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A DIFFUSIVE AGE-STRUCTURED SEIRS EPIDEMIC MODEL

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ABSTRACT. A model of an epidemic disease is analyzed with the following features: (1) the susceptible and infective populations are diffusing in a spatial region and (2) infected individuals pass through an incubation period before becoming infectives throughout the spatial region. The model consists of a system of nonlinear partial differential equations in which the infective population is structured by age since first infected. It is proved that the infective population converges to zero and the susceptible population converges to a constant positive value throughout the region. The extinction of the infective population is due to the uniformly positive rate of their removal.

1. Introduction

We analyze a system of partial differential equations modeling the spatial spread of infectious disease through a population structured by age of infection. The population is confined to a bounded region Ω in R^n. We divide the population into susceptible, exposed, and infective subclasses, having density functions $S(x,t)$, $E(x,t)$, and $I(x,t)$ respectively, for $x \in \Omega$ and $t \geq 0$. The total populations of the subclasses at time t are obtained by integrating the densities over Ω. We envision the following scenario. The susceptible class consists of individuals capable of becoming infected, the exposed class is made up of individuals who have contracted the disease but who are not yet capable of transmitting it, and the infective class consists of individuals capable of transmitting the disease. We shall assume that at a rate proportional to the product of the susceptible and the infective populations, individuals leave the susceptible class and enter the exposed class. We stipulate that individuals, after remaining in the exposed class for a fixed incubation or latency period of length τ, enter the infective class where they remain for the finite length of time σ. Individuals are removed from both the latent and infective phases of the disease with a constant mortality rate λ and the probability of mortality is independent of the length of time the individual has been in either of these phases. At the conclusion of the infective period, individuals pass back into the susceptible class and do not enter a removed or immune class. Properly speaking, this process may be described as an SEIRS (susceptible, exposed, infective, removed, susceptible) disease model which describes the progression of a potentially fatal disease offering the possibility of recovery but no immunity. However, because the individuals comprising the removed class no longer affect the dynamics of the spread of the disease, we do not need to include the removed class in the modeling process.

We note that in previous work of the authors, cf. [23] and [9], an SEIR model was considered and the following asymptotic behavior was observed: The susceptible population tends to a constant (which is nonzero if the initial population is positive

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on a set of positive measure), and the infective population tends to zero. It was noted in [9] that the infective population ultimately extinguishes even if the mortality rate λ is zero. The present paper addresses the question: If the infectives return to the susceptible population when their infection age reaches $\tau + \sigma$, does the extinction of the infectives still occur?

The traditional approach, cf. [3] and the references contained therein, would describe these kinetics with a system of delay-differential equations. Our previous work, [9–10, 23], has demonstrated the advantages of modeling the progression of an infection through different stages by means of a structural variable a which represents the age of infection. This age variable gives rise to a new state variable $\ell(x, t, a)$ representing the density of infection with respect to space and age. Spatial densities are obtained by integrating $\ell(x, t, a)$ over a positive age interval. For example, the density of the exposed class is obtained by integrating over the length of the period of incubation, namely,

$$E(x,t) = \int_0^\tau \ell(x,t,a)\,da. \tag{1.1}$$

The density of the infective class is obtained by integrating over the duration of the infectious period:

$$I(x,t) = \int_\tau^{\tau+\sigma} \ell(x,t,a)\,da. \tag{1.2}$$

Finally, we make the assumption that the population disperses by Brownian motion leading to the standard diffusion model of dispersion.

Many recent works have used distributed parameter systems to describe the spread of infectious disease. Diffusive epidemic models are treated in [4–10, 12, 23–24] and age structured epidemic models featuring incubation or latency periods appear in [2–3, 10, 15, 17–18].

