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Lagrange interpolation in the zeros of Bessel functions by entire functions of exponential type and mean convergence

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Abstract. Let J_α be the Bessel function of the first kind of order $\alpha > -1$. Then $G_\alpha(z) := z^{-\alpha} J_\alpha(z)$ is an entire function whose zeros are all real. We note that under appropriate conditions, the Lagrange interpolant of $f : \mathbb{R} \to \mathbb{C}$ in the zeros of $G_\alpha(\tau z)$ where $\tau > 0$ is an entire function of exponential type τ. We denote it by $L_{\tau,\alpha}(f; z)$ and study the mean convergence of $L_{\tau,\alpha}(f; \cdot)$ to f as $\tau \to \infty$. We obtain a theorem which is analogous to two well-known results, one due to J. Marcinkiewicz and another due to R. Askey. Some of the lemmas which we need for our proof of the theorem are results of independent interest; for example, Lemma 13 is an extension of the Whittaker-Shannon sampling theorem.

1. Introduction and statement of the main result

For each $n \in \mathbb{N}$, let a system of n distinct nodes

$$-1 \le x_{n,n} < \cdots < x_{n,k} < \cdots < x_{n,1} \le 1 \tag{1}$$

be specified. Let $L_{n-1}(f; \cdot)$ denote the polynomial of degree $\le n-1$ which interpolates f at the given nodes. It was shown by Faber [10] that for some $f \in C[-1,1]$, the sequence $\{L_n(f; x)\}$ does not converge uniformly to $f(x)$.

Let $Q_0, Q_1, \ldots$ be an orthogonal system of polynomials on $[-1,1]$ corresponding to some non-negative weight function w belonging to $L^1[-1,1]$. It was proved by Erdös and Turán [9] that if the points in (1) are the zeros of Q_n, then for each $f \in C[-1,1]$ and $p = 2$

$$\int_{-1}^{1} |f(x) - L_n(f; x)|^p w(x)\, dx \to 0 \quad \text{as} \quad n \to \infty\,. \tag{2}$$

At about the same time as Erdös and Turán, but independently of them, Marcinkiewicz [19] proved the following

Theorem A. *For each $n \in \mathbb{N}$, let*

$$\theta_{n,k} := \frac{2k\pi}{2n+1} \qquad k = 0, \pm 1, \pm 2, \ldots, \pm n,$$

and denote by $t_n(f; \cdot)$ the trigonometric interpolatory polynomial of degree not exceeding n with

$$t_n(f; \theta_{n,k}) = f(\theta_{n,k})\,.$$

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If $f : \mathbb{R} \to \mathbb{C}$ is a continuous, 2π-periodic function, then for every $p > 0$

$$\lim_{n\to\infty} \int_0^{2\pi} |f(\theta) - t_n(f;\theta)|^p d\theta = 0 . \tag{3}$$

It may be added that $\limsup_{n\to\infty} |f(\theta) - t_n(f;\theta)| = \infty$ for every θ if the continuous and 2π-periodic function f is suitably chosen [14, 20].

Note that the case $w(x) = (1 - x^2)^{-1/2}$ of (2) is contained in (3).

Askey [2, 3] looked for values of $p > 2$ for which (2) holds and observed that it was not possible to find such a p for all weights. In the positive direction, he proved the following

Theorem B. [3, Theorem 10] *Let $w(x) := (1 - x)^\alpha(1 + x)^\beta$ and $P_n^{\alpha,\beta}$ be the corresponding orthogonal (Jacobi) polynomial of degree n. If the points in* (1) *are the zeros of $P_n^{\alpha,\beta}$, then for each $f \in C[-1, 1]$,*

$$\lim_{n\to\infty} \int_{-1}^{1} |f(x) - L_n(f;x)|^p (1 - x)^\alpha (1 + x)^\beta dx = 0 \tag{4}$$

for $p < 4(\alpha + 1)/(2\alpha + 1)$ when
(i) $\alpha, \beta > -1/2$, *or*
(ii) $|\alpha - k| \le 1 + \beta, -1 < \beta < -\frac{1}{2}, 2k = 2, 3, \ldots$.

For further results about the mean convergence of Lagrange interpolation based on the zeros of Jacobi polynomials, see [23] and some of the papers cited therein.

We consider Lagrange interpolation of non-periodic functions in an infinite set of points on $\mathbb{R}$. Polynomials or trigonometric polynomials are clearly not suitable for such a purpose. According to a well-known result, a complex-valued function t is a trigonometric polynomial of degree n if and only if it is the restriction to $\mathbb{R}$ of an entire function of exponential type n which is periodic with period 2π. Thus, it is natural to use (non-periodic) entire functions of exponential type to interpolate non-periodic functions in an infinite set of points on $\mathbb{R}$. If so, what kind of points on $\mathbb{R}$ would be suitable for interpolation by entire functions of exponential type and for obtaining a convergence theorem like the one of Marcinkiewicz? In this paper, we shall see that multiples of the zeros of Bessel functions of the first kind of order $\alpha > -1$ are such points.

The Bessel function of the first kind of order $\alpha > -1$ can be defined by [33, p. 40]

$$J_\alpha(z) := \left(\frac{1}{2}z\right)^\alpha \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\left(\frac{1}{2}z\right)^{2\nu}}{\nu!\,\Gamma(\nu + \alpha + 1)} . \tag{5}$$

Note. Here and elsewhere in the text ζ^α, $\zeta \ne 0$ will mean $\exp(\alpha \log \zeta)$ where the logarithm has its principal value.

In view of (4), it is interesting to note that according to a classical formula [31, Theorem 8.1.1], if α, β are real, then

$$\lim_{n\to\infty} n^{-\alpha} P_n^{\alpha,\beta}(\cos \frac{z}{n}) = \left(\frac{1}{2}z\right)^{-\alpha} J_\alpha(z) , \tag{6}$$

uniformly in every bounded region of the complex plane .

From the coefficients in the expansion (5) for $J_\alpha(z)$ it is easily seen that

$$G_\alpha(z) := z^{-\alpha} J_\alpha(z)$$

is an even entire function of order 1 type 1 [5, Theorem 2.2.10] and so is of exponential type 1. It may be mentioned that

$$G_\alpha(z) = \begin{cases} \sqrt{\frac{2}{\pi}}\cos z & \text{if } \alpha = -\frac{1}{2}, \\ \sqrt{\frac{2}{\pi}}\dfrac{\sin z}{z} & \text{if } \alpha = \frac{1}{2}\,. \end{cases}$$

According to a theorem of Lommel [33, p. 482], the function $J_\alpha(z)$ has only real zeros if $\alpha > -1$. They are all simple with the possible exception of $z = 0$. Arranging the positive zeros of J_α in increasing order of magnitude, we shall denote the kth zero by $j_{\alpha,k}$ or simply by j_k, if there is no ambiguity. For each $k \in \mathbb{N}$, the zero $-j_{\alpha,k}$ of J_α will be denoted by $j_{\alpha,-k}$ or by j_{-k}.

From (6) it follows that if $-1 < x_{n,n} < \cdots < x_{n,1} < 1$ are the zeros of $P_n^{\alpha,\beta}$, and if we write $x_{n,k} = \cos\theta_{n,k}$, $0 < \theta_{n,k} < \pi$, then for a fixed $k \geq 1$

$$\theta_{n,k} \sim \frac{j_k}{n} \qquad \text{as } n \to \infty\,.$$

To $f : \mathbb{R} \to \mathbb{C}$, we formally associate

$$L_{\tau,\alpha}(f;z) = \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \frac{G_\alpha(\tau z)}{G'_\alpha(j_\nu)(\tau z - j_\nu)} f\left(\frac{j_\nu}{\tau}\right) \qquad \tau > 0, \tag{7}$$

which interpolates f at the points j_k/τ, $(k \in \mathbb{Z}\setminus\{0\})$. Note that τ does not have to be an integer.

In order to state our main theorem, we need to introduce a couple of definitions.

Definition 1. Given $p > 1$, we denote by $\mathcal{F}^{\alpha,p}(\delta)$ the set of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$f(x) = O\left(1/(|x|+1)^{\alpha+\frac{1}{2}+\frac{1}{p}+\delta}\right) \qquad x \in \mathbb{R}, \tag{8}$$

for some $\delta > 0$ and by $\mathcal{F}^{\alpha,p}$ the union $\bigcup_{\delta>0}\mathcal{F}^{\alpha,p}(\delta)$. It is clear that if $f \in \mathcal{F}^{\alpha,p}$, then $|x|^{\alpha+\frac{1}{2}}f(x) \in L^p(\mathbb{R})$ for all $p > 1$ when $\alpha \geq -\frac{1}{2}$ and for $1 < p < \frac{2}{|2\alpha+1|}$ when $-1 < \alpha < -1/2$.

Definition 2. Let $\mathcal{R}$ be the set of all functions $f : \mathbb{R} \to \mathbb{C}$ which are Riemann integrable on every finite interval.

Our analogue of Theorem A now may be stated as follows.

Theorem 1. *Let $\alpha \geq -1/2$, $p > 1$ or $-1 < \alpha < -1/2$, $1 < p < \frac{2}{|2\alpha+1|}$. Then*

$$\|f - L_{\tau,\alpha}(f;\cdot)\|_{\alpha,p} := \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}\left(f(x) - L_{\tau,\alpha}(f;x)\right)\right|^p dx\right)^{1/p} \to 0 \quad \text{as } \tau \to \infty \tag{9}$$

if $f \in \mathcal{F}^{\alpha,p}\bigcap\mathcal{R}$.

Note that the integral of $|x^{\alpha+\frac{1}{2}}\left(f(x) - L_{\tau,\alpha}(f;x)\right)|^p$ over $(-1,1)$, and so *a fortiori* over $(-\infty,\infty)$, may not be defined if $\alpha \in (-1,-1/2)$ and $p \geq \frac{2}{|2\alpha+1|}$.

Remark 1. We shall show that if $\alpha \geq -1/2$, then

$$\sup_{x\in\mathbb{R}}\left|x^{\alpha+\frac{1}{2}}\left(f(x) - L_{\tau,\alpha}(f;x)\right)\right|$$

may not tend to zero as $\tau \to \infty$ if f satisfies (8).

Remark 2. The case $\alpha = -1/2$ of Theorem 1 was treated in [27].

Remark 3. In the case $\alpha \geq -1/2$, we may replace (9) by

$$\int_{-\infty}^{\infty} |f(x) - L_{\tau,\alpha}(f;x)|^p |x|^{2\alpha+1} dx \to 0 \qquad \text{as } \tau \to \infty \tag{9'}$$

if $p \geq 2$.

1.1. Why the weight $|x|^{2\alpha+1}$ in (9′)? Let $\{\lambda_n\}$ be an increasing sequence of real numbers, $\lambda_n \neq 0$, $\lambda_{-n} = -\lambda_n$, and for large n, $0 < \lambda_n \leq n + \alpha/2 - 1/2$. According to a result of Boas and Pollard ([6] or [5, Corollary 9.6.14]), if f is an entire function of exponential type satisfying

$$|y|^\beta f(x+iy)e^{-\pi|y|} \to 0, \qquad |y| \to \infty,$$

and $\alpha \leq \beta$, then $f(z) \equiv 0$ if $f(\lambda_n) = 0$ for $n = \pm 1, \pm 2, \ldots$.

Using some of the facts about the zeros $j_{\alpha,k}$, which are listed in the next section, it easily can be concluded from the preceding result of Boas and Pollard that if an entire function of exponential type $\sigma < \tau$ vanishes at the zeros of $G_\alpha(\tau z)$, then it must be identically zero; here $\sigma = \tau$ is inadmissible as the example $G_\alpha(\tau z)$ shows. This means that an entire function of exponential type σ is completely determined by its values at the points $j_{\alpha,k}/\tau$ if $\sigma < \tau$, but not necessarily so if $\sigma = \tau$. However, the quadrature formula [11, 13]

$$\int_{-\infty}^{\infty} |x|^{2\alpha+1} f(x) dx = \frac{2}{\tau^{2\alpha+2}} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left| \frac{1}{G'_\alpha(|j_{\alpha,k}|)} \right|^2 f\left(\frac{j_{\alpha,k}}{\tau}\right) \tag{10}$$

holds for all entire functions of exponential type 2τ if the integral on the left exists in the sense of Lebesgue.

Since (10) correctly evaluates the integral for all entire functions of (exponential) type 2 times τ, we see it as a Gaussian quadrature formula. We find it interesting that the weight $|x|^{2\alpha+1}$ and the nodes $j_{\alpha,k}/\tau$ involved in (10) play the same role in (9′) as the corresponding quantities $(1-x)^\alpha(1+x)^\beta$ and the zeros of $P_n^{\alpha,\beta}$, involved in the Gauss-Jacobi quadrature formula, do in (4). It is for this reason that we consider (9′) to be an analogue of (4).

2. Auxiliary results

2.1. Relevant facts about Bessel functions and their zeros. The asymptotic formula [35, p. 368]

$$J_\alpha(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left\{ \left(\cos(z - \frac{\alpha\pi}{2} - \frac{\pi}{4})\right)\left(1 + O(z^{-2})\right) + \left(\sin(z - \frac{\alpha\pi}{2} - \frac{\pi}{4})\right) O(z^{-1}) \right\} \tag{11}$$

holds for $|z|$ large and $|\arg z| < \pi$. From (11) follows the auxiliary result [11, Lemma 1]

Lemma 1. *There exists a positive constant $c_1 = c_1(\alpha)$ depending only on α such that for all large R of the form $(N + \frac{\alpha}{2} + \frac{1}{4})\pi$ where $N \in \mathbb{N}$, we have*

$$\left|J_\alpha(Re^{i\theta})\right| > c_1 \frac{1}{\sqrt{R}} \cdot e^{R|\sin\theta|}, \qquad \theta \in \mathbb{R}. \tag{12}$$

Since $J_\alpha(z) - \left(\frac{2}{\pi z}\right)^{1/2}\cos(z - \alpha\frac{\pi}{2} - \frac{\pi}{4})$ is regular and of exponential type in $\Re z > 0$ and is $O(x^{-3/2})$ as $x \to \infty$ by (11), it follows from Lemma 2 below that

$$J'_\alpha(x) = -\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(x^{-3/2}\right) \quad \text{as} \quad x \to \infty\,. \tag{13}$$

Lemma 2. [15, p. 198, Lemma 14] *If f is of exponential type in $\Re z > 0$, and if for some $\lambda \in \mathbb{R}$,*

$$f(x) = O(x^\lambda) \qquad \text{as } x \to \infty\,,$$

then

$$f'(x) = O(x^\lambda) \qquad \text{as } x \to \infty\,.$$

For all large $m \in \mathbb{N}$, the number of zeros of J_α in the interval $(0, m\pi + (\frac{\alpha}{2} + \frac{1}{4})\pi)$ is exactly m [33, p. 497].

