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Generalizations of Montessus's Theorem on the Row Convergence of Rational Interpolations

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Abstract. In this paper, we investigate the convergence of the rows (with fixed denominator degrees) of the rational interpolations to a meromorphic function. We give an explicit algorithm for determining when convergence is guaranteed and obtain rates of convergence in the appropriate cases. Furthermore, we show some applications of our results to the zero distribution of orthogonal polynomials.

1. Introduction

Let Π_m denote the collection of all algebraic polynomials of degree at most m. A rational function $r_{m,n}(z)$ is said to be of type (m, n) if, for $p_m \in \Pi_m$ and $q_n \in \Pi_n$,

$$r_{m,n}(z) = p_m(z)/q_n(z), \qquad q_n(z) \not\equiv 0.$$

If $f(z)$ is analytic at $z = 0$ (in this paper, when we say $f(z)$ is analytic at z_0, we mean that $f(z)$ is analytic in a neighborhood of z_0), then for each pair of non-negative integers (n, μ), there exist polynomials $P_{n,\mu}(z) \in \Pi_n$ and $Q_{n,\mu}(z)$ $(\not\equiv 0) \in \Pi_\mu$ such that

$$Q_{n,\mu}(z)f(z) - P_{n,\mu}(z) = O(z^{n+\mu+1}) \qquad \text{as } z \to 0.$$

The rational function $P_{n,\mu}(z)/Q_{n,\mu}(z)$ is unique and is called the Padé approximant of type (n, μ) to $f(z)$. We denote this Padé approximant by $[n/\mu](z)$. Thus, for each $f(z)$, there corresponds a doubly-infinite array known as the Padé table. Concerning the row convergence of the Padé table, we have the following classical result of de Montessus de Ballore [4].

Theorem A. *Let $f(z)$ be analytic at $z = 0$ and meromorphic with precisely μ poles (counting multiplicity) in the disk $|z| < \rho$. Let D denote the domain obtained from $|z| < \rho$ by deleting the μ poles of $f(z)$. Then for n sufficiently large, the Padé approximant $[n/\mu](z)$ of type (n, μ) satisfies*

$$f(z) - [n/\mu](z) = O(z^{n+\mu+1}).$$

Each $[n/\mu](z)$, for n large, has precisely μ finite poles and, as $n \to \infty$, these poles approach, respectively, the μ poles of $f(z)$ in $|z| < \rho$. The sequence $\{[n/\mu](z)\}_{n=0}^{\infty}$ converges to $f(z)$ throughout D, uniformly on any compact subset of D.

Padé approximation is the interpolation at the point $z = 0$. The generalized Padé approximation is called a multipoint Padé approximation, i.e., rational interpolation. Let us have the definition first. Let

$$\{\ \beta_1^{(0)};\ \beta_1^{(1)}, \beta_2^{(1)};\ \dots\ ;\ \beta_1^{(n)}, \beta_2^{(n)}, \dots, \beta_{n+1}^{(n)};\ \dots\ \} \tag{1}$$

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be a triangular system of points (not necessarily distinct). Define the associated polynomials

$$\omega_l(z) := \prod_{j=1}^{l+1}\left(z-\beta_j^{(l)}\right) \quad \text{on} \quad \Lambda(l) := \left\{\beta_1^{(l)}, \beta_2^{(l)}, \dots, \beta_{l+1}^{(l)}\right\}, \qquad l \geq 0.$$

If $f(z)$ is analytic in $\Lambda(n+\mu)$ for each pair of non-negative integers (n, μ), then there exist polynomials $p_{n,\mu} \in \Pi_n$ and $q_{n,\mu}$ $(\not\equiv 0) \in \Pi_\mu$ such that

$$\frac{f(z)q_{n,\mu}(z) - p_{n,\mu}(z)}{\omega_{n+\mu}(z)}$$

is analytic in the set $\Lambda(n+\mu)$. We call the rational function $p_{n,\mu}(z)/q_{n,\mu}(z)$ a multipoint Padé approximant of type (n,μ). It is unique and denoted by $R_{n,\mu}(z) = R_{n,\mu}(f, \Lambda(n+\mu); z)$. For multipoint Padé approximant, we have an analogue of Theorem A due to Saff (cf. [8]).

In this paper, we will extend Theorem A and its analogue to the general case where the degree of the denominator of the interpolation does not equal the number of poles of the approximated function $f(z)$. We will state our main theorem and give notation in Section 2. The proof of the main theorem will be given in Section 3. The divergence in the complement set is discussed in Section 4. For the rational interpolation rows not satisfying our criteria, we will give an example in Section 5 to explain the complexity of convergence. Furthermore, an example of an application is shown in Section 6.

2. Results on the multipoint Padé approximants

Here we will study the behavior of rational interpolation on a more general point set E than on a disk. We assume that E is connected and equal to the closure of the interior of a finite number of mutually exterior Jordan curves $C_1, C_2, \dots, C_k$. Each C_i is of class A, i.e., each C_i can be represented parametrically in terms of arc length s by $x = x(s)$, $y = y(s)$ where $x(s)$ and $y(s)$ possess second derivatives with respect to s which satisfy a Lipschitz condition of some positive order in s. We summarize these assumptions by saying that E has **Property A**. Furthermore, let $G(z)$ be the Green's function of $K := \overline{\mathbf{C}} \setminus E$ having a pole at infinity, Δ be the logarithmic capacity for the point set E, and set

$$\Gamma_\rho := \{z; G(z) = \log\rho\}, \quad E_\rho := \text{int }(\Gamma_\rho), \quad H(z) := -\frac{\partial G(z)}{\partial x} + i\frac{\partial G(z)}{\partial y}.$$

Theorem 1. *Suppose E has property A. Let ρ be a fixed positive number, $f(z)$ be analytic on E, meromorphic with precisely μ (≥ 0) poles (counting multiplicity) in E_ρ, and analytic on Γ_ρ except for poles in the m distinct points $\alpha_1, \alpha_2, \dots, \alpha_m$ on Γ_ρ of respective orders $r_1, r_2, \dots, r_m$ $(r_1 \geq r_2 \geq \dots \geq r_m)$. Let the triangular schemes (1) have no limit point exterior to E and satisfy the relations*

$$\begin{gathered} |\omega_n(z)| \leq M\Delta^n \qquad \text{for } z \in E; \\ \left|G(z) + \log\Delta - n^{-1}\log|\omega_n(z)|\right| \leq Mn^{-1}, \end{gathered} \tag{2}$$

uniformly in z on each closed bounded subset of K. If none of $\{\alpha_i\}_{i=1}^{i=m}$ is a critical point of $G(z)$, then for "good" ν (see Definition 1 below) and for all n sufficiently large, there exists a unique multipoint Padé approximant $R_{n,\mu+\nu}(z) := R(f, \Lambda(n+\mu+\nu); z)$ of type $(n, \mu+\nu)$, which interpolates to $f(z)$ on the set $\Lambda(n+\mu+\nu)$. Each $R_{n,\mu+\nu}(z)$ has precisely $\mu+\nu$ finite poles, μ of which, as $n \to \infty$, approach the μ poles of $f(z)$ in E_ρ, respectively, and $p_{\nu,i}$ of which approach the points α_i, respectively (the $\{p_{\nu,i}\}$ are *defined in Definition 2 and satisfy* $\sum_{i=1}^{m} p_{\nu,i} = \nu$). *Let* D_ρ *denote the region obtained from* E_ρ *by deleting the* μ *poles of* $f(z)$ *in* E_ρ, *then the sequence* $\{R_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$ *converges to* $f(z)$ *throughout* D_ρ, *uniformly on any compact subset of* D_ρ, *and*

$$\left\|f(z) - R_{n,\mu+\nu}(z)\right\|_E \le \frac{An^\lambda}{\rho^n} \tag{3}$$

where $\lambda = \max_{1\le i\le m}\{r_i - 2p_i - 1\}$. *Furthermore, if we let* $q_n(z)$ *denote the monic denominator of* $R_{n,\mu+\nu}(z)$ *and let* $\pi(z)$ *denote the monic polynomial of degree* μ *having* μ *poles of* $f(z)$ *in* E_ρ *as its zeros, then*

$$\left\|q_n(z) - \pi(z)\prod_{i=1}^{m}(z-\alpha_i)^{p_{\nu,i}}\right\|_E \le C/n. \tag{4}$$

Before stating Definition 1, we introduce some needed notation. Here $[x]$ stands for the integer part of x and an empty sum is defined to be zero. Let

$$\begin{aligned}
u_{i,j} &:= \left[(r_i - r_{j+1})/2\right], && 1 \le i \le j+1 \le m;\\
w_{i,j} &:= r_i - r_{j+1} - 2u_{i,j}, && 1 \le i \le j+1 \le m;\\
s_l &:= \sum_{i=1}^{l-1} w_{i,l-1}, && 1 \le l \le m;\\
s_l &:= \sum_{i=1}^{2m-l} w_{i,2m-l}, && 1+m \le l < 2m;\\
v_0 &:= 0, &&\\
v_l &:= \sum_{i=1}^{l} u_{i,l}, && 1 \le l \le m-1;\\
v_l &:= \sum_{i=1}^{2m-l-1} u_{i,2m-l-1} + (2m-l-1)r_{2m-l} + \sum_{j=2m-l}^{m} r_j, && m \le l < 2m;\\
I_l &:= [v_{l-1}, v_l), && 1 \le l \le m-1;\\
I_m &:= [v_{m-1}, v_m], &&\\
I_l &:= (v_{l-1}, v_l], && m+1 \le l < 2m.
\end{aligned}$$

It is straightforward to verify that the points v_l satisfy $v_0 \le v_1 \le \cdots \le v_{2m-1}$, $v_{2m-1} = \sum_{i=1}^{m} r_i$, and $[0, v_{2m-1}] = \bigcup_{l=1}^{2m-1} I_l$, where the intervals are pairwise disjoint. We remark that some of the intervals I_l may be empty.

Definition 1. For a fixed ν, $0 \le \nu \le v_{2m-1}$, let $I_l, l = l(\nu)$, be the unique interval containing ν. Set

$$\begin{aligned}
k_l &:= \left[(\nu - v_{l-1})/l\right], & q_l &:= \nu - v_{l-1} - lk_l, && \text{if } 1 \le l \le m;\\
k_l &:= \left[(\nu - v_{l-1})/(2m-l)\right], & q_l &:= \nu - v_{l-1} - (2m-l)k_l, && \text{if } m+1 \le l \le 2m-1.
\end{aligned}$$

We say that ν is *good* if $q_l = 0$ or $q_l = s_l$. If $0 < \nu < v_{2m-1}$ is not good, we say that ν is *bad*.

Next, for good ν, we specify the quantities $\{p_{\nu,i}\}$ mentioned in Theorem 1.

Definition 2. Given a good ν, we first determine the number l such that $\nu \in I_l$; then, for this $l = l(\nu)$, we consider two cases in Definition 1.

Case 1. $q_l = 0$.
(a) If $1 \le l \le m$, then

$$p_{\nu,i} := \begin{cases} k_l + u_{i,l-1}, & 1 \le i \le l; \\ 0, & l < i \le m; \end{cases} \tag{5}$$

(b) if $m+1 \le l \le 2m-1$, then

$$p_{\nu,i} := \begin{cases} k_l + u_{i,2m-l} + r_{2m-l+1}, & 1 \le i \le 2m-l; \\ r_i, & 2m-l < i \le m. \end{cases} \tag{6}$$

Case 2. $q_l = s_l \ne 0$.
(a) If $1 \le l \le m$, then

$$p_{\nu,i} := \begin{cases} k_l + u_{i,l-1} + w_{i,l-1}, & 1 \le i \le l; \\ 0, & l < i \le m; \end{cases} \tag{7}$$

(b) if $m+1 \le l \le 2m-1$, then

$$p_{\nu,i} := \begin{cases} k_l + u_{i,2m-l} + r_{2m-l+1} + w_{i,2m-l}, & 1 \le i \le 2m-l; \\ r_i, & 2m-l < i \le m. \end{cases} \tag{8}$$

We have some remarks about our definitions.

