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LITTLEWOOD-PALEY DECOMPOSITION AND NAVIER-STOKES EQUATIONS

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ABSTRACT. Using the dyadic decomposition of Littlewood-Paley, we find a simple condition that, when tested on an abstract Banach space X, guarantees the existence and uniqueness of a local strong solution $v(t) \in \mathcal{C}([0,T);X)$ of the Cauchy problem for the Navier-Stokes equations in $\mathbb{R}^3$. Many examples of such Banach spaces are offered. We also prove some regularity results on the solution $v(t)$ and we illustrate, by means of a counterexample, that the above-mentioned sufficient condition is, in general, not necessary.

1. Introduction

In this paper, we prove some existence and uniqueness results for local strong solutions of the Cauchy problem for the Navier-Stokes equations in $\mathbb{R}^3$. We are primarily interested in the strong solutions $v(t)$ belonging to $\mathcal{C}([0,T);X)$, where X denotes an abstract Banach space of vector distributions on $\mathbb{R}^3$.

Several authors have addressed this problem. T. Kato [5] first presented a detailed $H^s(\mathbb{R}^3)$ theory and was able to prove such a result if $s > 5/2$. However, his method, which makes special use of some commutator and energy estimates, does not apply to the physical case $H^1(\mathbb{R}^3)$. Later, Kato and Ponce [8] generalized the same arguments in the case of $L^p_s(\mathbb{R}^3)$ spaces with $1 < p < \infty$ and $s > 1 + 3/p$.

More recently, considerable effort has been devoted to the analysis of Morrey-Campanato spaces $M^p_q(\mathbb{R}^3)$ [3, 4, 7, 11]. In particular, Federbush [2, 3] has obtained such an existence and uniqueness theorem in the $M^p_2(\mathbb{R}^3)$ setting when $p > 3$. His clever proof is essentially based upon a bilinear operator estimate which is expressed in a simple quadratic form using a wavelet frame. The paper of Taylor [11] deals with the more general case $M^p_q(\mathbb{R}^3)$, including the limiting case $p = 3$, and yields extensions of the results obtained in both [3] and [4]. His proof analyzes some general semigroup estimates in the Morrey-Campanato spaces, but does not deal with the *infrared* problem, which is, on the other hand, treated in [3].

All of the above-mentioned methods [3–5, 7, 8, 11] are rather specific and adapted to each different space. Therefore, it appears difficult to utilize them for more general applications. It remains desirable to have a general argument which can be used for an arbitrary Banach space X. This is exactly the purpose of the present paper, which has been partially inspired by the wavelet approach of Federbush, although a systematic use of the Littlewood-Paley decomposition in dyadic blocks Δ_j will be exploited.

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More generally, if X is a Banach space of vector distributions on $\mathbb{R}^3$ whose norm is translation invariant and if there is a sequence of reals $\eta_j > 0$, $j \in \mathbb{Z}$, such that

$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\eta_j < \infty \tag{1.1}$$

and that

$$\|\Delta_j(fg)\|_X \le \eta_j \|f\|_X \|g\|_X \tag{1.2}$$

for any scalar distributions f and g belonging to X, then we will show that for any initial data $v_0 \in X$ with $\nabla \cdot v_0 = 0$, there exists $T = T(\|v_0\|_X) > 0$ and a unique strong solution $v(t,x) \in \mathcal{C}\big([0,T);X\big)$ to the Navier-Stokes equations:

$$\begin{aligned} &\frac{\partial v}{\partial t} - \Delta v = -(v\cdot\nabla)v - \nabla p, \\ &\nabla \cdot v = 0, \qquad v(0) = v_0, \end{aligned} \tag{1.3}$$

where p is the usual pressure field that is automatically determined from v via (1.3) to within an inessential additive function of time.

The plan of the paper is the following. Section 2 contains the basic definitions and the proof of the main theorem. Section 3 presents several examples of Banach spaces X for which (1.1) and (1.2) are satisfied and allows us to recover many previous known results on Navier-Stokes. Finally, Section 4 is devoted to a counterexample which shows that (1.1) is, in general, not necessary to obtain an existence and uniqueness theorem for the Navier-Stokes equations and to offer concluding observations. For further applications and more general details on the subject treated below, we refer the reader to [2], where a systematic study of the existence and uniqueness of strong mild solutions to the Cauchy problem for the Navier-Stokes is performed.

2. The Navier-Stokes equations

2.1. The main theorem. We study the Cauchy problem for the Navier-Stokes equations (1.3) governing the time evolution of an incompressible fluid filling all of $\mathbb{R}^3$. We will focus our attention on the existence of solutions to (1.3) in $\mathcal{C}\big([0,T);X\big)$, the space of continuous functions of $t \in [0,T)$ with values in some Banach space X of vector distributions on $\mathbb{R}^3$.

Before stating the main hypotheses concerning X, let us recall some definitions.

