so that (13) remains satisfied and that (14a,b) jointly satisfy the principle (i), which is obvious, and also (ii), as will be seen presently. Under conditions (i) and (ii), this separation (14a,b,c) is unique.

For the wave elevation, we write

$$\tilde{\zeta}(x,t) = \epsilon[\zeta_+(\xi_+;\tau) + \zeta_-(\xi_-;\tau) + \epsilon\zeta_1(\xi_+,\xi_-,\tau) + \cdots], \qquad (15)$$

which to the leading term is related to ϕ by $\tilde{\zeta} = -\phi_t$, and hence by (9b) and (12),

$$\zeta_\pm = -\partial_\pm\phi_\pm. \qquad (16)$$

Now converting the characteristic variables back to the laboratory system in terms of x and t (by multiplying (14a,b) by $\epsilon^{5/2}$ and using (10a,b)), we can write (14a,b) as

$$(\pm\frac{1}{c}\partial_t + \partial_x)\zeta_\pm + \frac{3}{4}\partial_x\zeta_\pm^2 + \frac{1}{6}\partial_x^3\zeta_\pm \pm \frac{1}{2}(\zeta_+ - \zeta_-)\partial_x(\log bc) = -\frac{1}{2}\partial_x P_\pm \quad \text{(17a,b)}$$

in which the scaling parameter ϵ has been absorbed into $\zeta_\pm$ as understood (or by rescaling with $\epsilon = 1$ after having its purpose well served).

To determine ζ_1, we shall further introduce an assumption for varying channels with negligible convexity of the channel admittance (bc), i.e.

$$(bc)_{xx} \ll (bc)_x, \qquad (18)$$

which is expected to be valid for cases of practical interest since moderate departures from it would be a small, local, noncumulative effect. We then have the following relations:

$$\begin{aligned}\zeta_1 = &-(\partial_+ + \partial_-)\phi_1 - \partial_\tau(\phi_+ + \phi_-) + \zeta_+\zeta_- \\ &- \frac{1}{2}(\zeta_+^2 + \zeta_-^2) - \frac{1}{3}(\partial_+^2\zeta_+ + \partial_-^2\zeta_-) - (P_+ + P_-), \qquad (19)\end{aligned}$$

$$\partial_\tau\phi_\pm = -\frac{3}{4}\zeta_\pm^2 - \frac{1}{6}\partial_\pm^2\zeta_\pm - \partial_\mp\left[(\phi_+ + \phi_-)\int(\log bc)_x\,dx\right] - \frac{1}{2}P_\pm, \qquad \text{(20a,b)}$$

$$\phi_1 = -\frac{1}{4}(\phi_-\zeta_+ + \phi_+\zeta_-) + \frac{1}{2}(\phi_+ + \phi_-)\int(\log bc)_x\,dx + \Phi_+(\xi_+) + \Phi_-(\xi_-). \quad (21)$$

Here, (19) results from substituting (9), (10) and (15) in (7) and equating the terms of the highest order $(O(\epsilon^2))$ retained. Equations (20a,b) are the first integrals of (14a,b) and (21) is the integral of (14c) under condition (18), where Φ_+ and Φ_- in (21) are the integration constant functions which will remain indeterminate until the next higher order equation is considered, just as the ϕ_+ and ϕ_- were at the leading order stage of (12). To finalize our resulting equations, we must further assure that arbitrary initial conditions can be prescribed for ζ up to $O(\epsilon^2)$. This requirement is fulfilled if the terms in (19) with $\zeta_\pm^2$ and $\partial_\pm^2\zeta_\pm$ be cancelled by appropriately chosen complementary parts of $\Phi_\pm$. With this condition satisfied, we obtain the final expression for ζ_1 as

$$\zeta_1 = \frac{1}{2}\zeta_+\zeta_- + \frac{1}{4}(\phi_+\partial_-\zeta_- + \phi_-\partial_+\zeta_+) - \frac{1}{2}(\zeta_+ + \zeta_-)\int (\log bc)_x \, dx - \frac{1}{2}(P_+ + P_-). \tag{22}$$

We therefore have the final model equations by collecting (15), (17a,b) and (22) expressed in terms of (x,t), and finally with $\epsilon = 1$, as

$$\zeta(x,t) = \zeta_+(x,t) + \zeta_-(x,t) + \zeta_1(x,t), \tag{23}$$

$$(\pm\frac{1}{c}\partial_t + \partial_x)\zeta_\pm + \frac{3}{4}\partial_x\zeta_\pm^2 + \frac{1}{6}\partial_x^3\zeta_\pm \pm \frac{1}{2}(\zeta_+ - \zeta_-)\partial_x(\log bc) = -\frac{1}{2}\partial_x P_\pm, \tag{24a,b}$$

$$\zeta_1 = \frac{1}{2}\zeta_+\zeta_- + \frac{1}{4}(\phi_+\partial_x\zeta_- - \phi_-\partial_x\zeta_+) - \frac{1}{2}(\zeta_+ + \zeta_-)\int (\log bc)_x \, dx - \frac{1}{2}(P_+ + P_-). \tag{25}$$

The above system of equations constitute the desired model for evaluating generation and evolution of nonlinear dispersive waves moving bidirectionally in varying channels of negligible convexity of channel admittance.

