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INTERMEDIATE ROWS OF THE WALSH ARRAY OF BEST RATIONAL APPROXIMANTS TO MEROMORPHIC FUNCTIONS

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ABSTRACT. We investigate the convergence of the rows (fixed denominator degree) of the Walsh array of best rational approximants to a meromorphic function. We give an explicit algorithm for determining when convergence is guaranteed and obtain rates of convergence in the appropriate cases. This algorithm also provides the solution to an integer programming problem that arises in the study of Padé approximants.

1. Introduction

Let Π_m denote the collection of all algebraic polynomials of degree at most m. A rational function $r_{m,n}(z)$ is said to be *of type* (m,n) if for some $p_m \in \Pi_m$ and $q_n \in \Pi_n$,

$$r_{m,n}(z) = p_m(z)/q_n(z), \qquad q_n(z) \not\equiv 0.$$

If f is analytic at $z=0$, then for each pair of non-negative integers (n,μ), there exist polynomials $P_{n,\mu}(z) \in \Pi_n$ and $Q_{n,\mu}(z)(\not\equiv 0) \in \Pi_\mu$ such that

$$Q_{n,\mu}(z)f(z) - P_{n,\mu}(z) = O(z^{n+\mu+1}) \qquad \text{as } z \to 0.$$

The ratio $P_{n,\mu}(z)/Q_{n,\mu}(z)$ is unique and is called the *Padé approximant* of type (n,μ) to f. We denote this Padé approximant by $[n/\mu](z)$. Thus for each f there corresponds a doubly-infinite array indexed by n and μ which is known as the *Padé table*. Concerning the row convergence of this table, we have the following classical result of de Montessus de Ballore [5].

Theorem A. *Let f be analytic at $z=0$ and meromorphic with precisely μ poles (counting multiplicity) in the disk $|z|<\rho$. Let D denote the domain obtained from $|z|<\rho$ by deleting the μ poles of f. Then for n sufficiently large, the Padé approximant $[n/\mu](z)$ to f satisfies*

$$f(z) - [n/\mu](z) = O(z^{n+\mu+1}) \quad \textit{as } z \to 0.$$

Each $[n/\mu](z)$, for n large, has precisely μ finite poles, and as $n \to \infty$, these poles approach respectively, the μ poles of f in $|z|<\rho$. The sequence $\{[n/\mu](z)\}_{n=0}^{\infty}$ converges to f throughout D, uniformly on any compact subset of D.

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An analogue of Theorem A for best uniform rational approximants was established by J. L. Walsh [10]. To describe this theorem, we first introduce some needed notation. Let $E \subset \mathbf{C}$ be a compact set whose complement K (with respect to the extended plane $\bar{\mathbf{C}}$) is connected and possesses a classical Green's function $G(z)$ with a pole at infinity. Let $\Gamma_\sigma(\sigma > 1)$ denote generically the locus $G(z) = \log \sigma$, and let E_σ be the interior of Γ_σ. If the function f is continuous on E, then there exists for each pair (n, μ) a rational function $W_{n,\mu}(z)$ of type (n, μ) which is of best uniform approximation to f on E, in the sense that for all rational functions $r_{n,\mu}(z)$ of type (n, μ) we have

$$E_{n,\mu}(f) := \|f(z) - W_{n,\mu}(z)\|_E \leq \|f(z) - r_{n,\mu}(z)\|_E,$$

where $\|\cdot\|_E$ denotes the sup norm over E. The $W_{n,\mu}(z)$ need not be unique, but any particular determination of them will suffice for our purpose. The $W_{n,\mu}(z)$ form a table of double entries known as the *Walsh array.* J. L. Walsh [10] has established the following analogue of Theorem A for the rows of his array.

Theorem B. *Suppose that the function f is analytic on E and meromorphic with precisely μ poles (i.e., poles of total multiplicity μ) in E_ρ, $1 < \rho \leq \infty$. If $\{r_{n,\mu}(z)\}_{n=0}^\infty$ is a sequence of rational functions of respective types (n, μ) which satisfy*

$$\limsup_{n\to\infty} \|f(z) - r_{n,\mu}(z)\|_E^{1/n} \leq 1/\rho \tag{1}$$

(a condition which, in particular, is satisfied by the $(\mu + 1)$st row $\{W_{n,\mu}(z)\}_{n=0}^\infty$ of the Walsh array of f on E), then for n sufficiently large, each $r_{n,\mu}(z)$ has precisely μ finite poles, which approach the μ poles of f in E_ρ, respectively, and the sequence $\{r_{n,\mu}(z)\}_{n=0}^\infty$ converges uniformly to f on each compact subset of E_ρ that contains no pole of f.

We note that if f has multiple poles or several poles that lie on the same level curve Γ_σ, then there are certain rows of the Walsh array for which Theorem B provides no information on convergence. Consequently, there arises the following problem. Let E be a compact set as above. If f is analytic except for μ poles in E_ρ and is meromorphic with N (≥ 2) poles (counting multiplicity) on Γ_ρ, what can we say about the convergence of the rows $(\mu + 2)$ through $(\mu + N)$ of the Walsh array? For example, if f is analytic on E_ρ $(\mu = 0)$ and has poles at the six points $\alpha_1, \alpha_2, \ldots, \alpha_6$ on Γ_ρ with respective multiplicities $r_1 = 12$, $r_2 = 8$, $r_3 = 8$, $r_4 = 6$, $r_5 = 5$ and $r_6 = 1$, what can be said about the convergence of the sequences $\{W_{n,\nu}(z)\}_{n=0}^\infty$ for $\nu = 1, \ldots, 39$? In the second author's paper [7], the convergence of these "intermediate rows" of the Walsh array was investigated for the special case when f has poles at just two points on the boundary. Here we extend the method of [7] to the general situation when f has poles in m points on Γ_ρ. In a forthcoming paper [3], similar results for multipoint Padé approximants are obtained.

2. Statements of main results

The analysis of the intermediate rows of the Walsh table requires further assumptions on the compact point set E. Let Δ denote the logarithmic capacity (transfinite diameter) of the point set E and assume that there exists a sequence of monic polynomials $\omega_n(z)$ that have $n+1$ zeros, respectively, all belonging to the boundary of E, and that satisfy

$$\begin{gathered} |\omega_n(z)| \leq M\Delta^n, \quad z \text{ on } E, \\ \big|G(z) + \log\Delta - n^{-1}\log|\omega_n(z)|\big| \leq Mn^{-1}, \end{gathered} \tag{2}$$

for z on each compact set exterior to E, where M is a constant independent of n. We summarize these assumptions by saying that E has *Property A*. For example, if E is a line segment or E is the closed interior of a finite number of mutually exterior smooth Jordan curves $C_1, C_2, \ldots, C_k$, then E has Property A (cf. [9]).

We now can state our main result. Its proof is given in Section 3.

Theorem 1. *Suppose E has Property* A. *Let f be analytic on E, meromorphic with precisely μ (≥ 0) poles in E_ρ, and analytic on Γ_ρ except for poles in the m (≥ 1) distinct points $\alpha_1, \alpha_2, \ldots, \alpha_m$ on Γ_ρ with respective orders $r_1, r_2, \ldots, r_m$ ($r_1 \geq r_2 \geq \cdots \geq r_m \geq 1$). If none of $\{\alpha_i\}$ is a critical point of $G(z)$, then for "good" ν ($0 \leq \nu \leq r_1 + r_2 + \cdots + r_m$) defined in Definition 1 below, the $(\mu + \nu + 1)$st row of the Walsh array for f on E (i.e., the sequence $\{W_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$) converges uniformly to f on each compact subset of E_ρ that contains no poles of f.*

Furthermore, for fixed good ν and for n sufficiently large, each of the best approximating rational functions $W_{n,\mu+\nu}(z)$ has precisely $\mu + \nu$ finite poles, μ of which approach the μ poles of f in E_ρ, respectively, and $p_{\nu,i}$ of which approach the points α_i respectively, (the $\{p_{\nu,i}\}$ are defined below and satisfy $\sum_{i=1}^m p_{\nu,i} = \nu$), and

$$E_{n,\mu+\nu} := \|W_{n,\mu+\nu} - f\|_E \leq An^\lambda/\rho^n, \tag{3}$$

where $\lambda := \max_{1\leq i\leq m}(r_i - 2p_{\nu,i} - 1)$ and A is a constant independent of n.

