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Asymptotic behavior in a class of integrodifferential equations with diffusion

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Abstract. For a system of integrodifferential equations with diffusion, a sufficient condition is given for each solution to tend to a steady-state solution as $t \to \infty$.

1. Introduction

In this paper, we consider a system of integrodifferential equations with diffusion

$$\frac{\partial u_i}{\partial t}(t,x) = k_i \Delta u_i(t,x) + u_i(t,x)\Big\{a_i - b_i u_i(t,x) - \sum_{j=1}^{N} \int_0^t u_j(t-\tau,x)\,df_{ij}(\tau)\Big\},$$
$$t > 0, \quad x \in \Omega, \quad i = 1,\dots,N. \tag{1.1}$$

Here $\Delta = \sum_{i=1}^{\ell} \partial^2/\partial x_i^2$ and Ω is a bounded domain in $\mathbf{R}^\ell$ with smooth boundary $\partial\Omega$. Moreover, k_i, a_i, and b_i, $i = 1,\dots,N$, are positive constants, and the functions f_{ij}, $i,j = 1,\dots,N$, are of bounded variation on $\mathbf{R}^+ := [0,\infty)$ with $f_{ij}(0) = 0$. In mathematical ecology, system (1.1) describes the growth of N species alive in Ω whose i-th population density at time t and place x is $u_i(t,x)$, and the integral represents the effects of the past history on the present growth rate (cf. [2, 5–7, 9–11]).

Together with (1.1), we consider the initial-boundary conditions

$$u_i(0,\cdot\,) = u_{0i}(\cdot\,), \quad i = 1,\dots,N, \quad \text{in } \Omega, \tag{1.2}$$

$$\frac{\partial u_i}{\partial \mathbf{n}} = 0, \quad i = 1,\dots,N, \quad \text{on } (0,\infty)\times\partial\Omega, \tag{1.3}$$

where $\partial/\partial\mathbf{n}$ denotes the exterior normal derivative to $\partial\Omega$. A function $u(t,x) = (u_1(t,x),\dots,u_N(t,x))^T \in C(\mathbf{R}^+ \times \bar{\Omega}; \mathbf{R}^N)$ is called a (regular) solution of (1.1)–(1.3) if for each $i = 1,\dots,N$, $j = 1,\dots,\ell$, and $k = 1,\dots,\ell$, $\partial u_i/\partial t$, $\partial u_i/\partial x_j$, and $\partial^2 u_i/\partial x_j \partial x_k$ belong to the space $C((0,\infty)\times\Omega)$, $\partial u_i/\partial\mathbf{n}$ exists on $(0,\infty)\times\partial\Omega$, and, moreover, (1.1)–(1.3) are identically satisfied. In this article, we will investigate the asymptotic behavior of solutions of (1.1)–(1.3) as $t \to \infty$. Recently, Murakami and Yoshizawa [7] have obtained some results on the asymptotic behavior of solutions of (1.1)–(1.3) under the assumption that the diffusion coefficients k_i, $i = 1,\dots,N$, are sufficiently large. The purpose of this paper is to discuss the asymptotic behavior of solutions of (1.1)–(1.3) without any additional assumption on the size of k_i (cf. [5–6, 9–11]).

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2. Fundamental hypotheses and main results

Throughout this article, we employ the following notation. For any row vector $(x_1, \dots, x_N)$, let $(x_1, \dots, x_N)^T$ denote the transpose of $(x_1, \dots, x_N)$. Let $\mathbf{R}^N$ be the space of all real N-column vectors and denote by $|x|$ the norm of $x \in \mathbf{R}^N$. For any $x = (x_1, \dots, x_N)^T \in \mathbf{R}^N$ and $y = (y_1, \dots, y_N)^T \in \mathbf{R}^N$, $x > y$ $(y < x)$ and $x \geq y$ $(y \leq x)$ mean that $x_i > y_i$ and $x_i \geq y_i$ for $i = 1, \dots, N$, respectively. If a function f is locally of bounded variation on $\mathbf{R}^+$, we denote by $\mathrm{Var}\{f|_{[0,t]}\}$ the total variation of f on $[0, t]$, for each $t > 0$.

We now impose the following condition on (1.1).

(H1) For each $i, j = 1, \dots, N$, the constants k_i, a_i, and b_i are positive and the function f_{ij} is of bounded variation on $\mathbf{R}^+$ with $f_{ij}(0) = 0$.

For each $i, j = 1, \dots, N$, we define functions F_{ij}, F_{ij}^+, F_{ij}^- on $\mathbf{R}^+$ by $F_{ij}(0) = F_{ij}^+(0) = F_{ij}^-(0) = 0$ and

$$
\begin{aligned}
F_{ij}(t) &= \mathrm{Var}\{f_{ij}|_{[0,t]}\}, \\
F_{ij}^+(t) &= (F_{ij}(t) + f_{ij}(t))/2, \\
F_{ij}^-(t) &= (F_{ij}(t) - f_{ij}(t))/2,
\end{aligned}
$$

for $t > 0$. Then F_{ij}, F_{ij}^+, and F_{ij}^- are nondecreasing on $\mathbf{R}^+$, and

$$
C_{ij}^{\pm} := \lim_{t \to \infty} F_{ij}^{\pm}(t) \leq C_{ij}, \tag{2.1}
$$

where $C_{ij} := \lim_{t\to\infty} \mathrm{Var}\{f_{ij}|_{[0,t]}\}$. Moreover,

$$
F_{ij}^+ + F_{ij}^- = F_{ij}, \qquad F_{ij}^+ - F_{ij}^- = f_{ij},
$$

and, consequently,

$$
\begin{aligned}
C_{ij} &= C_{ij}^+ + C_{ij}^-, \\
f_{ij}(\infty) := \lim_{t\to\infty} f_{ij}(t) &= C_{ij}^+ - C_{ij}^-.
\end{aligned}
$$

We impose the following condition as well.