2.1. Integral invariants. The present model will now be shown to have two integral invariants. The first is the second-order invariant measure of excess mass:

$$m_e = \int_{-\infty}^{\infty} b(x)\zeta(x,t)\, dx. \tag{26}$$

To the leading order, m_e is found, by applying (24a,b), to vary at the rate

$$\dot{m}_e = \frac{dm_e}{dt} = \int b(\zeta_+ + \zeta_-)_t \, dx$$

$$= \int \frac{\partial}{\partial x}\left\{(bc)\left[(\zeta_- - \zeta_+) + \frac{3}{4}(\zeta_-^2 - \zeta_+^2) + \frac{1}{6}\partial_x^2(\zeta_- - \zeta_+) + \frac{1}{2}(P_- - P_+)\right]\right\} dx, \tag{27}$$

where variations of the admittance (bc) were observed for terms of $O(\epsilon)$ but not for terms of $O(\epsilon^2)$, thus making the last integral vanish under condition (5), with an error of $O(\epsilon^3)$. As this leaves $\dot{m}_e$ to be measured by the integral of $b(\zeta_1)_t$, which is of $O(\epsilon^2)$, hence there follows the statement for (26).

The variation of the energy, E, can be evaluated to the leading order by multiplying equations (24a,b) for $\zeta_\pm$ by $\zeta_\pm$ and summing their integrals, thereby giving

$$\begin{aligned}\dot{E} &= \frac{d}{dt}\int b(x)(\zeta_+^2 + \zeta_-^2)_t \, dx \\ &= \int \frac{\partial}{\partial x}\left\{(bc)\left[(\zeta_-^2 - \zeta_+^2) + (\zeta_-^3 - \zeta_+^3) + \frac{1}{12}(\zeta_{+x}^2 - \zeta_-^2)\right]\right\} dx \\ &\qquad + \int (bc)(\zeta_- P_{-x} - \zeta_+ P_{+x})\, dx \\ &= \int (bc)[P_+(\zeta_+)_x - P_-(\zeta_-)_x]\, dx \equiv \dot{W},\end{aligned} \tag{28}$$

where in the first step, $(bc)_x$ has been neglected for the highest order terms of $O(\epsilon^3)$ retained. This shows that with forcing being applied, the wave energy increases at the rate equal to the rate of external working, $\dot{W}$, by the x-component of surface pressure over the entire free surface. Therefore, the wave energy of any unforced system is a second-order integral invariant; it varies adiabatically with an error of $O(\epsilon^3)$.

2.2. Some general properties of the model equations. Aside from these integral invariants, this model has additional features that can be reckoned as conspicuous in connection with delineating wave-wave and wave-wall interactions. An inspection of the set of model equations shows that while $\zeta_\pm(x,t)$ are to be determined from (24a,b), (23) and (25) can be further combined to yield

$$\begin{aligned}\tilde{\zeta} =&\Big[\zeta_+\Big(x-\frac{1}{4}\phi_-,t\Big)+\zeta_-\Big(x+\frac{1}{4}\phi_+,t\Big)\Big]\Big(1-\frac{1}{2}\int(\log bc)_x\,dx\Big)\\ &+\frac{1}{2}\zeta_+\zeta_- -\frac{1}{2}(P_+ + P_-),\end{aligned} \tag{29}$$

since this recovers (23) and (25) by Taylor's expansion of $[\zeta_+ + \zeta_-]$ to leading terms. This combined form of $\tilde{\zeta}$ shows clearly that a head-on collision between a R-wave ζ_+ and a L-wave ζ_- results in a phase shift in x by $(-\frac{1}{4}\phi_-)$ for ζ_+ and by $(\frac{1}{4}\phi_+)$ for ζ_-. These phase shifts are clearly functions of the amplitudes of $\zeta_\pm$. It also indicates that the interaction between waves and channel wall will always give rise to reflection and transmission of waves in regions of varying channel admittance because there the R-waves and L-waves are coupled by (24a,b), and will further induce a variation in the wave amplitudes with the factor $(1-\frac{1}{2}\int(\log bc)_x\,dx)$ in addition to the variations in $\zeta_\pm$ due to the effects of varying admittance in (24a,b). The presence of the quadratic term $\frac{1}{2}\zeta_+\zeta_-$ in (29) indicates that the highest amplitude reached in wave-wave interaction will surpass the prediction on linear theory. The prediction of the phase shift and maximum wave height made by the present analysis is found in good agreement with Maxworthy's [8] experiment and compares well with higher-order theoretical results as collectively discussed by Power and Chwang [14].

