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Fusion of two solutions of a partial differential equation

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ABSTRACT. Several theorems are proved which give sufficient conditions for melding two solutions of a partial differential equation, or inequality, on "adjacent" domains. In particular, one obtains theorems concerning removable singularities for harmonic functions, solutions of the heat equation, subharmonic functions, and holomorphic functions of several variables.

1. Introduction

This work contains a number of recipes for melding two solutions of a linear partial differential equation or inequality along a common edge. This problem belongs to the field of removable singularities of solutions of PDE's, a subject which has attracted considerable interest in recent years (see the survey of J. Polking [11]). Some of our results are closely related to work of R. Harvey and J. Polking [3]. Our methods are elementary, modulo the use of standard results on smoothness of solutions of (hypo)elliptic equations. The main differences with [3] are that our operators are less general, which allows more to be said about their solutions, that our coefficients and hypersurfaces are not assumed to be C^∞-smooth, that our hypotheses are such that one gets additional smoothness of the solution, that various explicit examples are examined, and that some differential inequalities are treated as well.

In particular, we extend Theorem 5.2 of Harvey and Polking [3] for removable singularities on C^∞-smooth hypersurfaces to a result on removable singularities on a C^1-smooth hypersurface for solutions of partial differential equations as well as inequalities. As Harvey and Polking point out in [3], their Theorem 5.2 for C^∞-smooth hypersurfaces can be *reduced to* (or deduced from) the hyperplane case by means of local coordinate transformations. For our case of C^1-smooth hypersurfaces it is not so clear how to perform such a reduction, for in this context, the local coordinate transformations which straighten the hypersurface are only C^1-smooth so that the smoothness of the coefficients may be decreased by such coordinate transformations.

Most of the results of this paper were found while the first author was writing his thesis at the Université de Montréal. Theorem 2.1 concerns solutions of elliptic equations of second order and is a tool which the first author made use of in his thesis [2]. This theorem was the starting point of this work and its proof has led to the statements of the other results. These concern hypoelliptic equations (Section 3), subharmonic functions (Section 4) and solutions of the equation $\bar\partial u = f$ (Section 5).

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2. Elliptic equations of second order

We consider, first, solutions of elliptic equations with mild regularity assumptions on the coefficients. We now introduce these solutions.

Let D be a domain of $\mathbb{R}^m$, $m \geq 2$. Consider the following linear partial differential operator

$$L = \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} \beta_i(x) \frac{\partial}{\partial x_i} + c(x) \tag{2.1}$$

with real-valued coefficients $a_{ij} = a_{ji} \in C^{2,\alpha}(D)$, $\beta_i \in C^{1,\alpha}(D)$, $c \in C^1(D)$, $\alpha \in (0,1)$, and $i, j = 1, \ldots, m$.

We assume that L is of *elliptic* type in D. This means that

$$\sum_{i,j=1}^{m} a_{ij}(x) \xi_i \xi_j > 0 \tag{2.2}$$

for each $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m \backslash \{0\}$ and for each $x \in D$.

An operator L given by (2.1) and whose principal part satisfies (2.2) enjoys the following property.

Weyl's Lemma. [6, page 199] *Let $f \in C^{0,\alpha}(D)$. If u is locally integrable in D and satisfies*

$$\int_D u(x) L\varphi(x) dV = \int_D f(x) \varphi(x) dV \tag{2.3}$$

for every $\varphi \in C_c^\infty(D)$, then u coincides almost everywhere in D with a function v in $C^{2,\alpha}(D)$.

Therefore if $u \in C^0(D)$ satisfies (2.3) for every $\varphi \in C_c^\infty(D)$, then at each point $x \in D$, u satisfies the equation $Au(x) = f(x)$, where

$$Au(x) = \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} \big(a_{ij}(x) u(x)\big) - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \big(\beta_i(x) u(x)\big) + c(x) u(x). \tag{2.4}$$

Let S be a hypersurface of class C^1 in D. If $\vec{n}(x) = \big(n_1(x), \ldots, n_m(x)\big)$ is a continuous vector field normal to S, then, setting

$$\nu_i(x) = \sum_{j=1}^{m} a_{ij}(x) n_j(x), \quad i = 1, \ldots, m, \tag{2.5}$$

we define a continuous vector field $\vec{\nu}(x) = \big(\nu_1(x), \ldots, \nu_m(x)\big)$ on S and a first-order differential operator as follows:

$$\frac{\partial}{\partial \nu} = \vec{\nu} \cdot \nabla = \sum_{i,j=1}^{m} a_{ij} n_j \frac{\partial}{\partial x_i}. \tag{2.6}$$

If $a_{ij} = \delta_{ij}$, then $\vec{\nu} = \vec{n}$. By (2.2), $\vec{\nu} \cdot \vec{n} > 0$ and $\vec{\nu}$ is never tangent to S.

We shall say that two disjoint domains D_1 and D_2 are *adjacent* at a *free* hypersurface S of class C^1 if S is a hypersurface of class C^1 in $\partial D_1 \cap \partial D_2$ and if $\mathrm{dist}(x, \partial D_k \backslash S) > 0$ for each $x \in S$ and each $k = 1, 2$. If D_1 and D_2 are two such domains, then for $k = 1, 2$, $u_k \in C^1(D_k \cup S)$ will mean that $u_k \in C^1(D_k)$ and that u_k together with all its partial derivatives of first order extend continuously to $D_k \cup S$. Therefore, if $u_k \in C^1(D_k \cup S)$, then there exist $m+1$ functions ϑ_k^i, $i = 0, 1, \ldots, m$, continuous on $D_k \cup S$, such that $\vartheta_k^0 = u_k$ and $\vartheta_k^i = \partial u_k / \partial x_i$ for $i = 1, \ldots, m$ on D_k. In that case we set, by abuse of notation, $u_k = \vartheta_k^0$, $\partial u_k/\partial x_i = \vartheta_k^i$ for $i = 1, \ldots, m$, $\nabla u_k = (\vartheta_k^1, \ldots, \vartheta_k^m)$. We denote by $\vec{n}^k = (n_1^k, \ldots, n_m^k)$ the unit normal on S exterior to D_k, by $\vec{\nu}^k = (\nu_1^k, \ldots, \nu_m^k)$ the corresponding vector field with components

$$\nu_i^k = \sum_{j=1}^m a_{ij} n_j^k, \quad i = 1, \ldots, m, \tag{2.7}$$

and we set $\partial/\partial \nu^k = \vec{\nu}^k \cdot \nabla$, $\partial u_k / \partial \nu^k = \vec{\nu}^k \cdot (\partial u_k/\partial x_1, \ldots, \partial u_k / \partial x_m)$.

