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Asymptotic analysis of the peeling-off point of a French duck

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Dedicated to the memory of Charles Lange.

Abstract. The asymptotic theory of a relaxation oscillator displaying a duck (or canard) trajectory is studied. In a previous paper [18], the Poincaré part of the asymptotic approximation was constructed. In the present paper, the associated terms containing exponentially small factors are constructed, thereby giving a formal complete asymptotic approximation. All formulas are explicit and are compared with numerical experiments using Mathematica.

1. Introduction

This paper represents the second part of our asymptotic analysis of so-called duck solutions of the second-order nonlinear ordinary differential equation

$$\epsilon\frac{d^2x}{dt^2}+(x+x^2)\frac{dx}{dt}+x+\alpha=0. \tag{1.1}$$

It is well known that this equation exhibits a closed limit cycle for α values between 0 and 1, and positive ϵ. When ϵ is small, the trajectory displays a jerky motion, the so-called relaxation oscillations [23]. A few years ago, the surprising discovery was made that among the possible relaxation oscillations is a special class corresponding to a very narrow range of α-values [5, 6, 7, 9]. The discoverers of this class called the trajectories in this class "canards" or, in English, "ducks". We shall describe this class in detail below.

The first part of our theory appears in [18] which is based on results in B. Liu's thesis [17]. She constructed terms in the Poincaré (or power series) part of those asymptotic expansions needed to describe the duck phenomenon. (In the present case, this expansion is in powers of ϵ.) We summarize the conclusions from [17] and [18] in Section 2 below. Our theory makes explicit numerical predictions and, therefore, invites comparison with computer experimentation. Such comparisons are described in Section 5 of this paper. We feel the agreement between the predictions of our theory with the computer experiments gives strong support for our theory.

The Poincaré (power series) expansions constructed in [18] do not yield a *complete* asymptotic approximation [10, 21] since they do not include the exponentially small terms. However, construction of the Poincaré part is the necessary first phase in our theory. The present paper describes in detail the second phase wherein we construct the exponential terms in what we can call the *associated* expansion [10].

Let us explain briefly the duck phenomenon, as it pertains to (1.1), using specific examples. We begin with a case which does *not* display this phenomenon. Thus, suppose we set $\alpha = .5$ and $\epsilon = .001$. Using Mathematica [25], (1.1) is integrated.

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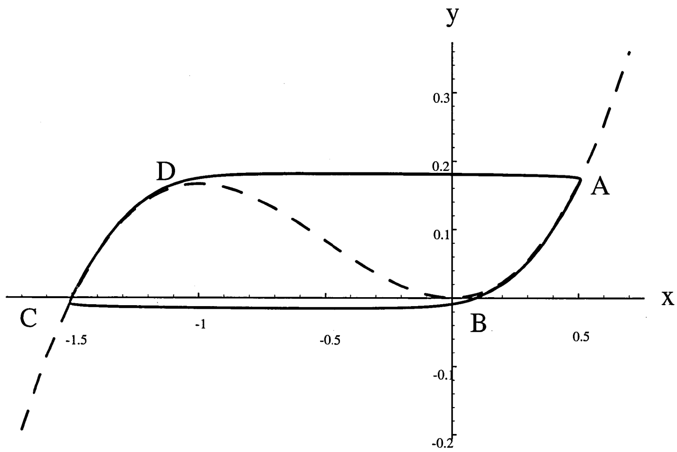

FIGURE 1.1. Plot of trajectory of limit cycle in the (x, y) plane for the system (1.2). $\alpha = .5$ and $\epsilon = .001$. AB and CD represent the slow motions, BC and DA represent the fast motions. The dashed graph is the cubic. Mathematica was used.

Figure 1.1 shows a plot of the resulting limit cycle in Liénard plane variables (x, y) which satisfy the familiar first-order system corresponding to (1.1)

$$\begin{aligned} \epsilon \frac{dx}{dt} &= y - F(x), \\ \frac{dy}{dt} &= -(x + \alpha), \end{aligned} \tag{1.2}$$

where

$$F(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2.$$

Again, the trajectory in Figure 1.1 does *not* display the duck behavior, and furthermore, the graph shown is typical for most values of α in the interval $(0, 1)$. This trajectory can be described [23] as follows: Beginning with a point A on the limit cycle and close to the right branch of the cubic, the trajectory descends, remaining close to the right branch (slow motion) until it reaches the vicinity of the local minimum B at the origin, at which point it rushes across to C on the left branch of the cubic (fast motion), whence it ascends remaining close to the left branch (slow motion again) until it reaches the vicinity of the local maximum D, whence it again displays a fast motion, rushing across to the right branch of the cubic again, closing the cycle. Notice that the middle branch of the cubic is never part of the trajectory.

Now suppose one sets $\alpha = .001012370$ with $\epsilon = .001$ as before. Again running this on Mathematica, one obtains the graph shown in Figure 1.2a. There are similarities between Figure 1.1 and Figure 1.2a, but there is also one striking difference, namely that the trajectory in Figure 1.2a, instead of leaving the cubic at the local minimum at B to rush across to the left branch, instead continues *past* the minimum and ascends part way up the middle branch to B' before rushing across to the left branch. In other words, the slow motion is extended somewhat before the fast motion begins. Notice that now part of the middle branch of the cubic is followed by the trajectory. The trajectory in Figure 1.2a has been called a "duck with head".

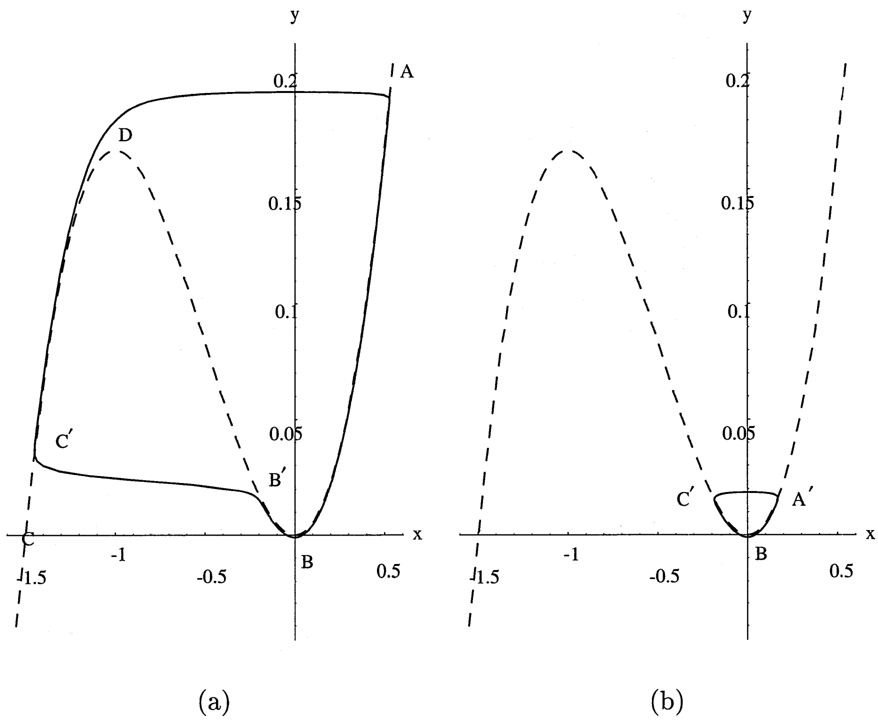

FIGURE 1.2. (a) Duck with head, $\alpha = .001012370$, $\epsilon = .001$. Instead of the rapid motion BC, there is the rapid motion B′C′. (b) Duck without head, $\alpha = .001012350$, $\epsilon = .001$. Only two branches of the cubic are involved. Mathematica was used.