For weakly varying channels, the present model reduces to Shuto's equation if the reflected wave ζ_- and interactive motion ζ_1 are neglected (with $P_\pm = 0$), i.e.

$$\Big(\frac{1}{c}\partial_t+\partial_x\Big)\zeta_+ +\frac{3}{4}\partial_x\zeta_+^2+\frac{1}{6}\partial_x^3\zeta_+ +\frac{1}{2}\zeta_+\partial_x(\log bc)=0. \tag{30}$$

Shuto's equation has one second-order invariant integral-measure of energy, as expected, but does not have the first-order invariant measure of excess mass unless bc is constant (see Miles [11]). Accordingly, for channels of variable admittance, Shuto's model has the drawback in losing (or gaining) mass for waves progressing in convergent (or divergent) channels, cumulatively to an extent which can be a considerable portion of the initial mass, as found numerically by Teng and Wu [18]. A remedy has been proposed by Kirby and Vengayil [4] who derived, by employing heuristic arguments and linear analogies, a set of coupled equations which is the same as (24a,b) with $P_\pm = 0$. Their model, however, does not consider the second-order interaction, ζ_1, between waves and channel walls. This term can nevertheless play a role of primary importance in producing accumulated phase shifts in waves throughout their mutual interaction as explained above, and in more accurately providing the highly transient wave profile of interactive waves.

3. Variable channels of arbitrary shape

The model equations (23)–(25) derived for varying rectangular channels can further be extended to cover the more general case of varying channels of arbitrary shape. For the latter case we adopt the channel Boussinesq equations developed by Teng and Wu [17] (eqs. 43, 44, 73, 74):

$$(b\tilde{\zeta})_t + [b(\tilde{h} + \tilde{\zeta})\bar{u}]_x = 0, \tag{31}$$

$$\bar{u}_t + \bar{u}\bar{u}_x + \tilde{\zeta}_x - \frac{1}{3}\kappa^2 \tilde{h}^2 \bar{u}_{xxt} = -(p_a)_x, \tag{32}$$

where $\bar{u}(x,t)$ and $\tilde{\zeta}(x,t)$ are defined as before, by (4a,b), now with the wetted cross-sectional area $A(x,t)$ given by

$$A(x,t) = A_h(x) + A_\zeta(x,t), \quad \tilde{h}(x) = A_h(x)/b(x), \tag{33}$$

$A_h(x)$ is the undisturbed channel cross-section of breadth $b(x)$ and mean depth $\tilde{h}(x)$, $A_\zeta(x,t) = b(x)\tilde{\zeta}(x,t)$ is the variation of $A(x,t)$ due to wave elevation. Here the coefficient $\kappa(x)$ denotes a channel-shape factor which depends solely on the geometry, unaffected by varying scales of the channel cross-section, with $\kappa = 1$ for rectangular channels as a standard reference (for other shapes, see Teng and Wu [17]), whereas the channel scale is measured by varying $b(x)$ and $\tilde{h}(x)$.

For channels with weakly varying shape and scale, we shall regard the channel as quasi-uniform, i.e. with $\kappa(x)$ and $\tilde{h}(x)$ taken as variables for the first-order terms and as constant for differential operators appearing in second-order terms. Then, by the similarity transformation:

$$\begin{aligned} dx = \kappa(x)\,dx', \quad t = \kappa t', \quad \tilde{h}(x) = \tilde{h}'(x'), \\ \bar{u}(x,t) = \bar{u}'(x',t'), \quad \tilde{\zeta}(x,t) = \tilde{\zeta}'(x',t'), \end{aligned} \tag{34}$$

the original model equations (31)–(32) for channels of arbitrary shape is transformed into the basic case of rectangular channels (2)–(3), now expressed in terms of the primed variables (also see Teng and Wu [17], eq.77). The corresponding rectangular channel will be called the analogous rectangular channel. It is therefore evident that under transformation (34), equations (23)–(25) provide (if written in terms of the primed variables) the generalized model equations for bidirectional long waves in the analogous rectangular channel and in turn for waves in varying channels of arbitrary shape upon transforming back by (34) into the original (unprimed) variables.

The simple relation (34) between the two corresponding channels presents another conspicuous feature of the present model which can be delineated more clearly by the simpler case of uniform channels, i.e. with κ = constant. In this simpler case, a system of waves in the arbitrary channel is related to its counterpart in the analogous rectangular channel through their amplitudes (a and a') and their phase functions ($\vartheta = k(x - ct), \vartheta' = k'(x' - c't')$, k and k' being wave numbers) such that

$$a' = a, \quad k'(x' - c't') = (k'/\kappa)(x - c't) = k(x - ct), \tag{35}$$

since they are both invariant by (34). Whence

$$k' = \kappa k \quad \text{and} \quad c' = c. \tag{36a}$$

Therefore, for the original arbitrary channel and the analogous rectangular channel of equal depth, the two corresponding wave systems will be equal in amplitude and wave velocity, but with the wavelength-ratio and the period-ratio given by

$$\lambda/\lambda' = k'/k = \kappa, \quad \text{and} \quad T/T' = \lambda/\lambda' = \kappa. \tag{36b}$$

The same length-ratio and time-ratio apply to other related quantities, such as for the quantities measuring wave-wave and wave-wall interactions.

In summary, (23)–(25) constitute the basic evolution equations for weakly nonlinear, weakly dispersive and weakly forced long waves in varying rectangular channels; they also apply to varying channels of arbitrary shape with the assistance of transformation (34).

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