Definition 2.1 (The operator $\mathbb{P}$). We let $\partial_j = -i\partial/\partial x_j$, $(i^2 = -1)$, and we indicate the Riesz transformation by $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$, for $j = 1, 2$, and 3.

For an arbitrary vector field $v(x) = (v_1(x), v_2(x), v_3(x))$ on $\mathbb{R}^3$, we set

$$z(x) = \sum_{j=1}^{3} (R_j v_j)(x), \tag{2.1}$$

and finally, we define the operator $\mathbb{P}$ by

$$(\mathbb{P}v)_k(x) = v_k(x) - (R_k z)(x), \qquad k = 1, 2, 3, \tag{2.2}$$

where $\mathbb{P}$ is a pseudo-differential operator of degree zero and is an orthogonal projection onto the kernel of the divergence operator.

Definition 2.2 (The Littlewood-Paley decomposition). Let us choose a real rotationally invariant function φ in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ whose Fourier transform is such that

$$0 \le \hat{\varphi}(\xi) \le 1, \quad \hat{\varphi}(\xi) = 1 \text{ if } |\xi| \le \frac{3}{4}, \quad \hat{\varphi}(\xi) = 0 \text{ if } |\xi| \ge \frac{3}{2}, \tag{2.3}$$

and let

$$\psi(x) = 8\varphi(2x) - \varphi(x), \tag{2.4}$$

$$\varphi_j(x) = 2^{3j}\varphi(2^j x), \ j \in \mathbb{Z}, \tag{2.5}$$

$$\psi_j(x) = 2^{3j}\psi(2^j x), \ j \in \mathbb{Z}. \tag{2.6}$$

We denote by S_j and Δ_j, respectively, the convolution operators with φ_j and ψ_j and, finally, the set $\{S_j, \Delta_j\}_{j\in\mathbb{Z}}$ (actually *a set*) is the Littlewood-Paley decomposition of unity, say,

$$I = S_0 + \sum_{j\ge 0} \Delta_j. \tag{2.7}$$

Definition 2.3 (Well-suited Banach spaces). We will say that a Banach space X of vector distributions on $\mathbb{R}^3$ is *well-suited* for the study of the Navier-Stokes equations (1.3) if the following conditions are satisfied:

(i) the X-norm is translation invariant, *i.e.*,

$$\|v(\cdot + k)\| = \|v(\cdot)\| \quad \text{for all } k \in \mathbb{R}^3, \tag{2.8}$$

(ii) there exists a sequence of reals $\eta_j > 0$, $j \in \mathbb{Z}$, such that

$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\eta_j < \infty \tag{2.9}$$

and that

$$\|\Delta_j(fg)\| \le \eta_j \|f\| \, \|g\| \tag{2.10}$$

for any scalar distributions f and g in X.

For every such space X, we also define the Besov-type space $\dot{B}_X^{0,1}(\mathbb{R}^3)$ by

$$\dot{B}_X^{0,1}(\mathbb{R}^3) = \Big\{ f \in X : \sum_{j\in\mathbb{Z}} \|\Delta_j(f)\| < \infty \Big\}. \tag{2.11}$$

We now can state the main theorem of the paper.

Theorem 2.1. *If X is a well-suited Banach space, then for any initial data $v_0 \in X$, $\nabla \cdot v_0 = 0$, there exists $T = T(\|v_0\|) > 0$ and a unique strong solution $v(t,x) \in \mathcal{C}([0,T);X)$ of the Navier-Stokes equations in the following "mild" form (where $S(t) = \exp(t\Delta)$ denotes the heat-semigroup)*

$$v(t) = S(t)v_0 - \int_0^t \mathbb{P}S(t-s)\nabla\cdot(v\otimes v)(s)ds. \tag{2.12}$$

Moreover, the fluctuation function w, defined by

$$w(t,x) =: v(t,x) - S(t)v_0(x), \tag{2.13}$$

belongs to $\dot{B}_X^{0,1}(\mathbb{R}^3)$ for $0 \le t < T$.

This last sentence signifies that $w(t)$ has a fast convergent development on a wavelet series, whereas the same is not necessarily true for $v(t)$ nor $S(t)v_0$. This argument could explain the intuitive choice of Federbush [3] to expand the velocity field $v(t)-v_0 = w(t)+S(t)v_0-v_0$ in terms of a divergence free wavelet basis. In fact, this amounts to a single condition to be verified on the initial data, say on $S(t)v_0-v_0$. Before passing to the proof Theorem 2.1, let us link (2.12) to the Navier-Stokes equations (1.3). We will see that

$$\int_0^4 \|\mathbb{P}S(4-s)\nabla\cdot(v\otimes v')(s)\|ds \le C\Big(\sum_{j\in\mathbb{Z}} 2^{-|j|}\eta_j\Big) \sup_{0\le s\le 4}\|v(s)\| \sup_{0\le s\le 4}\|v'(s)\| \tag{2.14}$$

for any pair of vector fields v and v' in X. This enables us to use a standard contraction mapping argument to solve (2.12). Finally a "mild " solution of the Navier-Stokes equation (2.12) is a "classical" solution of (1.3), and the reader is referred to [2] for a detailed discussion on the equivalence of these two problems.