The definitions of "good" ν and corresponding $p_{\nu,i}$, which were motivated by the method in [7], are somewhat technical and require the introduction of further notation. Let

$$\begin{aligned}
u_{i,j} &:= \lfloor (r_i - r_{j+1})/2 \rfloor, && 1 \leq i \leq j+1 \leq m;\\
w_{i,j} &:= r_i - r_{j+1} - 2u_{i,j}, && 1 \leq i \leq j+1 \leq m;\\[1ex]
s_l &:= \textstyle\sum_{i=1}^{l-1} w_{i,l-1}, && 1 \leq l \leq m;\\
s_l &:= \textstyle\sum_{i=1}^{2m-l} w_{i,2m-l}, && 1+m \leq l < 2m;\\[1ex]
v_0 &:= 0, &&\\
v_l &:= \textstyle\sum_{i=1}^{l} u_{i,l}, && 1 \leq l \leq m-1;\\
v_l &:= \textstyle\sum_{i=1}^{2m-l-1} u_{i,2m-l-1} &&\\
&\quad +(2m-l-1)r_{2m-l} &&\\
&\quad +\textstyle\sum_{j=2m-l}^{m} r_j, && m \leq l < 2m;\\[1ex]
I_l &:= [v_{l-1}, v_l), && 1 \leq l \leq m-1;\\
I_m &:= [v_{m-1}, v_m], &&\\
I_l &:= (v_{l-1}, v_l], && m+1 \leq l < 2m.
\end{aligned}$$

Here $\lfloor x \rfloor$ stands for the integer part of x, and any empty sum is understood to have the value zero. It is straightforward to verify that the integers v_l satisfy

$$v_0 \leq v_1 \leq \cdots \leq v_{2m-1} = \sum_{i=1}^{m} r_i$$

and $[0, v_{2m-1}] = \bigcup_{l=1}^{2m-1} I_l$. The intervals I_l are pairwise disjoint and some of them may be empty.

Definition 1. For fixed ν, $0 \le \nu \le v_{2m-1}$, let I_l, $l = l(\nu)$, be the unique interval containing ν. Set

$$k_l := \left\lfloor \frac{\nu - v_{l-1}}{l} \right\rfloor \quad \text{and} \quad q_l := \nu - v_{l-1} - lk_l, \qquad \text{if } 1 \le l \le m;$$

$$k_l := \left\lfloor \frac{\nu - v_{l-1}}{2m-l} \right\rfloor \quad \text{and} \quad q_l := \nu - v_{l-1} - (2m-l)k_l, \qquad \text{if } m+1 \le l \le 2m-1.$$

We say that ν is *good* if $q_l = 0$ or $q_l = s_l$. If ν $(0 \le \nu \le \nu_{2m-1})$ is not good, we say that ν is *bad*.

Next, for good ν, we specify the quantities $\{p_{\nu,i}\}$ mentioned in Theorem 1.

Definition 2. Given a good ν, we first determine to which interval I_l it belongs; then for this $l = l(\nu)$, we consider the two possibilities $q_l = 0$ or $q_l = s_l$ in Definition 1.
Case 1. $q_l = 0$
(a) If $1 \le l \le m$,

$$\begin{aligned} p_{\nu,i} &:= k_l + u_{i,l-1}, && 1 \le i \le l, \\ p_{\nu,i} &:= 0, && l < i \le m; \end{aligned} \tag{4}$$

(b) if $m+1 \le l \le 2m-1$,

$$\begin{aligned} p_{\nu,i} &:= k_l + u_{i,2m-l} + r_{2m-l+1}, && 1 \le i \le 2m-l, \\ p_{\nu,i} &:= r_i, && 2m-l < i \le m. \end{aligned} \tag{5}$$

Case 2. $q_l = s_l \ne 0$
(a) if $1 \le l \le m$,

$$\begin{aligned} p_{\nu,i} &:= k_l + u_{i,l-1} + w_{i,l-1}, && 1 \le i \le l, \\ p_{\nu,i} &:= 0, && l < i \le m; \end{aligned} \tag{6}$$

(b) if $m+1 \le l \le 2m-1$,

$$\begin{aligned} p_{\nu,i} &:= k_l + u_{i,2m-l} + r_{2m-l+1} + w_{i,2m-l}, && 1 \le i \le 2m-l, \\ p_{\nu,i} &:= r_i, && 2m-l < i \le m. \end{aligned} \tag{7}$$

Remark 1. Definition 1 gives explicit conditions for deciding whether ν is good. For example, $q_1 = 0$, so all $\nu \in I_1$ are good. The same is true for $\nu \in I_{2m-1}$, and if ν is one of the v_i's for $1 \le i \le m$, then ν is good. Furthermore, q_2 is either 0 or 1, so if $r_1 - r_2$ is odd, then all $\nu \in I_2$ are good. In fact, it is a simple exercise to write a program to compute all good ν for a given input $r_1, r_2, \ldots, r_m$.

Remark 2. For good ν, $\sum_{i=1}^{m} p_{\nu,i} = \nu$. Furthermore, for each t, $1 \le t \le m$, we have

$$0 \le p_{\nu,t} \le r_t, \tag{8}$$

and for $1 \le i \le m-1$, we have

$$p_{\nu,i} \ge p_{\nu,i+1}. \tag{9}$$

The proofs of (8) and (9) are straightforward but technical. The reader may consult [4] for the details.

As an illustration of Theorem 1, we consider the problem raised in the introduction.

Example. For $m = 6$, $r_1 = 12$, $r_2 = 8$, $r_3 = 8$, $r_4 = 6$, $r_5 = 5$, and $r_6 = 1$, the following table lists the good values of ν and the corresponding $p_{\nu,i}$'s.