According to a result of McMahon ([22]; also see [21], [33, p. 506]) the kth positive zero is given by

$$\begin{aligned} j_{\alpha,k} = (k + \tfrac{1}{2}\alpha - \tfrac{1}{4})\pi - \frac{4\alpha^2 - 1}{8\pi(k + \frac{1}{2}\alpha - \frac{1}{4})} - \frac{(4\alpha^2 - 1)(28\alpha^2 - 31)}{384\pi^3(k + \frac{1}{2}\alpha - \frac{1}{4})^3} \\ - \frac{(4\alpha^2 - 1)(1328\alpha^4 - 3928\alpha^2 + 3779)}{15360\pi^5(k + \frac{1}{2}\alpha - \frac{1}{4})^5} - \cdots\,. \end{aligned} \tag{14}$$

Thus, $j_{\alpha,k+1} - j_{\alpha,k} \sim \pi$ for all large k. Since for all $k \in \mathbb{Z} \setminus \{0\}$ the zeros $j_{\alpha,k}$ are simple, it follows that if $j_0 := 0$, then

$$j_{\alpha,k+1} - j_{\alpha,k} \geq \delta_1 = \delta_1(\alpha)\,, \tag{15}$$

$$j_{\alpha,k+1} - j_{\alpha,k} \leq \delta_2 = \delta_2(\alpha)\,, \tag{16}$$

where δ_1, δ_2 are positive constants depending only on α .

Formula (14) implies that for all large k

$$\left|\sin\left(j_k - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)\right| > \frac{1}{\sqrt{2}}\,.$$

Hence, by (13), there exists $k_0 \in \mathbb{N}$ such that

$$\left|J'_\alpha(|j_k|)\right| > \frac{1}{2\sqrt{\pi}\,|j_k|^{\frac{1}{2}}} \qquad \text{if } |k| \geq k_0\,. \tag{17}$$

The zeros of J_α are all simple and so $J'_\alpha(|j_k|) \neq 0$ for all $k \neq 0$. Therefore, from (17), it follows that for some positive constant $c_2 = c_2(\alpha)$ depending only on α, we have

$$\left|J'_\alpha(|j_k|)\right| > \frac{c_2}{|j_k|^{\frac{1}{2}}} \qquad k = \pm 1, \pm 2, \ldots\,, \tag{18}$$

and so

$$\left|G'_\alpha(j_k)\right| = \frac{|J'_\alpha(|j_k|)|}{|j_k|^\alpha} > \frac{c_2}{|j_k|^{\alpha+\frac{1}{2}}} \qquad k = \pm 1, \pm 2, \ldots\,. \tag{19}$$

In addition, we shall need

Lemma 3. *For all $\alpha > -1$,*

$$\int_0^\infty \frac{G_\alpha(x)}{x^2 - j_1^2}\,dx \neq 0\,.$$

Proof. According to a known formula ([1, formula 11.4.5] or [33, pp. 426–427, formula (11)]), we have

$$\int_0^\infty \frac{G_\alpha(x)}{x^2+\zeta^2}dx = \frac{\pi}{2\zeta^{\alpha+1}}\{I_\alpha(\zeta)-\mathrm{L}_\alpha(\zeta)\}, \quad \alpha > -\tfrac{5}{2}, \quad \Re\zeta > 0, \tag{20}$$

where I_α is related to J_α by ([1, formula 9.6.3] or [33, p. 77, formula (2)])

$$I_\alpha(z) = i^{-\alpha}J_\alpha(iz), \tag{21}$$

and L_α to the Struve function ([1, p. 496] or [33, p. 328])

$$\mathrm{H}_\alpha(z) = \left(\frac{1}{2}z\right)^{\alpha+1}\sum_{k=0}^\infty \frac{(-1)^k\left(\frac{1}{2}z\right)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\alpha+\frac{3}{2})}$$

by the formula ([1, formula 12.2.1] or [33, p. 329, formula (11)])

$$\mathrm{L}_\alpha(z) = i^{-\alpha-1}\mathrm{H}_\alpha(iz). \tag{22}$$

It is clear that if $\zeta = ij_1+\xi$ where $\xi > 0$, then

$$\frac{\pi}{2\zeta^{\alpha+1}}(I_\alpha(\zeta)-\mathrm{L}_\alpha(\zeta)) \to \frac{\pi}{2(ij_1)^{\alpha+1}}(I_\alpha(ij_1)-\mathrm{L}_\alpha(ij_1)) \qquad \text{as } \xi\to 0.$$

Besides, it is easily seen that as $\xi \to 0$,

$$\int_0^\infty \frac{G_\alpha(x)}{x^2+\zeta^2}dx \to \int_0^\infty \frac{G_\alpha(x)}{x^2-j_1^2}dx. \tag{23}$$

In order to prove this, we note that

$$\int_0^\infty \frac{G_\alpha(x)}{x^2+(ij_1+\xi)^2}dx - \int_0^\infty \frac{G_\alpha(x)}{x^2-j_1^2}dx = \int_0^\infty \frac{1}{x+j_1-i\xi}\cdot\frac{G_\alpha(x)}{x-j_1+i\xi}dx - \int_0^\infty \frac{G_\alpha(x)}{x^2-j_1^2}dx.$$

Let $\varepsilon > 0$ be given. There exists a $\delta_3 > 0$ such that for all $\delta \le \delta_3$,

$$\left|\int_{j_1-\delta}^{j_1+\delta} \frac{G_\alpha(x)}{x^2-j_1^2}dx\right| < \frac{\varepsilon}{3}. \tag{24}$$

Since G_α is an entire function, in the neighbourhood of j_1, we have

$$G_\alpha(z) = a_1(z-j_1)+O(|z-j_1|^2), \quad a_1 := G'_\alpha(j_1),$$

so that for all real ξ

$$\frac{|G_\alpha(x)|}{\sqrt{(x-j_1)^2+\xi^2}} \le |a_1| + O(|x-j_1|).$$

Hence, there exists a $\delta_4 > 0$ such that for all $\delta \le \delta_4$,

$$\left|\int_{j_1-\delta}^{j_1+\delta} \frac{1}{x+j_1-i\xi}\frac{G_\alpha(x)}{x-j_1+i\xi}dx\right| \le \int_{j_1-\delta}^{j_1+\delta}\frac{1}{|x+j_1|}\frac{|G_\alpha(x)|}{\sqrt{(x-j_1)^2+\xi^2}}dx < \frac{\varepsilon}{3}. \tag{25}$$

Furthermore,

$$\begin{aligned}\int_{[0,\infty)\setminus(j_1-\delta,j_1+\delta)} &\left|\frac{G_\alpha(x)}{x^2-j_1^2}-\frac{G_\alpha(x)}{x^2+\zeta^2}\right|dx \\ &= \xi\int_{[0,\infty)\setminus(j_1-\delta,j_1+\delta)}\left|\frac{G_\alpha(x)}{x^2-j_1^2}\right|\cdot\left|\frac{\xi+2ij_1}{x^2-j_1^2+\xi^2+2i\xi j_1}\right|dx \\ &\le \frac{\xi(\xi+2j_1)}{\delta^2}\int_{[0,\infty)\setminus(j_1-\delta,j_1+\delta)}\left|\frac{G_\alpha(x)}{x^2-j_1^2}\right|dx \\ &< \frac{\varepsilon}{3} \qquad (26)\end{aligned}$$

if $\delta := \min\{\delta_3, \delta_4\}$ and ξ is sufficiently small. The estimates (24), (25), and (26) readily imply (23). Hence, (20) holds also for $\zeta = ij_1$, i.e.,

$$\int_0^\infty \frac{G_\alpha(x)}{x^2-j_1^2}dx = \frac{\pi}{2(ij_1)^{\alpha+1}}\left(I_\alpha(ij_1) - \mathrm{L}_\alpha(ij_1)\right).$$

According to (21) and (22), respectively, we have

$$I_\alpha(ij_1) = i^{-\alpha}J_\alpha(j_1e^{\pi i}) = 0 \quad \text{and} \quad \mathrm{L}_\alpha(ij_1) = i^{-\alpha-1}\mathrm{H}_\alpha(j_1e^{\pi i}).$$

Consequently,

$$\int_0^\infty \frac{G_\alpha(x)}{x^2-j_1^2}dx = \frac{(-1)^{-\alpha}\pi}{2j_1^{\alpha+1}}\mathrm{H}_\alpha(j_1e^{\pi i}).$$

Hence, the Lemma holds if and only if

$$\mathrm{H}_\alpha(j_1e^{\pi i}) \neq 0. \qquad (27)$$

We shall show that (27) is indeed true.

Case 1. $-1 < \alpha \le -\frac{1}{2}$.

In this case, $0 < j_1 \le \frac{\pi}{2}$. According to a known formula [33, p. 328]

$$\mathrm{H}_\alpha(z) = \frac{\left(\frac{1}{2}z\right)^{\alpha+1}}{\Gamma(\frac{3}{2})\Gamma(\alpha+\frac{3}{2})}\left(1+\theta(z)\right)$$

where

$$|\theta(z)| < \frac{2}{3}\exp\left\{\frac{\frac{1}{4}|z|^2}{|\alpha+\frac{3}{2}|}-1\right\}.$$

Thus,

$$|\theta(-j_1)| < \frac{2}{3}\exp\left\{\frac{\frac{1}{4}|j_1|^2}{|\alpha+\frac{3}{2}|}-1\right\} \le \frac{2}{3}\exp\left(\frac{\pi^2}{8}-1\right) < 0.8422,$$

i.e., $|1+\theta(-j_1)| > 0.1588$ and so $\mathrm{H}_\alpha(j_1e^{\pi i}) \neq 0$.

Case 2. $-\dfrac{1}{2} < \alpha < \dfrac{1}{2}$.

In this case, $\frac{\pi}{2} < j_1 < \pi$ and ([1, formula 12.1.6] or [33, p. 328, formula (1)])

$$\mathrm{H}_\alpha(j_1e^{\pi i}) = -\frac{2(-\frac{1}{2}j_1)^\alpha}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^1(1-t^2)^{\alpha-\frac{1}{2}}\sin(j_1t)dt.$$

Since $\sin(j_1t) > 0$ for $0 < t < 1$, the integral is (strictly) positive and so $\mathrm{H}_\alpha(j_1e^{\pi i}) \neq 0$.

Case 3. $\alpha \ge \frac{1}{2}$.

It is known [33, Section 10.45] that

$$\mathrm{H}_\alpha(x) > 0 \qquad \text{for } x > 0.$$

Therefore,

$$\mathrm{H}_\alpha^*(x) := \frac{\mathrm{H}_\alpha(x)}{(\frac{1}{2}x)^{\alpha+1}} > 0 \qquad \text{for } x \geq 0.$$

However, H_α^* is an even function, so $\mathrm{H}_\alpha^*(x) > 0$ for all $x \in \mathbb{R}$. In particular, $\mathrm{H}_\alpha^*(j_1 e^{\pi i}) > 0$ and

$$\mathrm{H}_\alpha(j_1 e^{\pi i}) = \left(-\frac{1}{2}j_1\right)^{\alpha+1} \mathrm{H}_\alpha^*(j_1 e^{\pi i}) \neq 0\,.$$

□

2.2. Relevant facts about functions of exponential type. In addition to Lemma 2, we shall need some other facts about functions of exponential type. They are contained in Lemmas 4–9.

Lemma 4. *If g is regular and of exponential type for $x > 0$ and $\int_0^\infty |g(x)|^p dx < \infty$ for some positive p, then* [4]

$$\int_0^\infty |g(x+iy)|^p dx < \infty \tag{28}$$

for every real y, and $g(x) \to 0$ as $x \to \infty$. In case g is an entire function of exponential type τ and $\int_{-\infty}^\infty |g(x)|^p dx < \infty$, $p > 0$, then ([25] *or* [5, Theorem 6.7.1])

$$\left(\int_{-\infty}^\infty |g(x+iy)|^p dx\right)^{1/p} \leq e^{\tau|y|} \left(\int_{-\infty}^\infty |g(x)|^p dx\right)^{1/p}. \tag{29}$$

Lemma 5. [4], [13, Lemma 1] *Let $\{\lambda_k\}$ be an increasing sequence of positive numbers with $\lambda_{k+1} - \lambda_k \geq \delta > 0$. If g is regular and of exponential type in the open right half-plane such that*

$$\int_0^\infty |g(x)| dx < \infty\,, \tag{30}$$

then

$$\sum_{k=1}^\infty |g(\lambda_k)| \ < \ \infty\,. \tag{31}$$

Lemma 6. [5, Theorem 5.6.8], [15, Lemma 9] *Let $g(z)$ be regular and of exponential type $< \pi$, $|\arg z| \leq \beta \leq \frac{\pi}{2}$, $h(0) := \limsup_{x\to\infty} \frac{\log|g(x)|}{x} \leq 0$. Then we can write $g(z) = g_1(z) + g_2(z)$ where $g_1(z)$ is an entire function of exponential type less than π such that $g_1(x) = O(1/|x|)$ as $x \to -\infty$, while $g_2(z)$ is regular and of exponential type in $|\arg z| \leq \beta$ satisfying $g_2(x) = O(1/x)$ as $x \to +\infty$.*

The preceding result is really due to Macintyre [18], although it was not stated in this form by him.

Remark 4. The restriction on the exponential type of g is redundant. Indeed, if g is of exponential type $\tau \geq \pi$, then $g\left(\frac{\pi}{2\tau}z\right)$ is of exponential type $\frac{\pi}{2}$.

Lemma 7. *If g is an entire function of exponential type τ, and if $g \in L^p(\mathbb{R})$, $p > 0$, then*

$$\left(\int_{-\infty}^\infty |g'(x)|^p dx\right)^{1/p} \leq \tau \left(\int_{-\infty}^\infty |g(x)|^p dx\right)^{1/p}. \tag{32}$$

The case $p \geq 1$ of this inequality is classical [5, Theorem 11.3.3]. That it holds also for $0 < p < 1$ was proved in [26].

Lemma 8. *If g is holomorphic and of exponential type in the open right half-plane and belongs to $L^p(0,\infty)$ for some $p > 1$, then $g' \in L^p[\varepsilon,\infty)$ for every $\varepsilon > 0$.*

Proof. Since $g \in L^p(0,\infty)$, $g(x) \to 0$ as $x \to \infty$ by Lemma 4 and so, in particular, $h(0) \leq 0$. By Lemma 6 applied to $g(z+\varepsilon)$, $g(z) = g_1(z) + g_2(z)$ where g_1 is an entire function such that $g_1(x) = O(1/|x|)$ as $x \to -\infty$ and g_2 is regular and of exponential type in $\mathfrak{Re}\, z \geq \varepsilon$ satisfying $g_2(x) = O(1/x)$ as $x \to +\infty$. Hence, $g_1 \in L^p(-\infty,\varepsilon]$, $g_2 \in L^p[\varepsilon,\infty)$. Since $g \in L^p(0,\infty)$, by hypothesis, $g_1 = g - g_2$ belongs to $L^p[\varepsilon,\infty)$, so $g_1 \in L^p(\mathbb{R})$. By Lemma 7, g_1' belongs to $L^p(\mathbb{R})$. Next, we note that the function $zg_2(z)$, which is of exponential type in $\mathfrak{Re}\, z \geq \varepsilon$, is bounded on $[\varepsilon,\infty)$. Hence, by Lemma 2, its derivative $xg_2'(x) + g_2(x)$ is also bounded on $[\varepsilon,\infty)$, so $g_2'(x) = O(1/x)$ as $x \to +\infty$; in particular, $g_2' \in L^p[\varepsilon,\infty)$. It follows that $g' = g_1' + g_2' \in L^p[\varepsilon,\infty)$. □

We also need the following result due to Lindelöf.