Remark. We assume that $f(z)$ has μ poles in E_ρ and v_{2m-1} poles on Γ_ρ in our Theorem 1. The analogue of Theorem A guarantees that the rational interpolation rows $\{R_{n,\mu}(z)\}_{n=0}^{\infty}$ and $\{R_{n,\mu+v_{2m-1}}(z)\}_{n=0}^{\infty}$ converge. Here, for $0 < \nu < v_{2m-1}$, we are trying to find which "intermediate" row $\{R_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$ converges or which ν is good. We divide the integer interval $[0, v_{2m-1}]$ into $2m-1$ intervals $\{I_l\}$. Each non-empty interval has at least one good ν. Definition 1 gives explicit criteria for whether ν is good or not. For example, $q_1 \equiv 0$, so all $\nu \in I_1$ are good. The same is true for $\nu \in I_{2m-1}$. And if ν is one of v_i for $1 \le i \le m$, then ν is good. Furthermore, if $r_1 - r_2$ is odd, then all $\nu \in I_2$ are good. In fact, it is simple to write a program to compute all good ν and to verify if a given ν is good or bad for the input $r_1, r_2, \ldots, r_m$.

3. Proof of the main theorem

To prove Theorem 1, we need a Lemma first. We will let $L_{n+\mu+\nu}\big(h(z); z\big)$ denote the unique polynomial of degree $\le n+\mu+\nu$ which interpolates $h(z)$ on the set $\Lambda(n+\mu+\nu)$.

Lemma 1. *Let E be the same as in Theorem 1. Let $F(z)$ be analytic on E, meromorphic with precisely μ (≥ 0) poles in E_ρ, and analytic on Γ_ρ except for a pole at α of degree r, $H(\alpha) \ne 0$. Let $g_n(z) := L_{n+\mu+\nu}\big(F(z); z\big)$. Then*

$$\lim_{n\to\infty} n^{-(r+k)} g_n^{(k)}(\alpha) = \frac{(-1)^k H(\alpha)^{r+k} B_r}{(r-1)!(r+k)}, \qquad k = 0, 1, 2, \ldots, \tag{9}$$

$$\big|F(z) - g_n(z)\big| \le L(z) n^{r-1} \left| \frac{\omega_{n+\mu+\nu}(z)}{\omega_{n+\mu+\nu}(\alpha)} \right|, \qquad \text{for fixed } z \ne \alpha, \tag{10}$$

where $L(z)$ does not depend on n.

Proof. Write

$$F(z) =: u(z) + \sum_{i=1}^{r} B_i (z-\alpha)^{-i},$$

where $u(z)$ is analytic on Γ_ρ. Let $s_{n,i}(z) := L_{n+\mu+\nu}\big((z-\alpha)^{-i}; z\big)$, $i = 1, 2, \ldots, r$, and $\theta_{n+\mu+\nu}(z) := 1/\omega_{n+\mu+\nu}(z)$. From the definition of $s_{n,i}(z)$, we see that

$$(z-\alpha)^{-i} - s_{n,i}(z) = \frac{\omega_{n+\mu+\nu}(z) p_{n,i}(z)}{(z-\alpha)^i} \tag{11}$$

where $p_{n,i}(z)$ is a polynomial of degree $i-1$. The equivalence of (11) with

$$s_{n,i}(z) = \frac{\omega_{n+\mu+\nu}(z)\big[\theta_{n+\mu+\nu}(z) - p_{n,i}(z)\big]}{(z-\alpha)^i}$$

implies (since $s_{n,i}(z)$ has no finite pole and α is not a zero of $\omega_{n+\mu+\nu}(z)$) that $p_{n,i}(z)$ interpolates $\theta_n(z)$ in the point α considered of multiplicity i. This property of interpolation also is possessed by the polynomial

$$q_{n,i}(z) := \sum_{j=0}^{i-1} \theta_n^{(j)}(\alpha)(z-\alpha)^j/j!\ ,$$

and since both $p_{n,i}(z)$ and $q_{n,i}(z)$ are of degree $i-1$, we have $p_{n,i}(z) \equiv q_{n,i}(z)$. Thus

$$s_{n,i}(z) = \frac{\omega_{n+\mu+\nu}(z)\big(\theta_{n+\mu+\nu}(z) - \sum_{t=0}^{i-1} \theta_{n+\mu+\nu}^{(t)}(\alpha)(z-\alpha)^t/t!\big)}{(z-\alpha)^i}. \tag{12}$$

From (2), we get

$$\lim_{n\to\infty} n^{-1}\omega'_{n+\mu+\nu}(z)\theta_{n+\mu+\nu}(z) = -H(z),$$

uniformly on each compact subset of K, and by induction, we can prove (proof of Theorem 1, [7]) that

$$\begin{aligned} \lim_{n\to\infty} n^{-k}\omega_{n+\mu+\nu}(z)\theta_{n+\mu+\nu}^{(k)}(z) &= H(z)^k, \\ \lim_{n\to\infty} n^{-k}\omega_{n+\mu+\nu}^{(k)}(z)\theta_{n+\mu+\nu}(z) &= (-1)^k H(z)^k, \end{aligned} \tag{13}$$

uniformly on each compact subset of K. Thus, we get

$$\lim_{n\to\infty} n^{-(t+j)}\omega_{n+\mu+\nu}^{(t)}(\alpha)\theta_{n+\mu+\nu}^{(j)}(\alpha) = (-1)^t H(\alpha)^{t+j}. \tag{14}$$

Furthermore, by (12), we get for $k \geq 0$ that

$$\lim_{n\to\infty} n^{-(i+k)} s_{n,i}^{(k)}(\alpha) = \frac{(-1)^k H(\alpha)^{i+k}}{(i-1)!(i+k)}, \qquad i = 1, 2, \ldots, r. \tag{15}$$

Now let $s_{n,0}(z) := L_{n+\mu+\nu}\big(u(z); z\big)$ and

$$g_n(z) = s_{n,0}(z) + \sum_{i=1}^{r} B_i s_{n,i}(z);$$

then $g_n(z)$ interpolates $F(z)$ on the set $\Lambda(n+\mu+\nu)$. Using (15) for $i = 0, 1, \ldots, r$, one obtains

$$\lim_{n\to\infty} n^{-(r+k)} g_n^{(k)}(\alpha) = \frac{(-1)^k H(\alpha)^{r+k} B_r}{(r-1)!(r+k)}. \tag{16}$$

To prove (10), we use (12), (13) and get, for $z \neq \alpha$,

$$\begin{aligned}
|F(z) - g_n(z)| &\leq |u(z) - s_{n,0}(z)| + \sum_{i=1}^{r} |B_i| \, |(z-\alpha)^{-i} - s_{n,i}(z)| \\
&\leq \left| \frac{\omega_{n+\mu+\nu}(z)}{\omega_{n+\mu+\nu}(\alpha)} \right| \left[1 + \sum_{i=1}^{r} |B_i| \sum_{t=0}^{i-1} \frac{|\omega_{n+\mu+\nu}(\alpha)\theta^{(t)}_{n+\mu+\nu}(\alpha)(z-\alpha)^{t-i}|}{t} \right] \\
&\leq \left| \frac{\omega_{n+\mu+\nu}(z)}{\omega_{n+\mu+\nu}(\alpha)} \right| L(z) n^{r-1}
\end{aligned}$$

where $L(z)$ does not depend on n. □

Note. Define $g_{n,k}(z) := L_{n+\mu+\nu}\big((z-\alpha)^{k-1+\nu}F(z); z\big)$ for $k = 1, 2, \ldots$. Then, by the same reasoning, for $k = 1, 2, \ldots, r-\nu$,

$$\lim_{n\to\infty} n^{-(r-k+1-\nu+j)} g^{(j)}_{n,k}(\alpha) = \frac{(-1)^j H(\alpha)^{r-k+1-\nu+j} B_r}{(r-k+1-\nu-1)!(r-k+1-\nu+j)} \neq 0. \tag{17}$$

If $k > r - \nu$, then $(z-\alpha)^{k-1+\nu}F(z)$ has no pole at $z = \alpha$. If $k = r - \nu + 1$, then $(z-\alpha)^{k-1+\nu}F(z)$ is analytic and not zero at $z = \alpha$, thus we have

$$\lim_{n\to\infty} n^{-(r-k+1-\nu)} g_{n,k}(\alpha) = \lim_{n\to\infty} g_{n,k}(\alpha) = B_r \neq 0, \tag{18}$$

$$\lim_{n\to\infty} n^{-(r-k+1-\nu+j)} g^{(j)}_{n,k}(\alpha) = \lim_{n\to\infty} n^{-j} g^{(j)}_{n,k}(\alpha) = 0, \quad \text{for } j > 0. \tag{19}$$

If $k > r-\nu+1$, then $(z-\alpha)^{k-1+\nu}F(z)$ is analytic and has a zero of order $k-(r-\nu+1)$ at $z = \alpha$. Hence, there exits a $\sigma > \rho$ such that $(z-\alpha)^{k-1+\nu}F(z)$ is analytic in E_σ. By the Hermite formula,

$$g_{n,k}(z) - (z-\alpha)^{k-1+\nu}F(z) = \frac{1}{2\pi i} \int_{\Gamma_\sigma} \frac{\omega_{n+\mu+\nu}(z)(t-\alpha)^{k-1+\nu}F(t)}{\omega_{n+\mu+\nu}(t)(t-z)} dt, \quad z \in E_\sigma.$$

Thus

$$\lim_{n\to\infty} n^\tau g^{(j)}_{n,k}(\alpha) = 0 \qquad \text{for } j = 0, 1, \ldots, k-(r-\nu+1)-1, \text{ and any real } \tau. \tag{20}$$

Proof of Theorem 1. By the assumption of Theorem 1, $f(z)$ has μ poles inside E_ρ and m different poles $\alpha_1, \alpha_2, \ldots, \alpha_m$ on the Γ_ρ with multiplicities $r_1, r_2, \ldots, r_m$, respectively, $(H(\alpha_i) \neq 0,\ i = 1, 2, \ldots, m)$. Now, since $\pi(z)f(z)$ is analytic in E_ρ, write

$$\pi(z)f(z) =: f_0(z) + \sum_{i=1}^{m} \sum_{j=1}^{r_i} B_{j,i}(z-\alpha_i)^{-j};$$

$$\begin{aligned}
&(z-\alpha_j)^{k-1+p_j} \prod_{i=1, i\neq j}^{m} (z-\alpha_i)^{2p_i} \pi^2(z) f(z) \\
&=: u_j(z) + \sum_{l=1}^{r_j-k+1-p_j} C_{l,jj}(z-\alpha_j)^{-l} + \sum_{i=1,i\neq j}^{m} \sum_{l=1}^{r_i-2p_i} C_{l,ji}(z-\alpha_i)^{-l},
\end{aligned}$$

j is an integer such that $p_j > 0$ and $1 \leq j \leq m$, where $f_0(z)$, $u_j(z)$ are analytic on E_ρ; if any $r_j - k + 1 - p_j$, $r_i - 2p_i$ is less than 1, then the corresponding sum is empty. We define any empty sum to be zero.