(H2) There exist some positive constants $\delta_1, \dots, \delta_N$ such that

$$
b_i \delta_i > \sum_{j=1}^{N} C_{ij}^- \delta_j, \quad i = 1, \dots, N,
$$

where C_{ij}^- is the one in (2.1).

In what follows, we consider the $N \times N$ matrices B, C, C^+, C^-, $f(\infty)$ and the element $a \in \mathbf{R}^N$ defined by

$$
\begin{gathered}
B = \mathrm{diag}(b_1, \dots, b_N), \quad C = (C_{ij})_{N\times N}, \quad C^+ = (C_{ij}^+)_{N\times N}, \\
C^- = (C_{ij}^-)_{N\times N}, \quad f(\infty) = (f_{ij}(\infty))_{N\times N}, \quad \text{and} \quad a = (a_1, \dots, a_N)^T,
\end{gathered} \tag{2.2}
$$

respectively. In virtue of [1, Theorem 6.2.3], the condition (H2) is equivalent to the condition that the matrix $B - C^-$ is a nonsingular M-matrix; consequently, $B - C^-$

is inverse-positive [1, Theorem 6.2.3, $(M_{35}) \Rightarrow (N_{38})$]. Then $(B - C^-)^{-1}a > \mathbf{0} := (0, \ldots, 0)^T$. Indeed, from [1, Theorem 6.2.3, $(M_{35}) \Rightarrow (I_{28})$] it follows that $(B - C^-)x > \mathbf{0}$ for some $x > \mathbf{0}$; hence we may assume that $(B - C^-)x < a$. Then one gets $(B - C^-)^{-1}a \geq x > \mathbf{0}$, as required.

Together with (H1) and (H2), we consider the condition

$$\text{(H3)} \qquad a > C^+(B - C^-)^{-1}a.$$

Proposition 2.1. *Suppose that* (H1), (H2), *and* (H3) *are satisfied. Then there exists a unique* $\nu^* > \mathbf{0}$ *in* $\mathbf{R}^N$ *with the property*

$$[B + f(\infty)]\nu^* = a, \tag{2.3}$$

where B, $f(\infty)$, *and* a *are given in* (2.2).

The $\nu^* = (\nu_1^*, \ldots, \nu_N^*)^T$ ensured in the proposition is the positive steady-state solution of the system of ordinary differential equations

$$\frac{d\nu_i}{dt} = \nu_i\Big(a_i - b_i\nu_i - \sum_{j=1}^N f_{ij}(\infty)\nu_j\Big) \quad j = 1, \ldots, N.$$

We now state the main result of this paper, which provides sufficient conditions for the solution of (1.1)–(1.3) to tend to the steady-state solution as $t \to \infty$, uniformly on $\bar{\Omega}$.

Theorem 2.2. *Suppose that* (H1), (H2), *and* (H3) *are satisfied. If the initial function in* (1.2) *satisfies* $u_{0i} \in C^1(\bar{\Omega})$ *and* $u_{0i}(x) \geq 0$ $(\not\equiv 0)$ *in* $x \in \Omega$ *for* $i = 1, \ldots, N$, *then the solution* $u(t,x)$ *of* (1.1)–(1.3) *exists for all* $t \geq 0$ *and satisfies*

$$\limsup_{t\to\infty, x\in\bar{\Omega}} |u(t,x) - \nu^*| = 0, \tag{2.4}$$

where ν^* *is the one ensured in Proposition* 2.1.

We remark that a result similar to Theorem 2.2 has been obtained in [7] under a condition on the size of the diffusion coefficients k_i and a stronger condition than (H2), in the case where a_i and b_i are functions of t and $f_{ij}(t) = \int_0^t K_{ij}(s)ds$ for some $K_{ij} \in L^1(R^+)$. Also, Martin and Smith [6] have treated the equation with finite delay of the form

$$\frac{\partial u_i}{\partial t}(t,x) = k_i\Delta u_i(t,x) + u_i(t,x)\Big\{a_i - b_iu_i(t,x) - \sum_{j=1}^N \int_{-r}^0 u_j(t+\tau, x)dv_{ij}(\tau)\Big\},$$
$$t > 0, \quad x \in \Omega, \quad i = 1, \ldots, N,$$

instead of (1.1), where $r > 0$ is a constant and v_{ij} is a bounded Borel measure. By a method of Liapunov-Razumikhin type, they have obtained a result similar to Theorem 2.2 under the conditions that $B - C$ is a nonsingular M-matrix and that the ν^* in Proposition 2.1 is positive. We remark that (H2) and (H3) yield that $B - C$ is a nonsingular M-matrix together with $\nu^* > 0$, while the method in [6] is not valid for the equation with unbounded delay.