$\nu = 0,$	$p_{0,1} = 0,$	$p_{0,2} = 0,$	$p_{0,3} = 0,$	$p_{0,4} = 0,$	$p_{0,5} = 0,$	$p_{0,6} = 0,$
$\nu = 1,$	$p_{1,1} = 1,$	$p_{1,2} = 0,$	$p_{1,3} = 0,$	$p_{1,4} = 0,$	$p_{1,5} = 0,$	$p_{1,6} = 0,$
$\nu = 2,$	$p_{2,1} = 2,$	$p_{2,2} = 0,$	$p_{2,3} = 0,$	$p_{2,4} = 0,$	$p_{2,5} = 0,$	$p_{2,6} = 0,$
$\nu = 5,$	$p_{5,1} = 3,$	$p_{5,2} = 1,$	$p_{5,3} = 1,$	$p_{5,4} = 0,$	$p_{5,5} = 0,$	$p_{5,6} = 0,$
$\nu = 9,$	$p_{9,1} = 4,$	$p_{9,2} = 2,$	$p_{9,3} = 2,$	$p_{9,4} = 1,$	$p_{9,5} = 0,$	$p_{9,6} = 0,$
$\nu = 10,$	$p_{10,1} = 4,$	$p_{10,2} = 2,$	$p_{10,3} = 2,$	$p_{10,4} = 1,$	$p_{10,5} = 1,$	$p_{10,6} = 0,$
$\nu = 14,$	$p_{14,1} = 5,$	$p_{14,2} = 3,$	$p_{14,3} = 3,$	$p_{14,4} = 2,$	$p_{14,5} = 1,$	$p_{14,6} = 0,$
$\nu = 15,$	$p_{15,1} = 5,$	$p_{15,2} = 3,$	$p_{15,3} = 3,$	$p_{15,4} = 2,$	$p_{15,5} = 2,$	$p_{15,6} = 0,$
$\nu = 19,$	$p_{19,1} = 6,$	$p_{19,2} = 4,$	$p_{19,3} = 4,$	$p_{19,4} = 3,$	$p_{19,5} = 2,$	$p_{19,6} = 0,$
$\nu = 21,$	$p_{21,1} = 6,$	$p_{21,2} = 4,$	$p_{21,3} = 4,$	$p_{21,4} = 3,$	$p_{21,5} = 3,$	$p_{21,6} = 1,$
$\nu = 25,$	$p_{25,1} = 7,$	$p_{25,2} = 5,$	$p_{25,3} = 5,$	$p_{25,4} = 4,$	$p_{25,5} = 3,$	$p_{25,6} = 1,$
$\nu = 26,$	$p_{26,1} = 7,$	$p_{26,2} = 5,$	$p_{26,3} = 5,$	$p_{26,4} = 4,$	$p_{26,5} = 4,$	$p_{26,6} = 1,$
$\nu = 30,$	$p_{30,1} = 8,$	$p_{30,2} = 6,$	$p_{30,3} = 6,$	$p_{30,4} = 5,$	$p_{30,5} = 4,$	$p_{30,6} = 1,$
$\nu = 31,$	$p_{31,1} = 8,$	$p_{31,2} = 6,$	$p_{31,3} = 6,$	$p_{31,4} = 5,$	$p_{31,5} = 5,$	$p_{31,6} = 1,$
$\nu = 35,$	$p_{35,1} = 9,$	$p_{35,2} = 7,$	$p_{35,3} = 7,$	$p_{35,4} = 6,$	$p_{35,5} = 5,$	$p_{35,6} = 1,$
$\nu = 38,$	$p_{38,1} = 10,$	$p_{38,2} = 8,$	$p_{38,3} = 8,$	$p_{38,4} = 6,$	$p_{38,5} = 5,$	$p_{38,6} = 1,$
$\nu = 39,$	$p_{39,1} = 11,$	$p_{39,2} = 8,$	$p_{39,3} = 8,$	$p_{39,4} = 6,$	$p_{39,5} = 5,$	$p_{39,6} = 1,$
$\nu = 40,$	$p_{40,1} = 12,$	$p_{40,2} = 8,$	$p_{40,3} = 8,$	$p_{40,4} = 6,$	$p_{40,5} = 5,$	$p_{40,6} = 1.$

An immediate consequence of Theorem 1 is the following result which certainly agrees with one's expectations.

Corollary 1. *With the same assumptions on E and f as in Theorem 1, if $r_1 = r_2 = \cdots = r_m = r$, then only $\nu = mk$, $k = 1, 2, \ldots, r$, are good ν's in the sense of Definition 1. Furthermore, by Definition 2, for $\nu = mk$, there are $p_{mk,i} = k$ poles of $W_{n,\mu+mk}(z)$ that approach each of the points α_i, $i = 1, \ldots, m$, and*

$$E_{n,\mu+mk} = \|W_{n,\mu+mk} - f\|_E \le An^{r-2k-1}/\rho^n.$$

Remark 3. Lin [2] and Sidi [8] have investigated the row convergence of the Padé table. Under assumptions similar to those in Theorem 1, they obtained a sufficient condition for the convergence of an intermediate row of the Padé table; namely, that there is a *unique* solution to the following integer programming problem: Given positive integers $\nu, r_1, \ldots, r_m$,

$$\begin{aligned} &\text{maximize} \quad \sum_{i=1}^{m} (r_i\sigma_i - \sigma_i^2) \quad \text{over} \quad (\sigma_1, \sigma_2, \ldots, \sigma_m) \\ &\text{subject to} \quad \sum_{i=1}^{m} \sigma_i = \nu \quad \text{and } 0 \le \sigma_i \le r_i,\ \sigma_i \text{ integers},\ 1 \le i \le m. \end{aligned} \tag{10}$$

As we shall show, this uniqueness condition is actually the same as our conditions for good ν in Definition 1. In other words, the explicit formulas in Definition 1 give the values of ν for which this integer programming problem has a unique solution.

Proposition 1. *There is a unique solution to* (10) *if and only if ν is good in the sense of Definition* 1. *For good ν, the unique solution to* (10) *is $\sigma_i = p_{\nu,i}$, $1 \le i \le m$, where the $p_{\nu,i}$'s are given by Definition* 2.

The proof of Proposition 1 is given in Section 4.

Concerning the rates of convergence of the denominator polynomials for the best rational approximants, the following result is proved in [4].

Proposition 2. *With the assumptions of Theorem* 1, *for fixed good* ν *satisfying* $0 \le \nu \le v_{2m-1}$, *let* $q_n(z)$ *be the monic polynomial denominator of* $W_{n,\mu+\nu}(z)$, $\pi(z)$ *be the monic polynomial of degree* μ *whose zeros are the poles of* f *in* E_ρ, *and* $q(z) := \pi(z)\prod_{i=1}^{m}(z-\alpha_i)^{p_{\nu,i}}$. *Then*

$$\begin{aligned} \|q_n(z) - q(z)\| &= O(1/n) \quad \textit{for } 0 < \nu < v_{2m-1}; \\ \limsup_{n\to\infty} \|q_n(z) - q(z)\|^{1/n} &< 1 \quad \textit{for } \nu = 0 \textit{ or } \nu = v_{2m-1}. \end{aligned} \tag{11}$$

In (11) the norm can be taken over any compact subset of **C**.

We remark that although the possible divergence of rows corresponding to bad ν's is not investigated in the present paper, it is discussed in the forthcoming paper [3] where it is shown that there are bad ν's for which the limit points of poles of the corresponding row sequence can form an arc of a smooth curve.

3. Proof of Theorem 1

To prove Theorem 1, we apply some theorems from [7] along with two additional lemmas. Hereafter, $\{\omega_n(z)\}$ will denote a sequence of monic polynomials of respective degrees $n+1$, $n = 1, 2, \ldots$, that satisfies (2).

Lemma 1. *With the assumptions of Theorem* 1, *suppose that* ν *is any fixed integer* $(0 \le \nu \le r_1 + r_2 + \cdots + r_m)$ *and that* $r_{n,\mu+\nu}(z)$ *is a sequence of rational functions of types* $(n, \mu+\nu)$, *respectively, which satisfy for some real* τ

$$\|f(z) - r_{n,\mu+\nu}(z)\|_E = o(n^\tau/\rho^n) \quad \textit{as } n \to \infty. \tag{12}$$

Let $q_n(z)$ *be the monic polynomial whose zeros are the finite poles of* $r_{n,\mu+\nu}(z)$, *multiplicity included, and set* $p_n(z) := q_n(z) r_{n,\mu+\nu}(z)$. *If the finite poles of the* $r_{n,\mu+\nu}(z)$ *are uniformly bounded, then for any point* $\alpha \in \Gamma_\rho$ *and any integer* N,

$$\lim_{n\to\infty} \frac{p_n(z)\omega_{n-N}(\alpha)}{n^\tau \omega_{n-N}(z)} = 0$$

uniformly for z *on each closed set exterior to* Γ_ρ *(the* $r_{n,\mu+\nu}(z)$ *need not be defined for every* n*).*

The proof of this lemma is similar to the proof of Lemma 2 in [7] and therefore is omitted.