Let $s_{n,l,j}(z) := L_{n+\mu+\nu}\big((z-\alpha_j)^{-l}; z\big)$, $(j = 1,2,\ldots,m,\ l > 0)$. Then by (11), we get

$$(-l)(-l-1)\cdots(-l-q+1)(z-\alpha_j)^{-l-q} - s^{(q)}_{n,l,j}(z) = \sum_{s=0}^{q}\binom{q}{s}\omega^{(q-s)}_{n+\mu+\nu}(z)$$
$$\times\sum_{t=0}^{l-1}\theta^{(t)}_{n+\mu+\nu}(\alpha_j)(t-l)(t-l-1)\cdots(t-l-s+1)(z-\alpha_j)^{t-l-s}/t!\,.$$

We cannot guarantee the existence of the limit with $M_1 < |\omega_{n+\mu+\nu}(\alpha_i)/\omega_{n+\mu+\nu}(\alpha_j)| < M_2$ and $\alpha_i \neq \alpha_j$ when $i \neq j$; however, we have

$$\limsup_{n\to\infty} n^{-(l-1+q)}\big|s^{(q)}_{n,l,j}(\alpha_i)\big| = \big|M_{l,q,ji}H^{l-1+q}(\alpha_i)\big|, \qquad i \neq j. \tag{21}$$

For $i = j$, we expand the right side of (12) in a neighborhood of α_i and differentiate

$$s^{(q)}_{n,l,i}(z) = \sum_{s=0}^{q}\sum_{t=q-s}^{\infty}\binom{q}{s}$$
$$\times\frac{\omega^{(s)}_{n+\mu+\nu}(z)\theta^{(t)}_{n+\mu+\nu}(\alpha_i)t(t-1)\cdots(t-q+s+1)(z-\alpha_i)^{t-q+s}}{(t+l)!},$$

thus, we obtain

$$\lim_{n\to\infty} n^{-(l+q)}s^{(q)}_{n,l,i}(\alpha_i) = \lim_{n\to\infty} n^{-(l+q)}\sum_{s=0}^{q}\binom{q}{s}\frac{\omega^{(q-s)}_{n+\mu+\nu}(z)\theta^{(l+s)}_{n+\mu+\nu}(\alpha_i)s!}{(l+s)!}$$
$$= \frac{(-1)^q H^{l+q}(\alpha_i)}{(l-1)!(l+q)}. \tag{22}$$

For $j = 1,2,\ldots,m$, define

$$g_{n,k,j}(z) := L_{n+\mu+\nu}\Big((z-\alpha_j)^{k-1+p_j}\prod_{i=1,i\neq j}^{m}(z-\alpha_i)^{2p_i}\pi^2(z)f(z); z\Big),$$
$$\text{for } k = 1,2,\ldots,p_j,$$

$$g_{n,p_1+1}(z) := L_{n+\mu+\nu}\Big(\prod_{i=1}^{m}(z-\alpha_i)^{2p_i}\pi^2(z)f(z); z\Big).$$

Then, since j is an integer such that $p_j > 0$ and $1 \leq j \leq m$,

$$g_{n,k,j}(z) = s_{n,0,j}(z) + \sum_{l=1}^{r_j-k+1-p_j} C_{l,jj}s_{n,l,j}(z) + \sum_{i=1,i\neq j}^{m}\sum_{l=1}^{r_i-2p_i} C_{l,ji}s_{n,l,i}(z)$$

where $s_{n,0,j}(z) := L_{n+\mu+\nu}\big(u_j(z); z\big)$ $(j = 1,2,\ldots,m)$. Combining this with (18)–(22), if $r_j - k + 1 - p_j \geq r_i - 2p_i$, $i = 1,2,\ldots,m$, $1 \leq k \leq p_j$, then

$$\lim_{n\to\infty} n^{-(r_j-k-p_j+1+s)}g^{(s)}_{n,k,j}(\alpha_j) = N^{\{k\}}_{s,jj} < \infty, \tag{23}$$

$$\limsup_{n\to\infty} n^{-\max\{(r_j-k-p_j+s),(r_i-2p_i+s)\}}\big|g^{(s)}_{n,k,j}(\alpha_i)\big| = \big|N^{\{k\}}_{s,ji}\big| < \infty, \quad i \neq j. \tag{24}$$

With all these preparations, now we are ready to consider the multipoint Padé approximation of type (n,ν) for $f(z)$ and good ν, $0 \leq \nu \leq \sum_{i=1}^{m} r_i$. We will examine three cases separately.

Case 1. $0 \le \nu \le [(r_1 - r_2)/2]$. All ν's are good in this case. Set $p_1 := p_{\nu,1} = \nu$, $p_j := p_{\nu,j} = 0$, $j = 2, 3, \dots, m$. Then $r_1 - 2\nu \ge r_i$, $i = 2, 3, \dots, m$, and (23) becomes

$$\lim_{n\to\infty} n^{-(r_1-k-\nu+1+s)} g_{n,k,1}^{(s)}(\alpha_1) = B_{r_1,1} \frac{(-1)^{r_1-k-\nu+1} H^{r_1-k-\nu+1+s}(\alpha_1)}{(r_1-k-\nu+1)!(r_1-k-\nu+1+s)} =: N_{s,11}^{\{k\}} < \infty. \tag{25}$$

If $\nu = 0$, the theorem is true according to Saff (cf. [8]).

We now discuss the case $\nu \ge 1$. Suppose $\mu = 0$. Let

$$q_n(z) = \sum_{k=1}^{\nu} a_k^{(n)} (z-\alpha_1)^{k-1} + (z-\alpha_1)^\nu \quad \text{and} \quad h_n(z) = \sum_{k=1}^{\nu} a_k^{(n)} g_{n,k,1}(z) + g_{n,\nu+1}(z);$$

then $h_n(z) =: L_{n+\nu}\big(q_n(z)(z-\alpha_1)^\nu f(z); z\big)$. We shall choose the proper coefficients $a_k^{(n)}$, $1 \le k \le \nu$, so that the polynomial $(z-\alpha_1)^\nu$ is a factor of $h_n(z)$. For this purpose, we need to find a non-trivial solution $(a_1^{(n)}, a_2^{(n)}, \dots, a_\nu^{(n)})^T$ for the linear system

$$\sum_{k=1}^{\nu} a_k^{(n)} g_{n,k,1}^{(j)}(\alpha_1) = -g_{n,\nu+1}^{(j)}(\alpha_1), \qquad j = 0, 1, \dots, \nu-1. \tag{26}$$

To save space, we denote $M_{t\times s} = (m_{j,k})_{j=1,2,\dots,t;\, k=1,2,\dots,s}$ for a matrix with t rows and s columns. Let

$$\begin{aligned} A_{n,\nu\times\nu} &= \big(g_{n,k,1}^{(j-1)}(\alpha_1)\big)_{j=1,2,\dots,\nu;\, k=1,2,\dots,\nu}, \\ A_{n,\nu\times\nu}^\star &:= n^{-\nu(r_1-\nu)} A_{n,\nu\times\nu} \\ &= \big(n^{-(r_1-\nu+j-1-k+1)} g_{n,k,1}^{(j-1)}(\alpha_1)\big)_{j=1,2,\dots,\nu;\, k=1,2,\dots,\nu}, \\ Y_n &:= \big(g_{n,\nu+1,1}^{(j-1)}(\alpha_1)\big)_{j=1,2,\dots,\nu}, \\ Y_n^\star &:= \big(n^{-(r_1-2\nu+j-1)} g_{n,\nu+1,1}^{(j-1)}(\alpha_1)\big)_{j=1,2,\dots,\nu}. \end{aligned} \tag{27}$$

Noticing that $r_1 - 2\nu \ge r_2 > 0$ and $r_1 - \nu > \nu$ by (25), we obtain

$$\lim_{n\to\infty} n^{-\nu(r_1-\nu)} \det(A_{n,\nu\times\nu}) = \lim_{n\to\infty} \det(A_{n,\nu\times\nu}^\star) = d_\nu D_\nu \ne 0,$$
$$\lim_{n\to\infty} Y_n^\star = \big(N_{j-1,11}^{\{\nu+1\}}\big)_{j=1,2,\dots,\nu}$$

where the specific form of the constant d_ν and the non-zero determinant D_ν are given in [5]. Thus, for n large enough, $\det(A_{n,\nu\times\nu}) \ne 0$. Hence, the linear system (26) has the unique solution for sufficiently large n. Furthermore, $X_n^\star := \big(n^\nu a_1^{(n)}, n^{\nu-1} a_2^{(n)}, \dots, n^1 a_\nu^{(n)}\big)^T$ satisfying $A_{n,\nu\times\nu}^\star X_n^\star = Y_n^\star$, and it has a finite limit

$$n^{\nu-k+1} a_k^{(n)} \to \frac{d_\nu^{\{k\}} D_\nu^{\{k\}} H(\alpha_1)^{k-\nu-1}}{d_\nu D_\nu} < \infty, \qquad \text{for } k = 1, 2, \dots, \nu, \tag{28}$$

where $d_\nu^{\{k\}}$ and $D_\nu^{\{k\}}$ are constants depending only on k and ν. From this, we see that for each compact set Q of the plane

$$\big\| q_n(z) - (z-\alpha_1)^\nu \big\|_Q \le C/n. \tag{29}$$

Now set $R_{n,\nu}(z) := h_n(z)/\{q_n(z)(z-\alpha_1)^\nu\}$. Then, by our choice of the coefficients $a_k^{(n)}$'s, we know that $R_{n,\nu}(z)$ is a rational function of type (n, ν), and $q_n(z)$ is not zero inside E for n large enough from (29). Hence, $R_{n,\nu}(z)$ must interpolate $f(z)$ on the set $\Lambda(n+\nu)$. Moreover, from Lemma 1

$$\begin{aligned}\left|f(z)-\frac{h_n(z)}{q_n(z)(z-\alpha_1)^\nu}\right| &\le \frac{\sum_{k=1}^{\nu}\left|a_k^{(n)}\right|\left|(z-\alpha_1)^{k-1+\nu}f(z)-g_{n,k}(z)\right|+\left|(z-\alpha_1)^{2\nu}f(z)-g_{n,\nu+1}(z)\right|}{\left|q_n(z)(z-\alpha_1)^\nu\right|}\\ &\le Mn^{r-2\nu-1}\sigma^n/\rho^n \qquad \text{for } z \text{ on } E_\sigma\subset E_\rho.\end{aligned}$$

We have proved the results for $\mu=0$.

Next, suppose $\mu>0$ and $\gamma_1,\gamma_2,\dots,\gamma_\mu$ are poles of $f(z)$ inside E_ρ with $|\gamma_1|\le|\gamma_2|\le\cdots\le|\gamma_\mu|$. Define $\pi_j(z):=\prod_{t=1}^{j}(z-\gamma_t)$, $j=1,2,\dots,\mu$, and $\pi_0(z):=1$. It is obvious that $\pi_\mu(z)\equiv\pi(z)$. Define

$$w_{n,s}(z):=L_{n+\mu+\nu}\big(\pi_{s-1}(z)\pi(z)(z-\alpha_1)^{2\nu}f(z);z\big) \qquad \text{for } s=1,2,\dots,\mu.$$

Let

$$q_n(z)=\sum_{s=1}^{\mu}b_s^{(n)}\pi_{s-1}(z)(z-\alpha_1)^\nu+\sum_{k=1}^{\nu}a_k^{(n)}(z-\alpha_1)^{k-1}\pi(z)+(z-\alpha_1)^\nu\pi(z).$$

Then

$$\begin{aligned}h_n(z)&=\sum_{s=1}^{\mu}b_s^{(n)}w_{n,s}(z)+\sum_{k=1}^{\nu}a_k^{(n)}g_{n,k,1}(z)+g_{n,\nu+1}(z)\\ &=: L_{n+\mu+\nu}\big(q_n(z)\pi(z)(z-\alpha_1)^\nu f(z);z\big).\end{aligned}$$

Now we need $h_n(z)=\pi(z)(z-\alpha_1)^\nu d_n(z)$ where $d_n(z)$ is a polynomial of degree $\le n$. Thus, we get

$$\begin{pmatrix} C_{n,\mu\times\mu} & U_{n,\mu\times\nu}\\ V_{n,\nu\times\mu} & A_{n,\nu\times\nu}\end{pmatrix} X_n = Y_n \tag{30}$$

where the entries are as follows. Suppose we have ι different γ's, the multiplicities are $i_1,i_2,\dots,i_\iota$, respectively ($i_1+i_2+\cdots+i_\iota=\mu$). Let $y_1=-g_{n,\nu+1}(\gamma_1)$, $\dots$, $y_{i_1}=-g_{n,\nu+1}^{(i_1-1)}(\gamma_1)$, $\dots$, $y_{i_1+i_2+\cdots+i_{\iota-1}+1}=-g_{n,\nu+1}(\gamma_\mu)$, $\dots$, $y_{i_1+i_2+\cdots+i_\iota}=-g_{n,\nu+1}^{(i_\iota-1)}(\gamma_\mu)$; $y_{\mu+k}=-g_{n,\nu+1}^{(k-1)}(\alpha_1)$ for $k=1,2,\dots,\nu$;

$$\begin{aligned}X_n&=\big(b_1^{(n)},b_2^{(n)},\dots,b_\mu^{(n)},a_1^{(n)},a_2^{(n)},\dots,a_\nu^{(n)}\big)^{\mathrm T};\\ Y_n&:=(y_1,y_2,\dots,y_\mu,y_{\mu+1},\dots,y_{\mu+\nu})^{\mathrm T};\end{aligned}$$