Finally, suppose one sets $\alpha = .001012350$ with $\epsilon = .001$ as before. Running this on Mathematica, one obtains the limit cycle shown in Figure 1.2b. Once again the trajectory has an extended slow motion as it ascends part way up the middle branch, but this time the subsequent fast motion carries it across to the *right* branch of the cubic; the left branch is never involved as part of the trajectory. This limit cycle has been called a "duck without head". Figure 1.3 shows the corresponding graphs in the x,t-plane, illustrating clearly the slow-fast motion characteristic of a relaxation oscillation. Prior to our work in [18] and below, there were two asymptotic theories describing the duck phenomenon. The first is that of the French and Algerian mathematicians [5, 6, 7, 9] centered around Georg H. Reeb of Strasbourg University. They based their asymptotic analysis on nonstandard analysis [19]. In 1983, W. Eckhaus included a "standard" analysis of the problem in his paper [11], arriving at results which differed somewhat from the nonstandard analysis results. A description of Eckhaus' analysis appears on pages 94–98 of Grasman's monograph [12]. Eckhaus was apparently the first to attempt to analyze a duck trajectory in the traditional setting of hard analysis. More recent work on problems in this general area appear in [4], [8], and [15]. Much of the work is based on nonstandard analysis, but that in [15] uses a classical approach due to Pontryagin. See also [2].

It is natural for one to focus on the special values of α which yield duck trajectories. Such α-values depend on ϵ, as well as on the jumping-off point where the slow motion

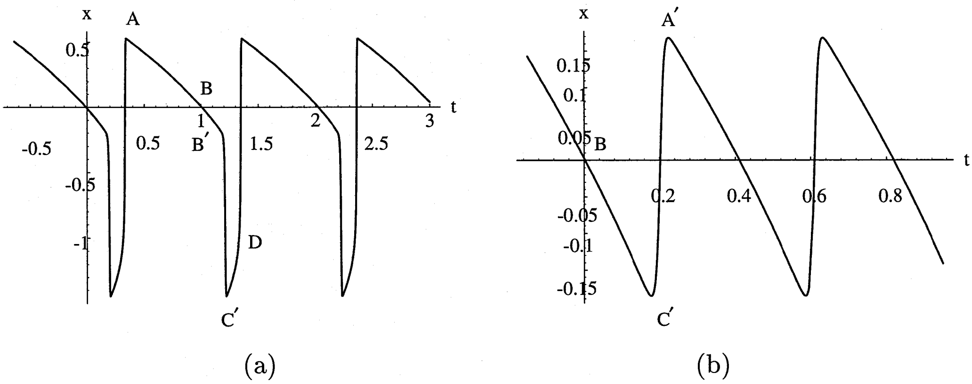

FIGURE 1.3. (a) Plot of $x(t)$ vs. t, $\alpha = .001012370$, $\epsilon = .001$. (b) Plot of $x(t)$ vs. t, $\alpha = .001012350$, $\epsilon = .001$. Letters correspond to those in Figure 1.2.

up the middle branch of the cubic changes into the fast motion onto one of the other two branches (depending on whether the duck has a head or not). Eckhaus proposes an expression for α, namely,

$$\alpha = \alpha_c(\epsilon) + \sigma\epsilon^{3/2}e^{-k^2/\epsilon}, \tag{1.3}$$

where $\alpha_c = \epsilon + o(\epsilon)$. His value of k depends on the jumping-off point which he denotes by x_j. A numerical value of the constant σ is not given.

Turning next to the results of the nonstandard analysis theory, we find (see [19; p. 176]) that this theory arrives at a rather different expression, namely,

$$\alpha = a_1\epsilon + \cdots + a_n\epsilon^n + \delta\epsilon^n, \tag{1.4}$$

where n is a natural number and $\delta \sim 0$ in the sense of nonstandard analysis. Lutz and Goze [19] and Zvonkin and Shubin [26] show that $\delta = O(\exp[-k/\epsilon])$ for some $k > 0$.

Turning finally to our theory, we propose

$$\alpha = \alpha_P + N(t_0)\epsilon^{-1/2}\exp[-I(t_0)/\epsilon], \tag{1.5}$$

where α_P was found [18; Appendix] to be

$$\alpha_P = \epsilon + 12\epsilon^2 + 346\epsilon^3 + 15186\epsilon^4 + \cdots, \tag{1.6}$$

and t_0 is the time of peeling off (and turns out to be related to Eckhaus' jumping-off position x_j; see Appendix). In our theory, both $N(t_0)$ and $I(t_0)$ are given by *explicit* formulas.

Expressions (1.3), (1.4), and (1.5) from the three theories have one attribute in common, which is that the right side of each can be regarded as the sum of two terms, the first being dominant and consisting of a Poincaré expansion in integer powers of ϵ, and the second being recessive due to the exponentially small factor. (For discussions on these ideas in the context of Stokes' rays in the complex plane, see [10, 21].)

However, there are also major differences among the three theories. We point out two of these:

(i) The theories leading to (1.3) and (1.4) cannot be directly compared with computer experiments since in both cases there are unspecified constants. More specifically, the Poincaré part of the formulas requires the computation of at least the first four coefficients of powers of ϵ in order to be useful when ϵ is as small as .001. (Only the coefficient of ϵ is computed in (1.3), and no coefficients are computed for (1.4).) On the other hand, four nonzero terms are displayed in (1.6), and the Appendix of [18] provides simple directions for computing the coefficients of higher-order terms. There is incomplete information in the terms in (1.3) and (1.4) containing the exponential factor because the constant coefficient, denoted by σ in Eckhaus' expression (1.3), is not calculated, and no attempt is made to estimate the implied constant in the δ in (1.4). In our theory, on the other hand, there are no undetermined constants in the right side of our (1.5); our results are in the form of explicit formulas and direct comparison with computer experiments is straightforward.

(ii) We wonder about the forms of the expressions in (1.3) and (1.4). Consider the form of the algebraic factor which multiplies the exponential term. In [18], we assumed for computational purposes that (1.3) has the correct form and determined the numerical value of σ which forces agreement with computer experiments using $\epsilon = .001$, a value which is easily in the range where duck trajectories appear. It was found that the value of σ needed to force agreement with numerical experiments exceeds 10^5. Further, the form (1.4) suggests that the implied constant [20] involved is even greater than 10^5. We wonder if these very large values needed to get agreement with experiment might not be due to the fact that the power of ϵ in the algebraic coefficients of the exponential terms in their theories are both positive. By contrast, our expression (1.5) predicts this algebraic factor is $\epsilon^{-1/2}$; the power is negative. So our algebraic factor is one million times larger than that of Eckhaus when $\epsilon = .001$. (Algebraic factors with negative powers are not uncommon; see for instance [1].)

The careful computer experiments we have carried out are described in Section 5. We have been able to use Mathematica [25] with success. One major advantage in using this particular software is its easy accessibility. All the numerical experiments we describe can be duplicated by the reader with Mathematica version 2.0 or higher.

Our theory uses the method of matched asymptotic expansions as expounded in [14, 16, 22, 24], with a natural extension to include transcendentally small terms. In this way we can explain the delicate mechanism whereby a special choice of α enables the trajectory to continue to follow the cubic until the peeling-off point is reached. The reader may suspect (e.g., from studying Figures 1.1 and 1.2a) that the choice of α depends essentially and delicately on what happens in the immediate vicinity of the minimum, and our analysis of the exponential terms in the asymptotic solution confirms and explains this suspicion. Finally, as was emphasized above, our analysis of the exponential terms in the asymptotic solution leads to an *explicit formula* for computing that special asymptotic value of α which causes the peeling off to occur at a predetermined time t_0; see (3.2), (3.5), and (4.22). All that is required is the evaluation of simple definite integrals, some of which can be evaluated in closed form (e.g., $I(t_0)$) and otherwise are routine calculations using a hand calculator and Simpson's rule. We used Mathematica.

The plan of the remainder of the paper is as follows. In Section 2, we summarize the facts needed from [18]. In Section 3, we introduce the transcendentally small terms which must be analyzed and we are led to the problem of finding the asymptotic behavior of the solution of a linear, nonhomogeneous differential equation, see (3.9). The remainder of Section 3 is devoted to finding asymptotic solutions of the corresponding homogeneous problem. These are used in Section 4 to find the asymptotic solution of the nonhomogeneous equation using the method of variation of parameters. A study of the required complementary solution then leads to our formula for α, (1.5). In Section 5, we present some numerical results to verify the asymptotic analysis. Closing remarks are presented in Section 6. The Appendix explains how one can relate Eckhaus' jumping-off point x_j to our peeling-off time t_0.