Before proving the theorem, we will provide some examples to compare our theorem with the results in [2, 5, 9–11].

Example 2.1. Consider a scalar equation

$$\frac{\partial u}{\partial t} = k\Delta u + u\Big(a - bu - \int_0^t u(t-s,x)\,df(s)\Big), \quad t > 0,\ x \in \Omega, \tag{2.5}$$

where k, a, and b are positive constants and the function f is of bounded variation on $\mathbf{R}^+$ with $f(0) = 0$. It is easy to see that (H1)–(H3) hold for (2.5) if and only if

$$b > \lim_{t\to\infty} \mathrm{Var}\{f|_{[0,t]}\}. \tag{2.6}$$

It follows from Theorem 2.2 that under the condition (2.6), the solution $u(t,x)$ of (2.5), (1.2), and (1.3) satisfies $\lim_{t\to\infty} u(t,x) = a/(b + f(\infty))$ uniformly on $\bar{\Omega}$. For the scalar equation

$$\frac{\partial u}{\partial t} = \Delta u + u\Big(a - bu - \int_0^t h(t-s)u(s)ds\Big), \quad t > 0,\ x \in \Omega, \tag{2.7}$$

which is a special case of (2.5), the condition (2.6) becomes

$$h \in L^1(\mathbf{R}^+) \quad \text{and} \quad b > \int_0^\infty |h(s)|ds. \tag{2.8}$$

Consequently, if (2.8) holds and if $u(0,\cdot) \in C^1(\bar{\Omega})$ with $u(0,x) \geq 0$ $(\not\equiv 0)$ for $x \in \Omega$, then the solution $u(t,x)$ of (2.7), (1.2), and (1.3) satisfies $\lim_{t\to\infty} u(t,x) = a/(b + \int_0^\infty h(s)ds)$ uniformly on $\bar{\Omega}$. Thus, as a special case of our theorem, one can obtain the result due to Redlinger [9] (cf. [10, 11]).

Example 2.2. Consider the system

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= k_1\Delta u_1 + u_1\Big(a - bu_1 - \int_0^t u_2(t-s,x)\,df_1(s) + \int_0^t u_2(t-s,x)\,df_2(s)\Big),\\ \frac{\partial u_2}{\partial t} &= k_2\Delta u_2 + u_2\Big(c - du_2 - \int_0^t u_1(t-s,x)dg_1(s) + \int_0^t u_1(t-s,x)dg_2(s)\Big),\end{aligned} \tag{2.9}$$

where k_1, k_2, a, b, c, d are positive constants, and f_1, f_2, g_1, and g_2 are bounded and nondecreasing functions on $\mathbf{R}^+$ with $f_1(0) = f_2(0) = g_1(0) = g_2(0) = 0$. One can easily see that (H1)–(H3) hold for (2.9) if and only if

$$\begin{gathered} bd > f_2(\infty)g_2(\infty), \quad f_1(\infty)(g_2(\infty)a + bc) < (bd - f_2(\infty)g_2(\infty))a,\\ g_1(\infty)(ad + f_2(\infty)c) < (bd - f_2(\infty)g_2(\infty))c. \end{gathered} \tag{2.10}$$

Then it follows from Theorem 2.2 that if (2.10) holds and if $u_i(0,\cdot) \in C^1(\bar{\Omega})$ with $u_i(0,x) \geq 0$ $(\not\equiv 0)$ in $x \in \Omega$ for $i = 1,2$, then the solution $(u_1(t,x), u_2(t,x))^T$ of (2.9), (1.2), and (1.3) satisfies

$$\lim_{t\to\infty} u_1(t,x) = \frac{ad - cf(\infty)}{bd - f(\infty)g(\infty)} \quad \text{and} \quad \lim_{t\to\infty} u_2(t,x) = \frac{bc - ag(\infty)}{bd - f(\infty)g(\infty)}$$

uniformly on $\bar{\Omega}$, where $f(\infty) = f_1(\infty) - f_2(\infty)$, and $g(\infty) = g_1(\infty) - g_2(\infty)$. If $f_1 = g_1 \equiv 0$ on $\mathbf{R}^+$, then the condition (2.10) is reduced to the condition

$$bd > f_2(\infty)g_2(\infty).$$

On the other hand, if $f_2 = g_2 \equiv 0$ on $\mathbf{R}^+$, then the condition (2.10) becomes

$$ad > cf_1(\infty), \qquad bc > ag_1(\infty),$$

which is identical to the one imposed by Brown [2, Theorem 3.1] in the case where (2.9) is the system without time delay.