Let us now concentrate our attention on the proof of Theorem 2.1.

2.2. The proof of the theorem. The proof of Theorem 2.1 is essentially based on the lemma:

Lemma 2.1. *If X is a well-suited Banach space, then there exists a function $\pi(t)\ge 0$ belonging to $L^1[0,4]$ such that, for any $t\in(0,T)$ and any pair of vector fields v and v' in X*

$$\|\mathbb{P}S(t)\nabla\cdot(v\otimes v')\| \le \pi(t)\|v\|\,\|v'\|. \tag{2.15}$$

In order to prove the lemma, let us first remark that we can limit ourselves, without loss of generality, to its scalar version, say, the estimate

$$\|\Lambda S(t)(fg)\| \le \pi(t)\|f\|\,\|g\|, \tag{2.16}$$

where $\Lambda = (-\Delta)^{-\frac{1}{2}}$ is the *scalar* Calderón operator and f and g are two arbitrary *scalar* functions in X. In fact, this is true if we recall that the *vector* operator $\mathbb{P}\nabla\cdot$ is a pseudo-differential operator of degree 1 whose symbol is given by the following coefficients:

$$c_j^{kl}(t) = \frac{\xi_j\xi_k\xi_l}{|\xi|^2}. \tag{2.17}$$

The proof of the lemma is based on the estimate

$$\pi(t) \le c\sum_{j\le m} 2^j\eta_j + c\sum_{j\ge m+1} 2^{-2j+3m}\eta_j, \tag{2.18}$$

if $t\in J_m := [4^{-m},4^{-m+1})$. The constants η_j are defined by (2.9) and (2.10) for any well-suited Banach space X.

Assuming (2.18) holds true, we find

$$\begin{aligned}\int_0^4 \pi(t)dt &\le 3\sum_{m=0}^{\infty}\Big[\sup_{t\in J_m}\pi(t)\Big]4^{-m}\\ &\le c\sum_{m=0}^{\infty}\sum_{j\le m} 2^{j-2m}\eta_j + c\sum_{m=0}^{\infty}\sum_{j\ge m+1} 2^{-2j+m}\eta_j\\ &\le 2c\sum_{j\in\mathbb{Z}} 2^{-|j|}\eta_j.\end{aligned} \tag{2.19}$$

Let us now return to (2.18). Keeping in mind the decomposition of unity (2.7), we arrive at the identity

$$\Lambda S(t) = \sum_{j\in\mathbb{Z}} \tilde{\Delta}_j \Lambda S(t) \Delta_j, \tag{2.20}$$

where

$$\tilde{\Delta}_j(f) = f * \tilde{\psi}_j, \tag{2.21}$$

$\tilde{\psi}_j$ being defined as ψ_j, except that $\tilde{\psi}_j$ is unity when $\frac{3}{4} \leq |\xi|$ and zero when $|\xi| \leq \frac{3}{8}$ or $|\xi| \geq 6$. All this implies that $\tilde{\Delta}_j \Delta_j = \Delta_j$. It should be noted that in (2.20) the different operators commute.

At this point, we let

$$2^j W_{j,t} = \tilde{\Delta}_j \Lambda S(t), \quad j \leq m, \tag{2.22}$$

and

$$2^{-2j+3m} W_{j,t} = \tilde{\Delta}_j \Lambda S(t), \quad j \geq m+1. \tag{2.23}$$

We observe that

$$W_{j,t}(f) = f * w_{j,t}, \tag{2.24}$$

where $w_{j,t} \in L^1(\mathbb{R}^3)$. More precisely, if $t \in J_m$, the following estimate can be deduced:

$$\|w_{j,t}\|_1 \leq c. \tag{2.25}$$

Finally, as the X-norm is translation invariant, we have, for $f \in X$ and $w \in L^1(\mathbb{R}^3)$,

$$\|f * w\|_X \leq \|w\|_1 \, \|f\|_X, \tag{2.26}$$

and the lemma follows. The proof of (2.25) is easy and left to the reader (see [2] for more details).

Let us now recall a classical lemma which enables us to conclude the proof of the existence and uniqueness part of Theorem 2.1.