Lemma 2. *With the assumptions of Theorem* 1, *suppose that* $r_{n,\mu+\nu}(z)$ *is a sequence of rational functions of types* $(n, \mu+\nu)$, *respectively, satisfying*

$$\limsup_{n\to\infty} \|f - r_{n,\mu+\nu}\|_E^{1/n} \le 1/\rho. \tag{13}$$

Let $q_n(z)$ *be the monic polynomials whose zeros are the finite poles, counting multiplicity, of* $r_{n,\mu+\nu}(z)$. *Then for any subsequence of* $q_n(z)$, *the limit points of their zeros must include all poles (counting multiplicity) of* f *in* E_ρ.

The proof is just a slight modification of the proof of Theorem 1 in [10], so we omit it.

Proof of Theorem 1. If $\nu = \sum_{i=1}^{m} r_i$, then the theorem follows from Theorem B. So suppose $\nu \le \sum_{i=1}^{m} r_i - 1$. If $\nu \in I_l$ and ν is good, set $p_s = p_{\nu,s}$ $(s = 1, 2, \ldots, m)$, where $p_{\nu,s}$ is defined in Definition 2. We first will assume and later will prove that

$$r_j - 2p_j + 1 > r_i - 2p_i - 1 \tag{14}$$

for $1 \leq j \leq l$ and $i = 1, 2, \ldots, m$ when $1 \leq l \leq m$, and for $1 \leq i \leq 2m - l$ and $j = 1, 2, \ldots, m$ when $m + 1 \leq l \leq 2m - 1$. We now can use an argument similar to the proof of Theorem 10 in [7] to establish Theorem 1.

Let $\pi(z)$ be the monic polynomial of degree μ whose zeros are the poles of f in E_ρ. Write

$$\pi(z)f(z) = f_0(z) + \sum_{i=1}^{m} S_i(z),$$

where $f_0(z)$ is analytic on $E_\rho \cup \Gamma_\rho$ and $S_i(z) = \sum_{k=1}^{r_i} B_{k,i}(z-\alpha_i)^{-k}$ is the singular part of $\pi(z)f(z)$ at the pole α_i. Let $R_{n,j}^{\{i\}}(z)$, $0 \leq j < r_i$, be the rational function of type (n, j) which is defined by Theorem 2 in [7] with $\alpha = \alpha_i$ and $F(z) = f_0(z) + S_1(z)$ for $i = 1, F(z) = S_i(z)$ for $2 \leq i \leq m$. Furthermore, let $R_{n,r_i}^{\{i\}}(z)$ be a best uniform rational approximation on E of type (n, r_i) to $F(z) = f_0(z) + S_1(z)$ for $i = 1$, $F(z) = S_i(z)$ for $2 \leq i \leq m$. Note that by Theorem 1 in [10], there exists $\theta > \rho$ such that

$$\|R_{n,r_i}^{\{i\}}(z) - F(z)\|_E \leq A_0/\theta^n, \tag{15}$$

where, as above, $F(z)$ is the corresponding function for each $i = 1, 2, \ldots, m$.

First, we consider the case $\nu \in I_l$, $1 \leq l \leq m$. From (14) and Theorem 2 in [7], we obtain the following inequalities (where π^{-1} denotes $1/\pi(z)$):

$$E_{n,\mu+\nu} := \|W_{n,\mu+\nu} - f\|_E \leq \left\|\pi^{-1}\sum_{i=1}^{m} R_{n-\nu,p_i}^{\{i\}} - f\right\|_E \leq An^\lambda/\rho^n, \tag{16}$$

where $\lambda := \max_{1\leq i\leq m}(r_i - 2p_i - 1)$;

$$\left\|W_{n,\mu+\nu} - \pi^{-1}\sum_{i=1}^{m} R_{n-\nu,p_i}^{\{i\}}\right\|_E \leq E_{n,\mu+\nu} + \left\|\pi^{-1}\sum_{i=1}^{m} R_{n-\nu,p_i}^{\{i\}} - f\right\|_E \leq 2An^\lambda/\rho^n, \tag{17}$$

$$\begin{aligned} &\left\|W_{n,\mu+\nu} - \pi^{-1}R_{n-\nu,p_j-1}^{\{j\}} - \pi^{-1}\sum_{i=1,i\neq j}^{m} R_{n-\nu,p_i}^{\{i\}}\right\|_E \\ &\leq E_{n,\mu+\nu} + \left\|f - \pi^{-1}R_{n-\nu,p_j-1}^{\{j\}}(z) - \pi^{-1}\sum_{i=1,i\neq j}^{m} R_{n-\nu,p_i}^{\{i\}}\right\|_E \\ &\leq A_1 n^{r_j-2p_j+1}/\rho^n, \end{aligned} \tag{18}$$

where we only consider those $j = 1, 2, \ldots, l$ such that $p_j \geq 1$. Then by [7, Theorem 2(ii)] and Lemma 1, we get

$$\begin{aligned} &\lim_{n\to\infty} \frac{(R_{n-\nu,p_j-1}^{\{j\}}(z) + \sum_{i=1,i\neq j}^{m} R_{n-\nu,p_i}^{\{i\}}(z))\omega_{n-\nu}(\alpha_j)}{n^{r_j-2p_j+1}\omega_{n-\nu}(z)} \\ &\qquad = C_j \sum_{k=0}^{p_j-1} B_{r_j-k,j}(z-\alpha_j)^{k-2p_j+1}, \end{aligned} \tag{19}$$

uniformly on each closed set exterior to Γ_ρ, where the constant C_j is not zero for $j = 1, 2, \ldots, l$.

Next suppose that $\nu \in I_l$ for $m < l < 2m$. Then (14) holds for $1 \leq i \leq 2m - l$, $j = 1, 2, \ldots, m$. Furthermore, for $2m - l < i \leq m$, we have $p_i = r_i$, so the estimate (15) holds. Combining these inequalities, we again get (16), (17), and (18). Thus (19) is still true by Theorem 2 in [7] and Lemma 1 for $j = 1, 2, \ldots, m$.

Now suppose that the finite poles of $\{W_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$ are uniformly bounded. Write $W_{n,\mu+\nu}(z) = p_n(z)/q_n(z)$ as in Lemma 1. Then the sequence $\{q_n(z)\}_{n=0}^{\infty}$ forms a normal family in the whole plane. Let $q(z)$ be any limit function of this sequence and note that $q(z)$ must be a polynomial of the form $q(z) = z^{\eta} + c_1 z^{\eta-1} + \cdots + c_{\eta}$, where $0 \le \eta \le \mu + \nu$. By Lemma 2, we know that $\pi(z)$ is a factor of $q(z)$, and we want to show that $q(z) = \pi(z)\prod_{j=1}^{m}(z-\alpha_j)^{p_j}$, where we only consider those p_j such that $p_j \ge 1$. Let

$$\phi_n(z) := \Big(\pi(z)W_{n,\mu+\nu}(z) - R^{\{j\}}_{n-\nu,p_j-1}(z) - \sum_{i=1,i\ne j}^{m} R^{\{i\}}_{n-\nu,p_i}(z)\Big) \times \frac{\omega_{n-\nu}(\alpha_j)q_n(z)Q^{\{j\}}_{n-\nu,p_j-1}(z)\prod_{i=1,i\ne j}^{m} Q^{\{i\}}_{n-\nu,p_i}(z)}{n^{r_j-2p_j+1}\omega_{n-\nu}(z)},$$