Lemma 9. [16, Theorem 18.3.5] *Let g be holomorphic and bounded in the upper half-plane. If g is continuous at all finite points of the real axis, and $g(x) \to a$ when $x \to +\infty$, then*

$$\lim_{z\to\infty} g(z) = a \tag{33}$$

uniformly in any sector $0 \leq \arg z \leq \pi - \delta$, $0 < \delta$.

2.3. Additional lemmas. The next two lemmas play an important role in our work.

Lemma 10. [32, Theorem A] *Let $\{a_k\}_{k\in\mathbb{Z}}$ denote any sequence of numbers such that the series*

$$\sum_{k=-\infty}^{\infty} |a_k|^p \qquad\qquad p > 1 \tag{34}$$

is convergent, and let

$$b_n := \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{a_k}{k+n+\frac{1}{2}} \tag{35}$$

for all values of n. Then, the series $\sum |b_n|^p$ is also convergent, and there is a number N_p depending only on p such that

$$\sum_{n=-\infty}^{\infty} |b_n|^p \leq N_p^p \sum_{n=-\infty}^{\infty} |a_n|^p\,. \tag{36}$$

Lemma 11. [25, p. 135 (Lemma 7)] *Let $\{u_k\}_{k\in\mathbb{Z}}$ denote any sequence of numbers such that the series*

$$\sum_{k=-\infty}^{\infty} |u_k|^p \qquad\qquad p > 1 \tag{37}$$

is convergent, and let $\{t_\mu\}_{\mu\in\mathbb{Z}}$ be a sequence of positive numbers such that $t := \sum_{\mu=-\infty}^{\infty} t_\mu < \infty$. If

$$v_n := \sum_{k=-\infty}^{\infty} t_{n-k}|u_k|\,, \tag{38}$$

for all values of n, then

$$\sum_{n=-\infty}^{\infty} v_n^p \le t^p \sum_{k=-\infty}^{\infty} |u_k|^p .$$

3. Proof of the main result

3.1. Preparatory lemmas.

3.1.1. *Properties of the interpolant $L_{\tau,\alpha}(f;\cdot)$.*

Lemma 12. *Let $\alpha > -1$, $p > 1$, $\tau > 0$. If*

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left| \left(\frac{j_k}{\tau}\right)^{\alpha+\frac{1}{2}} f\left(\frac{j_k}{\tau}\right)\right|^p < \infty , \tag{39}$$

then the series for $L_{\tau,\alpha}(f;z)$ converges absolutely and uniformly on all compact subsets of $\mathbb{C}$ and defines an entire function of exponential type τ.

Proof. Let E be an arbitrary compact subset of $\mathbb{C}$ and N_E the smallest integer such that $|z| \le \frac{1}{\tau} j_{N_E}$ for all $z \in E$. For $z \in E$, $N_1, N_2 \in \mathbb{Z}$, $N_1 < N_2$, we have

$$\left| \sum_{\substack{k=N_1 \\ k\neq 0}}^{N_2} \frac{G_\alpha(\tau z)}{G'_\alpha(j_k)(\tau z - j_k)} f\left(\frac{j_k}{\tau}\right)\right| \le A(N_1,N_2)\cdot B(N_1,N_2,z) \tag{40}$$

where

$$A(N_1,N_2) := \left(\sum_{\substack{k=N_1 \\ k\neq 0}}^{N_2} \left| \frac{f(j_k/\tau)}{G'_\alpha(j_k)} \right|^p \right)^{1/p} ,$$

$$B(N_1,N_2,z) := \left(\sum_{\substack{k=N_1 \\ k\neq 0}}^{N_2} \left| \frac{G_\alpha(\tau z)}{\tau z - j_k} \right|^q \right)^{1/q} , \quad q = \frac{p}{p-1} .$$

By (19)

$$A(N_1,N_2) \le \frac{1}{c_2} \left(\sum_{\substack{k=N_1 \\ k\neq 0}}^{N_2} \left| |j_k|^{\alpha+\frac{1}{2}} \left| f\left(\frac{j_k}{\tau}\right)\right| \right|^p \right)^{1/p} . \tag{41}$$

Clearly,

$$B(N_1,N_2,z) \le \left(\left(\sum_{\substack{|k|\le N_E \\ k\neq 0}} + \sum_{N_E+1\le |k|<\infty} \right) \left| \frac{G_\alpha(\tau z)}{\tau z - j_k} \right|^q \right)^{1/q} . \tag{42}$$

The first sum on the right-hand side of (42) may be estimated as follows. Let δ_1 be as in (15) and draw circles of radius $\frac{\delta_1}{3\tau}$ around each of the points $\frac{1}{\tau} j_l$, $|l| \le N_E$, $l \neq 0$. On each such circle $|z - \frac{1}{\tau} j_l| = \frac{\delta_1}{3\tau}$, we have

$$\begin{aligned} \left| \frac{G_\alpha(\tau z)}{\tau z - j_k} \right| &\le \frac{3}{\delta_1} \max_{|z-\frac{1}{\tau}j_l| = \frac{\delta_1}{3\tau}} |G_\alpha(\tau z)| \\ &\le \frac{3}{\delta_1} \max_{|z| \le \frac{(j_{N_E}+\frac{\delta_1}{3})}{\tau}} |G_\alpha(\tau z)| =: \frac{3M_E}{\delta_1} . \end{aligned} \tag{43}$$

Since $\frac{G_\alpha(\tau z)}{\tau z - j_k}$ is an entire function, it follows that the preceding estimate also holds inside each of the circles. If $z \in E$ but does not belong to any of the disks $|z - \frac{1}{\tau} j_l| \leq \frac{\delta_1}{3\tau}$, $|l| \leq N_E$, $l \neq 0$, then $|\tau z - j_k| > \frac{\delta_1}{3}$ and the estimate (43) holds trivially. Hence, for all $z \in E$

$$\sum_{\substack{|k|\leq N_E\\ k\neq 0}} \left|\frac{G_\alpha(\tau z)}{\tau z - j_k}\right|^q \leq (2N_E)\left(\frac{3M_E}{\delta_1}\right)^q . \tag{44}$$

Further, for $z \in E$

$$\begin{aligned}\sum_{N_E+1\leq|k|<\infty} \left|\frac{G_\alpha(\tau z)}{\tau z - j_k}\right|^q &\leq (M_E)^q \sum_{N_E+1\leq|k|<\infty} \frac{1}{(|j_k| - |j_{N_E}|)^q}\\ &\leq (M_E)^q \sum_{N_E+1\leq|k|<\infty} \frac{1}{(|k| - N_E)^q \delta_1^q}\\ &< \left(\frac{M_E}{\delta_1}\right)^q \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{1}{|k|^q} .\end{aligned}$$

Thus,

$$B(N_1, N_2, z) \leq \left(\frac{M_E}{\delta_1}\right)\left\{2N_E 3^q + \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{1}{|k|^q}\right\}^{1/q} . \tag{45}$$

In view of (39), it follows from (40), (41), and (45) that the series

$$\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{G_\alpha(\tau z)}{G'_\alpha(j_k)(\tau z - j_k)} f\left(\frac{j_k}{\tau}\right)$$

converges absolutely and uniformly on E, so its sum $L_{\tau,\alpha}(f;z)$ defines an entire function.

Since $G_\alpha(\tau z)$ is an entire function of exponential type τ, for each $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ depending on ε such that

$$|G_\alpha(\tau z)| < Ce^{(\tau+\varepsilon)|z|} \qquad z \in \mathbb{C} .$$

By the definition of N_E, we have $\tau|z| \leq j_{N_E}$ for all $z \in E$, so if $E := \{z \in \mathbb{C} : |z| \leq R\}$, then $\tau R \leq j_{N_E}$. Since $j_{k+1} - j_k \geq \delta_1$, there cannot be more than $[\frac{\tau R}{\delta_1}]$ positive zeros in $(0, \tau R]$, therefore, $N_E \leq \frac{\tau R}{\delta_1}$. From the above calculations, it follows that for $|z| \leq R$, $R > 0$, we have

$$\begin{aligned}|L_{\tau,\alpha}(f;z)| \leq \frac{1}{c_2}&\left(\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \left||j_k|^{\alpha+\frac{1}{2}} f\left(\frac{j_k}{\tau}\right)\right|^p\right)^{1/p}\\ &\times C\frac{e^{(\tau+\varepsilon)(R+\delta_2+\frac{\delta_1}{3\tau})}}{\delta_1}\left\{\left(\frac{2\tau R}{\delta_1}\right)3^q + \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{1}{|k|^q}\right\}^{1/q}\end{aligned}$$

where δ_2 is as in (16). This implies that $L_{\tau,\alpha}(f;\cdot)$ is of exponential type τ. □

Lemma 13. *Let f be an entire function of exponential type $\tau > 0$. If there exists $\delta > 0$ such that $|x|^{\alpha+\frac{1}{2}} f(x) \in L^p(\mathbb{R} \setminus (-\delta, \delta))$ for some $\alpha > -1$ and some $p > 1$, then*

$$f(z) \equiv L_{\tau,\alpha}(f; z) .$$

Remark 5. In particular, we can claim that if f is an entire function of exponential type τ satisfying the conditions of Lemma 13, then it cannot vanish at all the zeros of $G_\alpha(\tau z)$ without being identically zero.

Proof. Consider the integral

$$I_N(z) := \frac{1}{2\pi i} \oint_{C_N} \frac{f(\zeta)}{(z-\zeta) G_\alpha(\tau\zeta)} d\zeta$$

where C_N is the circle $|\zeta| = R_N := (N + \frac{\alpha}{2} + \frac{1}{4})\frac{\pi}{\tau}$, and N is a large positive integer. By the residue theorem,

$$I_N(z) = -\frac{f(z)}{G_\alpha(\tau z)} + \sum_{|\frac{j_k}{\tau}| < R_N} \frac{f\left(\frac{j_k}{\tau}\right)}{(z - \frac{j_k}{\tau})\tau G'_\alpha(j_k)}$$

if $|z| < R_N$, $z \neq j_k/\tau$. Hence, the desired result will follow if we show that as $N \to \infty$, $I_N(z) \to 0$ uniformly for all z belonging to any compact set $E \subset \mathbb{C}$.

By Lemma 1, there exists a positive constant $c_1 = c_1(\alpha)$ such that for all large N, say $N \geq N_0 = N_0(\alpha)$, we have

$$\left|I_N(z)\right| \leq \frac{1}{2\pi c_1} \int_{-\pi}^{\pi} \frac{|\tau R_N|^{\alpha+\frac{1}{2}} |f(R_N e^{i\theta})|}{e^{\tau R_N |\sin\theta|}} \frac{R_N}{\left|R_N - |z|\right|} d\theta . \tag{46}$$

By Lemma 4, the assumption $|x|^{\alpha+\frac{1}{2}} f(x) \in L^p(\mathbb{R} \setminus (-\delta, \delta))$ implies that

$$|x|^{\alpha+\frac{1}{2}} f(x) \longrightarrow 0 \qquad \text{as} \qquad x \to \pm\infty . \tag{47}$$

Hence, there exists a constant M_1 such that

$$|x|^{\alpha+\frac{1}{2}} |f(x)| \leq M_1 \quad \text{for} \quad |x| \geq 1 .$$

Let $M_2 := \max_{|z|\leq 1} |f(z)|$, $M := \max\{M_1, M_2\}$. Consider the function

$$\varphi(z) := (\tau(z + z_0))^{\alpha+\frac{1}{2}} f(z) \quad \text{where} \quad z_0 := 1 + i .$$

Clearly, φ is regular and of exponential type τ in the closed upper half-plane, and for $x \in \mathbb{R}$,

$$\begin{aligned} |\varphi(x)| &= \tau^{\alpha+\frac{1}{2}} ((x+1)^2 + 1)^{\frac{1}{2}(\alpha+\frac{1}{2})} |f(x)| \\ &\leq \begin{cases} (\sqrt{5}\tau)^{\alpha+\frac{1}{2}} |f(x)| & \text{if } |x| \leq 1, \\ (\sqrt{5}\tau)^{\alpha+\frac{1}{2}} |x|^{\alpha+\frac{1}{2}} |f(x)| & \text{if } |x| \geq 1. \end{cases} \\ &\leq M_0 \end{aligned}$$

where $M_0 := (\sqrt{5}\tau)^{\alpha+\frac{1}{2}} M$. By a well-known result [5, Theorem 6.2.4],

$$|\varphi(x+iy)| \leq M_0 e^{\tau y} \quad \text{for} \quad x \in \mathbb{R}, \quad 0 \leq y < \infty .$$

Now let

$$g(z) := \varphi(z) e^{i\tau z} .$$

It is regular and of exponential type in the upper half-plane. Since $|g(x)| = |\varphi(x)| \leq M_0$ for all $x \in \mathbb{R}$ and $h_g(\frac{\pi}{2}) = h_\varphi(\frac{\pi}{2}) - \tau \leq 0$, it follows [5, Theorem 6.2.4] that

$$|g(x+iy)| \leq M \qquad \text{for} \qquad x \in \mathbb{R}, \quad 0 \leq y < \infty .$$

By (47), we have

$$\lim_{x\to\infty}|g(x)| = \lim_{x\to\infty}|\varphi(x)| = 0\,.$$

Hence, the function g satisfies the conditions of Lemma 9 with $a = 0$, so

$$\lim_{z\to\infty} g(z) = 0$$

uniformly in the sector $0 \le \arg z \le \frac{3\pi}{4}$. Given $\varepsilon > 0$, there exists $N_1 \ge N_0(\alpha)$ such that for all $N > N_1$ and $\theta \in [0, \frac{3\pi}{4}]$, we have $|g(R_N e^{i\theta})| < \varepsilon$, i.e.,

$$\frac{\left|\tau(R_N e^{i\theta} + 1 + i)\right|^{\alpha+\frac{1}{2}}\left|f(R_N e^{i\theta})\right|}{e^{\tau R_N \sin\theta}} < \varepsilon\,.$$

Thus,

$$\frac{\left(\tau(R_N - \sqrt{2})\right)^{\alpha+\frac{1}{2}}\left|f(R_N e^{i\theta})\right|}{e^{\tau R_N|\sin\theta|}} < \varepsilon \tag{48}$$

for all $N > N_1$ and $\theta \in [0, \frac{3\pi}{4}]$. The same reasoning applied to the function $\varphi(-z)$ shows that (48) holds for all $\theta \in [\frac{\pi}{4}, \pi]$ as well, and therefore, for $\theta \in [0, \pi]$. Changing the value of z_0 to $1 - i$ in the definition of φ and applying the above considerations to the function $\overline{\varphi(\bar{z})}$, we conclude that (48) is true also for $\theta \in [-\pi, 0)$ and $N > N_1$. Hence, (48) is satisfied for $N > N_1$ without any restrictions on θ.

Now let E be an arbitrary compact subset of $\mathbb{C}$. For all sufficiently large N, say $N \ge N_2 \ge N_1$,

$$\frac{R_N}{|R_N - |z||} \le 2\,. \tag{49}$$

Using (48), (49) in (46), we see that as $N \to \infty$, $|I_N(z)| \to 0$ uniformly for all $z \in E$. □

Remark 6. The results contained in [7], [11, Lemma 3], [17], [28], and [34] may be seen as being relevant to Lemma 13, but they do not contain it.