We may now state our first result.

Theorem 2.1. *Let D_1 and D_2 be two disjoint domains of $\mathbb{R}^m$ $(m \geq 2)$ adjacent at a free hypersurface S of class C^1, and let A be a partial differential operator of the form (2.4) satisfying (2.2) in the domain $D = D_1 \cup D_2 \cup S$, with coefficients $a_{ij} \in C^{2,\alpha}(D)$, $\beta_i \in C^{1,\alpha}(D)$, $c \in C^1(D)$, $i, j = 1, \ldots, m$, $\alpha \in (0,1)$. Let $f \in C^{0,\alpha}(D)$, $u_1 \in C^2(D_1)$, and $u_2 \in C^2(D_2)$ satisfy $Au_1 = f$ in D_1 and $Au_2 = f$ in D_2. If $u_1 \in C^1(D_1 \cup S)$, $u_2 \in C^1(D_2 \cup S)$, and*

$$u_1 = u_2, \qquad \frac{\partial u_1}{\partial \nu^k} = \frac{\partial u_2}{\partial \nu^k} \tag{2.8}$$

on S with $k = 1$ or 2, then there exists $u \in C^{2,\alpha}(D)$ such that $Au = f$ in D, $u = u_1$ in D_1, and $u = u_2$ in D_2.

Proof. Let u be the function on $D_1 \cup D_2$ which equals u_k on D_k. Then by (2.8) u extends continuously to $D = D_1 \cup D_2 \cup S$. We show that u satisfies in D a relation of the form (2.3)

$$\int_D u(x) L\varphi(x) dV = \int_D f(x) \varphi(x) dV \tag{2.9}$$

for every $\varphi \in C_c^\infty(D)$. Let φ be an element of $C_c^\infty(D)$. If the support of φ is in $D_1 \cup D_2$, then (2.9) is clear since in that case $Au = f$ in a neighborhood of the support of φ. If the support of φ intersects S, then we consider a subdomain $W \subset \overline{W} \subset D$ containing the support of φ and such that, for $k = 1, 2$, the boundary of $W_k = W \cap D_k$ is of class C^1. By using the divergence theorem [12, page 100] and the hypothesis (2.8), we easily obtain that

$$\int_D u(x) L\varphi(x) dV = \sum_{k=1}^2 \int_{W_k} \varphi(x) Au_k(x) dV$$

and formula (2.9) follows. By applying Weyl's Lemma, we see that u satisfies all the conclusions of Theorem 2.1 and this completes the proof.

3. Hypoelliptic equations

In this section, we consider an operator A of the form (2.4) having *all its real-valued coefficients of class* C^∞, but we *no longer* assume that these satisfy (2.2). We denote by $\mathcal{D}'(D)$ the set of all distributions T in D and we let AT be the distribution defined by $AT(\varphi) = T(L\varphi)$ for every $\varphi \in C_c^\infty(D)$.

By definition, the *singular support* of $T \in \mathcal{D}'(D)$, denoted sing supp T, is the set of points in D having no open neighborhood to which the restriction of T is a C^∞ function. An operator A on $\mathcal{D}'(D)$ is then said to be *hypoelliptic* if

$$\operatorname{sing\,supp} T = \operatorname{sing\,supp} AT \tag{3.1}$$

for every $T \in \mathcal{D}'(D)$.

According to a theorem of L. Hörmander [8, page 151], (see also [10, page 139]), the principal part of a hypoelliptic operator A must be a positive or negative semi-definite quadratic form. More precisely this means that for any point x in D

$$\text{either} \quad \sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geq 0 \quad \text{or} \quad \sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \leq 0$$

for all $\xi \in \mathbb{R}^m$. We must beware of drawing the false conclusion (as Oleinik and Radkević did) that if the principal part is positive semi-definite at one point then it must be positive semi-definite at every point. In fact, Kannai gave an example of a hypoelliptic operator with principal part $x(\partial_x^2 + \partial_y^2)$! This makes results like Theorem 3.1 below more striking.

Set

$$a(x,\xi,\eta) = \sum_{i,j=1}^{m} a_{ij}(x)\xi_i\eta_j \tag{3.2}$$

for $\xi = (\xi_1, \ldots, \xi_m)$, $\eta = (\eta_1, \ldots, \eta_m)$ in $\mathbb{R}^m$ and assume (without loss of generality) that, at a fixed point x in D, $a(x,\xi,\xi) \geq 0$ for all $\xi \in \mathbb{R}^m$. Then $a(x,\cdot,\cdot)$ is a positive semi-definite symmetric bilinear form on $\mathbb{R}^m$ and therefore satisfies the inequality $a(x,\xi,\eta)^2 \leq a(x,\xi,\xi)a(x,\eta,\eta)$ for all $\xi,\eta \in \mathbb{R}^m$. In particular, if we choose $x \in S$, $\xi = \vec{n}(x) = \big(n_1(x), \ldots, n_m(x)\big)$ normal to S, and $\eta = e_j = (0, \ldots, 1, \ldots, 0)$, the unit vector along the j-th coordinate axis, this becomes

$$\Big(\sum_{i=1}^{m} a_{ij}(x)n_i(x)\Big)^2 \leq \Big(\sum_{i,k=1}^{m} a_{ik}(x)n_i(x)n_k(x)\Big)\big(a_{jj}(x)\big). \tag{3.3}$$

Inequality (3.3) shows that the vector field $\vec{\nu}$ defined by (2.5) on the hypersurface S in D in terms of the coefficients of a hypoelliptic operator A is transversal to S ($\vec{\nu}\cdot\vec{n} > 0$) at each point when $\vec{\nu}$ does not vanish.

We may now state the following result which is nothing but an immediate corollary of the proof of Theorem 2.1.