where $Q^{\{i\}}_{n-\nu,p_i}(z)$, $Q^{\{j\}}_{n-\nu,p_j-1}(z)$ are the monic denominator polynomials of $R^{\{i\}}_{n-\nu,p_i}(z)$ and $R^{\{j\}}_{n-\nu,p_j-1}(z)$, respectively. Since $|\omega_n(\alpha_j)/\omega_n(z)| \le A\rho^n/\sigma^n$ for z on Γ_σ, applying the inequalities (16), (18) and the Bernstein Lemma (cf. [12, §4.6]), we get that the sequence $\{\phi_n(z)\}_{n=0}^{\infty}$ is uniformly bounded on each Γ_σ ($\sigma > 1$) and hence on each compact subset of $K := \bar{\mathbf{C}}\setminus E$. By Lemma 1, Theorem 2 in [7], and (19), we see that some subsequence of $\{\phi_n(z)\}_{n=0}^{\infty}$ converges to the function

$$\phi(z) := -q(z)(z-\alpha_j)^{p_j-1}\prod_{i=1,i\ne j}^{m}(z-\alpha_i)^{p_i}C_j\sum_{k=0}^{p_j-1}B_{r_j-k,j}(z-\alpha_j)^{k-2p_j+1}$$

for z exterior to Γ_ρ. But the family $\{\phi_n(z)\}_{n=0}^{\infty}$ is normal in $K\setminus\{\infty\}$, and hence $\phi(z)$ must be analytic in this domain and, in particular, at $z = \alpha_j$. Therefore, since $B_{r_j,j} \ne 0$ and $C_j \ne 0$, $(z-\alpha_j)^{p_j}$ must be a factor of $q(z)$, $j = 1, 2, \ldots, m$. Because $\sum_{j=1}^{m} p_j = \nu$, it follows that

$$q(z) = \pi(z)\prod_{j=1}^{m}(z-\alpha_j)^{p_j}.$$

If the finite poles of $\{W_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$ are not uniformly bounded, then there is a subsequence of $\{W_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$, which we still denote by $\{W_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$, with the property that precisely τ finite poles of $W_{n,\mu+\nu}(z)$ approach infinity while the remaining $\eta - \tau$ (or fewer) poles are uniformly bounded in modulus less than some constant R. Let $\delta_{n,1}, \delta_{n,2}, \ldots, \delta_{n,\tau}$ be the finite poles of $W_{n,\mu+\nu}(z)$ which lie outside the circle of radius R. Set $q_n^\star(z) := q_n(z)\prod_{s=1}^{\tau}(-\delta_{n,s})^{-1}$, $p_n^\star(z) := p_n(z)\prod_{s=1}^{\tau}(-\delta_{n,s})^{-1}$. Then $W_{n,\mu+\nu}(z) = p_n^\star(z)/q_n^\star(z)$. Clearly the polynomials $\{q_n^\star(z)\}_{n=0}^{\infty}$ are uniformly bounded on compact subsets of $\mathbf{C}$, so they form a normal family. Let $q^\star(z)$ be any limit function of this sequence. Then $q^\star(z)$ must be a polynomial of the form $q^\star(z) = z^{\eta-\tau} + \cdots$, where $0 \le \eta \le \nu + \mu$. Using $q_n^\star(z)$ instead of $q_n(z)$ in the above reasoning, we again get that $\pi(z)$ and $(z-\alpha_j)^{p_j}$, $j = 1, 2, \ldots, m$, are all factors of $q^\star(z)$. So $q^\star(z) = c(z)\pi(z)\prod_{j=1}^{m}(z-\alpha_j)^{p_j}$, where $c(z)$ is a polynomial. But $\sum_{j=1}^{m} p_j + \mu = \nu + \mu$ is larger than the degree of $q^\star(z)$. This contradiction implies that the finite poles of $W_{n,\mu+\nu}(z)$ must be uniformly bounded.

We have shown that the only limit function of the sequence $\{q_n(z)\}$ is $\pi(z)\prod_{j=1}^{m}(z-\alpha_j)^{p_j}$. Hence

$$q_n(z) \to \pi(z)\prod_{j=1}^{m}(z-\alpha_j)^{p_j} \qquad \text{as } n \to \infty$$

uniformly for z on each compact subset of the plane. Now from Hurwitz's Theorem for fixed good ν and for n sufficiently large, each of the rational functions $W_{n,\mu+\nu}(z)$ has precisely $\mu+\nu$ finite poles, μ of which approach the μ poles of $f(z)$ in E_ρ, respectively, and $p_i = p_{\nu,i}$ of which approach the points α_i, respectively. Furthermore, since the sequence $\{W_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$ satisfies the conditions of Corollary 4 in [11] and has no limit points of poles other than the poles of $f(z)$, the first part of Theorem 1 follows.

What now remains is to prove (14) . Referring to the formulas for p_i in Definition 2, we will use that $\lfloor (r_i - r_l)/2 \rfloor = (r_i - r_l - w_{i,l-1})/2$ and $0 \le w_{i,l} \le 1$ for $0 \le i, l \le m$.

In part (a) of Cases 1 and 2 of Definition 2, when $1 \le l \le m-1$, we have by (4) and (6),

$$\begin{aligned} p_j &= k_l + \frac{r_j - r_l \mp w_{j,l-1}}{2} && \text{for } 1 \le j \le l, \\ p_j &= 0 && \text{for } l < j \le m, \end{aligned}$$

where the $\mp$ refers to Case 1 and Case 2, respectively. So, for $1 \le j \le l$ with $k_l \le u_{l,l}$, we obtain

$$\begin{aligned} r_j - 2p_j + 1 &= r_j - 2k_l - (r_j - r_l \mp w_{j,l-1}) + 1 \\ &= r_l - 2k_l \pm w_{j,l-1} + 1 \\ &\ge r_l - 2u_{l,l} > r_{l+1} - 1 \ge r_i - 1 = r_i - 2p_i - 1, \end{aligned}$$

for $l+1 \le i \le m$. For $1 \le i, j \le l$,

$$2(p_j - p_i) = (r_j - r_l \mp w_{j,l-1}) - (r_i - r_l \mp w_{i,l-1}),$$

so we get

$$2(p_j - p_i) \le r_j - r_i + 1 < r_j - r_i + 2.$$

This implies (14).

When $l = m$, $p_j = k_m + \dfrac{r_j - r_m \mp w_{j,m-1}}{2}$ for $1 \le j \le m$,

$$2(p_j - p_i) = (r_j - r_m \mp w_{j,m-1}) - (r_i - r_m \mp w_{i,m-1}) < r_j - r_i + 2,$$

and so (14) is still valid for $1 \le i, j \le m$.

In part (b) of Case 1 and Case 2 of Definition 2, it follows from (5) and (7) that

$$\begin{aligned} p_j &= k_l + \frac{r_j - r_{2m-l+1} \mp w_{j,2m-l}}{2} + r_{2m-l+1} && \text{for } 1 \le j \le 2m-l, \\ p_j &= r_j && \text{for } 2m-l < j \le m. \end{aligned}$$

Similarly for $1 \le i, j \le 2m-l$, we have $2(p_j - p_i) < r_j - r_i + 2$.

Next, for $1 \le i \le 2m-l < j \le m$, with $k_l \ge 0$,

$$\begin{aligned} r_i - 2p_i - 1 &= r_i - 2k_l - (r_i - r_{2m-l+1} \mp w_{i,2m-l}) - 2r_{2m-l+1} - 1 \\ &\le -2k_l + 1 - r_{2m-l+1} - 1 \\ &< 1 - r_{2m-l+1} \le 1 - r_j = r_j - 2p_j + 1. \end{aligned}$$

Thus (14) has been verified and the proof of Theorem 1 is complete. □

4. Proof of the equivalence to Lin and Sidi's condition

We shall refer to the integer programming problem in (10) as the "IP(ν) problem". Before proving Proposition 1, we state some results from [1] where the solution of IP(ν) was discussed. The following lemma combines Lemmas 2, 3, and 4 of [1].