Example 2.3. Consider the system

$$\frac{\partial u_i}{\partial t} = k_i \Delta u_i + u_i\Big(a_i - b_i u_i - \sum_{j=1}^{N} c_{ij} \int_0^t u_j(t-\tau, x)dg_{ij}(\tau)\Big), \quad i = 1, \ldots, N, \tag{2.11}$$

where k_i, a_i, and b_i, $i = 1, \ldots, N$, are positive constants, c_{ij}, $i, j = 1, \ldots, N$, are nonnegative constants, and g_{ij}, $i, j = 1, \ldots, N$, are nondecreasing functions on $\mathbf{R}^+$ with $g_{ij}(\infty) = g_{ij}(0) + 1$. It is easy to see that (H1)–(H3) hold for (2.11) if and only if

$$a_i > \sum_{j=1}^{N} \frac{c_{ij} a_j}{b_j}, \quad i = 1, \ldots, N. \tag{2.12}$$

It follows from Theorem 2.2 that if (2.12) holds and if $u_i(0, \cdot) \in C^1(\bar{\Omega})$ with $u_i(0, x) \geq 0$ ($\not\equiv 0$) for $x \in \Omega$, $i = 1, \ldots, N$, then the solution $u = (u_1, \ldots, u_N)^T$ of (2.11), (1.2), and (1.3) satisfies

$$\lim_{t \to \infty} |u(t, x) - \nu^*| = 0$$

uniformly on $\bar{\Omega}$, where $\nu^* = (\nu_1^*, \ldots, \nu_N^*)^T > \mathbf{0}$ is the one which satisfies the relation

$$b_i \nu_i^* + \sum_{j=1}^{N} c_{ij} \nu_j^* = a_i, \quad i = 1, \ldots, N.$$

Martin and Smith [5, Theorem 5.4] have also imposed the condition (2.12) to obtain the same result as (2.11) except with finite delay. We remark that our result is valid for (2.11) with unbounded delay, while the method in [5] is not.

3. Proof of the results

Proof of Proposition 2.1. Set $\nu^{(0)} = (B - C^-)^{-1} a$ and $\mu^{(0)} = B^{-1}(a - C^+ \nu^{(0)})$. Then $\nu^{(0)} > \mathbf{0}$, and, moreover, $\mu^{(0)} > \mathbf{0}$ by (H3). Define the sequences $\{\mu^{(m)}\}$ and $\{\nu^{(m)}\}$ in $\mathbf{R}^N$ by the relations

$$\begin{aligned} B\mu^{(m)} &= a - C^+ \nu^{(m-1)} + C^- \mu^{(m-1)}, \\ B\nu^{(m)} &= a + C^- \nu^{(m-1)} - C^+ \mu^{(m-1)}, \end{aligned} \quad m = 1, 2, \ldots . \tag{3.1}$$

Observe that $B\mu^{(1)} \geq a - C^+ \nu^{(0)} = B\mu^{(0)}$, $B\nu^{(0)} = (B - C^-)\nu^{(0)} + C^- \nu^{(0)} = a + C^- \nu^{(0)} \geq B\nu^{(1)}$, and $B\nu^{(0)} \geq (B - C^-)\nu^{(0)} = a \geq a - C^+ \nu^{(0)} = B\mu^{(0)}$; consequently, $\mu^{(1)} \geq \mu^{(0)}$, $\nu^{(0)} \geq \nu^{(1)}$, and $\nu^{(0)} \geq \mu^{(0)}$. Furthermore, one can prove that

$$\mathbf{0} < \mu^{(0)} \leq \mu^{(1)} \leq \cdots \leq \mu^{(m)} \leq \cdots \leq \nu^{(m)} \leq \cdots \leq \nu^{(1)} \leq \nu^{(0)}.$$

Hence there exist the limits $\mu = \lim_{m\to\infty}\mu^{(m)}$ and $\nu = \lim_{m\to\infty}\nu^{(m)}$, and μ and ν satisfy the relation

$$\begin{aligned} B\mu &= a - C^+\nu + C^-\mu, \\ B\nu &= a + C^-\nu - C^+\mu, \end{aligned} \tag{3.2}$$

together with $\nu \geq \mu > \mathbf{0}$ in $\mathbf{R}^N$. We claim that $\mu = \nu$. If the claim is true, then one can take $\mu = \nu$ as ν^* in the proposition. To establish the claim, we first note that

$$\begin{aligned} (B-C)\nu^{(0)} &= (B - C^- - C^+)(B - C^-)^{-1}a \\ &= a - C^+(B-C^-)^{-1}a > \mathbf{0}, \end{aligned}$$

by (H3). Since $\nu^{(0)} > \mathbf{0}$, it follows from [1, Theorem 6.2.3, $(I_{28}) \Rightarrow (A_1)$] that $\det(B-C) > 0$. Then $\nu - \mu = \mathbf{0}$, or $\mu = \nu$, because $B(\nu-\mu) = C(\nu-\mu)$ by (3.2).

Next, we prove the uniqueness of $\nu^* > \mathbf{0}$. To do this, it suffices to certify that $w = \mathbf{0}$ whenever $(B + f(\infty))w = \mathbf{0}$. Let $\nu^{(0)} = (d_1, \ldots, d_N)^T$. Then $b_i d_i > \sum_{j=1}^N C_{ij} d_j$ for $i = 1, \ldots, N$ because, as shown above, $(B-C)\nu^{(0)} > \mathbf{0}$. Set $\|w\| = \max\{|w_i|/d_i : i = 1, \ldots N\}$, where $w = (w_1, \ldots, w_N)^T$. Then $\|w\| = |w_i|/d_i$ for some i. Since $b_i w_i = \sum_{j=1}^N (C_{ij}^- - C_{ij}^+) w_j$, we obtain

$$\begin{aligned} \|w\| &= \frac{\left|\sum_{j=1}^N \left(C_{ij}^- - C_{ij}^+\right) w_j\right|}{(b_i d_i)} \\ &\leq \|w\| \frac{\sum_{j=1}^N \left|C_{ij}^- - C_{ij}^+\right| d_j}{(b_i d_i)} \\ &\leq \|w\| \frac{\sum_{j=1}^N C_{ij} d_j}{(b_i d_i)} \\ &\leq \Lambda \|w\|, \end{aligned}$$

where $\Lambda = \max\left\{ \left(\sum_{j=1}^N C_{ij} d_j\right)/(b_i d_i) : i = 1, \ldots, N \right\} < 1$. Consequently, $\|w\| = 0$ or $w = \mathbf{0}$, as required. □