$$C_{n,\mu\times\mu}:=\begin{pmatrix} w_{n,1}(\gamma_1) & w_{n,2}(\gamma_1) & \cdots & w_{n,\mu}(\gamma_1)\\ \vdots & \vdots & & \vdots\\ w_{n,1}^{(i_1-1)}(\gamma_1) & w_{n,2}^{(i_1-1)}(\gamma_1) & \cdots & w_{n,\mu}^{(i_1-1)}(\gamma_1)\\ \vdots & \vdots & & \vdots\\ w_{n,1}(\gamma_\mu) & w_{n,2}(\gamma_\mu) & \cdots & w_{n,\mu}(\gamma_\mu)\\ \vdots & \vdots & & \vdots\\ w_{n,1}^{(i_\iota-1)}(\gamma_\mu) & w_{n,2}^{(i_\iota-1)}(\gamma_\mu) & \cdots & w_{n,\mu}^{(i_\iota-1)}(\gamma_\mu)\end{pmatrix}$$

$$U_{n,\mu\times\nu} := \begin{pmatrix} g_{n,1}(\gamma_1) & g_{n,2}(\gamma_1) & \cdots & g_{n,\nu}(\gamma_1) \\ \vdots & \vdots & & \vdots \\ g_{n,1}^{(i_1-1)}(\gamma_1) & g_{n,2}^{(i_1-1)}(\gamma_1) & \cdots & g_{n,\nu}^{(i_1-1)}(\gamma_1) \\ \vdots & \vdots & & \vdots \\ g_{n,1}(\gamma_\mu) & g_{n,2}(\gamma_\mu) & \cdots & g_{n,\nu}(\gamma_\mu) \\ \vdots & \vdots & & \vdots \\ g_{n,1}^{(i_\iota-1)}(\gamma_\mu) & g_{n,2}^{(i_\iota-1)}(\gamma_\mu) & \cdots & g_{n,\nu}^{(i_\iota-1)}(\gamma_\mu) \end{pmatrix}$$

$$V_{n,\nu\times\mu} := \left(w_{n,k}^{(j-1)}(\alpha_1)\right)_{j=1,2,\ldots,\nu;\, k=1,2,\ldots,\mu},$$

and $A_{n,\nu\times\nu}$ is defined as in (27). It is obvious that $C_{n,\mu\times\mu}$ tend to a lower triangular matrix with non-zero diagonal elements as n tends to infinity, $\det(C_{n,\mu\times\mu}) \neq 0$ for n sufficiently large.

If we let $X_n^\star = \left(n^\tau b_1^{(n)}, n^\tau b_2^{(n)}, \ldots, n^\tau b_\mu^{(n)}, n^\nu a_1^{(n)}, n^{\nu-1} a_2^{(n)}, \ldots, n a_\nu^{(n)}\right)^{\mathrm{T}}$ for some positive constant τ, it satisfies

$$\begin{pmatrix} C_{n,\mu\times\mu} & U_{n,\mu\times\nu}^\star \\ V_{n,\nu\times\mu}^\star & A_{n,\nu\times\nu}^\star \end{pmatrix} X_n^\star = Y_n^\star,$$

where $U_{n,\mu\times\nu}^\star$ is obtained by multiplying the kth column of $U_{n,\mu\times\nu}$ with $n^{(\tau-\nu-1+k)}$; $V_{n,\nu\times\mu}^\star$ is obtained by multiplying the kth row of $V_{n,\nu\times\mu}$ with $n^{-(r_1-2\nu+\tau+k-1)}$; $A_{n,\nu\times\nu}^\star$ is defined as before; and

$$Y_n^\star = \left(n^\tau y_1, n^\tau y_2, \ldots, n^\tau y_\mu, n^{-(r_1-2\nu)} y_{\mu+1}, \ldots, n^{-(r_1-\nu-1)} y_{\mu+\nu}\right)^{\mathrm{T}}.$$

Because all $g_{n,k,1}(z)$'s are analytic and tend to zero at γ_t's, by (20), we see that $U_{n,\mu\times\nu}^\star$ tends to a zero matrix as $n \to \infty$ for ν and any real τ. Furthermore, by the same reasoning as the proof of (17), we know that

$$\lim_{n\to\infty} n^{-(r_1-2\nu+j)} \omega_{n,s}^{(j)}(\alpha_1) = \frac{(-1)^j H(\alpha_1)^{r_1-2\nu+j} B_{r_1} \pi_{s-1}(\alpha_1)\pi(\alpha_1)}{(r_1-2\nu-1)!(r_1-2\nu+j)} \quad \text{for } 2\nu < r_1.$$

Thus $\lim_{n\to\infty} V_{n,\nu\times\mu}^\star = V_{\nu\times\mu}$, which is a finite matrix for $\tau = 0$ and a zero matrix for $\tau > 0$ and

$$\lim_{n\to\infty} Y_n^\star = \left(0, 0, \ldots, 0, N_{0,11}^{\{\nu+1\}}, N_{1,11}^{\{\nu+1\}}, \ldots, N_{\nu-1,11}^{\{\nu+1\}}\right)^{\mathrm{T}}.$$

By the previous proof for $\mu = 0$, we know that for n large enough, $\det(A_{n,\nu\times\nu}^\star) \neq 0$, hence the linear system (30) has a unique solution and $\lim_{n\to\infty} X_n^\star$ exists and is bounded. That means $h_n(z)/\{\pi(z)(z-\alpha_1)^\nu q_n(z)\}$ is a rational function of type $(n, \mu+\nu)$ which interpolates $f(z)$ on the set $\Lambda(n+\mu+\nu)$ and (4) is true. Furthermore, we have the estimate

$$f(z) - \frac{h_n(z)}{q_n(z)\pi(z)(z-\alpha_1)^\nu} = \frac{\omega_{n+\mu+\nu}(z)}{q_n(z)\pi(z)(z-\alpha_1)^\nu} \times \frac{H(\alpha_1)^{r_1-2\nu-1} n^{r_1-2\nu-1} B_{r_1}}{\omega_{n+\mu+\nu}(\alpha_1) D_\nu (r_1-2\nu-1)!} \left\{C_\nu + o(1) + O(n^{-\tau})\right\} \tag{31}$$

with a non-zero determinant C_ν. Hence, (3) holds for both $\mu = 0$ and $\mu > 0$.

Case 2. $v_{l-1} \le \nu < v_l$ when $2 \le l \le m-1$ or $v_{m-1} \le \nu \le v_m$ when $l = m$. If ν is good, set $p_i = p_{\nu,i}$, $i = 1, 2, \ldots, m$; then by Definitions 1 and 2, we get (cf. [6] for the proof)

$$r_j - 2p_j + 1 > r_i - 2p_i - 1 \qquad \text{for } 1 \le j \le l \text{ and } i = 1, 2, \ldots, m. \tag{32}$$

Thus (17)–(20) become

$$\lim_{n\to\infty} n^{-(r_j-k-p_j+1+s)} g^{(s)}_{n,k,j}(\alpha_j) = N^{\{k\}}_{s,jj}, \qquad k = 1, 2, \ldots, p_j,\ j = 1, \ldots, l, \tag{33}$$

where, for $j = 1, \ldots, l$,

$$N^{\{k\}}_{s,jj} = \begin{cases} \dfrac{(-1)^s H(\alpha_j)^{r_j-k+1-p_j+s} B_{r_j,j}}{(r_j-k+1-p_j-1)!(r_j-k+1-p_j+s)}, & \text{if } k \le r_j - p_j,\ s = 0, 1, \ldots, p_j; \\ B_{r_j,j}, & \text{if } k = r_j - p_j + 1,\ s = 0; \\ 0, & \text{if } k = r_j - p_j + 1,\ s = 1, \ldots, p_j; \\ 0, & \text{if } k > r_j - p_j + 1,\ s = 0, 1, \ldots, p_j. \end{cases} \tag{34}$$

Notice that $p_i = 0$ for $l + 1 \le i \le m$; let

$$w_{n,s}(z) := L_{n+\mu+\nu}\Big(\prod_{i=1}^{m}(z-\alpha_i)^{2p_i}\pi_{s-1}(z)\pi(z)f(z)\,;\ z\Big), \qquad s = 1, 2, \ldots, \mu,$$

and construct

$$\begin{aligned} q_n(z) = &\sum_{j=1}^{l}\sum_{k=1}^{p_j} a^{(n)}_{k,j}(z-\alpha_j)^{k-1}\prod_{i=1,i\ne j}^{l}(z-\alpha_i)^{p_i}\pi(z) \\ &+ \sum_{s=1}^{\mu} b^{(n)}_s \pi_{s-1}(z)\prod_{i=1}^{l}(z-\alpha_i)^{p_i} + \pi(z)\prod_{i=1}^{l}(z-\alpha_i)^{p_i}, \\ h_n(z) = &\sum_{j=1}^{l}\sum_{k=1}^{p_j} a^{(n)}_{k,j} g_{n,k,j}(z) + \sum_{s=1}^{\mu} b^{(n)}_s w_{n,s}(z) + g_{n,p_1+1}(z). \end{aligned}$$

We want $h_n(z) = \prod_{i=1}^{l}(z-\alpha_i)^{p_i}\pi(z)d_n(z)$ where $d_n(z)$ is a polynomial of degree $\le n$, so we have

$$\tilde{M}_{n,l+1\times l+1} X_n = Y_n \tag{35}$$

where $\tilde{M}_{n,l+1\times l+1}$ is a matrix of matrices with entries

$$M_{n,j,i} := \big(g^{(s)}_{n,k,i}(\alpha_j)\big)_{s=0,1,\ldots,p_j-1;k=1,2,\ldots,p_i}, \qquad i, j = 1, \ldots, l;$$

$$M_{n,j,l+1} := \big(w^{(s)}_{n,k}(\alpha_j)\big)_{s=0,1,\ldots,p_j-1;k=1,2,\ldots,\mu}, \qquad j = 1, \ldots, l;$$

$$M_{n,l+1,i} := \begin{pmatrix} g_{n,1,i}(\gamma_1) & g_{n,2,i}(\gamma_1) & \cdots & g_{n,p_i,i}(\gamma_1) \\ \vdots & \vdots & & \vdots \\ g^{(i_1-1)}_{n,1,i}(\gamma_1) & g^{(i_1-1)}_{n,2,i}(\gamma_1) & \cdots & g^{(i_1-1)}_{n,p_i,i}(\gamma_1) \\ \vdots & \vdots & & \vdots \\ g_{n,1,i}(\gamma_\mu) & g_{n,2,i}(\gamma_\mu) & \cdots & g_{n,p_i,i}(\gamma_\mu) \\ \vdots & \vdots & & \vdots \\ g^{(i_\iota-1)}_{n,1,i}(\gamma_\mu) & g^{(i_\iota-1)}_{n,2,i}(\gamma_\mu) & \cdots & g^{(i_\iota-1)}_{n,p_i,i}(\gamma_\mu) \end{pmatrix};$$

$$M_{n,l+1,l+1} := C_{n,\mu\times\mu};$$

$$X_n = \left(a_{1,1}^{(n)}, a_{2,1}^{(n)}, \dots, a_{p_1,1}^{(n)}, \dots, a_{1,l}^{(n)}, a_{2,l}^{(n)}, \dots, a_{p_l,l}^{(n)}, b_1^{(n)}, b_2^{(n)}, \dots, b_\mu^{(n)}\right)^{\mathrm{T}},$$

$$\begin{aligned} Y_n = \big(&- g_{n,p_1+1}(\alpha_1), -g'_{n,p_1+1}(\alpha_1), \dots, -g^{(p_1-1)}_{n,p_1+1}(\alpha_1), \\ &\dots, -g_{n,p_1+1}(\alpha_l), -g'_{n,p_1+1}(\alpha_l), \dots, -g^{(p_l-1)}_{n,p_1+1}(\alpha_l), \\ &- g_{n,p_1+1}(\gamma_1), \dots, -g^{(i_1-1)}_{n,p_1+1}(\gamma_1), \dots, -g_{n,p_1+1}(\gamma_\mu), \dots, -g^{(i_1-1)}_{n,p_1+1}(\gamma_\mu)\big)^{\mathrm{T}}. \end{aligned}$$

Let $\tilde{N}_{l+1\times l+1}$ be a diagonal matrix of matrices with diagonal entries $N_{i,i}$, let $\tilde{W}_{l+1\times l+1}$ be another diagonal matrix of matrices with its diagonal elements $W_{i,i}$ where, for $i = 1, 2, \dots, l$,

$$N_{i,i} = \begin{pmatrix} n^{-(r_i-2p_i)} & & & \\ & n^{-(r_i-2p_i+1)} & & \\ & & \ddots & \\ & & & n^{-(r_i-p_i-1)} \end{pmatrix},$$

$$W_{i,i} = \begin{pmatrix} n^{-p_i} & & & \\ & n^{-p_i+1} & & \\ & & \ddots & \\ & & & n^{-1} \end{pmatrix},$$