Lemma 14. *Let* $\alpha \ge -\frac{1}{2}$, $p > 1$ *or* $-1 < \alpha < -\frac{1}{2}$, $1 < p < \frac{2}{|1+2\alpha|}$. *If*

$$\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\left|\left(\frac{j_k}{\tau}\right)^{\alpha+\frac{1}{2}} f\left(\frac{j_k}{\tau}\right)\right|^p < \infty\,, \tag{50}$$

then there exists a constant $B_{\alpha,p}$ *depending only on* α *and* p *such that*

$$\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}} L_{\tau,\alpha}(f;x)\right|^p dx \le B_{\alpha,p}^p \frac{\pi}{\tau}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\left|\frac{1}{\tau^{\alpha+\frac{1}{2}} G'_\alpha(j_k)} f\left(\frac{j_k}{\tau}\right)\right|^p\,. \tag{51}$$

Remark 7. This is an analogue of a classical inequality of J. Marcinkiewicz [19, Theorem 10] for trigonometric polynomials, and perhaps is the most nontrivial part of the paper.

Proof. Without loss of generality, we may assume $\tau = \pi$. Let

$$f_k := \frac{1}{G'_\alpha(j_k)} f\left(\frac{j_k}{\pi}\right), \qquad k = \pm 1, \pm 2, \dots\,.$$

Then, by (19),

$$\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} |f_k|^p \le \left(\frac{1}{c_2}\right)^p \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \left|(j_k)^{\alpha+\frac{1}{2}} f\left(\frac{j_k}{\pi}\right)\right|^p \tag{52}$$

$$= \left(\frac{\pi^{\alpha+\frac{1}{2}}}{c_2}\right)^p \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \left|\left(\frac{j_k}{\pi}\right)^{\alpha+\frac{1}{2}} f\left(\frac{j_k}{\pi}\right)\right|^p$$

$$< \infty$$

by (50), Hence, the right-hand side of (51) is finite.

Clearly $\int_{-\infty}^{\infty} |x^{\alpha+\frac{1}{2}} L_{\pi,\alpha}(f;x)|^p dx = \sum_{n=-\infty}^{\infty} \mathcal{I}_n$ where

$$\mathcal{I}_n := \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |x^{\alpha+\frac{1}{2}} L_{\pi,\alpha}(f;x)|^p dx, \qquad n = 0, \pm 1, \pm 2, \dots .$$

Denote by k_1 the smallest integer such that $j_k \ge 2$ for $k \ge k_1$. Then,

$$\mathcal{I}_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{\substack{k=-k_1+1\\k\neq 0}}^{k_1-1} x^{\alpha+\frac{1}{2}} \frac{J_\alpha(\pi x)}{(\pi x)^\alpha (\pi x - j_k)} f_k + x^{\alpha+\frac{1}{2}} \frac{J_\alpha(\pi x)}{(\pi x)^\alpha} \sum_{|k|\ge k_1} \frac{1}{\pi x - j_k} f_k \right|^p dx$$

$$\le 2^{p-1} \Bigg\{ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{\substack{k=-k_1+1\\k\neq 0}}^{k_1-1} x^{\alpha+\frac{1}{2}} \frac{J_\alpha(\pi x)}{(\pi x)^\alpha (\pi x - j_k)} f_k \right|^p dx$$

$$+ \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| x^{\alpha+\frac{1}{2}} \frac{J_\alpha(\pi x)}{(\pi x)^\alpha}\right|^p dx\right) \left(\sum_{|k|\ge k_1} \frac{1}{|j_k| - \frac{\pi}{2}} |f_k| \right)^p \Bigg\}$$

$$\le 2^{p-1} \Bigg\{ (2k_1 - 2)^{p-1} \sum_{\substack{k=-k_1+1\\k\neq 0}}^{k_1-1} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| x^{\alpha+\frac{1}{2}} \frac{J_\alpha(\pi x)}{(\pi x)^\alpha (\pi x - j_k)}\right|^p dx \right) |f_k|^p$$

$$+ \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| x^{\alpha+\frac{1}{2}} \frac{J_\alpha(\pi x)}{(\pi x)^\alpha}\right|^p dx\right) \left(\sum_{|k|\ge k_1} \left(\frac{1}{|j_k| - \frac{\pi}{2}}\right)^{\frac{p}{p-1}} \right)^{p-1} \sum_{|k|\ge k_1} |f_k|^p \Bigg\} .$$

If $\alpha \ge -\frac{1}{2}$, then for all $p > 1$, whereas if $-1 < \alpha < -\frac{1}{2}$, then for $1 < p < \frac{2}{|1+2\alpha|}$, there exists a constant $c_3 = c_3(\alpha, p)$ such that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| x^{\alpha+\frac{1}{2}} \frac{J_\alpha(\pi x)}{(\pi x)^\alpha (\pi x - j_k)}\right|^p dx \le c_3 \qquad \text{for} \qquad k = \pm 1, \pm 2, \dots, \pm(k_1 - 1) .$$

The condition on p also makes sure that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| x^{\alpha+\frac{1}{2}} \frac{J_\alpha(\pi x)}{(\pi x)^\alpha}\right|^p dx$$

exists. Its value depends on α and p; we denote it by $c_4 = c_4(\alpha, p)$. From (14), it follows that if $p > 1$, then the series

$$\sum_{|k|\ge k_1} \left(\frac{1}{|j_k| - \frac{\pi}{2}}\right)^{\frac{p}{p-1}}$$

converges. Denote its sum by $c_5 = c_5(\alpha, p)$. Thus

$$\mathcal{I}_0 \leq 2^{p-1}\left\{(2k_1-2)^{p-1}c_3 \sum_{\substack{k=-k_1+1 \\ k\neq 0}}^{k_1-1} |f_k|^p + c_4\cdot(c_5)^{p-1}\sum_{|k|\geq k_1}|f_k|^p\right\}$$
$$\leq c_6 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |f_k|^p \tag{53}$$

where

$$c_6 = c_6(\alpha,p) := \max\{2^{p-1}(2k_1-2)^{p-1}c_3, 2^{p-1}c_4\cdot(c_5)^{p-1}\}\,.$$

Now we proceed to estimate $\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\mathcal{I}_n$. Let

$$\mathcal{L}_n := \max_{n-\frac{1}{2}\leq x\leq n+\frac{1}{2}} |x^{\alpha+\frac{1}{2}}L_{\pi,\alpha}(f;x)|, \qquad n\neq 0\,.$$

Then, obviously

$$\mathcal{I}_n \leq \mathcal{L}_n^p, \qquad n\neq 0\,. \tag{54}$$

Let ξ_n be a point in $[-\frac{1}{2},\frac{1}{2}]$ such that

$$\mathcal{L}_n = \left|(n+\xi_n)^{\alpha+\frac{1}{2}}L_{\pi,\alpha}(f;n+\xi_n)\right|\,.$$

Then,

$$\mathcal{L}_n = \frac{1}{\pi^{\alpha+3/2}}\left|\sqrt{\pi(n+\xi_n)}J_\alpha(\pi(n+\xi_n))\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{f_k}{n+\xi_n-\frac{j_k}{\pi}}\right|\,. \tag{55}$$

At this stage we need to obtain estimates for $|\sqrt{\pi x}J_\alpha(\pi x)|$ and $|\sqrt{\pi x}\frac{J_\alpha(\pi x)}{\pi x - j_k}|$. From the asymptotic formula (11), we have

$$\frac{J_\alpha(z)}{z^\alpha} \sim \sqrt{\frac{2}{\pi}}\frac{1}{z^{\alpha+\frac{1}{2}}}\cos(z-\frac{\alpha\pi}{2}-\frac{\pi}{4}) + O\left(\frac{1}{z^{\alpha+\frac{3}{2}}}\right) \qquad \text{for} \quad z\in\mathbb{R}, \quad z\to\infty.$$

Consequently,

$$\left|\frac{J_\alpha(z)}{z^\alpha}\right| \leq \frac{2}{\sqrt{\pi}}\frac{1}{|z|^{\alpha+\frac{1}{2}}} \qquad \text{for} \qquad z\in\mathbb{R}, \quad z\to\pm\infty.$$

The function $\frac{J_\alpha(z)}{z^\alpha}$ is entire, so, if $d := \min\{\pi, j_1\}$, there exists a constant $c_7 = c_7(\alpha)$ such that

$$|x|^{\alpha+\frac{1}{2}}\left|\frac{J_\alpha(x)}{x^\alpha}\right| \leq c_7 \qquad \text{for} \quad x\in\mathbb{R}\setminus\left[-\frac{d}{2},\frac{d}{2}\right]\,.$$

Thus,

$$\left|\sqrt{\pi x}J_\alpha(\pi x)\right| \leq c_7 \qquad \text{for} \quad x\in\mathbb{R}\setminus\left[-\frac{d}{2\pi},\frac{d}{2\pi}\right]\,. \tag{56}$$

From (15), we have

$$j_{k+1}-j_k \geq 2\pi\delta', \qquad \delta' = \frac{\delta_1}{2\pi}\,. \tag{57}$$

Since $\sqrt{\pi z}J_\alpha(\pi z)$ is holomorphic and of exponential type in $\Re z \geq \frac{d}{2\pi}$, it follows from (56) that for $z\in\Delta_\alpha := \{z = x+iy : x\geq \frac{d}{2\pi}, |y|\leq\delta'\}$, we have [5, Theorem 6.2.4]

$$|\sqrt{\pi z}J_\alpha(\pi z)| \leq c_8$$

where $c_8 = c_8(\alpha)$ is a constant. For each $k \in \mathbb{N}$, let $D_k(\delta') := \{z : |z - \frac{j_k}{\pi}| < \delta'\}$. Then for all z in $\Delta_\alpha \setminus D_k(\delta')$, we have

$$\left|\sqrt{\pi z}\frac{J_\alpha(\pi z)}{\pi z - j_k}\right| \le \frac{c_8}{\pi\delta} =: c_9(\alpha) = c_9 . \tag{58}$$

In particular, (58) holds for z belonging to the boundary of the disk $D_k(\delta')$ on and inside which the function $\sqrt{\pi z}\frac{J_\alpha(\pi z)}{\pi z - j_k}$ is holomorphic. Hence, by the maximum modulus principle, (58) holds also for $z \in D_k(\delta')$, i.e., it is true for all $z \in \Delta_\alpha$.

Amongst the points $\{\frac{j_k}{\pi}\}_{k\in\mathbb{Z}}$, there are possibly two which are closest to $n + \xi_n$. Let $\frac{j_{k_n}}{\pi}$ be (any) one of them. In view of (57), we have

$$\left|n + \xi_n - \frac{j_k}{\pi}\right| \ge \delta' \qquad \text{for} \qquad k \in \mathbb{Z} \setminus \{k_n\} . \tag{59}$$

Now, let us return to (55) and write

$$\frac{1}{n + \xi_n - \frac{j_k}{\pi}} = \frac{1}{n - k + \frac{1}{2}} + \frac{\frac{j_k}{\pi} - k - \xi_n + \frac{1}{2}}{(n + \xi_n - \frac{j_k}{\pi})(n - k + \frac{1}{2})} .$$

So, by (56)

$$\begin{aligned}\mathcal{L}_n \le & \frac{c_7}{\pi^{\alpha+3/2}}\left|\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{f_k}{n - k + \frac{1}{2}}\right| \\ & + \left|\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{1}{\pi^{\alpha+\frac{1}{2}}} \cdot \frac{\sqrt{\pi(n+\xi_n)}J_\alpha(\pi(n+\xi_n))}{\pi(n+\xi_n) - j_k} \cdot \frac{\frac{j_k}{\pi} - k - \xi_n + \frac{1}{2}}{n - k + \frac{1}{2}} f_k\right| .\end{aligned}$$

Using, in the second sum on the right, the estimate (58) for $k = k_n$ and (56) for all other k, we obtain

$$\begin{aligned}\mathcal{L}_n \le & \frac{c_7}{\pi^{\alpha+3/2}}\left|\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{f_k}{n - k + \frac{1}{2}}\right| + \frac{c_9}{\pi^{\alpha+\frac{1}{2}}}\left|\frac{\frac{j_{k_n}}{\pi} - k_n - \xi_n + \frac{1}{2}}{n - k_n + \frac{1}{2}}\right| |f_{k_n}| \\ & + \frac{c_7}{\pi^{\alpha+3/2}} \sum_{\substack{k=-\infty\\k\neq 0,k_n}}^{\infty} \frac{|\frac{j_k}{\pi} - k - \xi_n + \frac{1}{2}|}{|n + \xi_n - \frac{j_k}{\pi}| \cdot |n - k + \frac{1}{2}|} |f_k| .\end{aligned} \tag{60}$$

From (16), it follows that $|n + \xi_n - \frac{j_{k_n}}{\pi}| \le \delta'' := \frac{\delta_2}{2\pi}$ and

$$\frac{|\frac{j_{k_n}}{\pi} - k_n - \xi_n + \frac{1}{2}|}{|n - k_n + \frac{1}{2}|} \le 1 + \frac{|n + \xi_n - \frac{j_{k_n}}{\pi}|}{|n - k_n + \frac{1}{2}|} \le 1 + 2\delta'', \tag{61}$$

since $n, k_n \in \mathbb{Z}$. Again, from (14), it follows that if $\beta_k := j_k/\pi - k$, then

$$|\beta_k| \le c_{10} \tag{62}$$

for some positive number $c_{10} = c_{10}(\alpha)$ and

$$\left|\frac{j_k}{\pi} - k - \xi_n + \frac{1}{2}\right| \le c_{10} + 1 . \tag{63}$$

Using the last three inequalities in (60), we obtain

$$\mathcal{L}_n \le |b_n| + |v_n| \tag{64}$$

where

$$b_n := \frac{c_7}{\pi^{\alpha+3/2}} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{f_k}{n-k+\frac{1}{2}} ,$$

$$v_n := \frac{c_9}{\pi^{\alpha+\frac{1}{2}}}(1+2\delta'')|f_{k_n}| + \frac{c_7(c_{10}+1)}{\pi^{\alpha+3/2}} \sum_{\substack{k=-\infty \\ k\neq 0,k_n}}^{\infty} \frac{1}{|n-k-\beta_k+\xi_n|\cdot|n-k+\frac{1}{2}|}|f_k| .$$

Defining $a_0 = 0$, $a_k := (c_7/\pi^{\alpha+3/2})f_{-k}$ for $k \neq 0$, we write b_n in the form $\frac{1}{\pi}\sum_{k=-\infty}^{\infty} \frac{a_k}{k+n+\frac{1}{2}}$ where, in view of (52), $\sum_{k=-\infty}^{\infty} |a_k|^p$ converges. Lemma 10 implies that

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |b_n|^p \leq N_p^p \left(\frac{c_7}{\pi^{\alpha+3/2}}\right)^p \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |f_k|^p \tag{65}$$

where N_p is a constant depending only on p. We write v_n in the form $\sum_{k=-\infty}^{\infty} t_{n-k}|u_k|$ by putting

$$u_0 = 0, \qquad u_k = f_k \quad \text{for} \quad k \neq 0; \qquad t_{n-k_n} = \frac{c_9(1+2\delta'')}{\pi^{\alpha+\frac{1}{2}}}, \qquad t_n = 1,$$

$$t_\mu := \frac{c_7(c_{10}+1)}{\pi^{\alpha+3/2}} \cdot \frac{1}{|\mu-\beta_{n-\mu}+\xi_n||\mu+\frac{1}{2}|} \qquad \text{for} \qquad \mu \neq n, n-k_n .$$