Theorem 3.1. *Let D_1 and D_2 be two disjoint domains of $\mathbb{R}^m$ ($m \geq 2$) adjacent at a free hypersurface S of class C^1. Let A be a partial differential operator of the form (2.4) with coefficients in $C^\infty(D)$ and which is hypoelliptic. Let $f \in C^\infty(D)$,*

$u_1 \in C^2(D_1)$, and $u_2 \in C^2(D_2)$ satisfy $Au_1 = f$ in D_1 and $Au_2 = f$ in D_2. If $u_1 \in C^1(D_1 \cup S)$, $u_2 \in C^1(D_2 \cup S)$, and

$$u_1 = u_2, \qquad \frac{\partial u_1}{\partial \nu^k} = \frac{\partial u_2}{\partial \nu^k} \tag{3.4}$$

on S with $k = 1$ or 2, then there exists $u \in C^\infty(D)$ such that $Au = f$ in D, $u = u_1$ in D_1, and $u = u_2$ in D_2.

Proof. Let u be the continuous function on D which equals u_k on D_k. By taking f this time in $C^\infty(D)$, we may repeat all the steps leading to formula (2.9) in the proof of Theorem 2.1, and we obtain that $T_u(L\varphi) = T_f(\varphi)$ for every $\varphi \in C_c^\infty(D)$, where T_u and T_f are the distributions defined by u and f. This shows that sing supp $AT_u = \phi$. Therefore (3.1) implies that u is of class C^∞ in a neighborhood of each point of D. Thus u satisfies all the conclusions of Theorem 3.1 and this completes the proof.

A new feature of Theorem 3.1 with respect to Theorem 2.1 is that it applies to solutions of the *heat equation*

$$\Delta_x u(x,t) = \partial u(x,t)/\partial t \tag{3.5}$$

where $x = (x_1, \dots, x_{m-1})$ and $\Delta_x = \sum_{i=1}^{m-1} \partial^2/\partial x_i^2$. In this case, (3.4) becomes

$$u_1 = u_2, \qquad \sum_{i=1}^{m-1} n_i^k \frac{\partial u_1}{\partial x_i} = \sum_{i=1}^{m-1} n_i^k \frac{\partial u_2}{\partial x_i}.$$

If D_1 and D_2 are adjacent at S_0 defined by $t = 0$, then by Theorem 3.1, we can extend two solutions u_1 and u_2 of (3.5) in D_1 and D_2 to a solution u of (3.5) in $D_1 \cup D_2 \cup S_0$ under the *sole* hypothesis that $u_1 = u_2$ on S_0, since then $n_i^k = 0$ for each $i = 1, \dots, m-1$.

The above remark, obtained from Theorem 3.1, is a corollary of a result of Harvey and Polking [3, Theorem 5.2] as well, since the heat operator (3.5) has normal order one with respect to S_0 (see [3, page 48]). It serves as a foretaste for our next result. In order to state this result, we now briefly recall some basic concepts.

A vector $\xi \in \mathbb{R}^m$ is said to be *characteristic* for A at $x \in D$ if $a(x,\xi,\xi) = 0$, where $a(x,\cdot,\cdot)$ is the bilinear form (3.2). We denote by $\mathrm{char}_x(A)$ the set of all such ξ. A hypersurface in D is called characteristic for A at x if its normal vector $\vec{n}(x)$ is in $\mathrm{char}_x(A)$. A hypersurface is called a *characteristic hypersurface* for A if it is characteristic at each of its points.

We again consider two disjoint domains of R^m adjacent at a free hypersurface S, and we set $D = D_1 \cup D_2 \cup S$. We assume that S is given (locally) by an equation of the form $p(x) = 0$ with

$$\begin{gathered} p \in C^1(D)\ ,\ \nabla p(x) \neq 0 \text{ on } S, \\ D_1 = \{\, x : p(x) < 0 \,\}\ ,\ D_2 = \{\, x : p(x) > 0 \,\}, \end{gathered} \tag{3.6}$$

and that S is a characteristic hypersurface for A. This means that

$$\sum_{i,j=1}^{m} a_{ij}(x) n_i(x) n_j(x) = 0 \tag{3.7}$$

for each $x \in S$ with $\vec{n}(x) = (n_1(x), \ldots, n_m(x))$ the normal vector. By inequality (3.3), this implies that the components

$$\nu_i(x) = \sum_{j=1}^{m} a_{ij}(x) n_j(x)$$

of the vector field $\vec{\nu}(x)$ vanish on S. We assume, moreover, the stronger condition (see [1, page 74])

$$a(x, \nabla p(x), \nabla p(x)) \equiv 0, \text{ in } D. \tag{3.8}$$

The following result generalizes the preceding example.

Theorem 3.2. *Let D_1 and D_2 be two disjoint domains of $\mathbb{R}^m$ $(m \geq 2)$ adjacent at a free hypersurface S of class C^1, and set $D = D_1 \cup D_2 \cup S$. Let A be a partial differential operator of the form* (2.4) *with coefficients in $C^\infty(D)$ satisfying* (3.1). *Assume that S is a characteristic hypersurface for A defined by an equation $p(x) = 0$ with p satisfying* (3.6) *and* (3.8). *Let $f \in C^\infty(D)$, $u_1 \in C^2(D_1)$, and $u_2 \in C^2(D_2)$ satisfy $Au_1 = f$ in D_1 and $Au_2 = f$ in D_2. If $u_1 \in C^0(D_1 \cup S)$, $u_2 \in C^0(D_2 \cup S)$, and $u_1 = u_2$ on S, then there exists $u \in C^\infty(D)$ such that $Au = f$ in D, $u = u_1$ in D_1, and $u = u_2$ in D_2.*

Using the terminology of Harvey and Polking [4, page 185], Theorem 3.2 can be rephrased by simply saying that these characteristic hypersurfaces are removable for continuous functions with respect to the hypoelliptic operator A.

In the proof of Theorem 3.2, we shall make use of the following identity [10, page 20].