Proof of Theorem 2.2. It suffices to prove the theorem in the case where $\delta_i = 1$ for $i = 1, \ldots, N$ in (H2). Indeed, if we set $\tilde{u}_i = u_i/\delta_i$, $i = 1, \ldots, N$, then the equation (1.1) is transformed into the equation

$$\frac{\partial \tilde{u}_i}{\partial t}(t,x) = k_i \Delta \tilde{u}_i(t,x) + \tilde{u}_i(t,x)\Big\{a_i - \tilde{b}_i \tilde{u}_i(t,x) - \sum_{j=1}^N \int_0^t \tilde{u}_j(t-\tau, x)\, d\tilde{f}_{ij}(\tau)\Big\},$$
$$t > 0, \quad x \in \Omega, \quad i = 1, \ldots, N, \tag{3.3}$$

with $\tilde{b}_i = b_i \delta_i$ and $\tilde{f}_{ij}(\tau) = f_{ij}(\tau)\delta_j$. Clearly, the conditions (H1) and (H2) with $\delta_i = 1$ are satisfied for (3.3). Moreover, if $\tilde{B}$, $\tilde{C}$, $\tilde{C}^+$, $\tilde{C}^-$, and $\tilde{f}(\infty)$, respectively, denote the ones corresponding to the B, C, C^+, C^-, $f(\infty)$ in (2.2), then $\tilde{B} = B\delta$, $\tilde{C}^+ = C^+\delta$, $\tilde{C}^- = C^-\delta$, $\tilde{f}(\infty) = f(\infty)\delta$, and $(\tilde{B} - \tilde{C}^-)^{-1} = \delta^{-1}(B - C^-)^{-1}$, where $\delta = \mathrm{diag}(\delta_1, \ldots, \delta_N)$. Then (H3) is also satisfied for (3.3), and the $\tilde{\nu}^*$ ensured in Proposition 2.1 for (3.3) is given by $\tilde{\nu}^* = \delta^{-1}\nu^*$. Therefore, if one can prove the theorem for (3.3), then the theorem would hold also for (1.1).

In the following, assuming that

$$(\mathrm{H2}')\qquad b_i > \sum_{j=1}^{N} C_{ij}^{-} =: C_i \quad i = 1, \dots N,$$

together with (H1) and (H3), we will prove the theorem. By virtue of (H1) and (H2′), we see from [8, Theorem 3.4] that there exists a unique solution $u(t,x)$ of (1.1)–(1.3) defined for all $t \geq 0$, and that the relation

$$\mathbf{0} \leq u(t,x) \leq (\beta, \dots, \beta)^T, \quad t > 0, \quad x \in \bar{\Omega} \tag{3.4}$$

holds true, where $\beta = \max\{a_i/(b_i - C_i),\ \sup_{x \in \bar{\Omega}} u_{0i}(x) :\ i = 1, \dots, N\}$.

Now, employing the argument in [9, pp. 138-141], we will establish (2.4). The proof will be divided into the five steps.

Step 1. For each $i = 1, \dots N$, we consider the solution $p_{1i}(t)$ of the ordinary differential equation

$$\frac{d}{dt} p_{1i} = p_{1i}\left(a_i - b_i p_{1i} + \beta C_i\right)$$

with $p_{1i}(0) = \beta$. It is easy to see that p_{1i} exists on $\mathbf{R}^+$ and it satisfies $0 < p_{1i}(t) \leq \beta$ for all $t \geq 0$. We claim that

$$u_i(t,x) \leq p_{1i}(t), \quad t \geq 0, \quad x \in \bar{\Omega}, \quad i = 1, \dots, N. \tag{3.5}$$

Indeed, this claim follows from [8, Theorem 3.4] because $(\mathbf{0}, p_1)$, where $p_1(t) = (p_{11}(t), \dots,\ p_{1N}(t))$, is a pair of lower and upper solutions for (1.1)–(1.3) (in the sense of [8]).

Step 2. Observe that $\lim_{t\to\infty} p_{1i}(t) = (a_i + \beta C_i)/b_i =: \beta_{1i}$. From (3.5) it follows that $\limsup_{t\to\infty} \{\max_{x\in\bar{\Omega}} u_i(t,x)\} \leq \beta_{1i}$ for all $i = 1, \dots, N$. Let $\epsilon > 0$. There exist a $t_1 > 0$ and a $t_2 > t_1$ such that

$$0 \leq u_i(t,x) < \beta_{1i} + \epsilon, \quad i = 1, \dots, N, \quad \text{for all } t \geq t_1 \text{ and } x \in \bar{\Omega}, \tag{3.6}$$

and

$$0 \leq F_{ij}^{-}(t) - F_{ij}^{-}(t - t_1) < \epsilon, \quad i,j = 1, \dots, N, \quad \text{for all } t \geq t_2. \tag{3.7}$$