Referring to (52), we see that $\sum_{k=-\infty}^{\infty} |u_k|^p < \infty$. Further, using (59) and (62), we conclude that $t := \sum_{\mu=-\infty}^{\infty} t_\mu < \infty$, so Lemma 11 may be applied to obtain

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |v_n|^p \leq t^p \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |f_k|^p . \tag{66}$$

By (64), we have

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathcal{L}_n^p \leq 2^p \left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |b_n|^p + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |v_n|^p \right) .$$

Combined with (54), this gives

$$\begin{aligned} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathcal{I}_n &\leq 2^p \left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |b_n|^p + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |v_n|^p \right) \\ &\leq 2^p \left(N_p^p \left(\frac{c_7}{\pi^{\alpha+3/2}}\right)^p + t^p \right) \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |f_k|^p \end{aligned} \tag{67}$$

by (65) and (66). Now recalling the bound for $\mathcal{I}_0$ from (53), we get

$$\begin{aligned} \int_{-\infty}^{\infty} \left| x^{\alpha+\frac{1}{2}} L_{\pi,\alpha}(f;x) \right|^p dx &= \sum_{n=-\infty}^{\infty} \mathcal{I}_n \\ &\leq \left(c_6 + 2^p \left(N_p^p \left(\frac{c_7}{\pi^{\alpha+3/2}}\right)^p + t^p \right) \right) \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |f_k|^p , \end{aligned}$$

which is equivalent to (51). □

3.1.2. *Approximation by step functions.*

Lemma 15. (i) *Let $|x|^{\alpha+\frac{1}{2}}f(x) \in L^p(\mathbb{R})$ for some $\alpha \geq -\frac{1}{2}$ and some $p > 1$. If $f \in L^p(-1,1)$, then for every $\varepsilon > 0$, there exists a step function Ω with compact support such that*

$$\left(\int_{-1}^{1} |f(x) - \Omega(x)|^p dx\right)^{1/p} < \varepsilon, \quad \left(\int_{|x|\geq 1} \left|x^{\alpha+\frac{1}{2}}(f(x) - \Omega(x))\right|^p dx\right)^{1/p} < \varepsilon\,. \tag{68}$$

(ii) *Let $-1 < \alpha < -\frac{1}{2}$, $1 < p < \dfrac{2}{|1+2\alpha|}$. If $f \in \mathcal{F}^{\alpha,p}(\delta)$ for some $\delta > 0$, then for every $\varepsilon > 0$, there exists a step function Ω with compact support such that*

$$\begin{aligned} \left(\int_{-1}^{1} |x^{\alpha+\frac{1}{2}}(f(x) - \Omega(x))|^p dx\right)^{1/p} &< \varepsilon\,, \\ \left(\int_{|x|\geq 1} \left|x^{\alpha+\frac{1}{2}}(f(x) - \Omega(x))\right|^p dx\right)^{1/p} &< \varepsilon\,, \\ \max_{u\in\mathbb{R}} \left(\int_{-3}^{3} |x^{\alpha+\frac{1}{2}}(f(x-u) - \Omega(x-u))|^p dx\right)^{1/p} &< \varepsilon\,. \end{aligned} \tag{69}$$

Proof. (i) Since $f \in L^p(-1,1)$, there exists a step function Ω_1 vanishing outside $(-1,1)$ such that [29, p. 118, problem 14]

$$\left(\int_{-1}^{1} |f(x) - \Omega_1(x)|^p dx\right)^{1/p} < \varepsilon\,.$$

The assumption $|x|^{\alpha+\frac{1}{2}}f(x) \in L^p(\mathbb{R})$ implies that for some $Y_0 > 1$ and all $Y \geq Y_0$,

$$\int_Y^\infty |x^{\alpha+\frac{1}{2}}f(x)|^p dx < \frac{\varepsilon^p}{4}, \qquad \int_{-\infty}^{-Y} |x^{\alpha+\frac{1}{2}}f(x)|^p dx < \frac{\varepsilon^p}{4}\,.$$

Clearly, $f \in L^p(1, Y_0)$ and $f \in L^p(-Y_0, -1)$. Hence, there exists a step function Ω_2 vanishing outside $[-Y_0,-1] \cup [1, Y_0]$ such that

$$\begin{aligned} \int_1^{Y_0} \left|f(x) - \Omega_2(x)\right|^p dx &< \frac{\varepsilon^p}{4Y_0^{p(\alpha+\frac{1}{2})}}\,, \\ \int_{-Y_0}^{-1} \left|f(x) - \Omega_2(x)\right|^p dx &< \frac{\varepsilon^p}{4Y_0^{p(\alpha+\frac{1}{2})}}\,. \end{aligned}$$

Thus, if

$$\Omega(x) := \begin{cases} \Omega_1(x) & \text{for } x \in (-1,1), \\ 0 & \text{for } x \in (-\infty, -Y_0) \cup (Y_0, \infty), \\ \Omega_2(x) & \text{for } x \in [-Y_0, -1] \cup [1, Y_0], \end{cases}$$

then clearly

$$\left(\int_{-1}^{1}\left|f(x)-\Omega(x)\right|^{p}dx\right)^{1/p}<\varepsilon\,,$$

$$\begin{aligned}\int_{|x|\geq 1}\left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega(x))\right|^{p}dx &= \int_{-\infty}^{-Y_0}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx+\int_{Y_0}^{\infty}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\\ &\quad+\int_{-Y_0}^{-1}\left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega(x))\right|^{p}dx+\int_{1}^{Y_0}\left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega(x))\right|^{p}dx\\ &< \frac{\varepsilon^p}{4}+\frac{\varepsilon^p}{4}+\frac{\varepsilon^p}{4}+\frac{\varepsilon^p}{4}\,,\end{aligned}$$

i.e.,

$$\left(\int_{|x|\geq 1}\left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega(x))\right|^{p}dx\right)^{1/p}<\varepsilon\,.$$

(ii) Let $\beta_p := \frac{|p(\alpha+\frac{1}{2})|+1}{|p(2\alpha+1)|}$ and note that $\beta_p > 1$, $|p(\alpha+\frac{1}{2})|\beta_p < 1$. Further, let

$$\gamma_{\alpha,p} := \left(\int_{-3}^{3} s^{p(\alpha+\frac{1}{2})\beta_p}ds\right)^{1/\beta_p}$$

and set

$$\varepsilon_1 := \min\left\{\left(\frac{\varepsilon}{(\gamma_{\alpha,p})^{1/p}M^{1/\beta_p}}\right)^{\frac{\beta_p}{\beta_p-1}},\varepsilon\right\}$$

where $M := \sup_{x\in\mathbb{R}}|f(x)| < \infty$.

There exists a $Y_0 > 1$, $\delta_0 \in (0,1)$ such that

$$\int_{-\infty}^{-Y_0}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx<\frac{\varepsilon_1^p}{5}\,,\quad \int_{Y_0}^{\infty}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx<\frac{\varepsilon_1^p}{5}\,, \tag{70}$$

$$\int_{Y-3}^{Y+3}\left|f(x)\right|^{p}dx<\frac{\varepsilon_1^p}{5}\qquad \text{for } |Y|>Y_0, \tag{71}$$

$$\int_{-\delta_0}^{\delta_0}\left|f(x)\right|^{p}dx\leq\int_{-\delta_0}^{\delta_0}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx<\frac{\varepsilon_1^p}{5}\,. \tag{72}$$

Clearly, f belongs to $L^p(-Y_0,-1)$ and to $L^p(1,Y_0)$. Hence, there exists a step function Ω_1 vanishing outside $[-Y_0,-1]\cup[1,Y_0]$ such that

$$\int_{-Y_0}^{-1}\left|f(x)-\Omega_1(x)\right|^{p}dx<\frac{\varepsilon_1^p}{5}\,,\quad \int_{1}^{Y_0}\left|f(x)-\Omega_1(x)\right|^{p}dx<\frac{\varepsilon_1^p}{5}\,, \tag{73}$$

so *a fortiori*

$$\int_{-Y_0}^{-1}\left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega_1(x))\right|^{p}dx<\frac{\varepsilon_1^p}{5}\,,\quad \int_{1}^{Y_0}\left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega_1(x))\right|^{p}dx<\frac{\varepsilon_1^p}{5}\,. \tag{74}$$

Since f belongs to $L^p(-1,-\delta_0)$ and to $L^p(\delta_0,1)$, there exists a step function Ω_2 vanishing outside $(-1,-\delta_0]\cup[\delta_0,1)$ such that

$$\int_{-1}^{-\delta_0}\left|f(x)-\Omega_2(x)\right|^{p}dx<\frac{\varepsilon_1^p}{5}\delta_0^{p|\alpha+\frac{1}{2}|}\,,\quad \int_{\delta_0}^{1}\left|f(x)-\Omega_2(x)\right|^{p}dx<\frac{\varepsilon_1^p}{5}\delta_0^{p|\alpha+\frac{1}{2}|}\,. \tag{75}$$

Consequently,

$$\int_{-1}^{-\delta_0} \left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega_2(x))\right|^p dx < \frac{\varepsilon_1^p}{5}\,, \quad \int_{\delta_0}^{1} \left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega_2(x))\right|^p dx < \frac{\varepsilon_1^p}{5}\,. \tag{76}$$

The step functions Ω_1, Ω_2 may be chosen such that $0 \le \Omega_j(x) \le M$, $j = 1,2$, at points where $f(x) \ge 0$, and $-M \le \Omega_j(x) \le 0$ when $f(x) \le 0$. Now put

$$\Omega(x) := \begin{cases} 0 & \text{for } |x| > Y_0 \text{ or } |x| < \delta_0, \\ \Omega_1(x) & \text{for } 1 \le |x| \le Y_0, \\ \Omega_2(x) & \text{for } \delta_0 \le |x| < 1\,. \end{cases}$$

From (72) and (76), it follows that

$$\left(\int_{-1}^{1} \left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega(x))\right|^p dx\right)^{1/p} < \left(\frac{3}{5}\right)^{1/p} \varepsilon_1 < \varepsilon\,,$$

whereas (70) and (74) imply

$$\left(\int_{|x|\ge 1} \left|x^{\alpha+\frac{1}{2}}(f(x)-\Omega(x))\right|^p dx\right)^{1/p} < \varepsilon\,.$$

For all $u \in \mathbb{R}$,

$$\begin{aligned}
&\int_{-3}^{3} \left|x^{\alpha+\frac{1}{2}}(f(x-u)-\Omega(x-u))\right|^p dx \\
&\quad = \int_{-u-3}^{-u+3} |u+t|^{p(\alpha+\frac{1}{2})} |f(t)-\Omega(t)|^p dt \\
&\quad \le \left(\int_{-u-3}^{-u+3} |u+t|^{p(\alpha+\frac{1}{2})\beta_p} dt\right)^{1/\beta_p} \cdot \left(\int_{-u-3}^{-u+3} |f(t)-\Omega(t)|^{\frac{p\beta_p}{\beta_p-1}} dt\right)^{\frac{\beta_p-1}{\beta_p}} \\
&\quad \le \gamma_{\alpha,p} \cdot \max_{u\in\mathbb{R}} \left(\int_{u-3}^{u+3} |f(t)-\Omega(t)|^{\frac{p\beta_p}{\beta_p-1}} dt\right)^{\frac{\beta_p-1}{\beta_p}}\,.
\end{aligned}$$

Since $|f(t)-\Omega(t)| \le M$ for all t, we have

$$\int_{u-3}^{u+3} |f(t)-\Omega(t)|^{\frac{p\beta_p}{\beta_p-1}} dt \le M^{\frac{p}{\beta_p-1}} \int_{u-3}^{u+3} |f(t)-\Omega(t)|^p dt < M^{\frac{p}{\beta_p-1}} \varepsilon_1^p$$

by (71)–(73) and (75). Hence,

$$\max_{u\in\mathbb{R}} \left(\int_{-3}^{3} \left|x^{\alpha+\frac{1}{2}}(f(x-u)-\Omega(x-u))\right|^p dx\right)^{1/p} < \gamma_{\alpha,p}^{\frac{1}{p}} M^{\frac{1}{\beta_p}} \varepsilon_1^{\frac{\beta_p-1}{\beta_p}} \le \varepsilon\,.$$

□

3.1.3. *An auxiliary function and its properties.* Given $f : \mathbb{R} \to \mathbb{C}$ and $\alpha > -1$, we define for each $\tau > 0$ the function

$$S_{\tau,\alpha}(f;z) := \frac{\tau}{A} \int_{-\infty}^{\infty} f(t) \frac{G_\alpha(\tau(z-t))}{(\tau(z-t))^2 - j_1^2} dt$$

where $A := \int_{-\infty}^{\infty} G_\alpha(t)/(t^2 - j_1^2) dt$; $A \ne 0$ by Lemma 3.

For real z, we also may write

$$S_{\tau,\alpha}(f;z) = \frac{\tau}{A} \int_{-\infty}^{\infty} f(z+t) \frac{G_\alpha(\tau t)}{(\tau t)^2 - j_1^2} dt\,.$$

Lemmas 16–19 contain facts about $S_{\tau,\alpha}(f;\cdot)$ which we need for the proof of our theorem on the mean convergence of $L_{\tau,\alpha}(f;\cdot)$.

Lemma 16. (i) *Given $\alpha \geq -\frac{1}{2}$, $p > 1$, let $|x|^{\alpha+\frac{1}{2}}f(x) \in L^p(\mathbb{R})$ and $f \in L^p(-1,1)$. Then, for $\tau \geq 1$*

$$\|S_{\tau,\alpha}(f;\cdot)\|_{\alpha,p} := \left(\int_{-\infty}^{\infty} |x^{\alpha+\frac{1}{2}} S_{\tau,\alpha}(f;x)|^p dx\right)^{1/p}$$
$$\leq C_{\alpha,p}\left\{\left(\int_{-1}^{1} |f(x)|^p dx\right)^{1/p} + \left(\int_{|x|\geq 1} |x^{\alpha+\frac{1}{2}} f(x)|^p dx\right)^{1/p}\right\} \tag{77}$$

where $C_{\alpha,p}$ is a constant depending only on α and p.

(ii) *If $-1 < \alpha < -\frac{1}{2}$, $1 < p < \frac{2}{|2\alpha+1|}$, and $f \in \mathcal{F}^{\alpha,p}(\delta)$ for some $\delta > 0$, then*

$$\|S_{\tau,\alpha}(f;\cdot)\|_{\alpha,p} \leq C'_{\alpha,p}\left\{\left(\int_{-1}^{1} |x^{\alpha+\frac{1}{2}} f(x)|^p dx\right)^{1/p}\right. \tag{78}$$
$$\left. + \left(\int_{|x|\geq 1} |x^{\alpha+\frac{1}{2}} f(x)|^p dx\right)^{1/p} + \max_{u\in\mathbb{R}}\left(\int_{-3}^{3} |x^{\alpha+\frac{1}{2}} f(x-u)|^p dx\right)^{1/p}\right\}$$

where $C'_{\alpha,p}$ is a constant depending only on α and p.