2. The equations

Throughout the following, Ω is a bounded region in R^n with smooth boundary, Δ denotes the n-dimensional Laplacian operator, and $\partial/\partial n$ denotes the outward normal derivative. We shall select positive constants d_1 and d_2 for the diffusivity or dispersion rates of the susceptible and infected populations, respectively, a positive constant $\lambda > 0$ for the mortality rate, and a positive constant $r > 0$ for the contact rate between susceptibles and infectives. We recall that $\tau > 0$ and $\sigma > 0$ represent the length of the incubation period and the period of disease duration. The equations for $S(x, t)$ and $\ell(x, t, a)$ are

$$\frac{\partial S(x,t)}{\partial t} = d_1 \Delta S(x,t) - rS(x,t)\int_\tau^{\tau+\sigma} \ell(x,t,a)\,da + \ell\,(x,t,\tau+\sigma)\,, \tag{2.1a}$$

$$\frac{\partial \ell\,(x,t,a)}{\partial t} + \frac{\partial \ell\,(x,t,a)}{\partial a} = d_2 \Delta \ell\,(x,t,a) - \lambda \ell\,(x,t,a)\,, \tag{2.1b}$$

for $x \in \Omega$, $t > 0$, and $a \in [0, \tau + \sigma]$. Homogeneous Neumann boundary conditions,

$$\frac{\partial S\,(x,t)}{\partial n} = \frac{\partial \ell\,(x,t,a)}{\partial n} = 0 \quad \text{for } x \in \partial\Omega,\ t > 0,\ a \in [0, \tau+\sigma], \tag{2.1c}$$

insure that the populations remain confined to Ω for all time.

We need initial conditions for S and ℓ:

$$S(x,0) = S_o(x) \quad \text{for } x \in \overline{\Omega}, \tag{2.1d}$$

$$\ell(x,0,a) = \ell_o(x,a) \quad \text{for } x \in \overline{\Omega},\ a \in [0, \tau+\sigma], \tag{2.1e}$$

$$\ell(x,t,0) = rS(x,t) \int_{\tau}^{\tau+\sigma} \ell(x,t,a)\, da \quad \text{for } x \in \overline{\Omega},\ t \geq 0. \tag{2.1f}$$

We remark that (2.1f) defines a birth function which describes input into the population at disease age $a = 0$.

By a *classical solution* to (2.1a–f), we shall mean a pair $\{S(x,t), \ell(x,t,a)\}$ which is twice continuously differentiable in x for $t > 0$ and $a \in [0, \tau+\sigma]$, continuously differentiable in t and a except along the lines $t - a = 0$, τ, $\tau + \sigma$, and satisfies the differential equations and the boundary and initial conditions. We point out that a jump discontinuity will be propagated along the characteristic line $t - a = 0$ unless the compatibility condition

$$\ell_o(x,0) = rS_o(x) \int_{\tau}^{\tau+\sigma} \ell(x,a)\, da, \quad x \in \overline{\Omega}, \tag{2.2}$$

holds.

In our analysis, we use the theory of analytic semigroups. Toward this end, we introduce operators A_1 and A_2 on $C(\overline{\Omega})$ having pointwise definitions:

$$(A_1 u)(x) = (d_1 \Delta u)(x) \quad \text{for } x \in \Omega, \tag{2.3a}$$

$$(A_2 u)(x) = [(d_2 \Delta - \lambda)(u)](x), \tag{2.3b}$$

with (for $i = 1, 2$),

$$\mathbb{D}(A_i) = \left\{ u \in C(\overline{\Omega}) \,\middle|\, u \in C^2(\overline{\Omega}) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}. \tag{2.4}$$

It is well-known, cf. [19], that for $i = 1, 2$, A_i is the infinitesimal generator of an analytic semigroup $\{T_i(t) \mid t \geq 0\}$ on $C(\overline{\Omega})$. In the second case, the operator norm is exponentially decaying. Arguments given in the first chapter of [24] may be modified to produce the following representation for $\ell(x,t,a)$ which we state without proof:

Proposition 2.1. *If* $\{S(x,t), \ell(x,t,a)\}$ *is the classical solution pair for* (2.1a–f), *then the solution of* (2.1b) *has the representation,*

$$\ell(x,t,a) = T_2(t)\ell_o(x, a-t) \; \mathit{for}\, x \in \Omega, t \in [0,a], \tag{2.5a}$$

$$\begin{aligned} \ell(x,t,a) &= T_2(a)\ell(x, t-a, 0) \\ &= rT_2(a)S(x,t-a)I(x,t-a) \; \mathit{for}\, x \in \Omega, a \in [0,t). \end{aligned} \tag{2.5b}$$