Lemma 2.2. *Let X be an abstract Banach space and $B\colon X \times X \longrightarrow X$ a bilinear operator, $\| \ \|$ being the X-norm, such that for any $x_1 \in X$ and $x_2 \in X$, we have*

$$\|B(x_1, x_2)\| \leq \eta \|x_1\| \, \|x_2\|. \tag{2.27}$$

Then for any $y \in X$ such that

$$4\eta\|y\| < 1, \tag{2.28}$$

the equation

$$x = y + B(x, x) \tag{2.29}$$

has a solution x in X. Moreover, this solution x is the only one such that

$$\|x\| \leq \frac{1 - \sqrt{1 - 4\eta\|y\|}}{2\eta}. \tag{2.30}$$

2.3. Some regularity results. Here we show that the solution $v(t) \in \mathcal{C}([0,T);X)$ given by Theorem 2.1 is such that

$$v(t) - S(t)v_0 \in \dot{B}_X^{0,1}, \quad 0 \le t < T. \tag{2.31}$$

In other words, we must find a decomposition $\{d_j(t)\}_{j\in\mathbb{Z}}$ of functions in X such that

$$v(t) - S(t)v_0 = \sum_{j\in\mathbb{Z}} d_j(t), \tag{2.32}$$

$$\text{supp } \hat{d}_j \subseteq \{\, \xi \in \mathbb{R}^3 : \alpha 2^j \le |\xi| \le \beta 2^j,\ \beta > \alpha > 0 \,\}, \tag{2.33}$$

and

$$\sum_{j\in\mathbb{Z}} \|d_j(t)\| < \infty, \quad 0 \le t < T. \tag{2.34}$$

To see this, we recall (2.12) and (2.20) to obtain

$$\begin{aligned} v(t) - S(t)v_0 &= \sum_{j\in\mathbb{Z}} \int_0^t \tilde{\Delta}_j \mathbb{P} S(t-s) \Delta_j \nabla \cdot (v \otimes v)(s)ds \\ &= \sum_{j\in\mathbb{Z}} d_j(t), \end{aligned} \tag{2.35}$$

where

$$d_j(t) = \int_0^t \tilde{d}_j(t-s,s)ds. \tag{2.36}$$

The following estimates hold for $\tilde{d}_j$ when $t - s \in J_m$:

$$\|\tilde{d}_j(t-s,s)\| \le c2^j \eta_j \Big(\sup_{0\le s\le T} \|v(s)\| \Big)^2, \quad j \le m, \tag{2.37}$$

and

$$\|\tilde{d}_j(t-s,s)\| \le c2^{-2j+3m} \eta_j \Big(\sup_{0\le s\le T} \|v(s)\| \Big)^2, \quad j \ge m+1; \tag{2.38}$$

thus implying that

$$\|d_j(t)\| \le c(2^{-|j|}\eta_j) \Big(\sup_{0\le s\le T} \|v(s)\| \Big)^2. \tag{2.39}$$

The proof of Theorem 2.1 now is complete.

Before providing the reader with some examples of well-suited Banach spaces, let us comment here on the strong continuity of the solution $v(t,x)$ so far obtained.

In the previous pages, we have tacitly assumed that the heat semi-group $S(t)$ is C_0-continuous in the well-suited Banach space X, for otherwise the flow $S(t)v_0$, $v_0 \in X$, would not be an element of $\mathcal{C}([0,T);X)$. More clearly, if X is a non-separable well-suited Banach space, then $S(t)$ is not necessarily a C_0 semi-group; consequently, the solution $v(t,x)$ given by Theorem 2.1 does not necessary belong to $\mathcal{C}([0,T);X)$.

On the other hand, it is not difficult to prove (using the translation invariant property of the X-norm eq. (2.8)) that $S(t)$ is a bounded flow, say

$$\|S(t)v_0\| \le \|S(t)\|_1 \|v_0\| = \|v_0\| \tag{2.40}$$

and that $S(t)v_0$ is weakly continuous in the sense of distributions.

All this is to say that, even if the heat semi-group is not *necessarily* C_0-continuous in X, Theorem 2.1 still applies and gives a solution $v(t,x)$ which belongs *at least* to $\mathcal{C}_*([0,T);X)$ in the sense made precise by the following definition.

Definition 2.4. A vector flow $v(t,x)$ belongs $\mathcal{C}_*\big([0,T);X\big)$ if and only if $v(t,x)$ is weakly continuous in the sense of distributions and is a bounded flow in X, viz. $v(t,x) \in L^\infty([0,T);X)$.

3. Examples

This section is devoted to the study of some classical Banach spaces (Hölder, Lebesgue, Sobolev, Morrey-Campanato) that are well-suited for the Navier-Stokes equations, according to the Definition 2.3. More general examples (including, for example, Besov and Triebel-Lizorkin spaces) are treated in [2].