2. The Poincaré terms; review of results from [18]

In this section, we review results we shall need from Liu's thesis [17] and reported in [18]. This gives us the terms in the asymptotic approximation that consist of integral powers of ϵ. This Poincaré part of the expansion lays the basis for construction of the exponential contribution to be discussed in Sections 3 and 4 below.

Recall that the differential equation under consideration is

$$\epsilon\frac{d^2x}{dt^2} + (x+x^2)\frac{dx}{dt} + x + \alpha = 0. \tag{2.1}$$

Liu assumes an outer expansion, uniformly valid on an interval containing $t = 0$, as follows

$$\begin{aligned} x(t;\epsilon) &= x_0(t) + \epsilon x_1(t) + \cdots, \\ \alpha &= \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 \cdots. \end{aligned} \tag{2.2}$$

Substitution into (2.1) and extraction of powers of ϵ yields a sequence of equations, the first two of which are

$$(x_0 + x_0^2)\frac{dx_0}{dt} + x_0 = -\alpha_0 \tag{2.3}$$

and

$$(x_0 + x_0^2)\frac{dx_1}{dt} + \left[(2x_0+1)\frac{dx_0}{dt} + 1\right]x_1 + \frac{d^2x_0}{dt^2} = -\alpha_1. \tag{2.4}$$

With no loss of generality, the zero for t is chosen so that $x(0) = 0$ and, therefore, $x_0(0) = 0$. One also wishes the outer asymptotic approximation to pass smoothly through $t = 0$ (i.e., the motion remains slow) and Liu found this forces $\alpha_0 = 0$ and $\alpha_1 = 1$. Solving for $x_0(t)$ and $x_1(t)$, she then found

$$x_0(t) = -1 + \sqrt{1-2t} \tag{2.5}$$

and

$$x_1(t) = \frac{1}{\sqrt{1-2t}}\left[(-1+\sqrt{1-2t}) + \frac{1}{2}\log|1-2t| - \frac{1}{\sqrt{1-2t}} + 1\right]. \tag{2.6}$$

We shall be needing the first few terms of the Taylor expansion of $x_0(t)$ about $t = 0$, namely,

$$x_0(t) = -t - \frac{1}{2}t^2 - \frac{1}{2}t^3 \cdots. \tag{2.7}$$

Next, choose the value of t between 0 and 1/2 where the peeling off is to take place and where a boundary layer exists to describe the fast motion from the middle branch of the cubic to the left (stable) branch. Denoting this value of t by t_0, the boundary layer is described by an inner asymptotic approximation written in the stretched variable

$$t^* = \frac{t - t_0 - \delta_0(\epsilon)}{\epsilon}, \tag{2.8}$$

with the inner expansion being assumed to be of the form

$$x_{\text{inner}}(t^*) = g_0(t^*) + \beta_1(\epsilon)g_1(t^*) + \beta_2(\epsilon)g_2(t^*) + \cdots . \tag{2.9}$$

As usual, this is substituted into the differential equation (2.1). Extracting different orders of the gauge functions $1, \beta_1(\epsilon), \beta_2(\epsilon), \cdots$, one obtains a sequence of differential equations whose solutions must be matched [14, 16, 24] with terms in two outer expansions to all appropriate orders. One of these outer expansions is the outer approximation (2.2), while the other corresponds to the next slow motion behavior on the left branch of the cubic. The detailed analysis of this latter approximation is in both [17] and [18]. Here we need review only the main results. The differential equation for the leading term was found to be (see (2.19) in [18])

$$\frac{dg_0}{dt^*} = -\frac{1}{3}(g_0 - B)(g_0 - B_1)(g_0 - B_2), \tag{2.10}$$

and its solution was given implicitly by

$$A\log(g_0 - B) + A_1 \log\left|\frac{g_0 - B_1}{B - B_1}\right| + A_2 \log\left|\frac{g_0 - B_2}{B - B_2}\right| = -\frac{1}{6}t^*, \tag{2.11}$$

where a constant of integration has been absorbed into $(-\frac{1}{6}t^*)$ and

$$B = -1 + \sqrt{1 - 2t_0}. \tag{2.12}$$

The remaining constants are expressed in terms of B:

$$A = \frac{1}{6B(B+1)}, \tag{2.13a}$$

$$A_1 = -\frac{1}{12B(B+1)} - \frac{2B+1}{4B(B+1)} \frac{1}{\sqrt{3(3+2B)(1-2B)}}, \tag{2.13b}$$

$$A_2 = -\frac{1}{12B(B+1)} + \frac{2B+1}{4B(B+1)} \frac{1}{\sqrt{3(3+2B)(1-2B)}}, \tag{2.13c}$$

$$B_1 = \frac{-(2B+3) + \sqrt{3(3+2B)(1-2B)}}{4}, \tag{2.13d}$$

$$B_2 = \frac{-(2B+3) - \sqrt{3(3+2B)(1-2B)}}{4}. \tag{2.13e}$$

Notice that

$$\begin{gathered} -1 < B < 0, \quad A < 0, \quad A_1 > 0, \quad A_2 > 0, \\ B_1 > 0, \quad \text{and} \quad B_2 < -1. \end{gathered} \tag{2.13f}$$

In Section 4, we shall need the behavior of $g_0(t^*)$ for large, negative t^*. For the duck-with-head solution, analysis of (2.11) yields

$$g_0(t^*) \sim B - e^{-(B+B^2)t^*} + \cdots . \tag{2.14}$$

It is obvious that matching to $O(1)$ is achieved if $\delta(\epsilon) = o(1)$. The details of matching of $x_0(t) + \epsilon x_1(t)$ and $g_0(t^*) + \epsilon g_1(t^*)$ in an overlap domain is described in [18]. The analysis which follows uses only $x_0(t)$ and $x_1(t)$ given in (2.5) and (2.6) and the asymptotic behavior of $g_0(t^*)$ given in (2.14). Higher-order terms in the inner expansion will not be needed. For these, the interested reader is referred to [17, 18].

3. Analysis of the transcendental terms, part I

We return to the original differential equation (1.1), but we modify the asymptotic expansions assumed in (2.2) for $x(t)$ and α to include exponentially small terms:

$$x(t;\epsilon) = (x_0(t) + \epsilon x_1(t) + \cdots) + \left(\delta f_1(t) + \delta^2 f_2(t) + \cdots\right) \tag{3.1a}$$

and

$$\alpha = \alpha_P + \alpha_{TST}, \tag{3.1b}$$

where α_P is given by (1.6) and

$$\alpha_{TST} = \delta a_1 + \delta^2 a_2 + \cdots , \tag{3.2}$$

where it is assumed $\delta = o(\epsilon^n)$, $n = 0, 1, 2, \ldots$. Thus the terms with a factor δ are the transcendentally small terms. (These are the T.S.T.s referred to in [14], hence the subscript on α_{TST}.) Substituting (3.1) and (3.2) into (2.1) and collecting like terms without factors of δ produces the sequence of problems already analyzed by Liu, so we pass to those terms which have δ as a factor. In particular, gathering all terms with δ raised to the first power yields

$$\frac{d^2 f_1}{dt^2} + \frac{1}{\epsilon}\left(X + X^2\right)\frac{df_1}{dt} + \frac{1}{\epsilon}\left(X' + 2XX' + 1\right) f_1 = -\frac{a_1}{\epsilon}, \tag{3.3}$$

where the symbol X stands for the outer asymptotic expansion described in the previous section, that is,

$$X(t) \equiv x_0(t) + \epsilon x_1(t) + \cdots , \tag{3.4}$$

and where a_1 will be allowed to depend on $\epsilon, a_1 = a_1(\epsilon)$. The prime denotes differentiation with respect to t. It turns out that the equation for $f_2(t)$ will not be needed for our purposes.