$N_{l+1,l+1} := n^{\tau} I_\mu$, ($I_\mu$ is the identity matrix of order μ), and $W_{l+1,l+1} := N^{-1}_{l+1,l+1}$. Then

$$\tilde{M}^\star_{n,l+1\times l+1} := \tilde{N}_{l+1\times l+1} \cdot \tilde{M}_{n,l+1\times l+1} \cdot \tilde{W}_{l+1\times l+1},$$

$$X^\star_n = \tilde{W}^{-1}_{l+1\times l+1} \cdot X_n, \quad \text{and} \quad Y^\star_n = \tilde{N}_{l+1\times l+1} \cdot Y_n$$

satisfy

$$\tilde{M}^\star_{n,l+1\times l+1} X^\star_n = Y^\star_n. \tag{36}$$

Let $M^\star_{i,i} = \lim_{n\to\infty} M^\star_{n,i,i}$. By (33), we obtain, for $i = 1, \dots, l$,

$$\det(M^\star_{i,i}) := \lim_{n\to\infty} \det(M^\star_{n,i,i}) = \lim_{n\to\infty} n^{-p_i(r_i-p_i)} \det(M_{n,i,i}) = d_{p_i,i} D_{p_i,i} \neq 0. \tag{37}$$

Similar to the reasoning in the proof of Case 1, we obtain for any $\tau > 0$,

$$\begin{aligned} M^\star_{j,l+1} &:= \lim_{n\to\infty} M^\star_{n,j,l+1} = V_{p_j\times\mu}, \qquad j = 1, \dots, l; \\ M^\star_{l+1,i} &:= \lim_{n\to\infty} M^\star_{n,l+1,i} = O_{\mu\times p_i}, \qquad i = 1, \dots, l. \end{aligned} \tag{38}$$

As for the limits of $M_{n,i,j}$ where $i \neq j$, we need further discussions. If $r_i - 2p_i \le 0$ for $1 \le i \le m$ and $r_j - p_j - k + 1 \le 0$, then $g_{n,k,j}(z)$ interpolates an analytic function. Furthermore, the interpolated function $(z-\alpha_j)^{k-1+p_j} \prod_{t=1,t\neq j}^m (z-\alpha_t)^{2p_t} \pi(z) f(z)$ has a zero of order $k - 1 - r_j + p_j$ at $z = \alpha_j$ and a zero of order $2p_i - r_i$ at $z = \alpha_i$, $i \neq j$, $1 \le i \le m$. Hence, we have

$$\lim_{n\to\infty} n^{-(r_i-2p_i+p_j-k+1+s)} g^{(s)}_{n,k,j}(\alpha_i) = 0, \tag{39}$$

by (20) for $s = 0, 1, \dots, 2p_i - r_i - 1$ and by (18),(19) for $s \ge 2p_i - r_i$. On the other hand, if $r_i - 2p_i \le 0$ for $1 \le i \le m$ and $r_j - p_j - k + 1 > 0$, then the interpolated function has a pole of order $k - 1 - r_j - p_j$ at $z = \alpha_j$. If one of $r_i - 2p_i > 0$, then all $r_j - p_j - k + 1 > 0$ for $1 \le k \le p_j$ according to (32). The interpolated function has at least one pole. Hence, the elements of $M^\star_{n,j,i}$ may not tend to zero; however, they are bounded. By (32), we get

$$r_i - 2p_i + 1 \geq r_j - 2p_j \geq r_i - 2p_i - 1 \quad \text{for } 1 \leq i, j \leq l,$$

so we have three possibilities: $r_i - 2p_i + 1 = r_j - 2p_j$; $r_i - 2p_i = r_j - 2p_j$; $r_i - 2p_i - 1 = r_j - 2p_j$. Let us study them one by one.

(a) $r_i - 2p_i + 1 = r_j - 2p_j$. In this case

$$\max\{(r_i - 2p_i + s), (r_j - p_j - k + s)\} = r_j - p_j - k + s,$$
$$\max\{(r_i - p_i - k + s), (r_j - 2p_j + s)\} = \begin{cases} r_i - p_i - k + s & \text{for } 1 \leq k \leq p_i - 1, \\ r_j - 2p_j + s = r_i - 2p_i + 1 + s & \text{for } k = p_i. \end{cases}$$

Thus by (24), the elements of $M^\star_{n,j,i}$ tend to a limit:

$$\lim_{n\to\infty} n^{-(r_j-2p_j+p_i-k+1+s)} g^{(s)}_{n,k,i}(\alpha_j) = \lim_{n\to\infty} n^{-(r_i-2p_i+1+p_i-k+1+s)} g^{(s)}_{n,k,j}(\alpha_j)$$
$$= \lim_{n\to\infty} n^{-2} n^{-(r_i-p_i-k+s)} g^{(s)}_{n,k,i}(\alpha_j) = 0, \qquad 1 \leq k \leq p_i, \quad i \neq j;$$

on the other hand, the elements of $M^\star_{n,i,j}$ are bounded:

$$\limsup_{n\to\infty} n^{-(r_i-2p_i+p_j-k+1+s)} \left|g^{(s)}_{n,k,j}(\alpha_i)\right| = \limsup_{n\to\infty} n^{-(r_j-2p_j-1+p_j-k+1+s)} \left|g^{(j)}_{n,k,j}(\alpha_i)\right|$$
$$= \limsup_{n\to\infty} n^{-(r_j-p_j-k+s)} \left|g^{(s)}_{n,k,j}(\alpha_i)\right| = \left|N^{\{k\}}_{s,ji}\right|, \qquad 1 \leq k \leq p_j, \quad i \neq j.$$

This means that

$$\lim_{n\to\infty} M^\star_{n,j,i} = O_{p_j\times p_i}, \quad \limsup_{n\to\infty} M^\star_{n,i,j} = U^\star_{p_i\times p_j}, \quad \liminf_{n\to\infty} M^\star_{n,i,j} = V^\star_{p_i\times p_j}, \tag{40}$$

where $U^\star_{p_i\times p_j}$ and $V^\star_{p_i\times p_j}$ are finite matrices and $O_{p\times q}$ is a zero matrix.

(b) For the case $r_i - 2p_i - 1 = r_j - 2p_j$, we only need to exchange the positions of i and j.

(c) $r_i - 2p_i = r_j - 2p_j$. In this case

$$\max\{(r_i - p_i - k + s), (r_j - 2p_j + s)\} = r_i - p_i - k + s \quad \text{for } 1 \leq k \leq p_i,$$
$$\max\{(r_i - 2p_i + s), (r_j - p_j - k + s)\} = r_j - p_j - k + s \quad \text{for } 1 \leq k \leq p_j.$$

Thus,

$$\lim_{n\to\infty} n^{-(r_j-2p_j+p_i-k+1+s)} g^{(s)}_{n,k,i}(\alpha_j) = \lim_{n\to\infty} n^{-(r_i-2p_i+p_i-k+1+s)} g^{(s)}_{n,k,i}(\alpha_j) = 0, \ i \neq j,$$
$$\lim_{n\to\infty} n^{-(r_i-2p_i+p_j-k+1+s)} g^{(s)}_{n,k,j}(\alpha_i) = \lim_{n\to\infty} n^{-(r_j-2p_j+p_j-k+1+s)} g^{(s)}_{n,k,j}(\alpha_i) = 0, \ i \neq j.$$

Hence, if $r_i - 2p_i = r_j - 2p_j$ and $i \neq j$, then

$$\lim_{n\to\infty} M^\star_{n,j,i} = O_{p_j\times p_i}, \qquad \lim_{n\to\infty} M^\star_{n,i,j} = O_{p_i\times p_j}. \tag{41}$$

Generally speaking, we are not sure of the existence of $\lim_{n\to\infty} M^\star_{n,i,j}$ for $i \neq j$. However, for every subsequence there exists a convergent sub-subsequence. Let $M^\star_{i,j}$ be one limit of $\{M^\star_{n,i,j}\}$. Furthermore, let $\tilde{M}^\star_{l+1\times l+1}$ be one limit of $\{\tilde{M}^\star_{n,l+1\times l+1}\}$. Then all entries of $\tilde{M}^\star_{l+1\times l+1}$ are bounded; some are zero matrices. We state that

$$\det(\tilde{M}^\star_{l+1\times l+1}) = \prod_{i=1}^{l} \det(M^\star_{i,i}) \cdot \det C_{\mu\times\mu}. \tag{42}$$

To verify this statement, we let $c_i = r_i - 2p_i$ for $1 \le i \le l$, and by (32), they satisfy

$$-1 \le c_i - c_j \le 1 \qquad \text{for } 1 \le i, j \le l.$$

Without loss of generality, we assume that (we can always rearrange $c_1, c_2, \ldots, c_l$ in such an order that)

$$c_2 - c_1 = 1, \ldots, c_t - c_1 = 1, c_{t+1} - c_1 = 0, \ldots, c_l - c_1 = 0,$$

where t is a number between 1 and l and $t = 1$ means all c_i's are the same. In fact, if $c_2 - c_1 = 1$, then $c_i - c_1 \neq -1$ because $c_2 - c_i \neq 2$. Thus, the above group of relations is valid. If $t = 1$, then all $M^\star_{i,j} = O_{p_i \times p_j}$ for $i \neq j$; we are done with our verification. If $t > 1$, for this arrangement, $M^\star_{j,1} = O_{p_j \times p_1}$ for $j = 2, \ldots, l$ by (40) and (41). Thus

$$\det(\tilde{M}^\star_{l+1\times l+1}) = \det M^\star_{1,1} \det \begin{pmatrix} M^\star_{2,2} & \cdots & M^\star_{2,l} \\ \vdots & & \vdots \\ M^\star_{l,2} & \cdots & M^\star_{l,l} \end{pmatrix} \det C_{\mu\times\mu}.$$

Since

$$c_3 - c_2 = 0, \ldots, c_t - c_2 = 0, c_{t+1} - c_2 = -1, \ldots, c_l - c_2 = -1,$$

$M^\star_{2,j} = O_{p_2 \times p_j}$ for $j = 3, \ldots, l$ by (40) and (41). Thus

$$\det(\tilde{M}^\star_{l+1\times l+1}) = \det M^\star_{1,1} \det M^\star_{2,2} \det \begin{pmatrix} M^\star_{3,3} & \cdots & M^\star_{3,l} \\ \vdots & & \vdots \\ M^\star_{l,3} & \cdots & M^\star_{l,l} \end{pmatrix} \det C_{\mu\times\mu}.$$

By repeating these steps, we get (42). Hence,

$$\lim_{n\to\infty} \det(\tilde{M}^\star_{n,l+1\times l+1}) = \lim_{n\to\infty} \prod_{i=1}^{l} \det(M^\star_{n,i,i}) \cdot \det C_{n,\mu\times\mu} \neq 0\,. \tag{43}$$