For each $i = 1, \dots, N$, we consider the solution $p_{2i}(t)$ of the ordinary differential equation

$$\frac{d}{dt} p_{2i} = p_{2i}\Big\{a_i - b_i p_{2i} + \sum_{j=1}^{N} C_{ij}^{-}(\beta_{1j} + \epsilon) + N\epsilon\beta\Big\}$$

with $p_{2i}(t_2) = \beta_{1i} + \epsilon$. We claim that

$$u_i(t,x) < p_{2i}(t), \quad t \geq t_2, \quad x \in \bar{\Omega}, \quad i = 1, \dots, N. \tag{3.8}$$

Suppose (3.8) is false for some i. Then there exists some $(t_3, x_3) \in (t_2, \infty) \times \bar{\Omega}$ such that $u_i(t_3, x_3) = p_{2i}(t_3)$ and $u_i(t,x) < p_{2i}(t)$ on $[t_2, t_3) \times \bar{\Omega}$. Set $w_i(t,x) =$

$u_i(t,x) - p_{2i}(t)$. Then $w_i(t_3,x_3) = 0$ and $w_i(t,x) < 0$ on $[t_2,t_3) \times \bar{\Omega}$. Moreover, we get $k_i \Delta w_i - \partial w_i/\partial t + f_1(t,x) w_i = g_1(t,x)$ in $(t_2,t_3] \times \Omega$, where

$$f_1(t,x) = a_i - b_i u_i(t,x) - \sum_{j=1}^{N} \int_0^t u_j(t-s,x)\, df_{ij}(s) - b_i p_{2i}(t)$$

and

$$g_1(t,x) = p_{2i}(t) \sum_{j=1}^{N} \left\{ C_{ij}^-(\beta_{1j}+\epsilon) + \int_0^t u_j(t-s,x)\, df_{ij}(s) + \epsilon\beta \right\}.$$

Note that $f_1(t,x)$ is bounded on $[t_2,t_3] \times \bar{\Omega}$. Also, $g_1(t,x) \geq 0$ on $(t_2,t_3] \times \Omega$, because

$$\begin{aligned} -\int_0^t u_j(t-s,x)\, df_{ij}(s) &= -\int_0^t u_j(t-s,x)\, dF_{ij}^+(s) + \int_0^t u_j(t-s,x)\, dF_{ij}^-(s) \\ &\leq \int_0^{t-t_1} u_j(t-s,x)\, dF_{ij}^-(s) + \int_{t-t_1}^t u_j(t-s,x)\, dF_{ij}^-(s) \\ &\leq (\beta_{1j}+\epsilon) F_{ij}^-(t-t_1) + \beta\big(F_{ij}^-(t) - F_{ij}^-(t-t_1)\big) \\ &\leq (\beta_{1j}+\epsilon) C_{ij}^- + \beta\epsilon \end{aligned}$$

by virtue of (3.4), (3.6), and (3.7). Then we get a contradiction by the strong maximum principle (e.g., [4, p.14, Corollary 1.1–9 and p.19, Remark 1.2–2]). Indeed, if $x_3 \in \Omega$, then $w_i(t,x) \equiv 0$ on $[t_2,t_3] \times \bar{\Omega}$ by the strong maximum principle, which is a contradiction because of $u_i(t_2,x) < \beta_{1i} + \epsilon = p_{2i}(t_2)$. We thus obtain $x_0 \in \partial\Omega$ and $w_i(t,x) < 0$ on $[t_2,t_3] \times \Omega$, and hence $\partial w_i/\partial \mathbf{n} > 0$ at (t_3,x_3) by the strong maximum principle again. But this is also a contradiction because of $\partial w_i/\partial \mathbf{n} = \partial u_i/\partial \mathbf{n} = 0$ at (t_3,x_3). Thus we must obtain (3.8).

Observe that $\lim_{t\to\infty} p_{2i}(t) = \big(a_i + N\epsilon\beta + \sum_{j=1}^N C_{ij}^-(\beta_{1j}+\epsilon)\big)/b_i$. From (3.8) it follows that

$$\limsup_{t\to\infty} \{ \max_{x\in\bar{\Omega}} u_i(t,x) \} \leq \frac{a_i + N\epsilon\beta + \sum_{j=1}^N C_{ij}^-(\beta_{1j}+\epsilon)}{b_i}.$$

Consequently,

$$\limsup_{t\to\infty} \{ \max_{x\in\bar{\Omega}} u_i(t,x) \} \leq \frac{a_i + \sum_{j=1}^N C_{ij}^- \beta_{1j}}{b_i} =: \beta_{2i}, \quad i = 1, \ldots, N,$$

because ϵ is arbitrary.

Step 3. For each $i = 1, \ldots, N$, we define the sequence $\{\beta_{mi}\}$ by

$$\beta_{mi} = \frac{a_i + \sum_{j=1}^N C_{ij}^- \beta_{m-1,j}}{b_i}, \quad m = 1, 2, \ldots,$$
$$\beta_{0i} = \beta.$$

Notice that $\beta \geq a_i/(b_i - C_i)$ for $i = 1, \ldots, N$. Then it is easy to see that the sequence $\{\beta_{mi}\}$ is nonincreasing and nonnegative. Hence there exists the limit $\gamma_i := \lim_{m\to\infty} \beta_{mi}$, which satisfies $\gamma_i \geq a_i/b_i > 0$ and $b_i\gamma_i = a_i + \sum_{j=1}^N C_{ij}^- \gamma_j$ for $i = 1, \ldots, N$. Hence $\gamma := (\gamma_1, \ldots, \gamma_n)^T = (B - C^-)^{-1} a$.