We obtain the following three-component system of weakly-coupled semilinear parabolic equations by formally differentiating $E(x,t)$ and $I(x,t)$, as given by (1.1) and (1.2) and applying (2.1b) and Proposition 2.1:

$$\begin{aligned} \frac{\partial S(x,t)}{\partial t} &= d_1 \Delta S(x,t) - rS(x,t)I(x,t) \\ &\quad + rT_2(\tau+\sigma)S(x, t-(\tau+\sigma))I(x, t-(\tau+\sigma)), \end{aligned} \tag{2.6a}$$

$$\begin{aligned} \frac{\partial E(x,t)}{\partial t} &= d_2 \Delta E(x,t) + \ell(x,t,0) - \ell(x,t,\tau) - \lambda E(x,t), \\ &= d_2 \Delta E(x,t) + rS(x,t)I(x,t) \\ &\quad - rT_2(\tau)S(x,t-\tau)I(x,t-\tau) - \lambda E(x,t), \end{aligned} \tag{2.6b}$$

$$\begin{aligned}\frac{\partial I(x,t)}{\partial t} &= d_2 \Delta I(x,t) + \ell(x,t,\tau) - \ell(x,t,\tau+\sigma) - \lambda I(x,t) \\ &= d_2 \Delta I(x,t) + rT_2(\tau) S(x,t-\tau) I(x,t-\tau) \\ &\quad - rT_2(\tau+\sigma) S(x,t-(\tau+\sigma)) I(x,t-(\tau+\sigma)) - \lambda I(x,t), \qquad (2.6c)\end{aligned}$$

for $x \in \Omega$ and $t > \tau + \sigma$. The appropriate initial data for these equations are

$$\begin{aligned} S(x,0) &= S_o(x), \\ E(x,0) &= E_o(x) = \int_0^\tau \ell_0(x,a)\, da, I(x,0) \\ &= I_o(x) = \int_\tau^{\tau+\sigma} \ell_o(x,a)\, da, \qquad (2.6d)\end{aligned}$$

for $x \in \overline{\Omega}$. Once again, we have homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial E}{\partial n} = 0 \quad \text{for } x \in \partial\Omega,\ t > 0. \qquad (2.6e)$$

Thus, we have derived a new system, equivalent to (2.1a–f), which could be termed a reaction-diffusion system with delays. Our analysis will take advantage of both formulations.

3. The main results

We begin by applying the method of steps to obtain a global existence result.

Theorem 3.1. *For each pair $(S_0(x), \ell_0(x,a))$, there exists a unique, nonnegative, classical, globally-defined solution pair $(S(x,t), \ell(x,t,a))$ satisfying* (2.1a–f).

Proof. Suppose that $0 \le t \le \tau$. If $a \ge \tau$, this implies that $a \ge t$, which yields, from (2.5a), $\ell(x,t,a) = T_2(t)\ell_0(x,a-t) \ge 0$. Thus, from (1.2), $I(x,t)$ is a known nonnegative function on $0 \le t \le \tau$. From (2.1a), we have

$$S(x,t) = T_1(t) S_0(x) - \int_0^t T_1(t-s)(rS(x,s)I(x,s) - \ell(x,s,\tau+\sigma))\, ds. \qquad (3.1)$$

For $s \in [0,\tau]$, $s < \tau+\sigma$, so, by (2.5a), $\ell(x,s,\tau+\sigma) = T_2(s)\ell_0(x,\tau+\sigma-s) \ge 0$. Since $I(x,s)$ is known on $[0,\tau]$, by [19, Theorem 2.2], there exists a unique continuous solution of (3.1) on $[0,\tau]$. Also, since $T_2(t)\left(C\left(\overline{\Omega}\right)\right) \subset \mathcal{D}(A_2)$ for all $t > 0$ and ℓ_0 is continuously differentiable in its second variable (except possibly at $t = a$), $I(x,t)$ is continuously differentiable on $(0,\tau]$. We can apply [19, Proposition 3.1] to conclude that $S(x,t)$ also is continuously differentiable on $(0,\tau]$ and satisfies (2.1a). By standard maximum principle arguments, $S(x,t)$ is nonnegative.