Remark 8. The estimate in (78) holds if $|x|^{\alpha+\frac{1}{2}}f(x) \in L^p(\mathbb{R})$ except that

$$\max_{u\in\mathbb{R}}\left(\int_{-3}^{3} \left|x^{\alpha+\frac{1}{2}} f(x-u)\right|^p dx\right)^{1/p}$$

may possibly be $+\infty$.

Proof. (i) Since $|x|^{\alpha+\frac{1}{2}}f(x) \in L^p(\mathbb{R})$ and $f \in L^p(-1,1)$, the function f must belong to $L^p(\mathbb{R})$. Set $B(z) := G_\alpha(z)/(z^2 - j_1^2)$. Then

$$\begin{aligned}
\|S_{\tau,\alpha}(f;\cdot)\|_{\alpha,p} &= \frac{\tau}{A}\left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}} \int_{-\infty}^{\infty} f(t)B(\tau(x-t))dt\right|^p dx\right)^{1/p} \\
&= \frac{\tau}{A}\left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}} \int_{-\infty}^{\infty} f(x-t)B(\tau t)dt\right|^p dx\right)^{1/p} \\
&\leq \frac{\tau}{A}\int_{-\infty}^{\infty} |B(\tau t)| \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}} f(x-t)\right|^p dx\right)^{1/p} dt
\end{aligned}$$

by the generalized Minkowski inequality [24, p. 21, Section 1.3.2]. Now, using the fact that

$$(|a| + |b|)^\lambda \leq \begin{cases} |a|^\lambda + |b|^\lambda & \text{if } 0 < \lambda \leq 1, \\ 2^{\lambda-1}(|a|^\lambda + |b|^\lambda) & \text{if } 1 \leq \lambda < \infty, \end{cases}$$

we obtain

$$\begin{aligned}
\|S_{\tau,\alpha}(f;\cdot)\|_{\alpha,p} &\le \frac{\tau}{A}\int_{-\infty}^{\infty}|B(\tau t)|\left(\int_{-\infty}^{\infty}\left|(x+t)^{\alpha+\frac12}f(x)\right|^p dx\right)^{1/p} dt\\
&\le \frac{\tau}{A}\max\{1,2^{\alpha-\frac12}\}\int_{-\infty}^{\infty}|B(\tau t)|\left\{\int_{-\infty}^{\infty}\left(\left(|x|^{\alpha+\frac12}+|t|^{\alpha+\frac12}\right)|f(x)|\right)^p dx\right\}^{1/p} dt\\
&\le \frac{\tau}{A}\max\{1,2^{\alpha-\frac12}\}2^{1-\frac1p}\int_{-\infty}^{\infty}|B(\tau t)|\left(\int_{-\infty}^{\infty}|x^{\alpha+\frac12}f(x)|^p dx\right.\\
&\qquad\left.+\,|t|^{p(\alpha+\frac12)}\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{1/p} dt\\
&\le \frac{\tau}{A}\max\{1,2^{\alpha-\frac12}\}2^{1-\frac1p}\int_{-\infty}^{\infty}|B(\tau t)|\left\{\left(\int_{-\infty}^{\infty}|x^{\alpha+\frac12}f(x)|^p dx\right)^{1/p}\right.\\
&\qquad\left.+\,|t|^{\alpha+\frac12}\left(\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{1/p}\right\}dt\,. \qquad (79)
\end{aligned}$$

If $\tau \ge 1$, then

$$\int_{-\infty}^{\infty}\tau|B(\tau t)|\cdot|t|^{\alpha+\frac12}dt = \int_{-\infty}^{\infty}|B(u)|\left|\frac{u}{\tau}\right|^{\alpha+\frac12}du \le \int_{-\infty}^{\infty}\left|u^{\alpha+\frac12}B(u)\right|du\,,$$

so

$$\begin{aligned}
\|S_{\tau,\alpha}(f;\cdot)\|_{\alpha,p} \le&\frac{1}{A}\max\{1,2^{\alpha-\frac12}\}\cdot 2^{1-\frac1p}\max\left\{\int_{-\infty}^{\infty}|B(u)|du,\int_{-\infty}^{\infty}|u^{\alpha+\frac12}B(u)|du\right\}\\
&\times\left\{\left(\int_{-\infty}^{\infty}|x^{\alpha+\frac12}f(x)|^p dx\right)^{1/p}+\left(\int_{-\infty}^{\infty}|f(x)|^p dx\right)^{1/p}\right\}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\int_{-\infty}^{\infty}|f(x)|^p dx &\le \int_{-1}^{1}|f(x)|^p dx+\int_{|x|\ge1}\left|x^{\alpha+\frac12}f(x)\right|^p dx\,,\\
\int_{-\infty}^{\infty}\left|x^{\alpha+\frac12}f(x)\right|^p dx &\le \int_{-1}^{1}|f(x)|^p dx+\int_{|x|\ge1}\left|x^{\alpha+\frac12}f(x)\right|^p dx,
\end{aligned}$$

so

$$\begin{aligned}
\|S_{\tau,\alpha}(f;\cdot)\|_{\alpha,p} \le&\frac{2}{A}\max\{1,2^{\alpha-\frac12}\}\cdot 2^{1-\frac1p}\max\left\{\int_{-\infty}^{\infty}|B(u)|du,\int_{-\infty}^{\infty}|u^{\alpha+\frac12}B(u)|du\right\}\\
&\times\left\{\left(\int_{|x|\ge1}\left|x^{\alpha+\frac12}f(x)\right|^p dx\right)^{1/p}+\left(\int_{-1}^{1}|f(x)|^p dx\right)^{1/p}\right\},
\end{aligned}$$

i.e., (77) holds.

(ii) From (79), we have

$$\|S_{\tau,\alpha}(f;\cdot)\|_{\alpha,p} \le \frac{\tau}{A}\left(\int_{-1}^{1}+\int_{|t|\ge1}\right)|B(\tau t)|\left(\int_{-\infty}^{\infty}\left|(x+t)^{\alpha+\frac12}f(x)\right|^p dx\right)^{1/p} dt\,. \qquad (80)$$

Now, we note that

$$\int_{-1}^{1}|B(\tau t)|\left(\int_{-\infty}^{\infty}\left|(x+t)^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right)^{1/p}dt$$
$$\leq\int_{-1}^{1}|B(\tau t)|\left(\int_{-2}^{2}\left|(x+t)^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right)^{1/p}dt$$
$$+\int_{-1}^{1}|B(\tau t)|\left(\int_{|x|\geq 2}\left(\frac{|x|^{|\alpha+\frac{1}{2}|}}{(|x|-1)^{|\alpha+\frac{1}{2}|}}\right)^{p}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right)^{1/p}dt$$
$$\leq\int_{-1}^{1}|B(\tau t)|\left(\int_{t-2}^{t+2}\left|x^{\alpha+\frac{1}{2}}f(x-t)\right|^{p}dx\right)^{1/p}dt$$
$$+2\int_{-1}^{1}|B(\tau t)|\left(\int_{|x|\geq 2}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right)^{1/p}dt$$
$$\leq\left\{\max_{|u|\leq 1}\left(\int_{-3}^{3}\left|x^{\alpha+\frac{1}{2}}f(x-u)\right|^{p}dx\right)^{1/p}\right.$$
$$\left.+2\left(\int_{|x|\geq 2}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right)^{1/p}\right\}\int_{-1}^{1}|B(\tau t)|dt\,. \tag{81}$$

Furthermore,

$$\int_{|t|\geq 1}|B(\tau t)|\left(\int_{-\infty}^{\infty}\left|(x+t)^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right)^{1/p}dt$$
$$=\int_{|t|\geq 1}|B(\tau t)|\left(\int_{|x+t|\leq\frac{1}{2}}\left|(x+t)^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right.$$
$$\left.+\int_{\mathbb{R}\setminus[-t-\frac{1}{2},-t+\frac{1}{2}]}\left(\left|\frac{x}{x+t}\right|^{|\alpha+\frac{1}{2}|}|x|^{\alpha+\frac{1}{2}}|f(x)|\right)^{p}dx\right)^{1/p}dt$$
$$\leq\int_{|t|\geq 1}|B(\tau t)|\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|x^{\alpha+\frac{1}{2}}f(x-t)\right|^{p}dx\right.$$
$$\left.+(2|t|+1)^{p|\alpha+\frac{1}{2}|}\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right)^{1/p}dt$$
$$\leq\int_{|t|\geq 1}|B(\tau t)|\left\{\left(\int_{-3}^{3}\left|x^{\alpha+\frac{1}{2}}f(x-t)\right|^{p}dx\right)^{1/p}\right.$$
$$\left.+(2|t|+1)^{|\alpha+\frac{1}{2}|}\left(\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right)^{1/p}\right\}dt$$
$$\leq\max_{|u|\geq 1}\left(\int_{-3}^{3}\left|x^{\alpha+\frac{1}{2}}f(x-u)\right|^{p}dx\right)^{1/p}\int_{|t|\geq 1}|B(\tau t)|dt$$
$$+\left(\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}}f(x)\right|^{p}dx\right)^{1/p}\int_{|t|\geq 1}(2|t|+1)^{|\alpha+\frac{1}{2}|}|B(\tau t)|dt\,. \tag{82}$$

The estimates (80)–(82) imply (78). □

Lemma 17. *If $f:\mathbb{R}\to\mathbb{C}$ is measurable and $f(x)/(|x|+1)^{\alpha+\frac{3}{2}}\in L^p(\mathbb{R})$ for some $\alpha>-1$ and some $p>1$, then $S_{\tau,\alpha}(f;\cdot)$ is an entire function. Furthermore, if $\alpha\geq-\frac{1}{2}$,*

$|x|^{\alpha+\frac{1}{2}}f(x) \in L^p(\mathbb{R})$, and $f \in L^p(-1,1)$, then $S_{\tau,\alpha}(f;\cdot)$ is of exponential type τ. The same can be said if $-1 < \alpha < -\frac{1}{2}$, $1 < p < \frac{2}{|2\alpha+1|}$, and $f \in \mathcal{F}^{\alpha,p}(\delta)$ for some $\delta > 0$.

Proof. Let

$$K_\alpha(w) := \frac{\tau}{A} \cdot \frac{G_\alpha(\tau w)}{(\tau w)^2 - j_1^2},$$

which is an entire function of exponential type τ. Then,

$$S_{\tau,\alpha}(f;z) = \int_{-\infty}^{\infty} f(\xi)K_\alpha(z-\xi)d\xi \,.$$

Now, consider the function

$$s_{\tau,\alpha}(z) = s_{\tau,\alpha}(z;a,b) := \int_a^b f(\xi)K_\alpha(z-\xi)d\xi$$

where $-\infty < a < b < \infty$. Take any point $z \in \mathbb{C}$ and choose $R := 2 + \max\{|z-a|, |z-b|\}$. The function K_α is entire, so there exists a constant M_R such that $|K_\alpha(w)| \le M_R$ on $|w| = R$. Hence, for $|h| \le 1$, we have [8, pp. 72–73]

$$\begin{aligned}
&\left|\frac{s_{\tau,\alpha}(z+h) - s_{\tau,\alpha}(z)}{h} - \int_a^b f(\xi)K'_\alpha(z-\xi)d\xi\right| \\
&\quad = \left|\int_a^b f(\xi)\left(\frac{K_\alpha(z-\xi+h) - K_\alpha(z-\xi)}{h} - K'_\alpha(z-\xi)\right)d\xi\right| \\
&\quad = \left|\frac{h}{2\pi i}\int_a^b f(\xi)\left(\int_{|w|=R} \frac{K_\alpha(w)}{(w-z+\xi)^2(w-z+\xi-h)}dw\right)d\xi\right| \\
&\quad \le |h|M_R R\int_a^b |f(\xi)|d\xi,
\end{aligned}$$

which tends to zero as $h \to 0$ since, clearly, $f \in L^1(a,b)$. Hence, $s_{\tau,\alpha}(\cdot;a,b)$ is entire.

Consider the sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ where

$$\Psi_n(z) := \int_0^n f(\xi)K_\alpha(z-\xi)d\xi \,.$$

Note that for any given $z, K_\alpha(z-\zeta)$ is an entire function of exponential type τ in the variable ζ. From the asymptotic formula (11) for J_α, it follows that, if L is any given positive number, then for some constant C_L depending only on L,

$$|K_\alpha(x-\xi)| \le \frac{C_L}{(|\xi|+1)^{\alpha+\frac{5}{2}}}$$

for all $x \in [-L, L]$, $\xi \in \mathbb{R}$. The function $F(\zeta) := (\zeta+i)^{\alpha+\frac{5}{2}}K_\alpha(x-\zeta)$ is holomorphic and of exponential type τ in the upper half plane. Besides, $|F(\xi)| \le C_L$ for all $x \in [-L, L]$, $\xi \in \mathbb{R}$. Hence [5, Theorem 6.2.4],

$$|F(\xi+iy)| \le C_L e^{\tau y}, \qquad x \in [-L,L], \quad \xi \in \mathbb{R}, \quad 0 \le y < \infty \,,$$

i.e., for $x \in [-L, L]$, $y \le 0$, and all $\xi \in \mathbb{R}$, we have

$$|K_\alpha(x+iy-\xi)| \le \frac{C_L e^{\tau|y|}}{|\xi + i(1+|y|)|^{\alpha+\frac{5}{2}}} \,. \tag{83}$$

Clearly, the same estimate also holds for $x \in [-L, L]$, $\xi \in \mathbb{R}$, $y > 0$. Hence, there exists a constant C_L^* such that

$$\left|K_\alpha(z-\xi)\right| \le \frac{C_L^*}{(|\xi|+1)^{\alpha+\frac{5}{2}}} \qquad \text{for} \qquad |\mathfrak{Re}\, z| \le L, \quad |\mathfrak{Im}\, z| \le L, \quad \xi \in \mathbb{R}\,. \tag{84}$$

Since $f(\xi)/(|\xi|+1)^{\alpha+3/2} \in L^p(\mathbb{R})$, it follows from Hölder's inequality that $\int_0^\infty f(\xi)K_\alpha(z-\xi)d\xi$ is convergent and defines a function Ψ. Given any compact set $E \subset \mathbb{C}$, we can find $L > 0$ such that $E \subset \{z = x+iy : -L \le x, y \le L\}$, so (84) holds for all $z \in E$ and all $\xi \in \mathbb{R}$. This allows us to conclude that $\Psi_n(z) \to \Psi(z)$ uniformly on E. The functions Ψ_n being all entire, the same can be said about $\Psi(z) := \int_0^\infty f(\xi)K_\alpha(z-\xi)d\xi$. Similarly, $\int_{-\infty}^0 f(\xi)K_\alpha(z-\xi)d\xi$ defines an entire function, so $S_{\tau,\alpha}(f;\cdot)$ must be entire also.