Lemma 3. *Let r_i, $i = 1, 2, \ldots, m$, be as in Theorem* 1 *and ν be a given integer satisfying $0 < \nu < W := \sum_{i=1}^m r_i$. Let $\Sigma := (\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m)$ be a solution to* IP(ν). *Then*

(a) *$\Sigma' := (r_1 - \hat{\sigma}_1, r_2 - \hat{\sigma}_2, \ldots, r_m - \hat{\sigma}_m)$ is a solution to* IP(ν') *where $\nu' = W - \nu$. (Thus it is sufficient to treat* IP(ν) *for $0 < \nu \le \lfloor W/2 \rfloor$.)*

(b) *With the change of variables*

$$\xi_i := r_i/2 - \sigma_i, \quad 1 \le i \le m; \quad \eta := W/2 - \nu,$$

the problem IP(ν) *is equivalent to*

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^m \xi_i^2 \quad \text{over } (\xi_1, \xi_2, \ldots, \xi_m),\ \xi_i \text{ integers or half-integers,} \\ &\text{subject to} \quad \sum_{i=1}^m \xi_i = \eta \quad \text{and } -r_i/2 \le \xi_i \le r_i/2,\ 1 \le i \le m. \end{aligned} \tag{20}$$

Let $\Xi := (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_m)$ be a solution to (20).

(i) *If $r_m \le \cdots \le r_{s+1} \le 2\eta/m < r_s \le \cdots \le r_1$, then for the components of Ξ we have $\hat{\xi}_i = r_i/2$, for $s < i \le m$. Furthermore, we can complete the solution by solving the reduced problem:*

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^s \xi_i^2 \quad \text{over } (\xi_1, \xi_2, \ldots, \xi_s),\ \xi_i \text{ integers or half-integers,} \\ &\text{subject to} \quad \sum_{i=1}^s \xi_i = \eta - \frac{1}{2}\sum_{i=s+1}^m r_i =: \eta' \ \text{ and } -r_i/2 \le \xi_i \le r_i/2,\ 1 \le i \le s. \end{aligned} \tag{21}$$

(ii) *If $\eta/m < r_i/2$ for $1 \le i \le m$, then the components of Ξ must satisfy $|\hat{\xi}_k - \hat{\xi}_j| \le 1$ for all indices $1 \le k,\ j \le m$.*

(iii) *Under the condition of* (ii), *define $a := \max_{1 \le i \le m}\{\hat{\xi}_i\}$. Then $\hat{\xi}_i$ can take only the values a, $a - 1/2$, or $a - 1$, for $1 \le i \le m$. Let us denote the total number of $\hat{\xi}_i$'s that equal a, $a - 1/2$, and $a - 1$ by x, y, and z, respectively. Then a, x, y and z are unique. If $z = 0$, the solution to* IP(ν) *is unique. If $z \ne 0$,* IP(ν) *does not have a unique solution.*

Proof of Proposition 1. By Lemma 3, we need only consider $0 < \nu \le W/2$. Since

$$\begin{aligned} v_m &= \sum_{i=1}^{m-1} u_{i,m-1} + (m-1)r_m + r_m \\ &= \frac{1}{2}\sum_{i=1}^m (r_i - r_m - w_{i,m-1}) + \sum_{i=1}^m r_m \\ &= \frac{1}{2}\sum_{i=1}^m (r_i + r_m - w_{i,m-1}) > \frac{1}{2}\sum_{i=1}^m r_i = \frac{W}{2}, \end{aligned} \tag{22}$$

we need only deal with $\nu \in (0, v_m) \subset \bigcup_{l=1}^m I_l$.

First, assume that $\nu \le W/2$ is good in the sense of Definition 1. We will show that, for the $p_{\nu,i}$'s given by Definition 2, $\mathbf{p} := (p_{\nu,1}, p_{\nu,2}, \ldots, p_{\nu,m})$ is the unique solution to IP(ν).

Let l be an integer such that $1 \le l \le m$ and $\nu \in I_l \cap (0, \lfloor W/2 \rfloor]$. We now make the same change of variables as in Lemma 3(b), namely,

$$\xi_i := r_i/2 - p_{\nu,i}, \quad 1 \le i \le m, \quad \eta := W/2 - \nu.$$

Then $0 \le \eta < W/2$, and by the definition, when $q_l := \nu - v_{l-1} - lk_l = 0$,

$$\begin{aligned} \xi_i &= r_i/2 - u_{i,l-1} - k_l \\ &= r_i/2 - (r_i - r_l - w_{i,l-1})/2 - k_l \\ &= r_l/2 + w_{i,l-1}/2 - k_l, && \text{for } 1 \le i \le l, \\ \xi_i &= r_i/2, && \text{for } l < i \le m; \end{aligned} \tag{23}$$

or, when $q_l := \nu - v_{l-1} - lk_l = \sum_{i=1}^{l-1} w_{i,l-1} \ne 0$,

$$\begin{aligned} \xi_i &= r_i/2 - u_{i,l-1} - w_{i,l-1} - k_l \\ &= r_i/2 - (r_i - r_l + w_{i,l-1})/2 - k_l \\ &= r_l/2 - w_{i,l-1}/2 - k_l, && \text{for } 1 \le i \le l, \\ \xi_i &= r_i/2, && \text{for } l < i \le m. \end{aligned} \tag{24}$$

If $l < m$,

$$\begin{aligned} \frac{2\eta}{m} &= \frac{W}{m} - \frac{2\nu}{m} > \frac{W}{m} - \frac{2v_l}{m} = \frac{1}{m}\sum_{i=1}^{m} r_i - \frac{1}{m}\sum_{i=1}^{l}(r_i - r_{l+1} - w_{i,l}) \\ &= \frac{1}{m}\sum_{i=l+1}^{m} r_i + \frac{1}{m}\sum_{i=1}^{l}(r_{l+1} + w_{i,l}) \ge r_{l+1} \ge r_i \text{ for } l < i \le m, \end{aligned} \tag{25}$$

by Lemma 3(i), the $(m-l)$-tuple $\overline{\Xi} := (\xi_{l+1}, \dots, \xi_m)$ given by (23) or (24) is a part of a solution to (20); if $l = m$, $\overline{\Xi}$ does not exist and we can go directly to the next step.

The problem now is reduced to (21) with $s = l$. We need to consider two separate cases: $k_l \ne 0$ and $k_l = 0$, where $k_l := \lfloor (\nu - v_{l-1})/l \rfloor$.

When $k_l \ne 0$, we have $\nu \ge v_{l-1} + l$ and

$$\begin{aligned} \eta' &:= \frac{W}{2} - \nu - \frac{1}{2}\sum_{i=l+1}^{m} r_i \le \frac{1}{2}\sum_{i=1}^{l} r_i - v_{l-1} - l \\ &= \frac{1}{2}\sum_{i=1}^{l} r_i - \frac{1}{2}\sum_{i=1}^{l}(r_i - r_l - w_{i,l-1}) - l \\ &= \frac{1}{2}\sum_{i=1}^{l}(r_l + w_{i,l-1}) - l < lr_l/2, \end{aligned} \tag{26}$$

where the last step follows from $w_{l,l-1} = 0$ and $w_{i,l-1} \le 1$ $(1 \le i \le l-1)$. Furthermore, $|\xi_j - \xi_k| \le 1$ for $1 \le j, k \le l$ from (23) or (24). By Lemma 3(ii), we know that $(\xi_1, \dots, \xi_l)$ is a solution to (21). Noticing that only one of (23) and (24) is valid for a given ν, we see that $z = 0$ in the sense defined in Lemma 3(iii). Thus $(\xi_1, \dots, \xi_l)$ is the unique solution to (21); consequently, Ξ is the unique solution to (20).