If $\tau < t \le \tau+\sigma$, then $I(x,t)$ satisfies

$$\begin{aligned} I(x,t) &= \int_\tau^{\tau+\sigma} \ell(x,t,a)\, da = \int_\tau^t \ell(x,t,a)\, da + \int_t^{\tau+\sigma} \ell(x,t,a)\, da \\ &= \int_\tau^t \ell(x,t,a)\, da + \int_t^{\tau+\sigma} T_2(t)\ell_0(x,a-t)\, da. \qquad (3.2)\end{aligned}$$

For $t > a$, we have from (2.5b),

$$\ell(x,t,a) = T_2(a)\ell(x,t-a,0) = T_2(a) rS(x,t-a) I(x,t-a). \qquad (3.3)$$

If $\tau < t \le 2\tau$ and $a > \tau$, then $t - a \le 2\tau - a < 2\tau - \tau = \tau$. Thus, $S(x,t-a)$ and $I(x,t-a)$ are known, and $\ell(x,t,a)$ is known from (3.3). This yields $I(x,t)$ on $(\tau, 2\tau]$ from (3.2) and then $S(x,t)$ on $(\tau,2\tau]$ from (3.1), assuming that $2\tau \le \tau + \sigma$.

Let k be the smallest positive integer such that $(k-1)\tau < \sigma \leq k\tau$. The entire process described above can be continued up to $k\tau < t \leq \tau + \sigma$ (since $k\tau < \tau + \sigma \leq \tau + k\tau = (k+1)\tau$) to find $I(x,t)$ and $S(x,t)$.

If $t > \tau + \sigma$,

$$I(x,t) = \int_{\tau}^{\tau+\sigma} \ell(x,t,a)\, da = \int_{\tau}^{\tau+\sigma} T_2(a)\, rS(x,t-a)\, I(x,t-a)\, da. \tag{3.4}$$

If $\tau + \sigma < t \leq 2\tau + \sigma$ and $a > \tau$, then $t - a \leq 2\tau + \sigma - \tau = \tau + \sigma$, and $S(x,t-a)$ and $I(x,t-a)$ are known. Thus $I(x,t)$ is known on $(\tau + \sigma, 2\tau + \sigma]$ from (3.4). If $t \geq \tau + \sigma$,

$$S(x,t) = T_1(t)\, S(x,\tau+\sigma) - \int_{\tau+\sigma}^{t} T_1(t-s)\, (rS(x,s)\, I(x,s) - \ell(x,s,\tau+\sigma))\, ds. \tag{3.5}$$

For $s > \tau + \sigma$,

$$\begin{aligned} \ell(x,s,\tau+\sigma) &= T_2(\tau+\sigma)\ell(x,s-(\tau+\sigma),0) \\ &= T_2(\tau+\sigma)\big(rS(x,s-(\tau+\sigma))I(x,s-(\tau+\sigma))\big). \end{aligned}$$

Suppose $\tau+\sigma < t \leq 2\tau+\sigma$. For $s \leq t \leq 2\tau+\sigma$, $s-(\tau+\sigma) \leq 2\tau+\sigma-\tau-\sigma = \tau$. Thus, $\ell(x,s,\tau+\sigma)$ is known. Since $I(x,s)$ also is known on $(\tau+\sigma, 2\tau+\sigma]$, the preceding analysis can be used to show the existence of a unique, continuously differentiable, nonnegative function $S(x,t)$ on $(\tau+\sigma, 2\tau+\sigma]$. We note that for $(n-1)\tau + \sigma < t \leq n\tau + \sigma$ and $\tau + \sigma < s \leq t \leq n\tau + \sigma$, we have $s - (\tau+\sigma) \leq n\tau + \sigma - \tau - \sigma = (n-1)\tau < (n-1)\tau + \sigma$, so $\ell(x,s,\tau+\sigma)$ is known. Thus, this entire process can be repeated on $(2\tau+\sigma, 3\tau+\sigma]$, ... to yield the existence of unique, nonnegative functions $S(x,t)$, $I(x,t)$, $\ell(x,t,a)$ satisfying (2.1a–f), (1.2). □

Before examining the asymptotic behavior of solutions of (2.1a-f), we first obtain uniform *a priori* bounds for S, E, I, and ℓ.