Now, let $\alpha \ge -\frac{1}{2}$ and suppose that for some $p > 1$, $|x|^{\alpha+\frac{1}{2}}f(x) \in L^p(\mathbb{R})$ and $f \in L^p(-1,1)$. Then, $f \in L^p(\mathbb{R})$. Note that $K_\alpha(w) \in L^q(\mathbb{R})$ for all $q > \frac{2}{3}$. Hence, for any fixed $z = x+iy$, Hölder's inequality gives $\left(q := \frac{p}{p-1}\right)$

$$\begin{aligned}\left|S_{\tau,\alpha}(f;z)\right| &= \left|\int_{-\infty}^{\infty} f(\xi)K_\alpha(x-\xi+iy)d\xi\right| \\ &\le \left(\int_{-\infty}^{\infty} |f(\xi)|^p d\xi\right)^{1/p}\left(\int_{-\infty}^{\infty} |K_\alpha(\xi+iy)|^q d\xi\right)^{1/q} \\ &\le \left(\int_{-\infty}^{\infty} |f(\xi)|^p d\xi\right)^{1/p}\left(\int_{-\infty}^{\infty} |K_\alpha(\xi)|^q d\xi\right)^{1/q} e^{\tau|y|}\end{aligned}$$

by Lemma 4, i.e., $S_{\tau,\alpha}(f;\cdot)$ is of exponential type τ.

Finally, let $-1 < \alpha < -\frac{1}{2}$, $1 < p < \frac{2}{|2\alpha+1|}$, and $f \in \mathcal{F}^{\alpha,p}(\delta)$ for some $\delta > 0$. Then, for $|x| \to \infty$,

$$f(x) = O\left(\frac{1}{(|x|+1)^\Delta}\right)$$

where $\Delta := \alpha+\frac{1}{2}+\frac{1}{p}+\delta > 0$. Hence, $f \in L^\rho(\mathbb{R})$ for all $\rho > \frac{1}{\Delta}$. Choose $\rho > \max\{1, \frac{1}{\Delta}\}$. Then for any fixed $z = x+iy$

$$\left|S_{\tau,\alpha}(f;z)\right| \le \left(\int_{-\infty}^{\infty} |f(\xi)|^\rho d\xi\right)^{1/\rho}\left(\int_{-\infty}^{\infty} |K_\alpha(\xi+iy)|^q d\xi\right)^{1/q}$$

where $q := \frac{\rho}{\rho-1}$, from which it follows that $S_{\tau,\alpha}(f;\cdot)$ is of exponential type τ, as above. □

Lemma 18. *If $\alpha \ge -\frac{1}{2}$, $|x|^{\alpha+\frac{1}{2}}f(x) \in L^p(\mathbb{R})$ for some $p > 1$, and $f \in L^p(-1,1)$, then*

$$\lim_{\tau\to\infty} \int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(f(x) - S_{\tau,\alpha}(f;x))\right|^p dx = 0\,. \tag{85}$$

The same conclusion also holds when $\alpha \in (-1, -\frac{1}{2})$ and $p \in \left(1, \frac{2}{|2\alpha+1|}\right)$ if $f \in \mathcal{F}^{\alpha,p}(\delta)$ for some $\delta > 0$.

Proof. First, we prove (85) for the characteristic function χ of the interval $[0,1]$. Clearly,

$$\begin{aligned}S_{\tau,\alpha}(\chi;x) &= \frac{\tau}{A}\int_{-x}^{-x+1}\frac{G_\alpha(\tau t)}{(\tau t)^2-j_1^2}dt\\ &= \frac{1}{A}\int_0^{\tau x}\frac{G_\alpha(t)}{t^2-j_1^2}dt+\frac{1}{A}\int_0^{\tau(1-x)}\frac{G_\alpha(t)}{t^2-j_1^2}dt\,.\end{aligned}$$

There exists a constant γ_α such that $|J_\alpha(t)|\le\gamma_\alpha$ if $|t|\ge j_1$. Hence, for $x\ge j_1+1$ and $\tau>1$, we have

$$\begin{aligned}\left|\chi(x)-S_{\tau,\alpha}(\chi;x)\right| &= \left|S_{\tau,\alpha}(\chi;x)\right|\\ &= \frac{1}{|A|}\int_{\tau(x-1)}^{\tau x}\frac{G_\alpha(t)}{t^2-j_1^2}dt\\ &\le \frac{\gamma_\alpha}{|A|}\cdot\frac{1}{(\tau(x-1))^\alpha\,(\tau(x-1)-j_1)}\int_{\tau(x-1)}^{\tau x}\frac{1}{t+j_1}dt\\ &\le \frac{\gamma_\alpha}{|A|}\cdot\frac{\tau}{(\tau(x-1))^\alpha\,((\tau(x-1))^2-j_1^2)}\,.\end{aligned}$$

Given $\varepsilon>0$, there exists, therefore, a number $Y_{\alpha,1}(\varepsilon)\ge j_1+1$ such that

$$\int_X^\infty\left|x^{\alpha+\frac12}(\chi(x)-S_{\tau,\alpha}(\chi;x))\right|^p dx<\frac{\varepsilon}{4} \tag{86}$$

if $X\ge Y_{\alpha,1}(\varepsilon)$. Similarly, there exists a number $Y_{\alpha,2}(\varepsilon)\ge j_1+1$ such that

$$\int_{-\infty}^{-X}\left|x^{\alpha+\frac12}(\chi(x)-S_{\tau,\alpha}(\chi;x))\right|^p dx<\frac{\varepsilon}{4} \tag{87}$$

if $X\ge Y_{\alpha,2}(\varepsilon)$. Let $Y_\alpha:=\max\{Y_{\alpha,1},\,Y_{\alpha,2}\}$.

From the definition of A, it follows that for every $\delta>0$, there exists a positive number $T_0(\delta)$ such that

$$\left|\frac{1}{A}\int_0^T\frac{G_\alpha(t)}{t^2-j_1^2}dt-\frac12\right|<\frac{\delta}{2}\qquad\text{for all } T>T_0(\delta)\,.$$

Since

$$\frac{1}{A}\int_0^u\frac{G_\alpha(t)}{t^2-j_1^2}dt$$

is a continuous function of u which tends to $\frac12$ as $u\to\infty$, it follows that for all $u\in\mathbb{R}$,

$$\left|\frac{1}{A}\int_0^u\frac{G_\alpha(t)}{t^2-j_1^2}dt\right|\le\mu_\alpha$$

where μ_α is a constant depending only on α.

Now, let η be any given positive number not exceeding $\frac{1}{2}$ such that

$$\left(\int_{-\eta}^{\eta}+\int_{1-\eta}^{1+\eta}\right)\left|x^{\alpha+\frac{1}{2}}\right|^{p}dx<\frac{\varepsilon}{4(1+2\mu_\alpha)^p}\,.$$

Then, for $x\in[\eta,1-\eta]$ and $\tau>\frac{1}{\eta}T_0(\delta)$, both τx and $\tau(1-x)$ are larger than $T_0(\delta)$, so

$$\left|S_{\tau,\alpha}(\chi;x)-1\right|\le\left|\frac{1}{A}\int_0^{\tau x}\frac{G_\alpha(t)}{t^2-j_1^2}dt-\frac{1}{2}\right|+\left|\frac{1}{A}\int_0^{\tau(1-x)}\frac{G_\alpha(t)}{t^2-j_1^2}dt-\frac{1}{2}\right|<\delta\,.$$

Similarly, if $x\ge 1+\eta$ or if $x\le-\eta$, then for $\tau>\frac{1}{\eta}T_0(\delta)$

$$\left|S_{\tau,\alpha}(\chi;x)\right|\le\left|\frac{1}{A}\int_0^{\tau|x|}\frac{G_\alpha(t)}{t^2-j_1^2}dt-\frac{1}{2}\right|+\left|\frac{1}{A}\int_0^{\tau|1-x|}\frac{G_\alpha(t)}{t^2-j_1^2}dt-\frac{1}{2}\right|<\delta\,.$$

Thus, if $E_\eta:=\{x:|x|<\eta \text{ or } |x-1|<\eta\}$, then

$$\left|\chi(x)-S_{\tau,\alpha}(\chi;x)\right|<\delta\qquad\text{for all } x\in\mathbb{R}\setminus E_\eta$$

if $\tau>\frac{1}{\eta}T_0(\delta)$. Now, setting

$$\delta:=\left(i\frac{\varepsilon}{4\int_{-Y_\alpha}^{Y_\alpha}|x^{\alpha+\frac{1}{2}}|^p dx}\right)^{1/p},$$

we obtain

$$\int_{[-Y_\alpha,Y_\alpha]\setminus E_\eta}\left|x^{\alpha+\frac{1}{2}}(\chi(x)-S_{\tau,\alpha}(\chi;x))\right|^p dx<\delta^p\cdot\int_{-Y_\alpha}^{Y_\alpha}|x^{\alpha+\frac{1}{2}}|^p dx=\frac{\varepsilon}{4}\tag{88}$$

for all $\tau>\frac{1}{\eta}T_0(\delta)=:T_1(\varepsilon)$. Furthermore,

$$\int_{E_\eta}\left|x^{\alpha+\frac{1}{2}}(\chi(x)-S_{\tau,\alpha}(\chi;x))\right|^p dx\le(1+2\mu_\alpha)^p\int_{E_\eta}|x^{\alpha+\frac{1}{2}}|^p dx<\frac{\varepsilon}{4}\,.\tag{89}$$

It follows from (86)–(89) that (85) holds for the characteristic function on $[0,1]$. A similar argument applies to the characteristic function on $[1,2]$. The result then easily extends to the characteristic function of any finite interval, and indeed to any step function with compact support.

Now, let $\alpha\ge-\frac{1}{2}$ and $|x|^{\alpha+\frac{1}{2}}f(x)\in L^p(\mathbb{R})$ for some $p>1$. Furthermore, let $f\in L^p(-1,1)$. Given $\varepsilon>0$, there exists, by Lemma 15, a step function Ω with compact support such that (68) holds. As explained above, it is possible to choose τ_1 such that for all $\tau>\tau_1$, we have

$$\left(\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}}(\Omega(x)-S_{\tau,\alpha}(\Omega;x))\right|^p dx\right)^{1/p}<\varepsilon\,.\tag{90}$$

Now, we note that

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(f(x) - S_{\tau,\alpha}(f;x))\right|^p dx\right)^{1/p} \\
&\leq \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(\Omega(x) - S_{\tau,\alpha}(f;x))\right|^p dx\right)^{1/p} + \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(f(x) - \Omega(x))\right|^p dx\right)^{1/p} \\
&\leq \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}S_{\tau,\alpha}(f - \Omega;x)\right|^p dx\right)^{1/p} + \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(\Omega(x) - S_{\tau,\alpha}(\Omega;x))\right|^p dx\right)^{1/p} \\
&\qquad + \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(f(x) - \Omega(x))\right|^p dx\right)^{1/p} \\
&\leq C_{\alpha,p}\left\{\left(\int_{-1}^{1} |f(x) - \Omega(x)|^p dx\right)^{1/p} + \left(\int_{|x|\geq 1} \left|x^{\alpha+\frac{1}{2}}(f(x) - \Omega(x))\right|^p dx\right)^{1/p}\right\} \\
&\quad + \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(\Omega(x) - S_{\tau,\alpha}(\Omega;x))\right|^p dx\right)^{1/p} \\
&\quad + \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(f(x) - \Omega(x))\right|^p dx\right)^{1/p}
\end{aligned}$$

since $f - \Omega$ clearly satisfies the conditions of part (i) of Lemma 16. By (68) and (90), we conclude that

$$\left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(f(x) - S_{\tau,\alpha}(f;x))\right|^p dx\right)^{1/p} \leq C_{\alpha,p} \cdot 2\varepsilon + \varepsilon + 2\varepsilon\ .$$

Since ε is arbitrary, (85) holds for the case under consideration.

Finally, let $-1 < \alpha < -\frac{1}{2}$, $1 < p < \frac{2}{|2\alpha+1|}$, and $f \in \mathcal{F}^{\alpha,p}(\delta)$ for some $\delta > 0$. This time, by Lemma 15, there exists a step function Ω with compact support such that (69) holds. Since $f - \Omega$ satisfies the conditions of part (ii) of Lemma 16, using (78), we obtain

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}(f(x) - S_{\tau,\alpha}(f;x))\right|^p dx\right)^{1/p} \\
&\leq C'_{\alpha,p}\left\{\left(\int_{-1}^{1} \left|x^{\alpha+\frac{1}{2}}(f(x) - \Omega(x))\right|^p dx\right)^{1/p} + \left(\int_{|x|\geq 1} \left|x^{\alpha+\frac{1}{2}}(f(x) - \Omega(x))\right|^p dx\right)^{1/p}\right. \\
&\qquad\qquad \left. + \max_{u\in\mathbb{R}}\left(\int_{-3}^{3} \left|x^{\alpha+\frac{1}{2}}(f(x-u) - \Omega(x-u))\right|^p dx\right)^{1/p}\right\} \\
&\quad + \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}\left(\Omega(x) - S_{\tau,\alpha}(\Omega;x)\right)\right|^p dx\right)^{1/p} \\
&\quad + \left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}\left(f(x) - \Omega(x)\right)\right|^p dx\right)^{1/p} .
\end{aligned}$$

By (69) and (90), we get

$$\left(\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}\left(f(x) - S_{\tau,\alpha}(f;x)\right)\right|^p dx\right)^{1/p} \leq C'_{\alpha,p} \cdot 3\varepsilon + \varepsilon + 2\varepsilon\ ,$$

which completes the proof of Lemma 18. □

Remark 9. Lemma 18 seems to be a result of independent interest.

Lemma 19. *According as α belongs to $[-\frac{1}{2}, \infty]$ or to $(-1, -\frac{1}{2})$, let f be in $\mathcal{F}^{\alpha,p} \cap \mathcal{R}$ for some p in $(1, \infty)$ or in $(1, \frac{2}{|2\alpha+1|})$, respectively. If $f^*(x) := f(x) - S_{\sigma,\alpha}(f; x)$ where $\sigma > 0$, then*

$$\lim_{\tau\to\infty} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{j_k - j_{k-1}}{\tau} \left| \left(\frac{j_k}{\tau}\right)^{\alpha+\frac{1}{2}} f^*\left(\frac{j_k}{\tau}\right)\right|^p = \int_{-\infty}^{\infty} |x^{\alpha+\frac{1}{2}} f^*(x)|^p dx\,. \tag{91}$$

Proof. By Lemma 17, the function $S_{\sigma,\alpha}(f; \cdot)$ is entire and of exponential type. By Lemma 16, $x^{\alpha+\frac{1}{2}} S_{\sigma,\alpha}(f; x)$ belongs to $L^p(\mathbb{R})$ and, therefore to $L^p(0, \infty)$. Since $W(z) := z^{\alpha+\frac{1}{2}} S_{\sigma,\alpha}(f; z)$ is holomorphic and of exponential type in the open right half-plane, it follows from Lemma 8 that $W' \in L^p[a, \infty)$ for every $a > 0$.