When $k_l = 0$, we cannot guarantee the strict inequality in (26). If $\eta' < lr_l/2$ is true, then the above reasoning and conclusion still hold. So suppose that $\eta' \ge lr_l/2$. This situation requires careful analysis. We define t and t' to be two integers such that $l > t' \geq t \geq 0$ and

$$r_t \geq r_l + 2, \quad r_{t+1} = r_{t+2} = \cdots = r_{t'} = r_l + 1, \quad r_{t'+1} = r_{t'+2} = \cdots = r_l \tag{27}$$

(where the case $t' = t > 0$ means that $r_i \neq r_l + 1$ for all i; the case $t' > t = 0$ means $r_1 = \cdots = r_{t'} = r_l + 1$, $r_{t'+1} = \cdots = r_l$; the case $t' = t = 0$ means $r_1 = \cdots = r_l$). Then

$$w_{t+1,l-1} = w_{t+2,l-1} = \cdots = w_{t',l-1} = 1,$$
$$w_{t'+1,l-1} = w_{t'+2,l-1} = \cdots = w_{l,l-1} = 0.$$

By (23) or (24),

$$\xi_{t'+1} = \xi_{t'+2} = \cdots = \xi_l = r_l/2 = r_i/2, \quad \text{for } t'+1 \leq i \leq l.$$

Again by Lemma 3(i), the $(l-t')$-tuple $(\xi_{t'+1}, \ldots, \xi_l)$ is a part of the solution and the original problem is reduced to (21) with $s = t'$. If $t' = 0$, $(\xi_1, \ldots, \xi_m)$ is the unique solution to (21), so we can stop here. Now, suppose $t' > 0$. Because $\nu \geq v_{l-1}$, we have

$$\begin{aligned}
\eta' &= \frac{W}{2} - \nu - \frac{1}{2}\sum_{i=t'+1}^{m} r_i \leq \frac{1}{2}\sum_{i=1}^{t'} r_i - v_{l-1} \\
&= \frac{1}{2}\sum_{i=1}^{t'} r_i - \frac{1}{2}\sum_{i=1}^{l}(r_i - r_l - w_{i,l-1}) = -\frac{1}{2}\sum_{i=t'+1}^{l} r_i + \frac{1}{2}\sum_{i=1}^{l}(r_l + w_{i,l-1}) \\
&= \frac{1}{2}\sum_{i=1}^{t'}(r_l + w_{i,l-1}) \leq t'\frac{r_l+1}{2} = t'\frac{r_{t'}}{2}.
\end{aligned}$$

If $\eta' < t'r_{t'}/2$, using t' instead of l, by the same reasoning as before, we arrive at Ξ being the unique solution to (20).

Suppose $\eta' = t'r_{t'}/2$. If $q_l = s_l \neq 0$, $\nu = v_{l-1} + s_l$, then

$$\begin{aligned}
\eta' &= \frac{W}{2} - \nu - \frac{1}{2}\sum_{i=t'+1}^{m} r_i \\
&= \frac{1}{2}\sum_{i=1}^{t'} r_i - \frac{1}{2}\sum_{i=1}^{l}(r_i - r_l - w_{i,l-1}) - s_l = -\frac{1}{2}\sum_{i=t'+1}^{l} r_i + \frac{1}{2}lr_l - \frac{1}{2}s_l \\
&= -\frac{1}{2}(l-t')r_l + \frac{1}{2}l\,r_l - \frac{1}{2}s_l = \frac{1}{2}t'r_{t'} - \frac{1}{2}s_l \neq \frac{1}{2}t'r_{t'},
\end{aligned}$$

which is a contradiction. Thus, in this case, $q_l = 0$. Furthermore, the ξ_i's are determined by (23), so we have

$$\xi_{t+1} = \xi_{t+2} = \cdots = \xi_{t'} = r_l/2 + 1/2 = r_i/2, \qquad \text{for } t+1 \leq i \leq t'. \tag{28}$$

Again by Lemma 3(i), the $(t'-t)$-tuple $(\xi_{t+1}, \ldots, \xi_{t'})$ is a part of the solution and the original problem is reduced to (21) with $s = t$. If $t = 0$, we are done. Now, assume $t > 0$. Because $\nu \geq v_{l-1}$, we come to a strict inequality:

$$\begin{aligned}
\eta' &= \frac{W}{2} - \nu - \frac{1}{2}\sum_{i=t+1}^{m} r_i \leq \frac{1}{2}\sum_{i=1}^{t} r_i - v_{l-1} \\
&= \frac{1}{2}\sum_{i=1}^{t}(r_l + w_{i,l-1}) \leq t\frac{r_l+1}{2} < t\frac{r_i}{2}
\end{aligned}$$

for $1 \le i \le t$. Using t instead of l, by the same reasoning as before, we arrive at Ξ being the unique solution to (20).

Conversely, assume that, for a given ν, IP(ν) has the unique solution $\Sigma := (\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m)$ (so (20) has a unique solution). We shall show that ν is good and $\hat{\sigma}_i = p_{\nu,i}$, $i = 1, 2, \ldots, m$.

Let the solution of (20) be $\Xi := (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_m)$ and set $\eta := \sum_{i=1}^m \hat{\xi}_i \equiv W/2 - \nu$. Because we only consider $\nu \in (0, W/2]$, it follows from (22) that we can find an l such that $1 \le l \le m$ and $\nu \in I_l$. When $\nu \in I_l$, (25) still holds. So $\hat{\xi}_i = r_i/2$ for $l < i \le m$ by Lemma 3 (i). Let $\eta' := \sum_{i=1}^l \hat{\xi}_i \equiv W/2 - \nu - \sum_{i=l+1}^m r_i/2 = \sum_{i=1}^l r_i/2 - \nu$.

To prove our statement, we need to examine two cases.

Case A. If $\nu - v_{l-1} \ge l$, then (26) is still true. This means $\eta'/l < r_l/2$. Hence by (ii) and (iii) of Lemma 3, $\hat{\xi}_i$ can only take the values $a := \max_{1 \le i \le l}\{\hat{\xi}_i\}$ or $a - \frac{1}{2}$ for $1 \le i \le l$. We need only to determine a. Since

$$\eta' = \sum_{i=1}^{l} \hat{\xi}_i = xa + y(a - \frac{1}{2}) = la - \frac{y}{2}$$

and $l \ge x \ge 1$, $l > y \ge 0$, we get

$$\frac{\eta'}{l} \le a < \frac{\eta'}{l} + \frac{1}{2}. \tag{29}$$

Let $k_l := \lfloor (\nu - v_{l-1})/l \rfloor$. Then

$$\frac{r_l}{2} - \frac{(\nu - v_{l-1})}{l} \le \frac{r_l}{2} - k_l < \frac{r_l}{2} - \frac{(\nu - v_{l-1})}{l} + 1.$$

Furthermore, since

$$\begin{aligned}
\frac{r_l}{2} - \frac{(\nu - v_{l-1})}{l} &= \frac{r_l}{2} - \frac{\left(\nu - \sum_{i=1}^l (r_i - r_l - w_{i,l-1})/2\right)}{l} \\
&= \sum_{i=1}^{l} \frac{(r_i - w_{i,l-1})}{2l} - \frac{\nu}{l} = \frac{1}{l}\Big(\frac{1}{2}\sum_{i=1}^{l} r_i - \nu\Big) - \frac{1}{2l} s_l \\
&= \frac{\eta'}{l} - \frac{s_l}{2l},
\end{aligned}$$

we get

$$\frac{\eta'}{l} - \frac{1}{2} \le \frac{r_l}{2} - \frac{\nu - v_{l-1}}{l} \le \frac{\eta'}{l}.$$

Thus

$$\frac{\eta'}{l} - \frac{1}{2} \le \frac{r_l}{2} - k_l < \frac{\eta'}{l} + 1.$$

Combining with (29), we see that

$$-1 < a - \big(\frac{r_l}{2} - k_l\big) < 1. \tag{30}$$

Recall that a is an integer or half-integer and the same is true for $r_l/2 - k_l$. So there are three situations:

$$a = \frac{r_l}{2} - k_l + \frac{1}{2}, \quad a = \frac{r_l}{2} - k_l, \quad a = \frac{r_l}{2} - k_l - \frac{1}{2}.$$

We need to consider the original solution $\Sigma = (\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m)$, $\hat{\sigma}_i = (r_i/2) - \hat{\xi}_i$ for $i = 1, 2, \ldots, m$ in order to determine a.