Let $\|u(\cdot)\|_{\infty,\Omega} = \sup_{x\in\overline{\Omega}} |u(x)|$ for $u \in C(\overline{\Omega})$. The norm on $L^p(\Omega)$, $p \geq 1$, is denoted by $\|\cdot\|_{p,\Omega}$. Adding the equations (2.6a–c) (where we note that $t > \tau + \sigma$) and integrating on $\Omega \times [0,t)$, we obtain

$$\begin{aligned} &\|S(\cdot,t)\|_{1,\Omega} + \|E(\cdot,t)\|_{1,\Omega} + \|I(\cdot,t)\|_{1,\Omega} + \lambda \int_0^t \|E(\cdot,s)\|_{1,\Omega}\, ds \\ &\quad + \lambda \int_0^t \|I(\cdot,s)\|_{1,\Omega}\, ds = \|S_0(\cdot)\|_{1,\Omega} + \|E_0(\cdot)\|_{1,\Omega} + \|I_0(\cdot)\|_{1,\Omega}. \end{aligned} \tag{3.6}$$

Lemma 3.1. *Let S, ℓ be the unique solution pair of* (2.1a–f), *and let E, I be defined by* (1.1) *and* (1.2), *respectively. Then there exists an $M_0 > 0$ which depends on $\|S_0\|_{\infty,\Omega}$, $\|\ell_0\|_{\infty,\Omega\times[0,\tau+\sigma]}$, $\|E_0\|_{\infty,\Omega}$ and $\|I_0\|_{\infty,\Omega}$, so that*

$$\max\{\|S(\cdot,t)\|_{\infty,\Omega},\ \|E(\cdot,t)\|_{\infty,\Omega},\ \|I(\cdot,t)\|_{\infty,\Omega},\ \|\ell(\cdot,t,\cdot)\|_{\infty,\Omega\times[0,\tau+\sigma]}\} \leq M_0$$

for all $t \geq 0$.

Proof. We first establish the bound on $S(x,t)$.

Let g be the Green's function associated with the semigroup $\{T_2(t) : t \geq 0\}$ generated by A_2, as defined by (2.3b), in the sense that

$$(T_2(t)\phi)(x) = \int_{\Omega} g(x,y,t)\, \phi(y)\, dy \tag{3.7}$$

for ϕ in $C(\overline{\Omega})$.

Let $t_1 \geq \tau + \sigma$ and $t \geq t_1$. By the nonnegativity of S and I, we have, from (2.1a),

$$S(\cdot, t) \leq T_1(t - t_1) S(\cdot, t_1) + \int_{t_1}^{t} T_1(t - s)\ell(\cdot, s, \tau + \sigma)\, ds.$$

By (2.5b) and (2.1f), we have

$$\begin{aligned} S(x,t) &\leq (T_1(t - t_1) S(\cdot, t_1))(x) + \int_{t_1}^{t} T_1(t-s)\left(T_2(\tau+\sigma)\ell(\cdot, s - \tau - \sigma, 0)\right)(x)\, ds \\ &= (T_1(t - t_1) S(\cdot, t_1))(x) + \int_{t_1}^{t} T_1(t-s) \\ &\quad \times \left(\int_{\Omega} g(\cdot, y, \tau + \sigma)\, rS(y, s - \tau - \sigma)\, I(y, s - \tau - \sigma)\, dy\right)(x)\, ds. \end{aligned} \tag{3.8}$$

Since $\|T_1(t)\| \leq 1$ for all $t \geq 0$ (where $\|\cdot\|$ denotes the operator norm on $C(\overline{\Omega})$), and since for a fixed $\bar{t} > 0$, there exists $C > 0$ such that

$$\sup_{x,y\in\overline{\Omega},\ t\geq\bar{t}} |g(x,y,t)| \leq C, \tag{3.9}$$

(3.8) yields

$$\begin{aligned} |S(x,t)| &\leq \|S(\cdot, t_1)\|_{\infty,\Omega} \\ &\quad + r\int_{t_1}^{t} \sup_{x\in\overline{\Omega}} \left|\int_{\Omega} g(x, y, \tau+\sigma)\, S(y, s-\tau-\sigma)\, I(y, s-\tau-\sigma)\, dy\right| ds \\ &\leq \|S(\cdot, t_1)\|_{\infty,\Omega} \\ &\quad + rC\int_{t_1}^{t} \left(\|S(\cdot, s-\tau-\sigma)\|_{\infty,\Omega} \cdot \int_{\Omega} I(y, s-\tau-\sigma)\, dy\right) ds. \end{aligned} \tag{3.10}$$