3.1. The Hölder spaces. The inhomogeneous Hölder spaces $C^s(\mathbb{R}^3)$, $s > 0$ are defined, in terms of the Littlewood-Paley decomposition, by the two conditions

$$\|S_0 f\|_\infty \le c, \tag{3.1}$$

$$\|\Delta_j f\|_\infty \le c 2^{-js}, \quad j \ge 0. \tag{3.2}$$

As is well known, $C^s(\mathbb{R}^3)$, $s > 0$, is a multiplicative algebra of functions, i.e.,

$$\|fg\| \le c\|f\|\,\|g\|, \tag{3.3}$$

and this easily implies that $C^s(\mathbb{R}^3)$ is a well-suited Banach space, as one can choose

$$\eta_j = \|\Delta_j\|_1 = c. \tag{3.4}$$

Concerning the regularity result (2.31), here one can prove (following the arguments given in Subsection 2.3) that

$$v(t) - S(t)v_0 \in C^{s+1}(\mathbb{R}^3), \tag{3.5}$$

whereas both v_0 and $v(t)$ belong to $C^s(\mathbb{R}^3)$ only.

As a consequence of Theorem 2.1, we thus obtain the result

Theorem 3.1. *Let X be the Hölder space $C^s(\mathbb{R}^3)$, $s > 0$. Then for any initial data $v_0 \in C^s(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$, there exists $T = T(\|v_0\|) > 0$ and a unique solution $v(t,x) \in \mathcal{C}_*\big([0,T);C^s(\mathbb{R}^3)\big)$ of the Navier-Stokes equations in the "mild" integral form. Moreover, the fluctuation function $w(t,x)$ satisfies the regularity property given by* (3.5).

3.2. The Lebesgue spaces. In this case, Young's inequality gives

$$\|\Delta_j(fg)\|_p \le \|\psi_j\|_r \, \|fg\|_{\frac{p}{2}}, \tag{3.6}$$

where, as usual,

$$\frac{1}{p} = \frac{2}{p} + \frac{1}{r} - 1. \tag{3.7}$$

Now,

$$\|\psi_j\|_r = c2^{3j/p}, \tag{3.8}$$

so that $\eta_j = c2^{3j/p}$ and $L^p(\mathbb{R}^3)$ is a well-suited Banach space as long as $p > 3$. This is, of course, a classical result (see, e.g., [11]). What we can infer here is that

$$v(t) - S(t)v_0 \in B^{s,p}_\infty(\mathbb{R}^3), \quad s = 1 - \frac{3}{p}, \tag{3.9}$$

$B^{s,p}_\infty(\mathbb{R}^3)$ being the usual Besov space (see [2] for more details).

As a consequence of Theorem 2.1, we thus obtain the result

Theorem 3.2. *Let X be the Lebesgue space $L^p(\mathbb{R}^3)$, $3 < p < \infty$. Then for any initial data $v_0 \in L^p(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$, there exists $T = T(\|v_0\|) > 0$ and a unique strong solution $v(t,x) \in \mathcal{C}([0,T); L^p(\mathbb{R}^3))$ of the Navier-Stokes equations in the "mild" integral form. Moreover, the fluctuation function $w(t,x)$ satisfies the regularity property given by* (3.9).

3.3. The Sobolev spaces. The case of the Sobolev spaces $H^s(\mathbb{R}^3)$ gives us the opportunity to show the general algorithm to be used in order to evaluate the constants η_j which appear in Definition 2.3 of well-suited Banach spaces. The above-mentioned algorithm is the well-known Bony's paraproduct decomposition [1], which reads as follows: for any two arbitrary tempered distributions f and g,

$$fg = \sum_0^\infty \Delta_n f S_{n-2} g + \sum_0^\infty \Delta_n g S_{n-2} f + \sum_{|n-n'|\le 2} \Delta_{n'} f \Delta_n g + S_0 f S_0 g. \tag{3.10}$$

From this decomposition we deduce, modulo some nondiagonal terms that we are neglecting for simplicity, for $j \ge 0$,

$$\Delta_j(fg) = \Delta_j f S_{j-2} g + \Delta_j g S_{j-2} f + \Delta_j\Big(\sum_{j \ge j} \Delta_k f \Delta_k g\Big). \tag{3.11}$$

The three contributions to η_j now are treated separately. To this end, let us first recall that the Sobolev space $H^s(\mathbb{R}^3)$ is characterized, in terms of a Littlewood-Paley decomposition, by the equivalent norm

$$\|f\| = \|S_0 f\|_2 + \Big(\sum_{j=0}^\infty 2^{2js} \|\Delta_j f\|_2^2\Big)^{\frac{1}{2}}. \tag{3.12}$$

Now, if $j \ge 0$ and $0 \le s < \frac{3}{2}$, we find, by using Bernstein's inequalities,

$$\begin{aligned} \|S_{j-2} f \Delta_j g\|_2 &\le \|S_{j-2} f\|_\infty \|\Delta_j g\|_2 \\ &\le c 2^{-js} 2^{3j/2} \|f\| 2^{-js} \|g\| \end{aligned} \tag{3.13}$$

and, if $j \ge 0$ and $s = \frac{3}{2}$, the same argument shows that

$$\|S_{j-2} f \Delta_j g\|_2 \le cj \|f\| 2^{-js} \|g\|. \tag{3.14}$$

On the other hand, if $j \ge 0$ and $s \ge 0$, Young's inequality gives

$$\begin{aligned} \Big\|\Delta_j\Big(\sum_{k \ge j} \Delta_k f \Delta_j g\Big)\Big\|_2 &\le c 2^{3j/2} \Big\|\sum_{k \ge j} \Delta_k f \Delta_k g\Big\|_1 \\ &\le c 2^{3j/2} \sum_{k \ge j} \|\Delta_k f\|_2 \|\Delta_k g\|_2 \\ &\le c 2^{3j/2} 2^{-js} 2^{-js} \|f\| \, \|g\|. \end{aligned} \tag{3.15}$$