We first observe that if a_1 is nonzero and of order unity, then the second derivative, when evaluated at $t = 0$, becomes unbounded as $\epsilon \to 0$ since $x_0(0) = 0$, and this would prevent $f_1(t)$ from passing smoothly from negative to positive values of t. However, we can have $a_1 = O(\epsilon)$ and, in anticipation of this, we write

$$a_1 = \epsilon a_1^{(1)} + o(\epsilon). \tag{3.5}$$

In order to study the asymptotic behavior of (3.3), we proceed as in Jeffreys [13, p. 44] by first writing

$$\frac{p(t)}{\epsilon} = \frac{1}{\epsilon}(X + X^2), \tag{3.6}$$

$$\frac{q(t)}{\epsilon} = \frac{1}{\epsilon}(X' + 2XX' + 1), \tag{3.7}$$

and then by defining a new dependent variable, z_1,

$$f_1 = z_1 \exp\left[-\frac{1}{2\epsilon}\int_0^t p(\tau)\,d\tau\right]. \tag{3.8}$$

Substituting into (3.3), there results

$$\frac{d^2z_1}{dt^2} + \left(-\frac{1}{4\epsilon^2}p^2 - \frac{1}{2\epsilon}p' + \frac{1}{\epsilon}q\right)z_1 = -a_1^{(1)} \exp\left[\frac{1}{2\epsilon}\int_0^t p(\tau)\,d\tau\right]. \tag{3.9}$$

This is a linear, nonhomogeneous equation which we shall solve using the method of variation of parameters. For this, we shall need two linearly independent solutions of the homogeneous equation. It will turn out that these can be chosen so that one has an exponentially decaying behavior as $|t|$ increases and the other has an exponentially growing behavior as $|t|$ increases. Of course, we cannot hope to solve the homogeneous equation exactly. Instead, we find the asymptotic behavior of its Green-Liouville (WKB) solutions, valid away from $t = 0$, supplemented by an argument which connects the two approximations across the turning point at $t = 0$. The details, which are standard, are as follows.

Following the discussion of Jeffreys [13], we write the homogeneous form of (3.9) as

$$\frac{d^2z_1}{dt^2} = \left(\frac{1}{\epsilon^2}\chi_0 + \frac{1}{\epsilon}\chi_1 + \chi_2\right)z_1 \tag{3.10}$$

where

$$\chi_0(t) = \frac{1}{4}x_0^2(1+x_0)^2, \tag{3.11}$$

$$\chi_1(t) = \frac{3}{2}x_0^2x_1 + \frac{1}{2}x_0x_1 + x_0^3x_1 - \frac{1}{2}x_0' - x_0x_0' - 1, \tag{3.12}$$

and the asymptotic approximation to $O(1)$ of χ_2, which we shall not need, is constructed from the $O(\epsilon)$ part of $p'/2 - q$ and the $O(\epsilon^2)$ part of $p^2/4$.

Beginning with $t < 0$, and still working from Jeffreys, we select that Green-Liouville solution which has an exponential decay as t *decreases* from 0. It can be written in standard form as

$$\begin{aligned} z_1^-(t) \sim & \frac{\sqrt{2}A_1^-}{\sqrt{|x_0|(1+x_0)}} \exp\left[\frac{1}{2\epsilon}\int_0^t \left(\sqrt{x_0^2(1+x_0)^2}\right)d\tau\right] \\ & \times \exp\left[\frac{1}{2}\int_{-1/4}^t \left(\frac{\chi_1}{\sqrt{\chi_0}} - \frac{1}{\tau}\right)d\tau\right] \times 2|t|^{1/2}, \quad t < 0. \end{aligned} \tag{3.13}$$

The last exponential factor in (3.13) includes a $1/\tau$ term because, without it, the integral diverges as $t \to 0^-$; the explicit integration of the $1/\tau$ term accounts for the last factor in (3.13). (A similar arrangement appears in (3.22) and (3.23).) For $t > 0$, the Green-Liouville solution which has an exponential decay as t *increases* from 0, can be written in the form

$$z_1^-(t) \sim \frac{\sqrt{2}D_1^-}{\sqrt{|x_0||1+x_0|}} \exp\left[-\frac{1}{2\epsilon}\int_0^t \sqrt{x_0^2(1+x_0)^2}\, d\tau\right]$$
$$\times \exp\left[-\frac{1}{2}\int_{1/4}^t \left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} + \frac{1}{\tau}\right) d\tau\right] \times 2t^{1/2}, \quad t > 0. \tag{3.14}$$

Remark: We have not yet shown that the solution represented by (3.13) can be continued through the turning point to the solution represented by (3.14). Thus, strictly speaking, there is no reason at this stage to use the same symbol on the left sides of both (3.13) and (3.14). Its justification begins with the discussion to follow. A similar comment relates to (3.22) and (3.23), where we have written the symbol $z_1^+(t)$ on the left sides of both. Again, this anticipates the justification which begins immediately following (3.23).

The constants A_1^- and D_1^- are related by connection formulas which we shall now find. We begin by examining the behavior of (3.13) and (3.14) when $|t|$ is small (yet large enough so that the Green-Liouville approximation is still valid). Observe that, as $t \to 0^-$, $\chi_0 \sim x_0^2/4 \sim t^2/4$ (from (3.11) and (2.7)) and $\chi_1 \sim -1/2$. Thus, as $t \to 0^-$,

$$\exp\left[\frac{1}{2}\int_{-1/4}^t \left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} - \frac{1}{\tau}\right) d\tau\right]$$
$$= \exp\left[\frac{1}{2}\int_{-1/4}^0 \left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} - \frac{1}{\tau}\right) d\tau\right] \times (1 + o(1)). \tag{3.15}$$

These estimates lead directly to

$$z_1^-(t) \sim \left(2\sqrt{2}A_1^- \exp\left[\frac{1}{2}\int_{-1/4}^0 \left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} - \frac{1}{\tau}\right) d\tau\right]\right) \exp\left[-\frac{t^2}{4\epsilon}\right],$$
$$t \text{ negative and small.} \tag{3.16}$$

Similarly,

$$z_1^-(t) \sim \left(2\sqrt{2}D_1^- \exp\left[-\frac{1}{2}\int_{1/4}^0 \left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} + \frac{1}{\tau}\right) d\tau\right]\right) \exp\left[-\frac{t^2}{4\epsilon}\right],$$
$$t \text{ positive and small.} \tag{3.17}$$

Now we examine the differential equation (3.10) in a small interval which includes the origin. Using only the terms which make up χ_0 and χ_1, and using (3.11) and (3.12), we obtain

$$\frac{d^2 z_1^-}{dt^2} \sim \left(\frac{1}{\epsilon^2}\left[\frac{1}{4}x_0^2(1+x_0)^2\right] + \frac{1}{\epsilon}\left[-1 - \frac{1}{2}x_0' - x_0 x_0' + \frac{3}{2}x_0^2 x_1 + \frac{1}{2}x_0 x_1 + x_0^3 x_1\right]\right) z_1^-. \tag{3.18}$$

If $|t|$ is small, we can approximate the coefficient of $1/\epsilon^2$ by $t^2/4$ and the coefficient of $1/\epsilon$ by $-1/2$. This gives the approximate differential equation

$$\frac{d^2 z_1^-}{dt^2} \sim \left(\frac{t^2}{4\epsilon^2} - \frac{1}{2\epsilon}\right) z_1^-, \qquad |t| \text{ small.} \tag{3.19}$$

All solutions of this equation are analytic because $t = 0$ is an ordinary point. One solution has an exponential decay, and is the one we want to match with (3.16) and (3.17). Thus, as one can check by substitution,

$$z_1^-(t) \sim E_1^- \, e^{-t^2/4\epsilon}, \qquad |t| \text{ small.} \tag{3.20}$$

Comparing (3.20) with (3.16) and (3.17), we find the following relationship among the constants A_1^-, D_1^-, E_1^-

$$\begin{aligned} E_1^- &= 2\sqrt{2}A_1^- \, \exp\left[\frac{1}{2}\int_{-1/4}^{0}\left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} - \frac{1}{\tau}\right) d\tau\right] \\ &= 2\sqrt{2}D_1^- \, \exp\left[-\frac{1}{2}\int_{1/4}^{0}\left(\frac{\chi_1(\tau)}{\sqrt{\chi(\tau)}} + \frac{1}{\tau}\right) d\tau\right]. \end{aligned} \tag{3.21}$$