Let $\xi = s - \tau - \sigma$. Then (3.10) yields

$$\begin{aligned} \|S(\cdot,t)\|_{\infty,\Omega} &\leq \|S(\cdot, t_1)\|_{\infty,\Omega} + rC\int_{t_1-\tau-\sigma}^{t-\tau-\sigma} \left(\|S(\cdot,\xi)\|_{\infty,\Omega}\int_{\Omega} I(y,\xi)\, dy\right) d\xi \\ &\leq \|S(\cdot, t_1)\|_{\infty,\Omega} + rC\int_{0}^{t} \|S(\cdot,\xi)\|_{\infty,\Omega}\, \|I(\cdot,\xi)\|_{\infty,\Omega}\, d\xi. \end{aligned} \tag{3.11}$$

An application of Gronwall's inequality to (3.11) yields

$$\begin{aligned} \|S(\cdot,t)\|_{\infty,\Omega} &\leq \|S(\cdot, t_1)\|_{\infty,\Omega} \exp\left(rC\int_{0}^{t} \|I(\cdot,\xi)\|_{1,\Omega}\, d\xi\right) \\ &\leq \|S(\cdot, t_1)\|_{\infty,\Omega} \exp\left(rC\int_{0}^{\infty} \|I(\cdot,\xi)\|_{1,\Omega}\, d\xi\right), \end{aligned} \tag{3.12}$$

where we have used (3.6) to observe that $\int_0^\infty \|I(\cdot,s)\|_{1,\Omega}\, ds < \infty$.

We define $W(x,t) = E(x,t) + I(x,t)$, and note that, by adding (2.6b) and (2.6c), W satisfies the equation and inequalities

$$\begin{aligned} \frac{\partial W(x,t)}{\partial t} &= d_2\Delta W(x,t) + \ell(x,t,0) - \ell(x,t,\tau+\sigma) - \lambda W(x,t) \\ &\leq d_2\Delta W(x,t) + rS(x,t)I(x,t) - \lambda W(x,t) \\ &\leq d_2\Delta W(x,t) + rS(x,t)W(x,t). \end{aligned}$$

Since we have *a priori* bounds for $\|W(\cdot,t)\|_{1,\Omega}$ by (3.6) and for $\|S(\cdot,t)\|_{\infty,\Omega}$ by (3.12), we can apply Alikakos-Moser iteration arguments [1] to produce a bound for $\|W(\cdot,t)\|_{1,\Omega}$ in terms of $\|W_0(\cdot)\|_{\infty,\Omega}$ and $\|W(\cdot,t)\|_{1,\Omega}$. This produces a uniform bound for $\|E(\cdot,t)\|_{\infty,\Omega}$ and $\|I(\cdot,t)\|_{\infty,\Omega}$. The uniform bound for $\|\ell(\cdot,t,\cdot)\|_{\infty,\Omega\times[0,\tau+\sigma]}$ now follows from (2.5a) and (2.5b). □

In a similar fashion to Lemmas 3.4–3.8 of [9], we next establish the following lemmas, which are needed in our proof of the asymptotic behavior of solutions of (2.1a-f).