Green's Identity. *Let W be a subdomain with compact closure in D and with C^1 boundary ∂W. If $u, v \in C^2(\overline{W})$, then*

$$\int_W \{v(x)Lu(x) - u(x)Av(x)\} dV = \int_{\partial W} [v(x), u(x)] \tag{3.9}$$

with

$$[v(x), u(x)] = \left\{ v(x)\frac{\partial u(x)}{\partial \nu} - u(x)\frac{\partial v(x)}{\partial \nu} + e_n(x)u(x)v(x) \right\} ds,$$

where

$$e_n(x) = \sum_{i=1}^{m} e_i(x) n_i(x),$$

$$e_i(x) = \beta_i(x) - \sum_{j=1}^{m} \partial a_{ij}(x)/\partial x_j, \quad i = 1, \ldots, m,$$

and where $n_i(x)$ are the components of the unit normal exterior to W.

Proof of Theorem 3.2. Let u be the continuous function on $D = D_1 \cup D_2 \cup S$ which equals u_k on D_k. We show that $T_u(L\varphi) = T_f(\varphi)$ for every $\varphi \in C_c^\infty(D)$.

For $\varepsilon_1 < 0$ given, set $S_{\varepsilon_1} = \{ x : p(x) = \varepsilon_1 \}$ and $D_1^{\varepsilon_1} = \{ x : p(x) < \varepsilon_1 \}$. Let $\varphi \in C_c^\infty(D)$ and assume that the support of φ intersects S. On S_{ε_1} we consider the following surface integral:

$$\int_{S_{\varepsilon_1}} [u_1, \varphi]_{\varepsilon_1} \equiv \int_{S_{\varepsilon_1}} \left(u_1 \frac{\partial \varphi}{\partial \nu^1_{\varepsilon_1}} - \varphi \frac{\partial u_1}{\partial \nu^1_{\varepsilon_1}} + e_{n^1_{\varepsilon_1}} u_1 \varphi \right) ds_{\varepsilon_1},$$

where

$$\frac{\partial}{\partial \nu^1_{\varepsilon_1}} = \sum_{i,j=1}^m a_{ij} n^1_{\varepsilon_1 i} \frac{\partial}{\partial x_j}, \quad e_{n^1_{\varepsilon_1}} = \sum_{i=1}^m e_i n^1_{\varepsilon_1 i},$$

and where $n^1_{\varepsilon_1} = (n^1_{\varepsilon_1 1}, \ldots, n^1_{\varepsilon_1 m})$ is the unit normal on S_{ε_1} which is exterior to $D_1^{\varepsilon_1}$. If ε_1 is chosen sufficiently close to zero, then by (3.9) and the hypothesis, we have

$$\int_{S_{\varepsilon_1}} [u_1, \varphi]_{\varepsilon_1} = \int_{D_1^{\varepsilon_1}} (u_1 L \varphi - \varphi A u_1) dV = \int_{D_1^{\varepsilon_1}} u_1 L \varphi \, dV - \int_{D_1^{\varepsilon_1}} \varphi f \, dV. \quad (3.10)$$

By (3.8), S_{ε_1} is characteristic and (3.10) becomes

$$\int_{S_{\varepsilon_1}} e_{n^1_{\varepsilon_1}} u_1 \varphi \, ds_{\varepsilon_1} = \int_{D_1^{\varepsilon_1}} u_1 L \varphi \, dV - \int_{D_1^{\varepsilon_1}} \varphi f \, dV. \quad (3.11)$$

Letting ε_1 tend to zero in (3.11), we find

$$\int_{D_1} u_1 L \varphi \, dV = \int_S e_{n^1} u_1 \varphi \, ds + \int_{D_1} \varphi f \, dV. \quad (3.12)$$

Now let $\varepsilon_2 > 0$ be given and set $S_{\varepsilon_2} = \{ x : p(x) = \varepsilon_2 \}$, $D_2^{\varepsilon_2} = \{ x : p(x) > \varepsilon_2 \}$. Then, as above, we have for ε_2 sufficiently small,

$$\int_{S_{\varepsilon_2}} [u_2, \varphi]_{\varepsilon_2} = \int_{D_2^{\varepsilon_2}} u_2 L \varphi \, dV - \int_{D_2^{\varepsilon_2}} \varphi f \, dV.$$

Since S_{ε_2} is characteristic, this becomes

$$\int_{S_{\varepsilon_2}} e_{n^2_{\varepsilon_2}} u_2 \varphi \, ds_{\varepsilon_2} = \int_{D_2^{\varepsilon_2}} u_2 L \varphi \, dV - \int_{D_2^{\varepsilon_2}} \varphi f \, dV, \quad (3.13)$$

where $n^2_{\varepsilon_2}$ is the unit normal on S_{ε_2} which is exterior to $D_2^{\varepsilon_2}$. Letting ε_2 tend to zero in (3.13), we find

$$\int_{D_2} u_2 L \varphi \, dV = \int_S e_{n^2} u_2 \varphi \, ds + \int_{D_2} \varphi f \, dV. \quad (3.14)$$

Therefore by using (3.12), (3.14), the hypothesis, and the fact that $n^1 = -n^2$ on S, we have

$$\begin{aligned} T_u(L\varphi) &= \int_{D_1} u_1 L \varphi \, dV + \int_{D_2} u_2 L \varphi \, dV \\ &= \int_S e_{n^1} \varphi (u_1 - u_2) ds + T_f(\varphi) = T_f(\varphi). \end{aligned} \quad (3.15)$$

From (3.15) and the hypoellipticity of A, we conclude that u satisfies all the conclusions of Theorem 3.2. This completes the proof.

Before ending this section, we present two additional examples of equations to which the above theorems apply. The first one is in $\mathbb{R}^2$ and the second in $\mathbb{R}^3$.

By setting $x_1 = x - t$, $x_2 = x + t$, the equation $\partial^2 u/\partial x^2 = \partial u/\partial t$ becomes

$$\frac{\partial^2 u}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_1}. \tag{3.16}$$

In this case the corresponding vector field $\vec{\nu} = (\nu_1, \nu_2) = (n_1 + n_2,\ n_1 + n_2)$ will vanish on any straight line where $n_1 = -n_2$. In particular, $\nu_1 = \nu_2 = 0$ on the straight line $x_1 = x_2$ and we can meld two solutions u_1 and u_2 of (3.16) if $u_1 = u_2$ on this line.