We now turn our attention to the exponentially growing solution of the homogeneous equation (3.10). All we need is a sign change in (3.13) and in (3.14). The results are

$$\begin{aligned} z_1^+ =& A_1^+ \frac{\sqrt{2}}{\sqrt{|x_0|(1+x_0)}} \exp\left[-\frac{1}{2\epsilon}\int_0^t \sqrt{x_0^2(1+x_0)^2}\, d\tau\right] \\ &\times \exp\left[-\frac{1}{2}\int_{-1/4}^{t}\left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} - \frac{1}{\tau}\right) d\tau\right] \times \frac{1}{2|t|^{1/2}}, \qquad t < 0, \end{aligned} \tag{3.22}$$

$$\begin{aligned} z_1^+ =& D_1^+ \frac{\sqrt{2}}{\sqrt{|x_0|\,|1+x_0|}} \exp\left[\frac{1}{2\epsilon}\int_0^t \sqrt{x_0^2(1+x_0)^2}\, d\tau\right] \\ &\times \exp\left[\frac{1}{2}\int_{1/4}^{t}\left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} + \frac{1}{\tau}\right) d\tau\right] \times \frac{1}{2t^{1/2}}, \qquad t > 0. \end{aligned} \tag{3.23}$$

Next, in order to justify the use of the same symbol on the left sides of (3.22) and (3.23) and at the same time to relate the constants A_1^+ and D_1^+, we examine (3.22) and (3.23) for small t (again, not too small—we wish to remain in a region where the Green-Liouville approximation remains valid). We recall the observation that the limit of integration t in the second exponential factor in (3.22) and in the second exponential factor in (3.23) can be replaced by 0 if we introduce a factor $1 + o(1)$ into the right sides. With this observation, together with the behavior of $\chi_0(\tau)$ and $\chi_1(\tau)$

for small $|\tau|$, we arrive at

$$z_1^+ \sim \frac{A_1^+}{(-t)\sqrt{2}} \exp\left[-\frac{1}{2}\int_{-1/4}^{0}\left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} - \frac{1}{\tau}\right)d\tau\right] \times \exp\left[\frac{t^2}{4\epsilon}\right],$$
$$t \text{ negative and small,} \tag{3.24}$$

$$z_1^+ \sim \frac{D_1^+}{t\sqrt{2}} \times \exp\left[\frac{1}{2}\int_{1/4}^{0}\left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} + \frac{1}{\tau}\right)d\tau\right] \times \exp\left[\frac{t^2}{4\epsilon}\right],$$
$$t \text{ positive and small.} \tag{3.25}$$

To obtain the relationship between A_1^+ and D_1^+, we again examine the differential equation (3.10) when t is small. We have approximately (see (3.19)),

$$\frac{d^2 z_1^+}{dt^2} \sim \left(\frac{t^2}{4\epsilon^2} - \frac{1}{2\epsilon}\right) z_1^+. \tag{3.26}$$

This time we are interested in the solution whose dominant behavior is exponentially growing. Standard arguments (see, for example, Bender and Orszag, [3]) show that there is a solution which behaves, for large values of $|t/\sqrt{\epsilon}|$, like

$$z_1^+(t) \sim E_1^+ \frac{1}{t} e^{t^2/4\epsilon}. \tag{3.27}$$

With some care for signs, we can now relate the constants. With no loss of generality, set $E_1^+ = 1$. Then, using values of t which are on the one hand small, but on the other hand $t/\sqrt{\epsilon}$ large, we arrive at

$$A_1^+ = -\sqrt{2}\exp\left[\frac{1}{2}\int_{-1/4}^{0}\left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} - \frac{1}{\tau}\right)d\tau\right], \tag{3.28}$$

$$D_1^+ = \sqrt{2}\exp\left[\frac{1}{2}\int_{0}^{1/4}\left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} + \frac{1}{\tau}\right)d\tau\right]. \tag{3.29}$$

This completes the asymptotic analysis of solutions of the homogeneous equation corresponding to (3.9). In the next section, we use these results to obtain the asymptotic behavior of a particular solution of (3.9). As we shall shortly see, this asymptotic behavior of the particular solution is crucial, because it dictates the complementary solution required to produce a duck-with-head.

4. Analysis of the transcendental terms, part II

It will be convenient to choose $E_1^- = 2$ in (3.21). With this choice, we obtain the following expressions for A_1^- , D_1^-, and E_1^-

$$A_1^- = \frac{1}{\sqrt{2}}\exp\left[-\frac{1}{2}\int_{-1/4}^{0}\left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} - \frac{1}{\tau}\right)d\tau\right], \tag{4.1}$$

$$D_1^- = \frac{1}{\sqrt{2}}\exp\left[-\frac{1}{2}\int_{0}^{1/4}\left(\frac{\chi_1(\tau)}{\sqrt{\chi_0(\tau)}} + \frac{1}{\tau}\right)d\tau\right], \tag{4.2}$$

$$E_1^- = 2. \tag{4.3}$$

The method of variation of parameters gives the following expression for a particular solution of (3.9)

$$
\begin{aligned}
z_p(t) = &- z_1^-(t) \int_0^t \frac{z_1^+(\tau)}{W(z_1^-, z_1^+)(\tau)} \left(-a_1^{(1)} \exp\left[\frac{1}{2\epsilon} \int_0^\tau p(s) ds \right] \right) d\tau \\
&+ z_1^+(t) \int_0^t \frac{z_1^-(\tau)}{W(z_1^-, z_1^+)(\tau)} \left(-a_1^{(1)} \exp\left[\frac{1}{2\epsilon} \int_0^\tau p(s) ds \right] \right) d\tau, \qquad (4.4)
\end{aligned}
$$

where $W(z_1^-, z_1^+)$ is the Wronskian of the two solutions $z_1^-(t)$ and $z_1^+(t)$. Since (3.9) has no dz/dt term, the Wronskian is constant. To calculate this constant, we shall use the asymptotic approximations from (3.20) and (3.27), remembering that we have chosen $E_1^+ = 1$ and $E_1^- = 2$. Thus, we obtain, after a short calculation,

$$
\begin{aligned}
W &= z_1^-(t) \frac{dz_1^+}{dt} - z_1^+(t) \frac{dz_1^-}{dt} \\
&= \frac{2}{\epsilon} \left[1 - \frac{\epsilon}{t^2} \right]. \qquad (4.5)
\end{aligned}
$$

This result is not a constant, and must be interpreted carefully. We could, of course, go back to the two solutions for which (3.20) and (3.27) are the asymptotic approximations, but since ϵ/t^2 is small in the region where (4.5) is valid, we conclude that, for $\sqrt{\epsilon} \ll t \ll 1$,

$$
W = \frac{2}{\epsilon} \left(1 + o(1) \right), \qquad (4.6)
$$

and so the constant value of W must be $2/\epsilon$. The variation of parameters equation now reads

$$
\begin{aligned}
z_p(t) = &- z_1^-(t) \int_0^t \frac{\epsilon}{2} z_1^+(\tau) \left(-a_1^{(1)} e^{\frac{1}{2\epsilon} \int_0^\tau p(s) ds} \right) d\tau \\
&+ z_1^+(t) \int_0^t \frac{\epsilon}{2} z_1^-(\tau) \left(-a_1^{(1)} e^{\frac{1}{2\epsilon} \int_0^\tau p(s) ds} \right) d\tau. \qquad (4.7)
\end{aligned}
$$

Let us examine each of the terms on the right side of (4.7). The first term on the right will not, when the conversion to its contribution to f_1 is made (see (3.8)), contribute any exponentially growing term. The reasoning is as follows: in the integrand, $z_1^+(\tau)$ grows exponentially, but $e^{1/2\epsilon \int_0^\tau p(s) ds}$ decays exponentially at just the correct rate to make the integrand, and hence the integral, bounded. Then the integral is multiplied by the exponentially decaying factor z_1^- and, in the conversion to f_1, is multiplied by an exponentially growing factor which, as can easily be verified, is just nicely balanced by the decaying factor. Thus no exponentially growing contributions to $f_1(t)$ come from the first term in (4.7).