Finally, if $s > \frac{3}{2}$, it is well known that $H^s(\mathbb{R}^3)$ is a multiplicative algebra, i.e.,

$$\|\Delta_j(fg)\| \le c \|f\| \, \|g\|. \tag{3.16}$$

All this implies that $H^s(\mathbb{R}^3)$ is a well-suited Banach space if $s > \frac{1}{2}$. Moreover, it also is easy to show the following regularity properties on the fluctuation. More precisely, if $\frac{1}{2} < s \le \frac{3}{2}$, then for any $\epsilon > 0$ and any $0 \le t < T$,

$$v(t) - S(t) v_0 \in H^{2s - \frac{1}{2} - \epsilon}(\mathbb{R}^3) \tag{3.17}$$

and, if $s > \frac{3}{2}$, then for any $\epsilon > 0$ and any $0 \le t < T$,

$$v(t) - S(t)v_0 \in H^{s+1-\epsilon}(\mathbb{R}^3). \tag{3.18}$$

As a consequence of Theorem 2.1, we thus obtain the result

Theorem 3.3. *Let X be the Sobolev space $H^s(\mathbb{R}^3)$, $s > \frac{1}{2}$. Then for any initial data $v_0 \in H^s(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$, there exists $T = T(\|v_0\|) > 0$ and a unique strong solution $v(t,x) \in \mathcal{C}\big([0,T); H^s(\mathbb{R}^3)\big)$ of the Navier-Stokes equations in the "mild" integral form. Moreover, the fluctuation function $w(t,x)$ satisfies the regularity property given by* (3.17)–(3.18).

3.4. The Morrey-Campanato spaces. Given $f \in L^q_{loc}(\mathbb{R}^3)$, we say that f belongs to the Morrey-Campanato space $M^p_q(\mathbb{R}^3)$, $1 \le q \le p < \infty$, provided that

$$\|f\| = \sup_{\substack{x\in\mathbb{R}^3\\ 0<R\le 1}} R^{3/p}\Big(R^{-3}\int_{B_R}|f|^q\Big)^{\frac{1}{q}} < \infty, \tag{3.19}$$

where $B_R = B(x,R)$ is an arbitrary ball of radius R centered at x. In the case $q = 2$, the Morrey-Campanato space is characterized, in terms of the Littlewood-Paley decomposition, by the two conditions [2]

$$\|S_0 f\|_\infty \le c \tag{3.20}$$

and

$$\int_{Q_m}\Big(\sum_{j\ge m}|\Delta_j f|^2\Big) \le c2^{-3m(1-2/p)}, \tag{3.21}$$

$Q_m = Q(x,m)$ being an arbitrary cube with sides of length 2^{-m}, $m \ge 0$, centered at x.

Using this equivalence, it is quite easy to prove that $M^p_2(\mathbb{R}^3)$ is a well-suited Banach space when $p > 3$, which is in agreement with the results obtained in [3] and [11]. The extension to the general case $M^p_q(\mathbb{R}^3)$ is possible, but not pursued here.

In order to prove the result for $q = 2$, let us recall the paraproduct decomposition (3.11); this allows us to treat the three contributions to η_m separately.

We first use a standard Bernstein's argument to infer from (3.21) that

$$\|S_j f\|_\infty \le 2^{3j/p}\|f\|, \tag{3.22}$$

so that for any cube Q_m with $m \le j$,

$$\begin{aligned}\int_{Q_m}|S_{j-2}f\Delta_j g|^2 &\le \|f\|^2 2^{6j/p}\int_{Q_m}|\Delta_j g|^2\\ &\le \|f\|^2\|g\|^2 2^{6j/p}\,2^{-3m(1-2/p)},\end{aligned} \tag{3.23}$$

which finally gives

$$\eta_j = c2^{3j/p}, \qquad j \ge 0, \tag{3.24}$$

for the first and second terms in (3.11). Let us now examine $\Delta_j\big(\sum_{k\ge j}\Delta_k f\Delta_k g\big)$. Here, we make special use of the fact that $q = 2$ to deduce the estimate (see also [11, (3.83)]):

$$\int_{Q_m}\left|\Delta_j\Big(\sum_{k\geq j}\Delta_k f\Delta_k g\Big)\right|^2 \leq \left\|\Delta_j\Big(\sum_{k\geq j}\Delta_k f\Delta_k g\Big)\right\|_\infty \int_{Q_m}\left|\Delta_j\Big(\sum_{k\geq j}\Delta_k f\Delta_k g\Big)\right|. \tag{3.25}$$