Let us consider *Kolmogorov's equation*

$$\frac{\partial^2 u}{\partial x_1^2} + x_1 \frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial t} = f. \tag{3.17}$$

This equation is hypoelliptic (see [8, page 147]), and $\vec{\nu} = (n_1, 0, 0)$ will vanish on the planes $t = 0$ and $x_2 = 0$, but not on the plane $x_1 = 0$. We can therefore meld two solutions of (3.17) on $t = 0$ or on $x_2 = 0$ if these two solutions coincide there. However, $\vec{\nu} = (1, 0, 0)$ on the plane $x_1 = 0$, and (3.4) becomes $u_1 = u_2$, $\partial u_1/\partial x_1 = \partial u_2/\partial x_1$.

4. Subharmonic functions

We now present a result on the "fusion" of two solutions of a partial differential inequality. For simplicity, we consider only the case of *subharmonic functions*.

Theorem 4.1. *Let D_1 and D_2 be two disjoint domains of $\mathbb{R}^m$ $(m \geq 2)$ adjacent at a free hypersurface S of class C^1. Let $u_1 \in C^2(D_1)$ and $u_2 \in C^2(D_2)$ be subharmonic in D_1 and D_2. If $u_1 \in C^1(D_1 \cup S)$, $u_2 \in C^1(D_2 \cup S)$, and*

$$u_1 = u_2, \qquad \frac{\partial u_j}{\partial n^k} \geq \frac{\partial u_k}{\partial n^k} \tag{4.1}$$

on S with $j, k = 1, 2$ and $j \neq k$, then there exists a continuous function u subharmonic in $D = D_1 \cup D_2 \cup S$ such that $u = u_1$ in D_1 and $u = u_2$ in D_2.

In order to illustrate Theorem 4.1, let us consider the continuous function $u(x)$ defined in $\mathbb{R}^3$ by

$$u(x) = \begin{cases} -1 & \text{if } \ |x| \leq 1, \\ -1/|x| & \text{if } \ |x| \geq 1. \end{cases}$$

Setting $D_1 = \{\, x \in \mathbb{R}^3 : |x| < 1 \,\}$ and $D_2 = \mathbb{R}^3 \backslash \bar{D}_1$, we conclude from Theorem 4.1 that u is subharmonic. Even if u_1 and u_2 are in fact harmonic in D_1 and D_2, it is clear that there is no hope for u to be harmonic. From this point of view, Theorem 4.1 can be seen as a "back door" to be used when Theorem 2.1 fails.

In the complex plane $\mathbb{C}$, let us consider the continuous function $u(z)$ defined by

$$u(z) = \begin{cases} 0 & \text{if } \ |z| \leq 1, \\ \log|z| & \text{if } \ |z| \geq 1. \end{cases}$$

As in the preceding example, we see, by using Theorem 4.1, that $u(z)$ is subharmonic in $\mathbb{C}$.

We now prove Theorem 4.1.

As above we denote by ΔT the distribution defined by $\Delta T(\varphi) = T(\Delta\varphi)$ where $\Delta = \sum_{i=1}^{m} \partial^2/\partial x_i^2$. We shall say that $\Delta T \geq 0$ if $\Delta T(\varphi) \geq 0$ for every non-negative test function φ. We shall make use of the following result (see for example [9, pages 74 and 77]).

Theorem A. *If u is a real-valued locally integrable function in an open set D in $\mathbb{R}^m$ satisfying $\Delta T_u \geq 0$, then T_u is a subharmonic function. Conversely, if u is subharmonic in D, then u is locally integrable and $\Delta T_u \geq 0$.*

In particular, $u \in C^2(D)$ is subharmonic if and only if $\Delta u \geq 0$. The function which is $-\infty$ identically is excluded in Theorem A.

Proof of Theorem 4.1. Let u be the continuous function in $D = D_1 \cup D_2 \cup S$ which equals u_k in D_k. We show that

$$\Delta T_u \geq 0. \tag{4.2}$$

Let $\varphi \in C_c^\infty(D)$, $\varphi \geq 0$. If the support of φ is in $D_1 \cup D_2$, the inequality (4.2) is clear, since in that case, by Theorem A, $\Delta u \geq 0$ in a neighborhood of the support of φ. If the support of φ intersects S, then we consider a subdomain $W \subset \overline{W} \subset D$ containing the support of φ and such that for $k = 1, 2$, the boundary of $W_k = W \cap D_k$ is of class C^1. Then there exist functions $p_k \in C^1(\mathbb{R}^m)$ with $\nabla p_k \neq 0$ on ∂W_k such that $W_k = \{x \in \mathbb{R}^m : p_k(x) < 0\}$, (see for example [9, page 59]). We set $W_k^\varepsilon = \{x \in \mathbb{R}^m : p_k(x) < \varepsilon\}$, where $\varepsilon < 0$. Then $\overline{W}_k^\varepsilon \subset W_k$ for each $\varepsilon < 0$, $W_k^\varepsilon \nearrow W_k$ when $\varepsilon \nearrow 0$, and the family $\{W_k^\varepsilon\}_{\varepsilon<0}$ is an exhaustion of W_k by domains having boundaries of class C^1. We denote by $\vec{n}_\varepsilon^k$ the unit normal exterior to W_k^ε. Then we have

$$\Delta T_u(\varphi) = \lim_{\varepsilon\to 0} \int_{W_1^\varepsilon} u_1\, \Delta\varphi\, dV + \lim_{\varepsilon\to 0} \int_{W_2^\varepsilon} u_2\, \Delta\varphi\, dV, \tag{4.3}$$

and by Green's Identity

$$\int_{W_k^\varepsilon} u_k\, \Delta\varphi\, dV = \int_{W_k^\varepsilon} \varphi\, \Delta u_k\, dV + \int_{\partial W_k^\varepsilon} \left(u_k \frac{\partial\varphi}{\partial n_\varepsilon^k} - \varphi \frac{\partial u_k}{\partial n_\varepsilon^k} \right) ds_\varepsilon.$$