Lemma 3.2. $\displaystyle\int_0^\infty \|I(\cdot,s)\|_{\infty,\Omega}\,ds < \infty.$

Proof. Let $t_1 \geq \tau+\sigma$ and $t \geq t_1$. By (1.2), (2.5b), (3.7), and (3.9),

$$\begin{aligned}
0 \leq \int_{t_1}^t I(x,s)\,ds &= \int_{t_1}^t \int_\tau^{\tau+\sigma} \ell(x,s,a)\,da\,ds \\
&= \int_{t_1}^t \int_\tau^{\tau+\sigma} T_2(a)\,\ell(x,s-a,0)\,da\,ds \\
&= \int_{t_1}^t \int_\tau^{\tau+\sigma} \int_\Omega g(x,y,a)\,\ell(y,s-a,0)\,dy\,da\,ds \\
&\leq rC \int_{t_1}^t \int_\tau^{\tau+\sigma} \int_\Omega S\,(y,s-a)\,I\,(y,s-a)\;dy\,da\,ds \\
&\leq rC\sigma \int_0^\infty \int_\Omega S\,(y,s)\,I\,(y,s)\;dy\,ds \\
&\leq rC\sigma \sup_{t\in[0,\infty)} \|S\,(\cdot,t)\|_{\infty,\Omega} \int_0^\infty \|I\,(\cdot,s)\|_{1,\Omega}\,ds.
\end{aligned}$$

By (3.12) and (3.6), the claim of the lemma has been shown. □

Lemma 3.3. *$\{I\,(\cdot,t) : t \geq 0\}$ is pre-compact in $C\,(\overline{\Omega})$.*

Proof. From (2.6c) and (2.3b),

$$\begin{aligned}
\frac{\partial I(x,t)}{\partial t} &= d_2\Delta I(x,t) + \ell\,(x,t,\tau) - \ell\,(x,t,\tau+\sigma) - \lambda I(x,t) \\
&= A_2 I(x,t) + F(x,t), \qquad (3.13)
\end{aligned}$$

where $F(x,t) = \ell\,(x,t,\tau) - \ell\,(x,t,\tau+\sigma)$ and $\|F(\cdot,t)\|_{\infty,\Omega} \leq M_0$ for all $t \geq 0$ by Lemma 3.1.

Since A_2 is the infinitesimal generator of an analytic semigroup $\{T_2\,(t) : t \geq 0\}$ and $0 \in \rho\,(A_2)$, fractional powers A_2^β, $0 < \beta < 1$, may be obtained. From (3.13), we have

$$I\,(\cdot,t) = T_2\,(t)\,I\,(\cdot,0) + \int_0^t T_2\,(t-s)\,F\,(\cdot,s)\;ds.$$

Thus,

$$A_2^\beta I\,(\cdot,t) = A_2^\beta T_2\,(t)\,I\,(\cdot,0) + \int_0^t A_2^\beta T_2\,(t-s)\,F\,(\cdot,s)\;ds. \qquad (3.14)$$

From the known semigroup estimates, we have $\|T_2(t)\| \leq e^{-\lambda t}$ and $\left\|A_2^\beta T_2(t-s)\right\| \leq C_1\,(\beta)\,(t-s)^{-\beta} e^{-\lambda(t-s)}$ where $C_1\,(\beta) > 0$. If $\tau > 0$, then $\{I(\cdot,t)\,|0 \leq t \leq \tau\}$ is compact. Moreover, there exists $C\,(\beta) > 0$ so that if $\lambda' < \lambda$, then $\left\|A_2^\beta T_2(t)\right\| \leq \dfrac{C\,(\beta)}{t^\beta} e^{-\lambda' t}$.

Thus, if $t > \tau > 0$, $\left\|A_2^\beta T_2(t) I(\cdot,0)\right\|_{\infty,\Omega} \le \frac{C(\beta)}{\tau^\beta} \|I(\cdot,0)\|_{\infty,\Omega}$, (3.14) now yields

$$\begin{aligned}\left\|A_2^\beta I(\cdot,t)\right\|_{\infty,\Omega} &\le \frac{C(\beta)}{\tau^\beta} \|I(\cdot,0)\|_{\infty,\Omega} + \int_0^t C_1(\beta) e^{-\lambda(t-s)}(t-s)^{-\beta} M_0\, ds \\ &\le C_2 + C_3 \int_0^\infty e^{-\lambda s} s^{-\beta}\, ds \\ &= C_2 + C_3 \frac{\Gamma(1-\beta)}{\lambda^{1-\beta}}\end{aligned}$$

where $C_2 = \frac{C(\beta)}{\tau^\beta} \|I(\cdot,0)\|_{\infty,\Omega}$ and $C_3 = C_1(\beta) M_0$. Thus $I(\cdot,t) = A_2^{-\beta} A_2^\beta I(\cdot,t)$, which implies that the trajectory $I(\cdot,t)$ can be realized as the compact image of a bounded set, and hence the conclusion of the lemma follows. □