Now, repeat the argument in Step 2 to obtain

$$\limsup_{t\to\infty}\big\{\max_{x\in\bar\Omega} u_i(t,x)\big\} \le \beta_{mi}, \quad i=1,\ldots,N, \quad m=1,2,\ldots .$$

Letting $m\to\infty$ in the above, we obtain

$$\limsup_{t\to\infty}\big\{\max_{x\in\bar\Omega} u_i(t,x)\big\} \le \gamma_i, \quad i=1,\ldots,N. \tag{3.9}$$

Step 4. Take an $\epsilon_0>0$ so small that

$$a_i > \sum_{j=1}^{N} C_{ij}^{+}(\gamma_j+\epsilon_0) + N\epsilon_0\beta, \quad i=1,\ldots,N, \tag{3.10}$$

which is possible by (H3). Let $\epsilon\in(0,\epsilon_0)$. By (3.9) and (2.1), there exist a $t_4>0$ and a $t_5>t_4$ such that

$$0 \le u_i(t,x) < \gamma_i+\epsilon, \quad i=1,\ldots,N, \quad \text{for all } t\ge t_4 \text{ and } x\in\bar\Omega \tag{3.11}$$

and

$$0 \le F_{ij}^{+}(t) - F_{ij}^{+}(t-t_4) < \epsilon, \quad i=1,\ldots,N, \quad \text{for all } t\ge t_5. \tag{3.12}$$

Since $0<u_i(t,x)$, $i=1,\ldots,N$, for all $(t,x)\in(0,\infty)\times\bar\Omega$, by the strong maximum principle, one can choose $\eta>0$ so that

$$\min_{x\in\bar\Omega} u_i(t_5,x) \ge 2\eta, \quad i=1,\ldots,N. \tag{3.13}$$

Now, consider the solution of the ordinary differential equations

$$\frac{d}{dt}q_i = q_i\Big\{a_i - b_iq_i - \sum_{j=1}^{N} C_{ij}^{+}(\gamma_j+\epsilon) - N\epsilon\beta\Big\}$$

with $q_i(t_5)=\eta$. Repeating almost the same argument as in the Step 2, we can see, by (3.11)–(3.13), that

$$q_i(t) < u_i(t,x), \quad t\ge t_5, \quad x\in\bar\Omega, \quad i=1,\ldots,N. \tag{3.14}$$

Since $\lim_{t\to\infty} q_i(t) = \big(a_i - \sum_{j=1}^{N} C_{ij}^{+}(\gamma_j+\epsilon) - N\epsilon\beta\big)/b_i$ by (3.10), (3.14) implies that

$$\liminf_{t\to\infty}\big\{\min_{x\in\bar\Omega} u_i(t,x)\big\} \ge \frac{a_i - \sum_{j=1}^{N} C_{ij}^{+}(\gamma_j+\epsilon) - N\epsilon\beta}{b_i}, \quad i=1,\ldots,N.$$

Letting $\epsilon\to 0^+$ in the above, we obtain that

$$\liminf_{t\to\infty}\big\{\min_{x\in\bar\Omega} u_i(t,x)\big\} \ge \frac{a_i - \sum_{j=1}^{N} C_{ij}^{+}\gamma_j}{b_i}, \quad i=1,\ldots,N. \tag{3.15}$$

Step 5. Set $(B-C^-)^{-1}a = \nu^{(0)} = (\nu_1^{(0)},\ldots,\nu_N^{(0)})^T$ and $B^{-1}(a-C^+\nu^{(0)}) = \mu^{(0)} = (\mu_1^{(0)},\ldots,\nu_N^{(0)})^T$, and consider the sequences $\{\mu^{(m)}\}$ and $\{\nu^{(m)}\}$ defined by (3.1). From (3.9) and (3.15) we see that

$$\mu^{(0)} \le \liminf_{t\to\infty}\{\min_{x\in\bar\Omega} u(t,x)\} \le \limsup_{t\to\infty}\{\max_{x\in\bar\Omega} u(t,x)\} \le \nu^{(0)}, \tag{3.16}$$

where $\liminf_{t\to\infty}\{\min_{x\in\bar\Omega} u(t,x)\}$ and $\limsup_{t\to\infty}\{\max_{x\in\bar\Omega} u(t,x)\}$ denote the vectors in $\mathbf{R}^N$ whose i-th components are $\liminf_{t\to\infty}\{\min_{x\in\bar\Omega} u_i(t,x)\}$ and $\limsup_{t\to\infty}\{\max_{x\in\bar\Omega} u_i(t,x)\}$, respectively. Since $\mathbf{0} < \mu^{(0)} \le \mu^{(1)}$ as noted in the proof of Proposition 2.1, one can choose an $\epsilon_1 > 0$ so small that