Since

$$\lim_{\varepsilon\to 0} \int_{W_k^\varepsilon} u_k\, \Delta\varphi\, dV = \int_{W_k} u_k\, \Delta\varphi\, dV$$

and

$$\lim_{\varepsilon\to 0} \int_{\partial W_k^\varepsilon} \left(u_k \frac{\partial\varphi}{\partial n_\varepsilon^k} - \varphi \frac{\partial u_k}{\partial n_\varepsilon^k} \right) ds_\varepsilon = \int_{\partial W_k} \left(u_k \frac{\partial\varphi}{\partial n^k} - \varphi \frac{\partial u_k}{\partial n^k} \right) ds,$$

we see that the limit

$$\lim_{\varepsilon\to 0} \int_{W_k^\varepsilon} \varphi\, \Delta u_k\, dV \geq 0$$

exists and is equal to

$$\int_{W_k} u_k\, \Delta\varphi\, dV - \int_{\partial W_k} \left(u_k \frac{\partial\varphi}{\partial n^k} - \varphi \frac{\partial u_k}{\partial n^k} \right) ds.$$

Thus, by (4.3),

$$\Delta T_u(\varphi) \geq \sum_{k=1}^{2} \int_{\partial W_k} \left(u_k \frac{\partial \varphi}{\partial n^k} - \varphi \frac{\partial u_k}{\partial n^k} \right) ds, \tag{4.4}$$

and (4.2) follows from (4.4) and the fact that by (4.1) we have

$$\int_S \varphi \left(\frac{\partial u_j}{\partial n^k} - \frac{\partial u_k}{\partial n^k} \right) ds \geq 0$$

with $j, k = 1, 2$ and $j \neq k$. Therefore, by Theorem A, (4.2) implies that u coincides almost everywhere with a subharmonic function v in D. But since $u \in C^0(D)$, we have $v(x) = u(x)$ for each $x \in D$, and we see that u is subharmonic in D and equals u_1 in D_1 and u_2 in D_2. This completes the proof of Theorem 4.1.

A first corollary of the proof of Theorem 4.1 can be stated as follows. If D is a bounded domain in $\mathbb{R}^m$ having a boundary of class C^1 and if $u \in C^2(D) \cap C^1(\bar{D})$, then the limit

$$\lim_{\varepsilon \to 0} \int_{D_\varepsilon} \Delta u \, dV$$

exists. If moreover $\Delta u \geq 0$ in D, then, by the monotone convergence theorem, we have

$$\int_{D_\varepsilon} \Delta u \, dV \nearrow \int_D \Delta u \, dV$$

when ε tends to zero, and then Δu is integrable on D. Thus, for such a domain D, if $u \in C^2(D) \cap C^1(\bar{D})$ is subharmonic, then Δu is integrable on D.

5. Solutions of $\bar{\partial} u = f$

In this section, we consider complex-valued functions defined on subsets of the m-dimensional complex number space

$$\mathbb{C}^m = \{ z : z = (z_1, \ldots, z_m), z_j = x_j + i y_j \in \mathbb{C},\ 1 \leq j \leq m \}.$$

We first introduce some standard notations.

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad j = 1, \ldots, m,$$

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right), \quad j = 1, \ldots, m,$$

$$dz_j = dx_j + i\, dy_j, \quad j = 1, \ldots, m,$$

$$d\bar{z}_j = dx_j - i\, dy_j, \quad j = 1, \ldots, m,$$

$$\begin{aligned} dV(z) &= (1/2i)^m (d\bar{z}_1 \wedge dz_1) \wedge \cdots \wedge (d\bar{z}_m \wedge dz_m) \\ &= dx_1 \wedge dy_1 \wedge \cdots \wedge dx_m \wedge dy_m. \end{aligned}$$

If $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p$ and $\beta = (\beta_1, \ldots, \beta_q) \in \mathbb{N}^q$ are multi-indices, then

$$dz^\alpha = dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_p},$$

$$d\bar{z}^\beta = d\bar{z}_{\beta_1} \wedge \cdots \wedge d\bar{z}_{\beta_q}.$$

If A and B are two ordered subsets of $\{1,\dots,m\}$, then we set $\varepsilon^A_B = \operatorname{sign}\pi$ if $A = B$ as sets and π is a permutation which takes A into B, and $\varepsilon^A_B = 0$ in all other cases.

In order to state our result we now recall some definitions.

In a domain D of $\mathbb{C}^m$, $m \geq 1$, we consider the equation $\bar\partial u = f$, where

$$f = \sum_{\alpha,\gamma} f_{\alpha\gamma} dz^\alpha \wedge d\bar z^\gamma$$

is in the class $C^0_{p,q+1}(D)$ of all continuous differential forms of type $(p, q+1)$ on D. The sum is therefore taken over all strictly increasing multi-indices $\alpha = (\alpha_1,\dots,\alpha_p) \in \mathbb{N}^p$ and $\gamma = (\gamma_1,\dots,\gamma_{q+1}) \in \mathbb{N}^{q+1}$ with $\alpha_i, \gamma_i \in \{1,\dots,m\}$. If

$$u = \sum_{\alpha,\beta} u_{\alpha,\beta}\, dz^\alpha \wedge d\bar z^\beta$$

is in $C^1_{p,q}(D)$, then

$$\bar\partial u = \sum_{j=1}^{m} \sum_{\alpha,\beta} \frac{\partial u_{\alpha\beta}}{\partial \bar z_j} d\bar z_j \wedge dz^\alpha \wedge d\bar z^\beta,$$

where this time $\sum_{\alpha,\beta}$ is taken over all strictly increasing multi-indices $\alpha \in \mathbb{N}^p$ and $\beta \in \mathbb{N}^q$, and therefore

$$\bar\partial u = (-1)^p \sum_{\alpha,\gamma} \left(\sum_{j,\beta} \varepsilon^\gamma_{j\beta} \frac{\partial u_{\alpha\beta}}{\partial \bar z_j} \right) dz^\alpha \wedge d\bar z^\gamma \tag{5.1}$$

where $j = 1,\dots,m$ and $\gamma \in \mathbb{N}^{q+1}$.