The integral in the second term has an integrand which is exponentially small when τ is not small (both factors in the integrand decay exponentially for large values of t/ϵ). However, when τ is as small as $O(\epsilon)$, the integrand is of order unity, so the integral should give a contribution of order ϵ. Subsequent multiplication by the exponentially growing $z_1^+(t)$, followed by multiplication by the exponential factor which converts z_p to f_p, leaves us with an exponentially growing contribution.

Important remark: The fact that the only significant contribution to the integral comes from the vicinity of the point where the cubic has its local minimum explains why the delicate mechanism that makes a duck trajectory possible is located in a small neighborhood of this local minimum.

Let us examine our situation a little more closely. We first make the observation that whether t is positive or negative, the sign of the exponentially growing contribution from the second integral in (4.7) is the same. Note that the general solution is obtained by adding a complementary solution to z_p. The exponentially growing component comes from z_1^+, which we note has a *sign change* when one passes from positive to negative values of t. This introduces the very interesting and, in fact, crucial possibility of using part of the complementary solution to neutralize the growing exponential behavior for negative t, leaving exponentially growing behavior for positive t.

Let us see if this strategy leads to progress in the present problem. To do so, we must find the asymptotic approximation for the second integral appearing in (4.7). This is easy when we remember that the integrand decays exponentially, because then, as is customary in the asymptotic evaluation of integrals, we can first replace the integrand by its expression for small argument, then integrate out to infinity. This yields, from (3.20),

$$-a_1^{(1)} \int_0^t \frac{\epsilon}{2} z_1^-(\tau)\, e^{1/2\epsilon \int_0^\tau p(s)ds}\, d\tau \sim -a_1^{(1)} \int_0^{\pm\infty} \frac{\epsilon}{2}(2e^{-\frac{\tau^2}{2\epsilon}})\, d\tau. \tag{4.8}$$

The result is

$$-a_1^{(1)} \int_0^t \frac{\epsilon}{2} z_1^-(\tau) e^{1/2\epsilon \int_0^\tau p(s)ds} \sim \begin{cases} -\epsilon a_1^{(1)}\sqrt{\epsilon\pi} & \text{if } t > 0, \\ \epsilon a_1^{(1)}\sqrt{\epsilon\pi} & \text{if } t < 0. \end{cases}$$

To obtain the desired neutralization of the exponential growth for negative values of t, we simply add the complementary solution

$$z_c(t) = \left(-\epsilon a_1^{(1)}\sqrt{\epsilon\pi}\right) z_1^+(t). \tag{4.9}$$

Consequently, for t *positive* and $|t| \gg \sqrt{\epsilon}$, we have the asymptotic behavior for $z_1(t)$

$$z_1(t) \sim \left(-2\epsilon a_1^{(1)}\sqrt{\epsilon\pi}\right) z_1^+(t). \tag{4.10}$$

Substituting from (3.23) and (3.29), we find (4.10) gives

$$\begin{aligned} z_1(t) \sim & \frac{-\epsilon a_1^{(1)}\sqrt{\pi\epsilon}}{(\chi_0(t))^{1/4}} \sqrt{2} \exp\left[\frac{1}{2}\int_0^{1/4}\left(\frac{\chi_1}{\sqrt{\chi_0}} + \frac{1}{\tau}\right) d\tau\right] \times \exp\left[\frac{1}{2}\int_{1/4}^t\left(\frac{\chi_1}{\sqrt{\chi_0}} + \frac{1}{\tau}\right) d\tau\right] \\ & \times t^{-1/2} \times \exp\left[\frac{1}{2\epsilon}\int_0^t \sqrt{x_0^2(1+x_0)^2\, d\tau}\right], \quad t > 0. \end{aligned} \tag{4.11}$$

This yields the following expression for $f_1(t)$, using (3.8) in (4.11),

$$f_1(t) \sim \frac{-\epsilon a_1^{(1)}\sqrt{\pi\epsilon}}{(\chi_0(t))^{1/4}}\sqrt{2}\exp\left[\frac{1}{2}\int_0^{1/4}\left(\frac{\chi_1}{\sqrt{\chi_0}}+\frac{1}{\tau}\right)d\tau\right]\times\exp\left[\frac{1}{2}\int_{1/4}^{t}\left(\frac{\chi_1}{\sqrt{\chi_0}}+\frac{1}{\tau}\right)d\tau\right]$$
$$\times\exp\left[\frac{1}{2\epsilon}\int_0^t\sqrt{x_0^2(1+x_0)^2}\,d\tau\right]\times t^{-1/2}\times\exp\left[-\frac{1}{2\epsilon}\int_0^t p(\tau)\,d\tau\right],\quad t>0. \tag{4.12}$$

Now we rewrite the last two exponential factors in terms of the functions x_0 and x_1, recalling when taking square roots that $x_0(t)$ is negative when $t > 0$,

$$\begin{aligned}
&\exp\left[\frac{1}{2\epsilon}\int_0^t\sqrt{x_0^2(1+x_0)^2}\,d\tau\right]\times\exp\left[-\frac{1}{2\epsilon}\int_0^t p(\tau)\,d\tau\right]\\
&\quad=\exp\left[\frac{1}{2\epsilon}\int_0^t\sqrt{(x_0)^2(1+x_0)^2}\,d\tau\right]\\
&\qquad\times\exp\left[-\frac{1}{2\epsilon}\int_0^t\left((x_0+x_0^2)+\epsilon(x_1+2x_0x_1)+O(\epsilon^2)\right)d\tau\right]\\
&\quad=\exp\left[\frac{1}{\epsilon}\int_0^t -x_0(1+x_0)\,d\tau\right]\\
&\qquad\times\exp\left[-\frac{1}{2}\int_0^t(x_1+2x_0x_1)\,d\tau\right]\times(1+O(\epsilon)).
\end{aligned}\tag{4.13}$$

Then we have, still for $t \gg \sqrt{\epsilon} > 0$,

$$\begin{aligned}
f_1(t) \sim&\frac{-\epsilon a_1^{(1)}\sqrt{\pi\epsilon}}{(\chi_0(t))^{1/4}}\sqrt{2}\exp\left[\frac{1}{2}\int_0^{1/4}\left(\frac{\chi_1}{\sqrt{\chi_0}}+\frac{1}{\tau}\right)d\tau\right]\\
&\times\exp\left[\frac{1}{2}\int_{1/4}^{t}\left(\frac{\chi_1}{\sqrt{\chi_0}}+\frac{1}{\tau}\right)d\tau\right]\times t^{-1/2}\\
&\times\exp\left[-\frac{1}{2}\int_0^t(x_1+2x_0x_1)\,d\tau\right]\\
&\times\exp\left[\frac{1}{\epsilon}\int_0^t -x_0(1+x_0)\,d\tau\right].
\end{aligned}\tag{4.14}$$

Let us now choose the peeling-off point by choosing the time t_0 when the peeling takes place; see [18]. Since we shall be examining times which are $o(1)$ away from t_0, to leading order, we can replace t by t_0 in the second and third exponential factors in (4.14). Then we approximate the last exponential factor in (4.14) by the first two terms in its Taylor expansion in $(t - t_0)$ by rewriting it first as

$$\begin{aligned}
&\exp\left[\frac{1}{\epsilon}\int_0^t -x_0(1+x_0)\,d\tau\right]\\
&\quad=\exp\left[\frac{1}{\epsilon}\int_0^{t_0} -x_0(1+x_0)\,d\tau\right]\times\exp\left[\frac{1}{\epsilon}\int_{t_0}^{t} -x_0(1+x_0)\,d\tau\right].
\end{aligned}\tag{4.15}$$