Next, for an arbitrary cube R_j of length 2^{-j}, we have

$$\begin{aligned}\int_{R_j}\left|\sum_{k\geq j}\Delta_k f\Delta_k g\right| &\leq \int_{R_j}\Big(\sum_{k\geq j}|\Delta_k f|^2\Big)^{\frac12}\Big(\sum_{k\geq j}|\Delta_k g|^2\Big)^{\frac12}\\ &\leq \|f\|\ \|g\|2^{-3j(1-2/p)},\end{aligned} \tag{3.26}$$

which gives

$$\left\|\Delta_j\Big(\sum_{k\geq j}\Delta_k f\Delta_k g\Big)\right\|_\infty \leq \|f\|\ \|g\|2^{6j/p}. \tag{3.27}$$

Finally, we observe that if $m \leq j$, then

$$\int_{Q_m}|\Delta_j f| \leq c\sup_{Q'_m}\int_{Q'_m}|f|, \tag{3.28}$$

but

$$\begin{aligned}\int_{Q'_m}\left|\sum_{k\geq j}\Delta_k f\Delta_k g\right| &\leq \Big(\int_{Q'_m}\sum_{k\geq j}|\Delta_k f|^2\Big)^{\frac12}\Big(\int_{Q'_m}\sum_{k\geq j}|\Delta_k g|^2\Big)^{\frac12}\\ &\leq \|f\|\ \|g\|2^{-3j(1-2/p)},\end{aligned} \tag{3.29}$$

thus giving the same value of

$$\eta_j = c2^{3j/p} \tag{3.30}$$

for the third term on the left-hand side of (3.11). The estimate of η_j for $j \leq 0$ is trivial and leads to $\eta_j = c$ (see [2]); this allows us to deduce that

$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\eta_j < \infty \tag{3.31}$$

as long as $p > 3$.

In this case, Theorem 2.1 applies and gives a solution $v(t) \in \mathcal{C}_*\big([0,T);M_2^p(\mathbb{R}^3)\big)$ of the Navier-Stokes equations. Moreover, this solution is such that if $3 < p < 6$, then

$$v(t) - S(t)v_0 \in \mathcal{C}_*\big([0,T);M_2^{3p/(6-p)}(\mathbb{R}^3)\big), \tag{3.32}$$

if $p = 6$, then

$$v(t) - S(t)v_0 \in \mathcal{C}_*\big([0,T);BMO(\mathbb{R}^3)\big), \tag{3.33}$$

and if $p > 6$, then

$$v(t) - S(t)v_0 \in \mathcal{C}_*\big([0,T);C^{1-6/p}(\mathbb{R}^3)\big). \tag{3.34}$$

As a consequence of Theorem 2.1, we thus obtain the result

Theorem 3.4. *Let X be the Morrey-Campanato space $M_2^p(\mathbb{R}^3)$, $p > 3$. Then for any initial data $v_0 \in M_2^p(\mathbb{R}^3)$, $\nabla \cdot v_0 = 0$, there exists $T = T(\|v_0\|) > 0$ and a unique solution $v(t,x) \in C_*\big([0,T); M_2^p(\mathbb{R}^3)\big)$ of the Navier-Stokes equations in the "mild" integral form. Moreover, the fluctuation function $w(t,x)$ satisfies the regularity property given by* (3.32)–(3.34).

4. A counterexample

Here we exhibit a Banach space X such that

$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\eta_j = \infty, \tag{4.1}$$

η_j being the best constant appearing in (2.10) for which an existence and uniqueness theorem for the Navier-Stokes equations in $C\big([0,T);X\big)$ can be deduced.

The space X is defined, in the Littlewood-Paley framework, by the two conditions on $f \in \mathcal{S}'(\mathbb{R}^3)$,

$$\|S_0 f\|_3 \le c \tag{4.2}$$

and

$$\|\Delta_j f\|_3 \le c\frac{\epsilon_j}{j+1}, \quad j \ge 0, \quad \epsilon_j \in l^2(\mathbb{N}), \tag{4.3}$$

and the X-norm is assigned by

$$\|f\| = \|S_0 f\|_3 + \Big(\sum_{j=0}^{\infty}(j+1)^2\|\Delta_j f\|_3^2\Big)^{\frac{1}{2}}. \tag{4.4}$$

An easy calculation shows that (see [2])

$$\eta_j = c\frac{2^j}{j+1}, \qquad j \ge 0, \tag{4.5}$$

thus implying (4.1). On the other hand,

$$\Big(\sum_{j\ge 0} 2^{-2j}\eta_j^2\Big)^{\frac{1}{2}} < \infty \tag{4.6}$$

and this condition enables us to use the contraction mapping argument in the space $C\big([0,T);X\big)$ exactly in the same fashion as in Section 2. The main idea is to establish an estimate analogous to (2.14) for

$$\Big\|\int_0^4 \mathbb{P}S(4-s)\nabla\cdot(v\otimes v')ds\Big\|, \tag{4.7}$$

i.e., without utilizing the convexity inequality first.