We shall say that $u \in C^0_{p,q}(D)$ satisfies $\bar\partial u = f$ in D if

$$(-1)^{p+1} \sum_{j,\beta} \varepsilon^\gamma_{j\beta} \int_D u_{\alpha\beta} \frac{\partial \varphi}{\partial \bar z_j} dV = \int_D f_{\alpha\gamma}\, \varphi\, dV \tag{5.2}$$

for every $\varphi \in C^\infty_c(D)$ and each strictly increasing multi-index $\alpha \in \mathbb{N}^p$, $\gamma \in \mathbb{N}^{q+1}$. In particular, if $u \in C^1_{p,q}(D)$ satisfies (5.2), then

$$(-1)^p \sum_{j,\beta} \varepsilon^\gamma_{j\beta} \int_D \varphi \frac{\partial u_{\alpha\beta}}{\partial \bar z_j} dV = \int_D f_{\alpha\gamma}\, \varphi\, dV,$$

and therefore

$$(-1)^p \sum_{j,\beta} \varepsilon^\gamma_{j\beta} \frac{\partial u_{\alpha\beta}}{\partial \bar z_j}(z) = f_{\alpha\gamma}(z) \tag{5.3}$$

for each $z \in D$ and each $\alpha \in \mathbb{N}^p$, $\gamma \in \mathbb{N}^{q+1}$. The system (5.3) means (by (5.1)) that the two forms $\bar\partial u$ and f have the same coefficients and thus that u satisfies $\bar\partial u = f$ in D in the classical sense.

Our next result concerns solutions of $\bar\partial u = f$ defined on adjacent domains of $\mathbb{C}^m$.

Two disjoint domains D_1 and D_2 of $\mathbb{C}^m$ will be said to be *adjacent* at a *free* hypersurface S of class C^1 if S is a hypersurface of class C^1 in $\partial D_1 \cap \partial D_2$ and if $\operatorname{dist}(z, \partial D_k \backslash S) > 0$ for each $z \in S$, $k = 1, 2$. If D_1 and D_2 are two such domains, then we denote by $C^0_{p,q}(D_k \cup S)$ the class of all elements in $C^0_{p,q}(D_k)$ whose coefficients admit a continuous extension to $D_k \cup S$.

Theorem 5.1. *Let D_1 and D_2 be two disjoint domains of $\mathbb{C}^m$ adjacent at a free hypersurface S of class C^1, and set $D = D_1 \cup D_2 \cup S$. Let $f \in C^0_{p,q+1}(D)$, $u^1 \in C^1_{p,q}(D_1)$, and $u^2 \in C^1_{p,q}(D_2)$ be such that $\bar\partial u^1 = f$ in D_1 and $\bar\partial u^2 = f$ in D_2. If $u^1 \in C^0_{p,q}(D_1 \cup S)$, $u^2 \in C^0_{p,q}(D_2 \cup S)$, and $u^1 = u^2$ on S, then there exists $u \in C^0_{p,q}(D)$ such that $\bar\partial u = f$ in D, $u = u^1$ in D_1, and $u = u^2$ in D_2.*

It is not possible in general to draw the conclusion that the form u, in Theorem 5.1, belongs to $C^\infty_{p,q}(D)$ when $f \in C^\infty_{p,q+1}(D)$. To see this, let $v \in C^2_{p,q-1}(D)\backslash C^3_{p,q-1}(D)$ and set $u = \bar\partial v \in C^1_{p,q}(D)$. Then $\bar\partial u = \bar\partial\,\bar\partial v = 0$ but $u \notin C^\infty_{p,q}(D)$. However, when u is a *function* satisfying $\bar\partial u = f$, then this follows from a known result (see [7, Cor. 2.1.6]). This observation leads us to the following.

Corollary 5.2. *Let D_1 and D_2 be two disjoint domains of $\mathbb{C}^m$ adjacent at a free hypersurface S of class C^1, and set $D = D_1 \cup D_2 \cup S$. Let $f \in C^\ell_{0,1}(D)$ with $\ell \geq 0$, $u^1 \in C^1(D_1)$, and $u^2 \in C^1(D_2)$ be such that $\bar\partial u^1 = f$ in D_1 and $\bar\partial u^2 = f$ in D_2. If $u^1 \in C^0(D_1 \cup S)$, $u^2 \in C^0(D_2 \cup S)$, and $u^1 = u^2$ on S, then there exists $u \in C^{\ell+\alpha}(D)$ for each $0 < \alpha < 1$, such that $\bar\partial u = f$ in D, $u = u^1$ in D_1, and $u = u^2$ in D_2.*

The corollary contains the classical result that a continuous function on an open set D in the complex plane which is holomorphic in D off the real axis is holomorphic in D. We refer to [3, Theorems 5.1 and 5.2] for generalizations of this. In addition, the paper [5] contains results of this type for holomorphic functions of several variables.

As mentioned above, the corollary is an immediate consequence of Theorem 5.1. However, in order to illustrate a different approach for this kind of problem, we shall give a proof of the corollary which does not depend on Theorem 5.1 when $\ell \geq 1$.

Proof of Theorem 5.1. Let $u \in C^0_{p,q}(D)$ be the differential form whose coefficients $u_{\alpha\beta}$ coincide with those $(u^k_{\alpha\beta})$ of u^k on $D_k \cup S$. We show that u satisfies

$$(-1)^{p+1} \sum_{j,\beta} \varepsilon^\gamma_{j\beta} \int_D u_{\alpha\beta} \frac{\partial\varphi}{\partial\bar z_j} dV = \int_D f_{\alpha\gamma}\, \varphi\, dV \tag{5.4}$$

for every $\varphi \in C^\infty_c(D)$ and each strictly increasing multi-index $\alpha \in \mathbb{N}^p$ and $\gamma \in \mathbb{N}^{q+1}$. Let $\varphi \in C^\infty_c(D)$. If the support of φ is in $D_1 \cup D_2$ then (5.4) follows immediately, since in that case, $\bar\partial u = f$ on a neighborhood of the support of φ. If the support of φ intersects S, then we consider a subdomain $W \subset \overline{W} \subset D$ containing the support of φ such that, for $k = 1, 2$, the boundary of $W_k = W \cap D_k$ is of class C^1, and we show that

$$-\int_D u_{\alpha\beta} \frac{\partial\varphi}{\partial\bar z_j} dV = \sum_{k=1}^{2} \int_{W_k} \varphi \frac{\partial u^k_{\alpha\beta}}{\partial\bar z_j} dV \tag{5.5}$$

for each $j = 1, \ldots, m$ and each strictly increasing multi-index $\alpha \in \mathbb{N}^p$ and $\beta \in \mathbb{N}^q$. We have

$$-\int_D u_{\alpha\beta} \frac{\partial\varphi}{\partial\bar z_j} dV = -\sum_{k=1}^{2} \int_{W_k} u^k_{\alpha\beta} \frac{\partial\varphi}{\partial\bar z_j} dV = -\sum_{k=1}^{2} \int_{W_k} \left\{ \frac{\partial}{\partial\bar z_j}(u^k_{\alpha\beta}\varphi) - \varphi \frac{\partial u^k_{\alpha\beta}}{\partial\bar z_j} \right\} dV.$$