For the case $a = (r_l/2) - k_l$, $a - 1/2 = (r_l/2) - k_l - 1/2$, since the $\hat{\sigma}_i$'s are integers, they must have the following forms:

$$\begin{aligned}
\hat{\sigma}_i &= \frac{r_i}{2} - \frac{r_l}{2} + k_l = u_{i,l-1} + w_{i,l-1} + k_l && \text{if } r_i - r_l \text{ is even, } 1 \le i \le l; \\
\hat{\sigma}_i &= \frac{r_i}{2} - \frac{r_l}{2} + k_l + \frac{1}{2} = u_{i,l-1} + w_{i,l-1} + k_l && \text{if } r_i - r_l \text{ is odd, } 1 \le i \le l; \\
\hat{\sigma}_i &= 0 && \text{for } l < i \le m;
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{m} \hat{\sigma}_i &= \sum_{i=1}^{l} (u_{i,l-1} + w_{i,l-1} + k_l) \\
&= v_{l-1} + \sum_{i=1}^{l} w_{i,l-1} + lk_l = \nu + s_l - q_l,
\end{aligned}$$

where $q_l := \nu - v_{l-1} - lk_l$.

For the case $a = (r_1/2) - k_l + 1/2$, $a - 1/2 = (r_1/2) - k_l$, similarly, we have

$$\begin{aligned}
\hat{\sigma}_i &= \frac{r_i}{2} - \frac{r_l}{2} + k_l = u_{i,l-1} + k_l && \text{if } r_i - r_l \text{ is even, } 1 \le i \le l; \\
\hat{\sigma}_i &= \frac{r_i}{2} - \frac{r_l}{2} + k_l - \frac{1}{2} = u_{i,l-1} + k_l && \text{if } r_i - r_l \text{ is odd, } 1 \le i \le l; \\
\hat{\sigma}_i &= 0 && \text{for } l < i \le m;
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{l} \hat{\sigma}_i &= \sum_{i=1}^{l} (u_{i,l-1} + k_l) = v_{l-1} + lk_l \\
&= v_{l-1} + (\nu - v_{l-1} - q_l) = \nu - q_l.
\end{aligned}$$

For the case $a = (r_1/2) - k_l - 1/2$, $a - 1/2 = (r_1/2) - k_l - 1$,

$$\begin{aligned}
\hat{\sigma}_i &= \frac{r_i}{2} - \frac{r_l}{2} + k_l + 1 = u_{i,l-1} + k_l + 1 && \text{if } r_i - r_l \text{ is even, } 1 \le i \le l; \\
\hat{\sigma}_i &= \frac{r_i}{2} - \frac{r_l}{2} + k_l + \frac{1}{2} \\
&= u_{i,l-1} + k_l + 1 && \text{if } r_i - r_l \text{ is odd, } 1 \le i \le l; \\
\hat{\sigma}_i &= 0 && \text{for } l < i \le m;
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{m} \hat{\sigma}_i &= \sum_{i=1}^{l} (u_{i,l-1} + 1 + k_l) \\
&= v_{l-1} + l + lk_l > v_{l-1} + l + (\nu - v_{l-1} - l) = \nu.
\end{aligned}$$

This contradicts the fact that $\sum_{i=1}^{m} \hat{\sigma}_i = \nu$, so $a \ne (r_1/2) - k_l - 1/2$.

Since IP(ν) does have a unique solution, a must be one of $(r_1/2) - k_l$ or $(r_1/2) - k_l + 1/2$. If both a and $(r_1/2) - k_l$ are integers or half-integers, then $a = (r_1/2) - k_l$; if only one of them is an integer, then $a = (r_1/2) - k_l + 1/2$. Additionally, if $a = (r_1/2) - k_l$, then $\nu = \sum_{i=1}^{m} \hat{\sigma}_i = \nu - q_l + s_l$, and we conclude that $q_l = s_l$. In the other case, $a = (r_1/2) - k_l + 1/2$, so we have $\nu = \sum_{i=1}^{m} \hat{\sigma}_i = \nu - q_l$ and conclude that $q_l = 0$. Thus, by Definitions 1 and 2, ν is good in either, case and $\hat{\sigma}_i \equiv p_{\nu,i}$.

Case B. If $0 \le \nu - v_{l-1} < l$, then $k_l = 0$ and

$$\begin{aligned}\eta' &= \frac{1}{2}\sum_{i=1}^{l} r_i - \nu > \frac{1}{2}\sum_{i=1}^{l} r_i - v_{l-1} - l = \frac{lr_l}{2} + \frac{1}{2}\sum_{i=1}^{l-1} w_{i,l-1} - l = l(\frac{r_l}{2} - 1),\\ \eta' &\le \frac{1}{2}\sum_{i=1}^{l} r_i - v_{l-1} = \frac{lr_l}{2} + \frac{1}{2}\sum_{i=1}^{l-1} w_{i,l-1} = \frac{lr_l}{2} + \frac{s_l}{2} \le \frac{l}{2}(r_l + 1).\end{aligned} \tag{31}$$

If $\eta' < lr_1/2$, then the reasoning for Case A is still valid and we obtain the desired result.

If $\eta' \ge lr_1/2$, let t and t' be defined as in (27). Then, by Lemma 3(i),

$$\hat{\xi}_{t'+1} = \hat{\xi}_{t'+2} = \cdots = \hat{\xi}_l = r_l/2 = r_i/2, \quad \text{for } t'+1 \le i \le l.$$

If $t' = 0$, the desired conclusion follows as before. If $t' \ne 0$, define

$$\eta'' = \eta' - \frac{1}{2}\sum_{i=t'+1}^{l} r_i = \eta' - (l - t')r_l/2.$$

Then we have

$$\frac{t'r_l}{2} - l \le \eta'' \le \frac{t'r_l}{2} + \frac{s_l}{2} \le \frac{t'}{2} r_{t'},$$

where we have used the fact that $s_l = \sum_{i=1}^{l} w_{i,l-1} = \sum_{i=1}^{t'} w_{i,l-1} \le t'/2$.

If $\eta'' < t'r_{t'}/2$, then set $a := \max_{1 \le i \le t'}\{\hat{\xi}_i\}$. Now if we use t' instead of l in Case A, then the reasoning there is still valid except that (30) becomes

$$-\frac{l}{t'} < a - \frac{r_l}{2} < 1\,.$$

Hence the last situation for determining a is $a \le r_1/2 - 1/2$ instead of $a = r_1/2 - 1/2$. In this situation, we will still get $\sum_{i=1}^{m} \hat{\sigma}_i > \nu$, which is a contradiction. Therefore, we arrive at the same conclusion.

If $\eta'' = t'r_{t'}/2$, then by Lemma 3(i) again,

$$\hat{\xi}_{t+1} = \hat{\xi}_{t+2} = \cdots = \hat{\xi}_{t'} = r_{t'}/2 = (r_l + 1)/2 = r_i/2, \quad \text{for } t+1 \le i \le l.$$

We change t' to t in the above reasoning and still get the same conclusion.

Since IP(ν) does have a unique solution, we have checked all cases and verified that $\nu = v_{l-1}$, or $\nu = v_{l-1} + s_l$; so ν is good and $\hat{\sigma}_i \equiv p_{\nu,i}$ by Definitions 1 and 2. □

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