Notice that the limit is the same for every convergent subsequence and $\{\tilde{M}^\star_{n,l+1\times l+1}\}$ is a bounded sequence. Thus, the above limit is true for the whole sequence. We conclude from (43) that (35) has a unique solution for n large enough,

$$\frac{h_n(z)}{\prod_{i=1}^{l}(z-\alpha_i)^{p_i}\pi(z)q_n(z)},$$

which is the rational interpolation of type $(n, \mu+\nu)$ to $f(z)$ on the set $\Lambda(n+\mu+\nu)$. Furthermore, we have the estimations for the coefficients:

$$\begin{aligned} &\lim_{n\to\infty} n^\tau b_k^{(n)} = 0 && \text{for } k = 1, 2, \ldots, \mu, \text{ any } \tau, \\ &\lim_{n\to\infty} n^{p_i-k+1} a_{k,i}^{(n)} = \frac{d_{p_i}^{\{k\}} D_{p_i}^{\{k\}} H(\alpha_i)^{k-p_i-1}}{d_{p_i} D_{p_i}} && \text{for } k = 1, 2, \ldots, p_i, \end{aligned} \tag{44}$$

where d_{p_i} and D_{p_i} are constants only related to p_i, $H(\alpha_i)$, and $B_{r_i,i}$, and $d_{p_i}^{\{k\}}$ and $D_{p_i}^{\{k\}}$ are constants only related to k, p_i, $H(\alpha_i)$, and $B_{r_i,i}$; $i = 1, \ldots, l$. Thus (4) is true. Since $\nu = \sum_{i=1}^{l} p_i$, we have

$$
\begin{aligned}
f(z) - R_{n,\mu+\nu}(z) &= \frac{q_n(z)\pi(z)\prod_{i=1}^{l}(z-\alpha_i)^{p_i} f(z) - h_n(z)}{q_n(z)\pi(z)\prod_{i=1}^{l}(z-\alpha_i)^{p_i}} \\
&= \frac{1}{q_n(z)\pi(z)\prod_{i=1}^{l}(z-\alpha_i)^{p_i}} \\
&\quad \times \Bigg\{ \sum_{l=1}^{\mu} b_l^{(n)} \left[\prod_{i=1}^{l}(z-\alpha_i)^{2p_i} \pi_{l-1}(z)\pi(z) f(z) - w_{n,l}(z) \right] \\
&\qquad + \sum_{i=1}^{l}\sum_{k=1}^{p_i} a_{k,i}^{(n)} \left[(z-\alpha_i)^{k-1+p_i} \prod_{j=1, j\neq i}^{l} (z-\alpha_j)^{2p_j} \pi^2(z) f(z) - g_{n,k,1}(z) \right] \\
&\qquad + \prod_{i=1}^{l}(z-\alpha_i)^{2p_i}\pi^2(z) f(z) - g_{n,p_1+1}(z) \Bigg\} \\
&= \frac{\omega_{n+\mu+\nu}(z)}{q_n(z)\pi(z)\prod_{i=1}^{l}(z-\alpha_i)^{p_i}} \\
&\quad \times \left\{ \sum_{1\le i\le l} \frac{C_{p_i,i} L_{p_i,i} n^{r_i-2p_i-1}}{(z-\alpha_i)\omega_{n+\mu+\nu}(\alpha_i)} + \sum_{r_i-2p_i-1=\lambda\ge 0} \frac{C_{p_i,i} n^{\lambda}}{(z-\alpha_i)\omega_{n+\mu+\nu}(\alpha_i)} + o(1) \right\}.
\end{aligned}
$$

Hence, (3) is true for $\lambda = \max_{1\le i\le l}\{r_i - 2p_i - 1\} = \max_{1\le i\le m}\{r_i - 2p_i - 1\}$.

Case 3. $v_l < \nu \le v_{l+1}$, $m+1 \le l \le 2m-1$. Set $p_i = p_{\nu,i}$ for $i = 1, \ldots, m$. Then $p_i = r_i$ for $i = 2m-l+1, \ldots, m$, and (cf. [6])

$$r_i - 2p_i - 1 \le r_j - 2p_j \le r_i - 2p_i + 1 \qquad \text{for } 1 \le i, j \le 2m-l. \tag{45}$$

Construct the same $q_n(z)$ and $h_n(z)$ as in Case 2 for $l = m$. Similarly, we want $h_n(z) = \prod_{i=1}^{m}(z-\alpha_i)^{p_i}\pi(z)d_n(z)$, where $d_n(z)$ is a polynomial of degree $\le n$, so we have

$$\tilde{M}_{n,m+1\times m+1} X_n = Y_n \tag{46}$$

where $\tilde{M}_{n,m+1\times m+1}$, X_n, and Y_n are the same as in Case 2 for $l = m$. Furthermore,

$$
\begin{gathered}
\tilde{M}^{\star}_{n,m+1\times m+1} := \tilde{N}_{m+1\times m+1} \cdot \tilde{M}_{n,m+1\times m+1} \cdot \tilde{W}_{m+1\times m+1}, \\
X_n^{\star} = \tilde{W}^{-1}_{m+1\times m+1} \cdot X_n, \qquad Y_n^{\star} = \tilde{N}_{m+1\times m+1} \cdot Y_n
\end{gathered}
$$

satisfy

$$\tilde{M}^{\star}_{n,m+1\times m+1} X_n^{\star} = Y_n^{\star}. \tag{47}$$

Here $\det C_{n,\mu\times\mu}$ still tends to a nonzero limit, and we still have

$$
\begin{aligned}
M^{\star}_{j,m+1} &:= \lim_{n\to\infty} M^{\star}_{n,j,m+1} = V_{p_j\times\mu}, \qquad j = 1, \ldots, m, \\
M^{\star}_{m+1,i} &:= \lim_{n\to\infty} M^{\star}_{n,m+1,i} = O_{\mu\times p_i}, \qquad i = 1, \ldots, m.
\end{aligned} \tag{48}
$$

We only need to study $M^{\star}_{n,i,j}$ now. It is easy to check by Definition 2 that

$$r_i - 2p_i \le 0 \qquad \text{for } 1 \le i \le m.$$

Thus, if $2m - l + 1 \le j \le m$, the $g_{n,k,j}(z)$ interpolates an analytic function. Furthermore, the interpolated function $(z-\alpha_j)^{k-1+p_j}\prod_{t=1,t\neq j}^{m}(z-\alpha_t)^{2p_t}\pi(z)f(z)$ has a zero of order $k-1$ at $z=\alpha_j$ and a zero of order $2p_i - r_i$ at $z=\alpha_i$, $i \neq j$, $1 \le i \le m$. Hence, for elements of $M^{\star}_{n,i,j}$, (39) is true. This gives us

$$\lim_{n\to\infty} M^{\star}_{n,i,j} = O_{p_i \times p_j} \qquad \text{for } 2m-l+1 \le j \le m,\ 1 \le i \le m,\ i \neq j.$$

As for a fixed $j \le 2m-l$, $M^{\star}_{n,i,j}$ may not tend to a zero matrix, but they are bounded. In fact, $g^{(s)}_{n,k,j}(z)$ interpolates an analytic function when $r_j - p_j - k + 1 \le 0$. For this situation, the above reasoning is still valid. That means

$$\lim_{n\to\infty} n^{-(r_i-2p_i+p_j-k+1+s)} g^{(s)}_{n,k,j}(\alpha_i) = 0,$$
$$0 \le s \le p_j - 1, \quad 1 \le i \le m, \quad i \neq j, \quad k \ge r_j - p_j + 1.$$

When $1 \le k < r_j - p_j + 1$, the interpolated function

$$(z-\alpha_j)^{k-1+p_j} \prod_{t=1, t\neq j}^{m} (z-\alpha_t)^{2p_t} \pi(z) f(z)$$

has only one pole at $z=\alpha_j$ of order $r_j - p_j - k + 1$, and (23) and (24) are true here. For $1 \le i \le 2m-l$, (45) is true; for $2m-l < i \le m$,

$$\begin{aligned} r_j - 2p_j - 1 &\le r_j - 2[k_l + u_{j,2m-l} + r_{2m-l+1}] - 1 \\ &\le r_j - 2k_l - (r_j - r_{2m-l+1} - 1) - 2r_{2m-l+1} - 1 \\ &= -2k_l - r_{2m-l+1} \le -r_{2m-l+1} \le -r_i = r_i - 2p_i. \end{aligned}$$

Hence,

$$\begin{aligned} &\limsup\nolimits_{n\to\infty} n^{-(r_i-2p_i+p_j-k+1+s)} \left| g^{(s)}_{n,k,j}(\alpha_i) \right| \\ &\quad = \limsup_{n\to\infty} n^{-(r_i-2p_i-r_j+2p_j+1)} n^{-(r_j-p_j-k+s)} \left| g^{(s)}_{n,k,j}(\alpha_i) \right| < \infty, \end{aligned}$$

for $0 \le s \le p_j - 1, \quad 1 \le i \le m, \quad i \neq j, \quad k < r_j - p_j + 1$. We conclude that for any convergent subsequence,

$$\begin{aligned} \lim_{n\to\infty} \det(\tilde{M}^{\star}_{n,m+1\times m+1}) &= \lim_{n\to\infty} \det \begin{pmatrix} M^{\star}_{n,1,1} & \cdots & M^{\star}_{n,1,2m-l} \\ \vdots & & \vdots \\ M^{\star}_{n,2m-l,1} & \cdots & M^{\star}_{n,2m-l,2m-l} \end{pmatrix} \\ &\quad \times \det \begin{pmatrix} M^{\star}_{n,2m-l+1,2m-l+1} & & & \\ & \ddots & & \\ & & M^{\star}_{n,m,m} & \\ & & & C_{n,\mu\times\mu} \end{pmatrix}. \end{aligned}$$

Applying reasoning similar to that in Case 2 to the matrix in the first determinant, we deduce that (42) is still true. For the second determinant,

$$\lim_{n\to\infty} n^{-(r_j-p_j-k+1+s)} g^{(s)}_{n,k,j}(\alpha_j) = n^{-(-k+1+s)} g^{(s)}_{n,k,j}(\alpha_j) = \begin{cases} B_{r_j,j} & \text{when } k = s+1, \\ 0 & \text{when } k \neq s+1. \end{cases}$$

Thus, (37) still holds here. We come to the conclusion

$$\begin{aligned} \det(\tilde{M}^{\star}_{m+1\times m+1}) &= \lim_{n\to\infty} \det(\tilde{M}^{\star}_{n,m+1\times m+1}) \\ &= \lim_{n\to\infty} \prod_{i=1}^{m} \det(M^{\star}_{n,i,i}) \cdot \det C_{n,\mu\times\mu} \neq 0\,. \end{aligned}$$

Thus, all conclusions in case 2 are still valid for this case. This means that (46) has a unique solution for n large enough, $h_n(z)/\left(\prod_{i=1}^{m}(z-\alpha_i)^{p_i}\pi(z)q_n(z)\right)$, which is the rational interpolation of type $(n, \mu+\nu)$ to $f(z)$ on the set $\Lambda(n+\mu+\nu)$. The estimates (44) for the coefficients are still true. Since $\nu = \sum_{i=1}^{m} p_i$, we have

$$f(z) - R_{n,\mu+\nu}(z) = \frac{\omega_{n+\mu+\nu}(z)}{q_n(z)\pi(z)\prod_{i=1}^{m}(z-\alpha_i)^{p_i}} \times \left\{ \sum_{1\le i<2m-l} \frac{C_{p_i,i}L_{p_i,i}n^{r_i-2p_i-1}}{(z-\alpha_i)\omega_{n+\mu+\nu}(\alpha_i)} + \sum_{r_i-2p_i-1=\lambda\ge 0} \frac{C_{p_i,i}n^{\lambda}}{(z-\alpha_i)\omega_{n+\mu+\nu}(\alpha_i)} + o(1) \right\}.$$

Hence, (3) is valid for $\lambda = \max_{1\le i\le m}\{r_i - 2p_i - 1\}$.

The proof of Theorem 1 is completed. □

4. Other results

Corollary 1. *Let $f(z)$ be as in Theorem 1. Suppose that ν is good by Definition 1, and $R_{n,\mu+\nu}(z)$ is a rational function of type $(n, \mu+\nu)$ interpolating $f(z)$ on the set $\Lambda(n+\mu+\nu)$. Then, except for a finite number of points, the inequality*

$$\limsup_{n\to\infty} \left| \frac{R_{n,\mu+\nu}(z)\rho^n}{n^{\lambda}\exp(nG(z))} \right| > 0$$

holds for each z exterior to Γ_ρ. Consequently, the sequence $\{R_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$ can converge in at most a finite number of points exterior to Γ_ρ.