These integrals exist since $\partial u^k_{\alpha\beta}/\partial \bar{z}_j$ admits a continuous extension to $\overline{W}_k$. We denote by $\vec{n}^k$ the exterior normal to W_k. By the divergence theorem [12, page 100], we have

$$\sum_{k=1}^{2}\int_{W_k}\frac{\partial}{\partial \bar{z}_j}(u^k_{\alpha\beta}\,\varphi)dV=\sum_{k=1}^{2}\int_{\partial W_k}\frac{1}{2}\{\varphi\, u^k_{\alpha\beta}\cos(x_j,\vec{n}^k)-i\,\varphi\, u^k_{\alpha\beta}\cos(y_j,\vec{n}^k)\}ds.$$

This is zero since $n^1=-n^2$ on $\partial W_1\cap\partial W_2$, and (5.5) follows. Now (5.4) follows from (5.5) since

$$\begin{aligned}(-1)^{p+1}\sum_{j,\beta}\varepsilon^{\gamma}_{j\beta}\int u_{\alpha\beta}\frac{\partial\varphi}{\partial\bar{z}_j}dV&=(-1)^p\sum_{j,\beta}\varepsilon^{\gamma}_{j\beta}\sum_{k=1}^{2}\int_{W_k}\varphi\frac{\partial u^k_{\alpha\beta}}{\partial\bar{z}_j}dV\\&=\sum_{k=1}^{2}\int_{W_k}\varphi\, f_{\alpha\gamma}\,dV=\int_D\varphi\, f_{\alpha\gamma}\,dV,\end{aligned}$$

and therefore we see that u satisfies the conclusion of Theorem 5.1. This completes the proof.

Proof of Corollary. Our proof rests on the famous

Bochner-Martinelli Formula. [7, page 54] *Let D be a domain of $\mathbb{C}^m$ with piecewise C^1-boundary. If $u\in C^0(\bar{D})$ satisfies $\bar{\partial}u=f\in C^0_{0,1}(\bar{D})$, then for each $z\in D$*

$$u(z)=\int_{\partial D}u(\xi)K(\xi,z)-\int_D f(\xi)\wedge K(\xi,z)\tag{5.6}$$

where $K(\xi,z)$ is the Bochner-Martinelli kernel.

Let u be the continuous function on $D=D_1\cup D_2\cup S$ which equals u^k on D_k. Let $B\subset\bar{B}\subset D$ be a ball centered at a point of S with a radius so small that S divides B into two subdomains $B_1\subset D_1$ and $B_2\subset D_2$. Consider the function $\hat{u}$ defined in $\mathbb{C}^m$ by

$$\begin{aligned}\hat{u}(z)\equiv\int_{\partial B}u(\xi)K(\xi,z)-\int_B f(\xi)\wedge K(\xi,z)&\\=\sum_{k=1}^{2}\int_{\partial B_k}u^k(\xi)K(\xi,z)-\int_B f(\xi)\wedge K(\xi,z).&\end{aligned}\tag{5.7}$$

Since the kernel $K(\xi,z)$ is of class C^∞, we see that the integral over ∂B in (5.7) is a function of class C^∞ in B. Moreover, by a result on the regularity of transformations defined by certain singular integrals (see [7, Lemma 1.8.5]), the volume integral in (5.7) belongs to $C^{\ell+\alpha}(B)$ for each $0<\alpha<1$. This therefore shows that $\hat{u}\in C^{\ell+\alpha}(B)$ for each $0<\alpha<1$.

We now prove that $\hat{u}=u^k$ on B_k. To do this, we first observe that, by (5.6), we have

$$u^k(z)=\int_{\partial B_k}u^k(\xi)K(\xi,z)-\int_{B_k}f(\xi)\wedge K(\xi,z)\tag{5.8}$$

for each $z\in B_k$. In addition, it follows from Stokes' theorem that

$$\int_{\partial B_k}u^k(\xi)K(\xi,z)=\int_{B_k}d_\xi\left[u^k(\xi)K(\xi,z)\right]=\int_{B_k}f(\xi)\wedge K(\xi,z)\tag{5.9}$$

for each $z \in B_j$ with $j \neq k$, since

$$d_\xi\left[u^k(\xi)K(\xi,z)\right] = \bar{\partial}_\xi\left[u^k(\xi)K(\xi,z)\right] = \bar{\partial}u^k(\xi) \wedge K(\xi,z) = f(\xi) \wedge K(\xi,z).$$

Thus by (5.7), (5.9), and (5.8), we obtain

$$\begin{aligned}\hat{u}(z) &= \sum_{j=1}^{2}\left\{\int_{\partial B_j} u^j(\xi)K(\xi,z) - \int_{B_j} f(\xi) \wedge K(\xi,z)\right\} \\ &= \int_{\partial B_k} u^k(\xi)K(\xi,z) - \int_{B_k} f(\xi) \wedge K(\xi,z) = u^k(z)\end{aligned}$$

for each $z \in B_k$. This shows that u belongs to $C^{\ell+\alpha}(B)$, and therefore to $C^{\ell+\alpha}(D)$, for each $0 < \alpha < 1$. If $\ell \geq 1$, it is then clear that $\bar{\partial}u = f$ in D. If $\ell = 0$ then the argument in the proof of Theorem 5.1 applies and we see that $\bar{\partial}u = f$ in D in the generalized sense. In each case we have shown that u satisfies all the conclusions of the corollary and this completes the proof.

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