Let C be such that

$$\left|x^{\alpha+\frac{1}{2}} f(x)\right| < \frac{C}{(|x|+1)^{\delta+\frac{1}{p}}} \qquad \text{for some } \delta > 0 \text{ and all } x \in \mathbb{R}\,.$$

Given $\varepsilon > 0$, we choose X_ε in $\left[(12C^p/\delta p\varepsilon)^{1/\delta p}, \infty\right)$ large enough to have

$$\int_{|x|\geq X_\varepsilon} \left|x^{\alpha+\frac{1}{2}} f(x)\right|^p dx < \frac{\varepsilon}{12}\,, \tag{92}$$

$$\int_{|x|\geq X_\varepsilon} |W(x)|^p\, dx < \frac{1}{2^{p-1}} \cdot \frac{\varepsilon}{24}\,, \tag{93}$$

$$\int_{|x|\geq X_\varepsilon} |W'(x)|^p\, dx < \frac{1}{2^{p-1}} \cdot \frac{\varepsilon}{24}\,. \tag{94}$$

If $l = l(\tau)$ is the largest integer such that $j_l/\tau \leq X_\varepsilon$, then

$$\sum_{k=l+2}^{\infty} \frac{j_k - j_{k-1}}{\tau} \left|\left(\frac{j_k}{\tau}\right)^{\alpha+\frac{1}{2}} f\left(\frac{j_k}{\tau}\right)\right|^p < \frac{C^p}{\tau} \sum_{k=l+2}^{\infty} \frac{j_k - j_{k-1}}{(j_k/\tau)^{1+\delta p}}$$

$$< C^p \tau^{\delta p} \int_{j_{l+1}}^{\infty} \frac{1}{x^{1+\delta p}} dx = \frac{C^p}{\delta p} \cdot \left(\frac{\tau}{j_{l+1}}\right)^{\delta p} < \frac{\varepsilon}{12} \tag{95}$$

since $(\tau/j_{l+1})^{\delta p} < (1/X_\varepsilon)^{\delta p} < \delta p\varepsilon/12C^p$. Similarly,

$$\sum_{k=-\infty}^{-l-2} \frac{j_k - j_{k-1}}{\tau} \left|\left(\frac{j_k}{\tau}\right)^{\alpha+\frac{1}{2}} f\left(\frac{j_k}{\tau}\right)\right|^p < \frac{\varepsilon}{12}\,. \tag{96}$$

For each $k \in \mathbb{Z}$, let $\xi_k \in [j_{k-1}/\tau, j_k/\tau]$ be such that

$$\int_{\frac{j_{k-1}}{\tau}}^{\frac{j_k}{\tau}} |W(x)|^p dx = \frac{j_k - j_{k-1}}{\tau} \cdot |W(\xi_k)|^p\,. \tag{97}$$

Then

$$\sum_{k=l+2}^{\infty} \frac{j_k - j_{k-1}}{\tau} \left|W\left(\frac{j_k}{\tau}\right)\right|^p = \sum_{k=l+2}^{\infty} \frac{j_k - j_{k-1}}{\tau} \left|W\left(\frac{j_k}{\tau}\right) - W(\xi_k) + W(\xi_k)\right|^p$$

$$\leq 2^{p-1} \sum_{k=l+2}^{\infty} \frac{j_k - j_{k-1}}{\tau} \left(\left|W\left(\frac{j_k}{\tau}\right) - W(\xi_k)\right|^p + |W(\xi_k)|^p\right)$$

$$= 2^{p-1} \sum_{k=l+2}^{\infty} \left(\frac{j_k - j_{k-1}}{\tau} \left|\int_{\xi_k}^{\frac{j_k}{\tau}} W'(x) dx\right|^p + \int_{\frac{j_{k-1}}{\tau}}^{\frac{j_k}{\tau}} |W(x)|^p dx\right)$$

by (97). Clearly,

$$\left|\int_{\xi_k}^{\frac{j_k}{\tau}} W'(x)dx\right|^p \le \int_{\frac{j_{k-1}}{\tau}}^{\frac{j_k}{\tau}} |W'(x)|^p dx \cdot \left(\int_{\frac{j_{k-1}}{\tau}}^{\frac{j_k}{\tau}} 1dx\right)^{p-1}$$
$$= \left(\frac{j_k - j_{k-1}}{\tau}\right)^{p-1} \int_{\frac{j_{k-1}}{\tau}}^{\frac{j_k}{\tau}} |W'(x)|^p dx,$$

so

$$\begin{aligned}
&\sum_{k=l+2}^{\infty} \frac{j_k - j_{k-1}}{\tau}\left|W\left(\frac{j_k}{\tau}\right)\right|^p \\
&\le 2^{p-1} \sum_{k=l+2}^{\infty}\left(\left(\frac{j_k - j_{k-1}}{\tau}\right)^p \int_{\frac{j_{k-1}}{\tau}}^{\frac{j_k}{\tau}} |W'(x)|^p dx + \int_{\frac{j_{k-1}}{\tau}}^{\frac{j_k}{\tau}} |W(x)|^p dx\right) \\
&\le 2^{p-1}\left(\left(\frac{\delta_2}{\tau}\right)^p \int_{\frac{j_{l+1}}{\tau}}^{\infty} |W'(x)|^p dx + \int_{\frac{j_{l+1}}{\tau}}^{\infty} |W(x)|^p dx\right) \quad \text{[by (16)]} \\
&\le 2^{p-1}\left(\left(\frac{\delta_2}{\tau}\right)^p \int_{X_\varepsilon}^{\infty} |W'(x)|^p dx + \int_{X_\varepsilon}^{\infty} |W(x)|^p dx\right) < \frac{\varepsilon}{12}
\end{aligned} \tag{98}$$

by (93) and (94), if $\tau > \delta_2$. Similarly,

$$\sum_{k=-\infty}^{-l-2} \frac{j_k - j_{k-1}}{\tau}\left|W\left(\frac{j_k}{\tau}\right)\right|^p < \frac{\varepsilon}{12} \qquad \text{if } \tau > \delta_2\,. \tag{99}$$

It is clear that $|x^{\alpha+\frac{1}{2}} f(x)|^p$ and $|W(x)|^p$ both belong to $\mathcal{R}$. Hence, taking note of (92) and (93), we can claim that

$$\begin{aligned}
&\left|\sum_{k=-l-1}^{l+1} \frac{j_k - j_{k-1}}{\tau}\left|\left(\frac{j_k}{\tau}\right)^{\alpha+\frac{1}{2}}\left(f\left(\frac{j_k}{\tau}\right) - S_{\sigma,\alpha}\left(f;\frac{j_k}{\tau}\right)\right)\right|^p\right. \\
&\qquad \left. - \int_{-X_\varepsilon}^{X_\varepsilon}\left|x^{\alpha+\frac{1}{2}}\left(f(x) - S_{\sigma,\alpha}(f;x)\right)\right|^p dx\right| < \frac{7\varepsilon}{12}
\end{aligned} \tag{100}$$

for all large τ. The desired result follows from (92), (93), (95), (96), and (98)–(100). □

3.2. Proof of Theorem 1. Let $\sigma > 0$, and consider

$$f_\sigma^* := f - S_{\sigma,\alpha}(f;\cdot)\,.$$

By Lemma 17, $S_{\sigma,\alpha}(f;\cdot)$ is an entire function of exponential type σ and so $\left(x^{\alpha+\frac{1}{2}}\, S_{\sigma,\alpha}(f;x)\right)^p$ is regular and of exponential type in the open right half-plane. Furthermore, by Lemma 16, $\int_{-\infty}^{\infty} |x^{\alpha+\frac{1}{2}} S_{\sigma,\alpha}(f;x)|^p\, dx < \infty$. Hence, by Lemma 5,

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left|\left(\frac{j_k}{\tau}\right)^{\alpha+\frac{1}{2}} S_{\sigma,\alpha}\left(f;\frac{j_k}{\tau}\right)\right|^p < \infty, \qquad \tau > 0\,. \tag{101}$$

Since the function f obviously satisfies (39), it follows from (101) and Lemma 12 that the series for $L_{\tau,\alpha}(f_\sigma^* + S_{\sigma,\alpha}(f;\cdot);z)$ is absolutely convergent. Hence, for $\tau \geq \sigma$,

$$\begin{aligned} L_{\tau,\alpha}(f;z) &= L_{\tau,\alpha}(f_\sigma^* + S_{\sigma,\alpha}(f;\cdot);z) \\ &= L_{\tau,\alpha}(f_\sigma^*;z) + L_{\tau,\alpha}(S_{\sigma,\alpha}(f;\cdot);z) \\ &= L_{\tau,\alpha}(f_\sigma^*;z) + S_{\sigma,\alpha}(f;z) \end{aligned}$$

by Lemma 13. Therefore,

$$f(z) - L_{\tau,\alpha}(f;z) = f_\sigma^*(z) - L_{\tau,\alpha}(f_\sigma^*;z)$$

and

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} \left| x^{\alpha+\frac{1}{2}}(f_\sigma^*(x) - L_{\tau,\alpha}(f_\sigma^*;x))\right|^p dx\right)^{1/p} \\ &\leq 2^{(p-1)/p}\left\{\left(\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}} f_\sigma^*(x)\right|^p dx\right)^{1/p} + \left(\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}} L_{\tau,\alpha}(f_\sigma^*;x))\right|^p dx\right)^{1/p}\right\}. \end{aligned}$$

In view of the hypothesis on f and (101), we have

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left|\left(\frac{j_k}{\tau}\right)^{\alpha+\frac{1}{2}} f_\sigma^*\left(\frac{j_k}{\tau}\right)\right|^p < \infty,$$

so, by Lemma 14,

$$\begin{aligned} \int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}} L_{\tau,\alpha}(f_\sigma^*;x))\right|^p dx &\leq B_{\alpha,p}^p \frac{\pi}{\tau} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left|\frac{1}{\tau^{\alpha+\frac{1}{2}} G_\alpha'(j_k)} \cdot f_\sigma^*\left(\frac{j_k}{\tau}\right)\right|^p \\ &\leq B_{\alpha,p}^p \frac{\pi}{\delta_1 c_2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{j_k - j_{k-1}}{\tau}\left|\left(\frac{j_k}{\tau}\right)^{\alpha+\frac{1}{2}} f_\sigma^*\left(\frac{j_k}{\tau}\right)\right|^p \end{aligned}$$

by (19) and (15). Now, it follows from Lemma 19 that

$$\lim_{\tau\to\infty} \int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}} L_{\tau,\alpha}(f_\sigma^*;x))\right|^p dx \leq B_{\alpha,p}^p \frac{\pi}{\delta_1 c_2}\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}} f_\sigma^*(x)\right|^p dx .$$

Hence,

$$\begin{aligned} &\lim_{\tau\to\infty}\left(\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}}(f(x) - L_{\tau,\alpha}(f;x))\right|^p dx\right)^{1/p} \\ &= \lim_{\tau\to\infty}\left(\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}}(f_\sigma^*(x) - L_{\tau,\alpha}(f_\sigma^*;x))\right|^p dx\right)^{1/p} \\ &\leq 2^{(p-1)/p}\left\{1 + B_{\alpha,p}\left(\frac{\pi}{\delta_1 c_2}\right)^{1/p}\right\}\left(\int_{-\infty}^{\infty}\left|x^{\alpha+\frac{1}{2}} f_\sigma^*(x)\right|^p dx\right)^{1/p} . \qquad (102) \end{aligned}$$

Now let ε be any positive number. In view of Lemma 18, the right-hand side of (102) can be made less than ε by taking σ large enough. With this, the theorem is proved. □

3.3. Justification of Remark 1. If X denotes the Banach space of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ which vanish outside $[0,1]$, then $f \to L_{\tau,\alpha}(f;\cdot)$ defines a bounded linear transformation $\Lambda_{\tau,\alpha}$ from X to the normed linear space $C[-1,1]$ of all continuous functions ϕ with $\|\phi\| := \max_{-1\le x\le 1} |\phi(x)|$. For all large τ, we have

$$\begin{aligned}
\|\Lambda_{\tau,\alpha}\| &\geq \sum_{0<\frac{j_\nu}{\tau}\leq 1} \frac{|G_\alpha(\frac{1}{2}j_1)|}{|G'_\alpha(j_\nu)||j_\nu - \frac{1}{2}j_1|} \\
&> \frac{1}{c_2}|G_\alpha(\tfrac{1}{2}j_1)| \sum_{0<\frac{j_\nu}{\tau}\leq 1} \frac{|j_\nu|^{\alpha+\frac{1}{2}}}{|j_\nu - \frac{1}{2}j_1|} \qquad \text{[by (19)]} \\
&\geq \frac{1}{c_2}|G_\alpha(\tfrac{1}{2}j_1)| \sum_{0<\frac{j_\nu}{\tau}\leq 1} \frac{1}{|j_\nu - \frac{1}{2}j_1|}
\end{aligned}$$

since $j_1 \geq \pi/2$ for $\alpha \geq -1/2$. Using the asymptotic formula (14), we easily conclude that

$$\left\|\Lambda_{\tau,\alpha}\right\| > c_{11} \log \tau$$

where $c_{11} = c_{11}(\alpha)$ is a positive constant depending only on α. Thus, $\sup_\tau \left\|\Lambda_{\tau,\alpha}\right\| = \infty$. Hence, by the Banach-Steinhaus theorem [30, p. 98], there exists a function $f^* \in X$ and so satisfying (8) such that

$$\max_{-1\leq x\leq 1} \left|f^*(x) - L_{\tau,\alpha}(f^*;x)\right|$$

does not remain bounded as $\tau \to \infty$. This idea has been used before in a similar situation [12].

3.4. Justification of Remark 3. For $p \geq 2$, we obviously have

$$\int_{|x|\geq 1} \left|f(x) - L_{\tau,\alpha}(f;x)\right|^p |x|^{2\alpha+1}\, dx \leq \int_{|x|\geq 1} \left|x^{\alpha+\frac{1}{2}}(f(x) - L_{\tau,\alpha}(f;x))\right|^p dx,$$

so it is enough to check that

$$\int_{-1}^{1} \left|f(x) - L_{\tau,\alpha}(f;x)\right|^p |x|^{2\alpha+1}\, dx \to 0 \qquad \text{as } \tau \to \infty\,. \tag{103}$$

Going through the proof of Theorem 1, we need to show that if $\tau > \sigma$ and $\sigma \to \infty$, then

$$\int_{-1}^{1} |x|^{2\alpha+1} \left|f(x) - S_{\sigma,\alpha}(f;x)\right|^p dx \to 0\,, \tag{104}$$

$$\int_{-1}^{1} |x|^{2\alpha+1} \left|L_{\tau,\alpha}(f^*_\sigma;x)\right|^p dx \to 0\,. \tag{105}$$

As regards (104), it can be established in a manner analogous to Lemma 18 by first proving it for step functions, which involves verifying it for $[0,a]$, $[a,b]$ where $0 < a < b \leq 1$; the argument can be completed as before with the help of Lemma 15 and the following (easy to prove) modification of Part (i) of Lemma 16.

Lemma 16′. *If f satisfies the conditions of Part* (i) *of Lemma* 16, *then for $\tau \geq 1$, we have*

$$\int_{-1}^{1} |x|^{2\alpha+1} \big|S_{\tau,\alpha}(f;x)\big|^p dx \leq C''_{\alpha,p} \Bigg\{ \bigg(\int_{-1}^{1} |f(x)|^p dx \bigg)^{1/p} + \bigg(\int_{|x|\geq 1} |x|^{2\alpha+1} |f(x)|^p dx \bigg)^{1/p} \Bigg\}$$

where $C''_{\alpha,p}$ is a constant depending only on α and p.

In order to verify (105) we first observe that

$$\int_{-1}^{1} |x|^{2\alpha+1} |L_{\tau,\alpha}(f_\sigma^*;x)|^p dx \leq B_{\alpha,p}^p \frac{\pi}{\tau} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left| \frac{1}{\tau^{\alpha+\frac{1}{2}} G'_\alpha(j_k)} f_\sigma^* \left(\frac{j_k}{\tau} \right) \right|^p$$

which can be proved the same way as (53). Then we apply Lemmas 19 and 18 as in the proof of Theorem 1.

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