In what follows, only the "high-frequency" part of the X-norm will be examined since no problems arise for the $\|S_0 f\|_3$-part.

We again use the identity

$$\Lambda S(t) = \sum_{j\le m} 2^j \Delta_j W_{j,t} + \sum_{j\ge m+1} 2^{-2j+3m}\Delta_j W_{j,t}, \tag{4.8}$$

where

$$W_{j,t}(f) = f * w_{j,t} \tag{4.9}$$

with

$$\|w_{j,t}\|_1 \le 1, \qquad j \in \mathbb{Z}, \quad t \in J_m. \tag{4.10}$$

Then

$$\left\| \int_0^4 \Lambda S(s)[f(4-s)g(4-s)]ds \right\| = \left\| \sum_{m=0}^{\infty} I_m \right\| \tag{4.11}$$

where

$$I_m = \int_{4-m}^{4-m+1} \Big(\sum_{k\le m} 2^k \Delta_k W_{k,s} + \sum_{k\ge m+1} 2^{-2k+3m} \Delta_k W_{k,s} \Big)(fg)(4-s)ds. \tag{4.12}$$

Now, at the high frequencies,

$$\left[\sum_{j=0}^{\infty} (j+1)^2 \Big\| \Delta_j \Big(\sum_{m=0}^{\infty} I_m \Big) \Big\|_3^2 \right]^{\frac{1}{2}} \le \left[\sum_{j=0}^{\infty} (j+1)^2 \Big(\sum_{m=0}^{\infty} \|\Delta_j I_m\|_3 \Big)^2 \right]^{\frac{1}{2}} \tag{4.13}$$

and, modulo some nondiagonal terms,

$$\|\Delta_j I_m\|_3 \le c2^{-2m+j} \sup_{s\in J_m} \|\Delta_j (fg)(4-s)\|_3, \quad 0 \le j \le m, \tag{4.14}$$

and

$$\|\Delta_j I_m\|_3 \le c2^{m-2j} \sup_{s\in J_m} \|\Delta_j (fg)(4-s)\|_3, \quad j \ge m, \tag{4.15}$$

from which, utilizing the definition of η_j, we finally obtain

$$\left[\sum_{j=0}^{\infty} (j+1)^2 \Big(\sum_{m=0}^{\infty} \|\Delta_j I_m\|_3 \Big)^2 \right]^{\frac{1}{2}} \le c \Big(\sum_{j=0}^{\infty} 2^{-2j} \eta_j^2 \Big)^{\frac{1}{2}} \sup_{0\le s\le 4} \|f(s)\| \sup_{0\le s\le 4} \|g(s)\|, \tag{4.16}$$

and the contraction mapping argument applies as long as (4.5) is valid.

5. Concluding remarks

In this paper, we presented a general algorithm to solve the Cauchy problem for the Navier-Stokes equations in $\mathbb{R}^3$ with initial data in an abstract Banach space X.

Although this algorithm does not provide a *complete* characterization (i.e., a *necessary* and *sufficient* condition) of the problem, it turns out to give, for each functional family (such as $L^p(\mathbb{R}^3)$, $H^s(\mathbb{R}^3)$, or $M_2^p(\mathbb{R}^3)$), the "right" functional setting of the problem. More precisely, the above-mentioned algorithm applies as long as η_j (which is the best constant appearing in (2.10)) does not attain the *limit* value 2^j, which corresponds, in turn, to the *limit* space $\tilde{X}$ (such as $L^3(\mathbb{R}^3)$, $H^{1/2}(\mathbb{R}^3)$, or $M_2^3(\mathbb{R}^3)$) whose norm is characterized, in each family, by the property

$$\|v(\cdot)\| = \|\lambda v(\lambda\cdot)\| \qquad \text{for all } v \in \tilde{X} \text{ and } \lambda > 0. \tag{5.1}$$

Now, we note that if $v(t,x)$ is a solution of the Navier-Stokes equations, then the same holds true for $v_\lambda(t,x) = \lambda v(\lambda^2 t, \lambda x)$ for every $\lambda > 0$. In other words, the limit spaces have, as far as the space variable is concerned, the same scaling invariance as the Navier-Stokes equations.

T. Kato was the first author to consider this *limit* space and to circumvent the problems arising in the study of the Lebesgue space $L^p(\mathbb{R}^3)$. We have not treated this problem here and we refer the reader to Kato's original paper [6]. We recall as well that his clever method thereafter was applied by different authors in different frameworks (see, e.g., [11] for the Morrey-Campanato space, [2, 10] for the Besov spaces, and [9] for the Sobolev-Bessel space).

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