$$\epsilon_1 < \mu_i^{(0)} \quad \text{and} \quad a_i > \sum_{j=1}^N C_{ij}^+(\nu_j^{(0)} + \epsilon_1) - \sum_{j=1}^N C_{ij}^-(\mu_j^{(0)} - \epsilon_1) + 2N\epsilon_1\beta \tag{3.17}$$

for all $i = 1, \ldots, N$. Let $\epsilon \in (0, \epsilon_1)$. By (3.16) and (H1), there exist a $t_6 > 0$ and a $t_7 > t_6$ such that

$$\mu_i^{(0)} - \epsilon < u_i(t,x) < \nu_i^{(0)} + \epsilon, \quad i = 1, \ldots, N, \tag{3.18}$$

for all $(t,x) \in [t_6, \infty) \times \bar\Omega$, and

$$0 \le F_{ij}(t) - F_{ij}(t - t_6) < \epsilon, \qquad 0 \le C_{ij} - F_{ij}(t - t_6) < \epsilon, \tag{3.19}$$

$i, j = 1, \ldots, N$, for all $t \ge t_7$. For each $i = 1, \ldots, N$, we consider the solutions $\bar p_i(t)$ and $\bar q_i(t)$ of the ordinary differential equations

$$\frac{d}{dt}\bar p_i = \bar p_i\Big\{a_i - b_i\bar p_i + \sum_{j=1}^N C_{ij}^-(\nu_j^{(0)} + \epsilon) - \sum_{j=1}^N C_{ij}^+(\mu_j^{(0)} - \epsilon) + 2N\epsilon\beta\Big\},$$
$$\frac{d}{dt}\bar q_i = \bar q_i\Big\{a_i - b_i\bar q_i - \sum_{j=1}^N C_{ij}^+(\nu_j^{(0)} + \epsilon) + \sum_{j=1}^N C_{ij}^-(\mu_j^{(0)} - \epsilon) - 2N\epsilon\beta\Big\}$$

with $\bar p_i(t_7) = \nu_i^{(0)} + \epsilon$ and $\bar q_i(t_7) = \mu_i^{(0)} - \epsilon$, respectively. Repeating the argument in Step 2, we can see by (3.17)–(3.19) that

$$\bar q_i(t) < u_i(t,x) < \bar p_i(t), \quad t > t_7, \quad x \in \bar\Omega, \quad i = 1, \ldots, N. \tag{3.20}$$

Since

$$\begin{aligned} 0 &< a_i - \sum_{j=1}^N C_{ij}^+(\nu_j^{(0)} + \epsilon) + \sum_{j=1}^N C_{ij}^-(\mu_j^{(0)} - \epsilon) - 2N\epsilon\beta \\ &\le a_i + \sum_{j=1}^N C_{ij}^-(\nu_j^{(0)} + \epsilon) - \sum_{j=1}^N C_{ij}^+(\mu_j^{(0)} - \epsilon) + 2N\epsilon\beta \end{aligned}$$

by (3.17), we have

$$\lim_{t\to\infty} \bar p_i(t) = \Big(a_i + \sum_{j=1}^N C_{ij}^-(\nu_j^{(0)} + \epsilon) - \sum_{j=1}^N C_{ij}^+(\mu_j^{(0)} - \epsilon) + 2N\epsilon\beta\Big)\Big/ b_i$$

and

$$\lim_{t\to\infty} \bar q_i(t) = \Big(a_i - \sum_{j=1}^N C_{ij}^+(\nu_j^{(0)} + \epsilon) + \sum_{j=1}^N C_{ij}^-(\mu_j^{(0)} - \epsilon) - 2N\epsilon\beta\Big)\Big/ b_i.$$

Then it follows from (3.20) that

$$\limsup_{t\to\infty}\{\max_{x\in\bar\Omega} u_i(t,x)\} \le \Big(a_i + \sum_{j=1}^N C_{ij}^-(\nu_j^{(0)} + \epsilon) - \sum_{j=1}^N C_{ij}^+(\mu_j^{(0)} - \epsilon) + 2N\epsilon\beta\Big)\Big/ b_i$$

and

$$\liminf_{t\to\infty}\{\min_{x\in\bar\Omega} u_i(t,x)\} \geq \Big(a_i - \sum_{j=1}^{N} C_{ij}^{+}(\nu_j^{(0)}+\epsilon) + \sum_{j=1}^{N} C_{ij}^{-}(\mu_j^{(0)}-\epsilon) - 2N\epsilon\beta\Big)\Big/ b_i$$

for all $i = 1, \ldots, N$. Letting $\epsilon \to 0^+$ in the above, we have

$$\mu^{(1)} \leq \liminf_{t\to\infty}\{\min_{x\in\bar\Omega} u(t,x)\} \leq \limsup_{t\to\infty}\{\max_{x\in\bar\Omega} u(t,x)\} \leq \nu^{(1)}.$$

Repeating the above argument, we can derive that

$$\mu^{(m)} \leq \liminf_{t\to\infty}\{\min_{x\in\bar\Omega} u(t,x)\} \leq \limsup_{t\to\infty}\{\max_{x\in\bar\Omega} u(t,x)\} \leq \nu^{(m)} \tag{3.21}$$

for $m = 1, 2, \ldots$. Then (2.4) follows from (3.21) because, as noted in the proof of Proposition 2.1, $\lim_{m\to\infty}\mu^{(m)} = \lim_{m\to\infty}\nu^{(m)} = \nu^*$. □

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