Proof. Assume ν is good; according to the proof of Theorem 1, we get

$$f(z) - R_{n,\mu+\nu}(z) = \frac{\omega_{n+\mu+\nu}(z)}{q_n(z)\pi(z)\prod_{i=1}^{m}(z-\alpha_i)^{p_i}} \left\{ \sum_{i=1}^{m} \frac{C_{\nu,i}n^{r_i-2p_i-1}}{(z-\alpha_i)\omega_{n+\mu+\nu}(\alpha_i)} + o(1) \right\},$$

where, if $\nu < \sum_{i=1}^{m} r_i$, then at least one $C_{\nu,i} \ne 0$, $i = 1, 2, \ldots, m$. Thus,

$$\left| [f(z) - R_{n,\mu+\nu}(z)]q_n(z)\frac{\omega_{n+\mu+\nu}(\alpha_1)}{\omega_{n+\mu+\nu}(z)n^{\lambda}} \right| \le A$$

for any compact set exterior to Γ_ρ. We conclude that the functions inside the absolute value form a normal family. Consequently, there exists a convergent subsequence. Because (for at least a subsequence)

$$\lim_{n\to\infty} \sum_{i=1}^{m} \frac{C_{\nu,i}n^{r_i-2p_i-1-\lambda}\omega_{n+\mu+\nu}(\alpha_1)}{(z-\alpha_i)\omega_{n+\mu+\nu}(\alpha_i)} = C(z) \not\equiv 0,$$

and

$$\lim_{n\to\infty} f(z)q_n(z)\frac{\omega_{n+\mu+\nu}(\alpha_1)}{\omega_{n+\mu+\nu}(z)n^{\lambda}} = 0$$

for z exterior to Γ_ρ, we obtain

$$\limsup_{n\to\infty} \left| R_{n,\mu+\nu}(z)\rho^n \exp(-nG(z))n^{-\lambda} \right| > 0 \quad \text{except in a finite set.}$$

This means that $\{R_{n,\mu+\nu}(z)\}_{n=0}^{\infty}$ is divergent in $\mathbf{C}\backslash E_\rho \bigcup \Gamma_\rho$ except at finite points. □

Remark. If we let $\beta_i^{(n)} = 0$ for $1 \le i \le n+1$, all n, then $\omega_n(z) = z^{n+1}$, $E = \{z : |z| \le \delta\}$ satisfy the assumptions and the multipoint Padé approximations become Padé approximations. Hence, our results also can be applied to Padé approximations.

5. An example for "bad" ν's

In this section, we give a proposition about "bad" ν's. The investigation on the poles of $\{[n/1](z)\}_{n=0}^{\infty}$ for the function $f(z)$ with exactly two different simple poles (cf. [1], p.239) has been done. We point out that the limit point set of the poles could make up a continuous curve.

Proposition 1. *Let $E = \{z : |z| \le 1/2\}$, E_ρ be the open unit disk, and*

$$f(z) = \frac{g(z)}{(z-a)(z-b)}$$

where $|a| = |b| = 1$ and $a \ne b$, and $g(z) = \sum_{n=0}^{\infty} g_n z^n$ is analytic on $|z| \le 1$ with $g(a) \ne 0$ and $g(b) \ne 0$. Let $\phi = arg(b/a)$ and $[n/1](z)$ be the $(n,1)$ Padé approximant of $f(z)$. Then, if ϕ/π is a rational number t/s (t and s have no common factors), the sequence $\{[n/1](z)\}$ has at most $2s$ distinct poles; if ϕ/π is irrational, the limit point set of poles of $\{[n/1](z)\}$ make up an arc of a circle in $|z| < 1$. Let O be the closure of the set of poles of $\{[n/1](z)\}$, then $\{[n/1](z)\}$ is uniformly convergent on any compact subset of $E_\rho \setminus O$.

Proof. The function $f(z)$ is analytic on E_ρ and analytic on Γ_ρ except at 2 points $z_1 = a$, $z_2 = b$. The function $f(z)$ and E satisfy the assumptions, so we can apply Definitions 1 and 2. Since $m = 2$, $r_1 = 1$, $r_2 = 1$, good values of ν are 0 and 2. The value $\nu = 1$ is "bad". Now, let us calculate the Padé approximants $\{[n/1](z)\}$ and their poles.

We consider first $g(z) = 1$. Because the Taylor expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} \frac{1}{(b-a)} \left(\frac{1}{a^{n+1}} - \frac{1}{b^{n+1}} \right) z^n,$$

for each n, with $f_{-1} := 0$,

$$[n/1](z) = \frac{\sum_{j=0}^{n} z^j \det \begin{pmatrix} f_j & f_{j-1} \\ f_{n+1} & f_n \end{pmatrix}}{\det \begin{pmatrix} 1 & z \\ f_{n+1} & f_n \end{pmatrix}} = \frac{\sum_{j=0}^{n} (-f_{n+1} f_{j-1} + f_n f_j) z^j}{-f_{n+1} z + f_n}$$

$$= \frac{(-f_{n+1} z + f_n) \sum_{j=0}^{n-1} f_j z^j + f_n^2 z^n}{-f_{n+1} z + f_n}.$$

When $f_n \ne 0$, it is a non-degenerate case. Furthermore, when $f_{n+1} = 0$, there is no pole; when $f_{n+1} \ne 0$, the pole of $[n/1](z)$ should be

$$z_{n,p} = \frac{f_n}{f_{n+1}} = \frac{\frac{1}{b-a}\left(\frac{1}{a^{n+1}} - \frac{1}{b^{n+1}} \right)}{\frac{1}{b-a}\left(\frac{1}{a^{n+2}} - \frac{1}{b^{n+2}} \right)}$$

$$= \frac{a\left(1 - e^{i\phi} e^{-(n+2)i\phi}\right)}{1 - e^{-(n+2)i\phi}}$$

where $e^{-i\phi} := a/b$. If ϕ/π is rational, then there will be finite distinct poles $\{z_{n,p}\}$. For the special case $\phi = \pi$, the Padé approximant $[n/1](z)$ is either a degenerate case or has a constant denominator, so there is no pole. On the other hand, if ϕ/π is irrational, by a result due to Bohl, Sierpinski, and Weyl (cf. [10]), $\{e^{ik\phi}\}_{k=0}^{\infty}$ is uniformly distributed on the unit circle, i.e., the sequence is dense on the unit circle.

Thus, if ϕ/π is irrational, then the limit point set of $\{z_{n,p}\}$ consists of points of the form

$$z_\theta = \frac{a(1-e^{i\phi}e^{i\theta})}{1-e^{i\theta}}, \quad \theta \in [0, 2\pi].$$

Generally, if

$$f(z) = \frac{g(z)}{(z-a)(z-b)}$$

where $g(z) = \sum_{n=0}^{\infty} g_n z^n$ is analytic on $|z| \le 1$ and $g(a) \ne 0$, $g(b) \ne 0$, similar to before, then for the non-degenerate case and $f_{n+1} \ne 0$, the pole of the Padé approximant $[n/1](z)$ of $f(z)$ is

$$\begin{aligned} z_{n,p} = \frac{f_n}{f_{n+1}} &= \frac{\sum_{k=0}^{n} \frac{1}{b-a}\left(\frac{1}{a^{n+1-k}} - \frac{1}{b^{n+1-k}}\right) g_k}{\sum_{k=0}^{n+1} \frac{1}{b-a}\left(\frac{1}{a^{n+2-k}} - \frac{1}{b^{n+2-k}}\right) g_k} \\ &= \frac{a^{-n-1}g(a) + \epsilon_n - b^{-n-1}g(b) - \varepsilon_n}{a^{-n-2}g(a) + \epsilon_{n+1} - b^{-n-2}g(b) - \varepsilon_{n+1}} \\ &= \frac{a\big(r - e^{i\phi}e^{-(n+2)i\phi}\big) + a^{n+2}(\epsilon_n - \varepsilon_n)/g(b)}{r - e^{-(n+2)i\phi} + a^{n+2}(\epsilon_{n+1} - \varepsilon_{n+1})/g(b)} \end{aligned}$$

where $r := g(a)/g(b)$ (recall $g(b) \ne 0$) and

$$\epsilon_n := a^{-n-1}\left(\sum_{k=0}^{n} g_k a^k - g(a)\right) \to 0,$$

$$\varepsilon_n := b^{-n-1}\left(\sum_{k=0}^{n} g_k b^k - g(b)\right) \to \infty$$

as $n \to 0$, since $|a| = 1$, $|b| = 1$. For convergent subsequences,

$$z_{n_j,p} \to \frac{a(r - e^{i\phi}e^{i\theta})}{r - e^{i\theta}} \quad \text{for} \quad n_j \to \infty \quad \text{and some} \quad \theta \in [0, 2\pi).$$

Notice that if $|r| \ne 1$, then $f_n \ne 0$ for n large enough; thus $[n/1](z)$ is a non-degenerate case for n large enough. If $r = e^{i\gamma}$ for some $\gamma \in [0, 2\pi)$, then the only degenerate cases are in the subsequence $\{[n_j/1](z)\}$ for n_j's satisfying $e^{i\gamma} = e^{-(n_j+1)i\phi}$. For n_j's satisfying $e^{i\gamma} = e^{-(n_j+2)i\phi}$, the subsequence $\{[n_j/1](z)\}$ has no poles because all denominators are constants.

For the same reason, if ϕ/π is rational, then there only will be finite distinct poles $\{z_{n,p}\}$. Obviously, if we write $\phi/\pi := t/s$ where t and s are integers having no common factors, then there will be at most $2s$ distinct poles. On the other hand, if ϕ/π is irrational, then the limit point set of $\{z_{n,p}\}$ is made up of points of the form

$$z_\theta = \frac{a(r - e^{i\phi}e^{i\theta})}{r - e^{i\theta}}, \qquad \theta \in [0, 2\pi].$$

Since $w = \frac{a(r-e^{i\phi}z)}{r-z}$ is a Möbius transformation on the complex plane, it maps a circle not passing through its pole to a circle and a circle passing through its pole to a straight line. For our problem, the circle being mapped is the unit circle, so if $|r| = 1$, the image set or point set $\{(x(\theta) = \mathrm{Re}(z_\theta),\ y(\theta) = \mathrm{Im}(z_\theta))\}$ is a straight line passing through zero. For $|r| \neq 1$, since we can find a θ such that $|z_\theta| < 1$, $(e^{i\phi} \neq 1)$, a part of the circle $\{(x(\theta) = \mathrm{Re}(z_\theta),\ y(\theta) = \mathrm{Im}(z_\theta))\}$, or an arc, is inside $|z| < 1$. Now

$$O = \left\{\ z : |z| < 1;\ z = \frac{a(r - e^{i\phi}e^{i\theta})}{r - e^{i\theta}} \quad \text{for } 0 \le \theta \le 2\pi\ \right\}$$

is the closure of the set of poles of $\{[n/1](z)\}_{n=0}^{\infty}$. Because the Padé sequence $\{[n/1](z)\}_{n=0}^{\infty}$ converges on $|z| < 1$ except on the limit points of the poles, our proposition is true. □

6. An example of applications

One application of our results is to Szegö polynomials associated with Wiener-Levinson filters [3].

Definition 3. Doubly-infinite sequences $x = \{x(m)\}_{m=-\infty}^{\infty}$ of real numbers are called (discrete) signals. Here we consider N-truncated causal signals $x_N = \{x_N(m)\}$ since we can get only finite data, where

$$x_N(m) := \begin{cases} x(m), & \text{if } 0 \le m \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

and $x_N(0) \neq 0$. The Z-transform of x_N is given by $X_N(z) := \sum_{m=0}^{N-1} x_N(m) z^{-m}$.

For the absolutely continuous distribution function $\phi_N(\theta)$ given by

$$\phi_N'(\theta) := \frac{1}{2\pi}\left|X_N\left(e^{i\theta}\right)\right|^2, \quad -\pi \le \theta \le \pi,$$

the inner product is defined by

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\phi_N(\theta), \qquad f, g \in \mathbf{C}_0.$$

We denote by $\{\rho_n(z) = \rho_n(\phi_N; z)\}_{n=0}^{\infty}$ the sequence of monic Szegö polynomials orthogonal with respect to ϕ_N, and denote the reciprocal polynomials by

$$\rho_n^*(z) := \rho_n^*(\phi_N; z) := z^n\overline{\rho_n(\phi_N; 1/\bar{z})}, \ \ n = 0, 1, 2, \ldots .$$

With these definitions, we have a theorem by Jones et al. [3].

Theorem B. *Let*

$$D_N(z) = \pm x_N(0) \prod_{|z_k| \ge 1} (z - z_k) \prod_{|z_k| < 1} (1 - \bar{z}_k z)$$

where the z_k's are all zeros of the polynomial $z^{N-1}X_N(z)$, and the sign is chosen so that $D_N(0) > 0$. Define $F_N(z) := 1/D_N(z)D_N^*(z)$. Let M denote the degree of $D_N(z)$, $0 \le M \le N-1$, and let $[n/M]$ be the classical Padé approximant of type (n, M) to the rational function $F_N(z)$. Then there exists a polynomial $Q_{n,M}(z)$ (of degree $\le M$) such that

$$[n/M](z) = \frac{\rho_n^*(z)}{Q_{n,M}(z)}. \tag{49}$$

Now we can use our theorem for this case. Because all poles of $F_N(z)$ on $|z| = 1$ have even degrees, $r_1, r_2, \ldots, r_m$ in Theorem 1 are all even numbers. If the number of poles of $F_N(z)$ in $|z| < 1$ is μ, then the number of poles of $F_N(z)$ on $|z| = 1$ is $2(M - \mu)$. By our Definitions 1 and 2, we can verify that $\nu := M - \mu$ is good. Thus, we have a corollary.