Then, since $x_0(1+x_0)$ is given exactly by (see (2.5))

$$x_0(1+x_0) = -\sqrt{1-2t}+1-2t,$$

we have

$$\int_{t_0}^{t} -x_0(1+x_0)\,d\tau = \int_{t_0}^{t} \left[\sqrt{1-2\tau} - (1-2\tau)\right]\,d\tau \tag{4.16}$$

which can be integrated in closed form. However, for purposes of matching, we shall need the Taylor series expansion of the result in powers of $(t - t_0)$. Thus, we use

$$\begin{aligned}&\sqrt{1-2\tau} - (1-2\tau)\\&\quad = \left(\sqrt{1-2t_0} - (1-2t_0)\right) + \left(-\frac{1}{\sqrt{1-2t_0}} + 2\right)(\tau - t_0) + \cdots\end{aligned} \tag{4.17}$$

so that

$$\begin{aligned}&\exp\left[\frac{1}{\epsilon}\int_{t_0}^{t} -x_0(1+x_0)\,d\tau\right]\\&\quad = \exp\left[\frac{1}{\epsilon}\int_{t_0}^{t}\left(\left(\sqrt{1-2t_0} - (1-2t_0)\right) + \left(2 - \frac{1}{\sqrt{1-2t_0}}\right)(\tau - t_0) + \cdots\right)d\tau\right]\\&\quad \sim \exp\left[\frac{1}{\epsilon}\left(\sqrt{1-2t_0} - (1-2t_0)\right)(t-t_0)\right] \times \exp\left[\frac{1}{\epsilon}\left(2 - \frac{1}{\sqrt{1-2t_0}}\right)\frac{(t-t_0)^2}{2}\right].\end{aligned} \tag{4.18}$$

Now we are ready to carry out the matching of f_1 with the exponential term in (2.14). (The constant term already has been matched with x_0.) Writing (2.14) in terms of t and t_0 using (2.8) and recalling $\delta_0(\epsilon) = o(1)$, there results

$$\begin{aligned}g_0(t^*) &= B - e^{-(B+B^2)t^*} + \cdots\\&= -1 + \sqrt{1-2t_0} - \exp\left[\left(\sqrt{1-2t_0} - (1-2t_0)\right)\frac{t - t_0 - \delta_0}{\epsilon}\right],\end{aligned} \tag{4.19}$$

since

$$B \equiv -1 + \sqrt{1-2t_0}. \tag{4.20}$$

We now see immediately that the exponential part of the expansion (3.1) matches with the second term of (4.19) if we identify the coefficients of

$$\exp\left[\left(\sqrt{1-2t_0} - (1-2t_0)\right)\frac{t-t_0}{\epsilon}\right]$$

in the two terms $g_0(t^*)$ and $\delta f_1(t)$, namely

$$\begin{aligned}-1 &= \delta\frac{-\epsilon a_1^{(1)}\sqrt{\pi\epsilon}}{(\chi_0(t_0))^{1/4}}\sqrt{2}\exp\left[\frac{1}{2}\int_0^{1/4}\left(\frac{\chi_1}{\sqrt{\chi_0}} + \frac{1}{\tau}\right)d\tau\right]\\&\quad \times \exp\left[\frac{1}{2}\int_{1/4}^{t_0}\left(\frac{\chi_1}{\sqrt{\chi_0}} + \frac{1}{\tau}\right)d\tau\right] \times t_0^{-1/2} \times \exp\left[-\frac{1}{2}\int_0^{t_0}(x_1 + 2x_0x_1)\,d\tau\right]\\&\quad \times \exp\left[\frac{1}{\epsilon}\int_0^{t_0}|x_0|(1+x_0)\,d\tau\right].\end{aligned} \tag{4.21}$$

Recall that one of our primary goals was to obtain an asymptotic expression for $a = \epsilon + \ldots + \delta\epsilon a_1^{(1)} + \ldots$. From (4.21), the product $\delta\epsilon a_1^{(1)}$ is found to be

$$\delta\epsilon a_1^{(1)} = N(t_0) \times \epsilon^{-\frac{1}{2}} \exp\left[-\frac{I(t_0)}{\epsilon}\right] \tag{4.22}$$

where

$$N(t_0) \equiv \frac{(\chi_0(t_0))^{1/4}(t_0)^{1/2}}{\sqrt{2\pi}\exp\left[\frac{1}{2}\int_0^{t_0}\left(\frac{\chi_1}{\sqrt{\chi_0}} + \frac{1}{\tau}\right)d\tau\right]\exp\left[-\frac{1}{2}\int_0^{t_0}(x_1 + 2x_0x_1)\,d\tau\right]}, \tag{4.23}$$

and where

$$I(t_0) \equiv \int_0^{t_0} |x_0|(1 + x_0)\,d\tau. \tag{4.24}$$

This concludes the asymptotic analysis. A main result is (1.5), which is an explicit formula predicting the value of α that will produce a specific duck with head. In the next section, we shall compare this asymptotic result with computer experiments.

5. Comparison of the asymptotic theory with computer experiments

The comparison of our theory with computer experiments proceeds as follows. Begin by choosing a value for the peeling-off time t_0, and compute the corresponding value of B determined by (4.20). Then, determine the value of α from our theory, namely

$$\alpha = \alpha_P + N(t_0) \times \epsilon^{-1/2} \exp\left[-\frac{I(t_0)}{\epsilon}\right], \tag{5.1}$$

using $\epsilon = .001$ and using Mathematica's `NIntegrate` to compute $N(t_0)$ and $I(t_0)$ from (4.23) and (4.24). (The formulas are so simple that the calculations can be done with pencil and paper.)

Remark. For α_P, we use the value $\alpha_P = .0010123621265054$ determined from preliminary calculations using Mathematica's `NDSolve` to solve the full nonlinear problem. We do not use the result obtained using (1.6), namely .001012361186 which, although obviously an excellent approximation for most purposes, is not close enough to separate out the exponential terms. The numerical value was found by trial and error, the final steps being $\alpha = \alpha_p$ producing a duck *without* a head, and $\alpha = .0010123621265055$ producing a duck *with* a head. This brackets the "true" α_P.

Now, numerically integrate (1.1) using Mathematica's `NDSolve` with the option for `WorkingPrecision` set rather high; we generally used 26. The numerical solution thus computed is used as follows: First, write the inner equation (2.10) in terms of the outer variable t, and then replace g_0 by $x(t)$ as determined by numerical integration of (1.1). (That is, we accept the numerical solution as being "exact" in the sense used by Bender and Orszag [3] when they check the validity of asymptotic results derived in their book and, of course, we are assuming g_0 is a close approximation to the true exact solution in the boundary-layer region.) The result is

$$(-.003)\frac{dx}{dt} = (x - B)(x - B_1)(x - B_2). \tag{5.2}$$

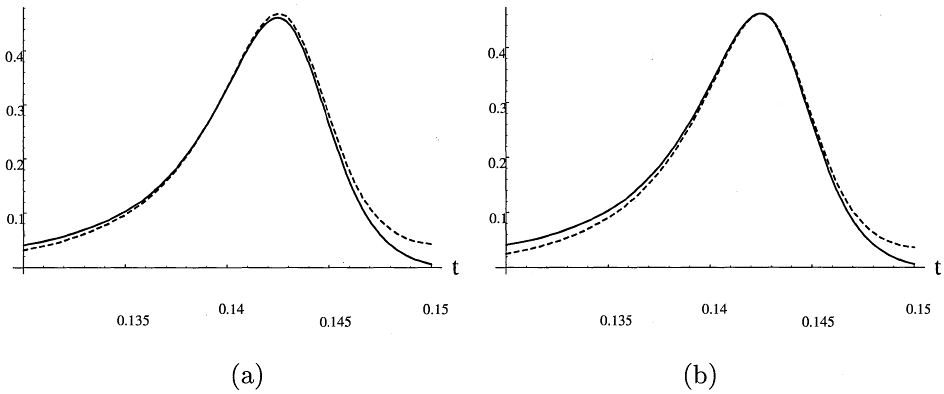

FIGURE 5.1. In both figures, the solid curve is the graph of the left side of (5.2) using $t_0 = .14$ and $\alpha = .0011339931265054$ computed using (5.1); see Table 1. The dashed curves depict the right side of (5.2), with (a) having $B = x_j = -.151$ and (b) having $B = B_b = -.1689$. For larger values of t_0, the agreement is even better.