Corollary 2. *With the hypotheses of Theorem B,*

$$\lim_{n\to\infty}[n/M](z) = \lim_{n\to\infty}\frac{\rho_n^*(z)}{Q_{n,M}(z)} = \frac{1}{D_N(z)D_N^*(z)}$$

locally uniformly on the set $\{\,|z|<1\,\}\setminus\{$*zeros of* $D_N^*(z)\}$. *Moreover, the degree of* $Q_{n,M}(z)$ *is exactly* M, *and the* M *zeros of* $Q_{n,M}(z)$ *approach the* M *zeros of* $D_N^*(z)$.

Next, let us study the limit function outside $\{\,z : |z| < 1\,\}$ when $D(z)$ has at least one zero on $|z| = 1$. Note here that $E = \{\,z : |z| \le 1\,\}$ and $H(z) = -z^{-1}$. Write $D_N^*(z) = a_\mu \pi(z)\prod_{i=1}^m (z-\alpha_i)^{s_i}$ where $\pi(z)$ is a polynomial with all zeros inside $|z| < 1$ and $\{\alpha_i\}_{i=1}^m$ are all on $|z| = 1$. Let $b_\mu = \bar{a}_\mu\prod_{i=1}^m(-\bar{\alpha}_i)^{s_i}$; then by calculating all constants in (31), we have for $z\overline{\in}\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$,

$$\rho_n^*(z) = \frac{Q_{n,M}(z)}{D_N(z)D_N^*(z)} + \frac{z^{n+M+1}}{cb_\mu D_N^*(z)n}\sum_{i=1}^m \frac{\pi(\alpha_i)(-1)^{s_i}\alpha_i s_i}{\pi^*(\alpha_i)(z-\alpha_i)\alpha_i^{n+M+1}} + o(n^{-1}).$$

From this estimate, we see that the Padé approximant sequence is divergent outside E. Because we are interested in the asymptotic behavior of Szegö orthogonal polynomials, we need the following estimate for $\rho_n(z)$ if $z\overline{\in}\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$,

$$\rho_n(z) = \frac{z^{n+M}Q_{n,M}^*(z)}{D_N^*(z)D_N(z)} + \frac{1}{\bar{c}\bar{b}_\mu D_N(z)n}\sum_{i=1}^m \frac{\pi^*(\alpha_i)(-1)^{s_i-1}s_i\alpha_i^{n+M+1}}{\pi(\alpha_i)(z-\alpha_i)} + o(n^{-1}). \quad (50)$$

With this asymptotic formula, we can study some interesting behavior of $\rho_n(z)$. We will focus the rest of this section on a discussion of the zero distribution of $\{\rho_n(z)\}_{n=1}^\infty$. We can get the same theorems as Szabados [9]. However, our proof is much simpler. We will use the notation $Z(F_N)$ for the set of cluster-points of the roots of the $\rho_n(z)$'s.

Proposition 2.

$$Z(F_N)\bigcap\{\,z : |z| > 1\,\} = \emptyset.$$

Proof. By (50), we get

$$z^{-n}\rho_n(z) \to \frac{z^M D_N^*(z)}{D_N(z)D_N^*(z)} \qquad \text{for } |z| > 1 \quad \text{as } n\to\infty.$$

The function $z^M/D_N(z)$ has no zero in $|z| \ge 1+\varepsilon$ for any $\varepsilon > 0$. Thus, by Hurwitz's theorem, $z^{-n}\rho_n(z)$ has no zeros in $|z|\ge 1+\varepsilon$ for n large enough. □

We need a deep and general theorem of Erdös and Turan [2] for the proof of next proposition.

Theorem C. *If* $\xi_1,\xi_2,\ldots,\xi_N$ *are roots of the polynomial* $a_0 + a_1 z + \cdots + a_N z^N$ $(a_0, a_N \ne 0)$, *then for any* $0 \le \alpha \le \beta \le 2\pi$, *we have*

$$\left|\sum_{\alpha\le\arg\xi\le\beta} 1 - N(\beta-\alpha)/2\pi\right| \le 16\sqrt{N\log\Big(\sum_{j=0}^N |a_j|/\sqrt{|a_0||a_N|}\Big)}. \quad (51)$$

Proposition 3.

$$\{\,z : |z| = 1\,\} \subset Z(F_N).$$

Proof. By the same reasoning as in the proof of Theorem 4 in [9], we know that for any $\varepsilon > 0$, there exists a subsequence $\{n_j\}_{j=0}^\infty$ of integers such that for some p, $0 \le p < m$,

$$\lim_{j\to\infty}\alpha_l^{n_j} = \alpha_l^{p-M-1} \qquad \text{for } l = 1,2,\ldots,m.$$

Hence (50) implies

$$\lim_{j\to\infty} n_j\rho_{n_j}(z)D_N(z) = \frac{1}{\bar{c}\bar{b}_\mu}\sum_{i=1}^{m}\frac{\pi^*(\alpha_i)(-1)^{s_i-1}s_i\alpha_i^p}{\pi(\alpha_i)(z-\alpha_i)} \qquad \text{for} \quad |z|\le 1-\varepsilon. \tag{52}$$

By Hurwitz's theorem, this limit means that for j sufficiently large ($j \ge j_1(\varepsilon)$), the polynomial $\rho_{n_j}(z)$ has at most $m-1$ zeros in $|z| \le 1-\varepsilon$. On the other hand, by Proposition 3, for sufficiently large j ($j \ge j_2(\varepsilon)$), $\rho_{n_j}(z)$ does not have zeros in $|z| \ge 1-\varepsilon$. Hence, $\rho_{n_j}(z)$ has at least $n_j - m + 1$ roots in $1-\varepsilon < |z| < 1+\varepsilon$ provided that $j \ge j_3(\varepsilon) := \max\{j_1(\varepsilon), j_2(\varepsilon)\}$.

The next step will be to prove that $n_j - m + 1$ roots of $\rho_{n_j}(z)$ are asymptotically uniformly distributed in angles with vertex at 0. Here we will apply Theorem C. Hence, we need to calculate $\sum_{j=0}^{N}|a_j|/\sqrt{|a_0||a_N|}$ for $\rho_n(z)$. Obviously, the value is the same for $\rho_n(z)$ and $\rho_n^*(z)$. Let $h_n(z) = c\prod_{i=1}^{m}(z-\alpha_i)^{s_i}\pi(z)\rho_n^*(z)$. Since $|\alpha_i| = 1$, $\sum_{j=0}^{N}|a_j|$ is bounded by $(n+M)^M$ multiplying the sum of the absolute value of coefficients of $h_n(z)$.

From the proof of Theorem 1, we have that

$$\begin{aligned} h_n(z) &= \sum_{l=1}^{\mu} b_l^{(n)} w_{n,l}(z) + \sum_{i=1}^{m}\sum_{k=1}^{s_i} a_{k,i}^{(n)} g_{n,k,i}(z) + g_{n,s_1+1}(z) \\ &=: \sum_{l=1}^{\mu} b_l^{(n)} \sum_{j=0}^{n+M} d_j^{(l)} z^j + \sum_{i=1}^{m}\sum_{k=1}^{s_i} a_{k,i}^{(n)} \sum_{j=0}^{n+M} e_{j,i}^{(k)} z^j + \sum_{j=0}^{n+M} e_{j,s_1+1} z^j =: \sum_{i=0}^{n+M} h_i^{(n)} z^i \end{aligned}$$

where the $b_l^{(n)}$'s and $a_{k,i}^{(n)}$'s satisfy (44). Furthermore, $w_{n,l}(z)$ and $g_{n,k,i}(z)$ are $n+M$th truncated Maclaurin's series of

$$\pi_{l-1}(z)\pi(z)\prod_{i=1}^{m}(z-\alpha_i)^{2s_i}F_N(z)$$

and

$$\pi^2(z)(z-\alpha_i)^{k-1+s_i}\prod_{j=1,j\ne i}^{m}(z-\alpha_j)^{2s_j}F_N(z),$$

respectively. Since $\pi_{l-1}(z)\pi(z)\prod_{i=1}^{m}(z-\alpha_i)^{2s_i}\, F_N(z)$ is analytic in $|z| < \sigma$ and $\sigma > 1$, $|d_j^{(l)}|$ are bounded for all j and l. Thus, because of (44), for any $1 > \delta > 0$,

$$\left|\sum_{l=1}^{\mu} b_l^{(n)} d_j^{(l)}\right| < \delta/(n+1) \qquad \text{for } n \text{ large enough, } 1 \le j \le n+M. \tag{53}$$

For the second sum, let

$$\begin{aligned} &\pi^2(z)(z-\alpha_i)^{k-1+s_i}\prod_{j=1,\,j\ne i}^{m}(z-\alpha_j)^{2s_j}F_N(z) \\ &\qquad = \frac{\pi(z)}{a_\mu b_\mu \pi^*(z)}\,\frac{1}{(z-\alpha_i)^{k-1+s_i}\prod_{j=1,j\ne i}^{m}(z-\alpha_j)^{s_j}} =: \sum_{j=0}^{\infty} e_{j,i}^{(k)} z^j. \end{aligned}$$

It is obvious that for some constant C,

$$|e_{j,i}^{(k)}| \le C(j+s_i)^{s_i-k+1} \le C(n+s_i)^{s_i-k+1} \qquad \text{for } 0 \le j \le n. \tag{54}$$

According to (53) and (54), for n large enough and a constant L,

$$\sum_{j=0}^{n}|h_j^{(n)}| \le \sum_{l=1}^{\mu}|b_l^{(n)}d_j^{(l)}| + \sum_{i=1}^{m}\sum_{k=1}^{s_i}|a_{k,i}^{(n)}e_{j,i}^{(k)}|$$
$$\le \delta + \sum_{i=1}^{m}\sum_{k=1}^{s_i}|a_{k,i}^{(n)}|C(n+s_i)^{s_i-k+1} \le L. \tag{55}$$

We are almost finished except for finding a lower bound for $h_{n+M}^{(n)}$. From (52), we see that there is a subsequence $\{n_j\}_{j=1}^{\infty}$ and an integer p such that

$$\lim_{j\to\infty} h_{n_j+M}^{(n_j)} = \lim_{j\to\infty} n_j\rho_{n_j}(0)D_N(0) = \frac{1}{\bar{c}\bar{b}_\mu}\sum_{i=1}^{m}\frac{\pi^*(\alpha_i)(-1)^{s_i}s_i\alpha_i^p}{\pi(\alpha_i)} \ne 0.$$

As for $h_0^{(n)}$, it goes to a non-zero constant $D_N^*(0)/D_N(0)$. Since $h_{n+M}^{(n)} = ca_n$ and $h_0^{(n)} = c\pi(0)(-\alpha)^s a_0$, combining these with (55), we obtain, for n large enough (at least for a subsequence $\{n_j\}_{j=1}^{\infty}$),

$$\log\left(\sum_{j=0}^{n}|a_j|/\sqrt{|a_0||a_n|}\right) \le \log(n+M)^{(M+1)} = (M+1)\log(n+M). \tag{56}$$

Let any $\xi_0 \in |z| = 1$, $\theta_0 = \arg\xi_0$, and choose N such that

$$16\sqrt{M+1}\sqrt{n\log(n+M)} < n\varepsilon/(2\pi) \qquad \text{whenever } n > N(\varepsilon).$$

Then

$$\frac{n2\varepsilon}{(2\pi)} - 16\sqrt{M+1}\sqrt{n\log(n+M)} \le \sum_{|\arg z-\theta_0|\le\varepsilon} 1$$
$$\le \frac{n2\varepsilon}{(2\pi)} + 16\sqrt{M+1}\sqrt{n\log(n+M)}.$$

Hence, if we let $j \ge N$, then $\rho_{n_j}^*(z)$ has at least $n_j\varepsilon/(2\pi) - m + 1$ roots in the "ε-neighborhood" of the point ξ_0: $\{z : 1-\varepsilon < |z| < 1+\varepsilon, |\arg z - \theta_0| \le \varepsilon\}$, which means that $\xi_0 \in Z(F_N)$. By the arbitrariness of $\xi_0 \in |z| = 1$, we conclude that $\{z : |z| = 1\} \subset Z(F_N)$. □

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