The accuracy criterion is essentially the degree to which (5.2) is satisfied. That is, we compute and graph the left side of (5.2) and compare the result with the computation and graphing of the right side of (5.2) for various values of B, including the value determined by (4.20). We shall refer to this special value as x_j for reasons explained in the Appendix. Of course, we look for good agreement when the value of B is set equal to x_j, but in any case we choose that value which gives the best fit, see Figure 5.1. We denote this best value by B_b and define the error to be the difference between the numerically determined B_b and the theoretically determined x_j expressed as a percentage, that is,

$$\%\text{ error } = \left|\frac{x_j - B_b}{B_b}\right| \times 100. \tag{5.3}$$

Some preliminary discussion of values of t_0 for which one might reasonably expect the above criterion to be a fair means of assessing the validity of our theory is in order. First of all, one wants to use a value of ϵ that is small enough to give good asymptotics (which improve as $\epsilon \to 0$), yet not so small as to make impractical the numerical integrations (which become more difficult as $\epsilon \to 0$). The choice $\epsilon = .001$ appears to be large enough for the numerical computations to be feasible, yet small enough for the asymptotics to be reasonably accurate. Having chosen ϵ, we recall that asymptotic theory is used in two other ways in deriving our formulas, namely, in the use of WKB theory, and in the asymptotic evaluation of some integrals. Specifically, we want t_0 to be large enough to keep us away from the turning point where the Green-Liouville (WKB) solutions break down. For this purpose, we have selected a lowest value of $t_0 = .1$, even though the resulting trajectory in the Liénard plane shown in Figure 5.2 requires some imagination to appear duck-like. (Rather surprisingly, the error for this marginal case is a respectable 16%.) On the other hand, serious numerical difficulties can be expected when t_0 is assigned values that lead to the second term on

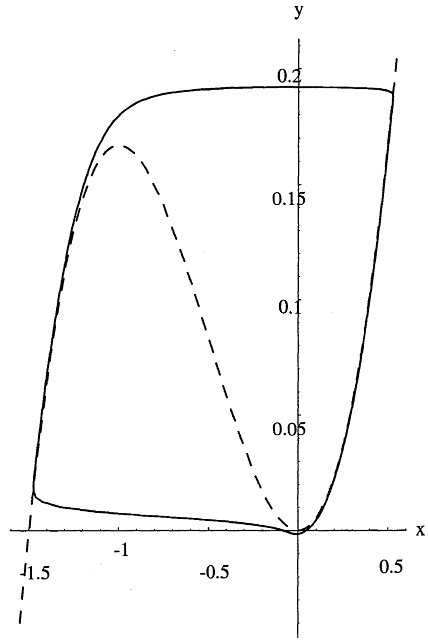

FIGURE 5.2. Liénard plane trajectory when $t_0 = .1$, $\epsilon = .001$, and (5.1) is used to calculate α. The dotted curve is the cubic. The presence of a duck is questionable for this small value of t_0. For this reason, the table starts with $t_0 = .1$

the right side of (5.2) taking on values that are on the edge of the precision to which we have numerically calculated $\alpha_P = .0010123621265054$. We feel safe with t_0 as large as .24, but are less sure for larger values, even though, as the table below shows, the agreement remains excellent for values up to .27.

6. Concluding remarks

The method of matched asymptotic expansions has been used to study the duck or canard phenomenon in nonlinear oscillation theory. Our success in this depends on a natural extension of the method of matched asymptotic expansions to include transcendentally small terms in the matching process. Such an extension is necessary because the sensitive dependence of duck trajectories on the parameter α is due to such transcendentally small terms. (By matching we mean that terms in two asymptotic expansions must have the same functional form on an overlap domain of validity, [14, 16, 22, 24].) As is typical of the method of matched asymptotic expansions, our analysis yields explicit formulas, including (1.5) which predicts, with easy calculations, the value of α required to produce a specific duck trajectory.

We have compared our asymptotic analysis with computer experiments using Mathematica. Our theory agrees very nicely with the computer experiments. In fact, based on the criterion used, our predictions agree with the numerics to within 6% for values of α_{TST} ranging over seven orders of magnitude; see the entries in rows 3–7 in the third column of the Table 1. We believe the agreement of our results with computer experiments attests to the robustness of the underlying principles of the method of

t_0	x_j	α_{TST}	α	B_b	$\left\lvert\frac{x_j - B_b}{B_b}\right\rvert \times 100$
.10	$-.106$	7.48911×10^{-3}	.0085014721265054	$-.12585$	15.78
.14	$-.151$	1.21631×10^{-4}	.0011339931265054	$-.16890$	10.60
.18	$-.200$	4.81001×10^{-7}	.0010128431275054	$-.21259$	5.92
.22	$-.252$	5.58847×10^{-10}	.0010123626853524	$-.26114$	3.50
.24	$-.279$	1.24354×10^{-11}	.0010123621389408	$-.2872$	2.86
.26	$-.307$	2.15161×10^{-13}	.001012362126720561	$-.3145$	2.54
.27	$-.322$	2.5947×10^{-14}	.001012362126531347	$-.3287$	2.13

TABLE 1. The first column is the peeling-off time. The second column is computed from (4.20). The third column is α_{TST} computed from (4.22), (4.23), (4.24). Column four is the sum of column three and $\alpha_P = .0010123621265054$. Column five is the value of B in (5.2) that gives the best fit. Column six compares the theoretical value in column two with the numerical value in column five.

matched asymptotic expansions upon which we have based our analysis.

Nicholas Kazarinoff passed away suddenly on November 21, 1991, a few months after work on this paper began. Our colleague, teacher, and dear friend is sadly missed.

We join in paying tribute to Charles Lange, who will always be remembered for his outstanding original contributions to science. His generosity and warm-hearted personality endeared him to his colleagues and friends as well as to those who knew him by reputation but were not fortunate enough to have known him personally. We believe that our colleague, Nick Kazarinoff, would consider it an honor to have his last paper included in this special memorial issue.

Appendix. Relationship between peeling-off time t_0 and Eckhaus' jumping-off point x_j

Recall Eckhaus' formula for α, namely

$$\alpha = \alpha_c + \sigma\epsilon^{3/2} \exp[-k^2/\epsilon]. \tag{A.1}$$

On page 481 of his paper, Eckhaus gives his formula (3.2.7) for k^2, which for the case $F(x) = x^3/3 + x^2/2$, reads

$$k^2 = \int_0^{x_j} z(a+z)^2\, dz, \tag{A.2}$$

and x_j is declared to be the jumping-off point. Thus, the exponential factor in (A.1) reads

$$\exp\left[-\int_0^{x_j} z(1+z)^2\, dz \Big/ \epsilon\right]. \tag{A.3}$$

On the other hand, the exponential factor in our theory is, from (4.22) and (4.24),

$$\exp\left[-\int_0^{t_0} |x_0|\,(1+x_0)\,d\tau \Big/ \epsilon\right]. \tag{A.4}$$

We recall from (2.5) that

$$x_0(\tau) = -1 + \sqrt{1-2\tau}, \tag{A.5}$$

so that

$$\begin{aligned} d\tau &= -\sqrt{1-2\tau}\,dx_0 \\ &= (-1-x_0)\,dx_0. \end{aligned} \tag{A.6}$$

Thus the change of variables (A.5) gives

$$\begin{aligned} \int_0^{t_0} |x_0|(1+x_0)\,d\tau &= \int_0^{-1+\sqrt{1-t_0}} |x_0|(1+x_0)[-(1+x_0)]dx_0 \\ &= +\int_0^{-1+\sqrt{1-2t_0}} x_0(1+x_0)^2 dx_0, \end{aligned} \tag{A.7}$$

where, in the second step, we recall $x_0(\tau)$ is negative for positive τ.

The integrals in (A.3) and (A.7) agree when

$$x_j = -1 + \sqrt{1-2t_0}. \tag{A.8}$$

This is the relationship we set out to establish between x_j and t_0.

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