Lemma 3.4. *$\frac{d}{dt} \|I(\cdot,t)\|_{2,\Omega}^2$ is bounded in t.*

Proof. Similar to (3.13) in the proof of Lemma 3.3, we have

$$\frac{\partial I(x,t)}{\partial t} = d_2 \Delta(x,t) + F(x,t) - \lambda I(x,t)$$

where $F(x,t) = \ell(x,t,\tau) - \ell(x,t,\tau+\sigma)$ and $\|F(\cdot,t)\|_{\infty,\Omega} \le M_0$. Denoting the inner product $L_2(\Omega)$ by $(\cdot,\cdot)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \|I(\cdot,t)\|_{2,\Omega}^2 &= (I(\cdot,t), I(\cdot,t)) \\ &= d_2 (\Delta I_2(\cdot,t), I(\cdot,t)) + (F(\cdot,t), I(\cdot,t)) - \lambda \|I(\cdot,t)\|_{2,\Omega}^2 \\ &\le (F(\cdot,t), I(\cdot,t)).\end{aligned}$$

The conclusion of the lemma follows from the boundedness of $\|F(\cdot,t)\|_{\infty,\Omega}$ and $\|I(\cdot,t)\|_{\infty,\Omega}$. □

Lemma 3.5. *$\int_0^\infty \|I(\cdot,s)\|_{2,\Omega}^2\, ds < \infty$.*

Proof. For positive constants C_4 and C_5,

$$\begin{aligned}\int_0^\infty \|I(\cdot,s)\|_{2,\Omega}^2\, ds &\le C_4 \int_0^\infty \|I(\cdot,s)\|_{2,\Omega}\, ds \\ &\le C_5 \int_0^\infty \|I(\cdot,s)\|_{\infty,\Omega}\, ds < \infty\end{aligned}$$

by Lemma 3.2. □

Lemma 3.6. *$\lim_{t\to\infty} \|I(\cdot,t)\|_{2,\Omega}^2 = 0$.*

Proof. The result follows from Lemmas 3.4 and 3.5. □

We now can give our result on the asymptotic behavior of solutions of (2.1a–f).

Theorem 3.2. *There exists a constant $S_\infty \ge 0$ so that*

$$\lim_{t\to\infty} \|S(\cdot,t) - S_\infty\|_{\infty,\Omega} = 0. \tag{3.15}$$

Moreover,

$$\lim_{t\to\infty} \|I(\cdot,t)\|_{\infty,\Omega} = 0,$$

and, if $a \in [0, \tau + \sigma]$, *then*

$$\lim_{t \to \infty} \|\ell(\cdot, t, a)\|_{\infty,\Omega} = 0.$$

Finally, if $S_0 > 0$ *on a set of positive Lebesgue measure, then* $S_\infty > 0$.

Proof. We first show that $\lim_{t\to\infty} \|I(\cdot, t)\|_{\infty,\Omega}$ exists and

$$\lim_{t \to \infty} \|I(\cdot, t)\|_{\infty,\Omega} = 0. \tag{3.16}$$

By Lemma 3.3, every sequence $\{I(\cdot, t_n)\}$ has a convergent subsequence. Assume (3.16) is false. Then, for the sequence t_n, there exists a subsequence θ_n such that $\lim_{n\to\infty} I(\cdot, \theta_n) = q \neq 0$ $\left(q \in C(\overline{\Omega})\right)$. By Lemma 3.6,

$$\lim_{n \to \infty} \int_\Omega |I(x, \theta_n)|^2 \, dx = \int_\Omega [q(x)]^2 \, dx = 0,$$

which is a contradiction. Our arguments demonstrate that if t_n is a sequence such that $t_n \to \infty$, we can extract a subsequence θ_n, so that $\lim_{n\to\infty} I(\cdot, \theta_n) = 0$. Moreover, because the limits of each of the subsequences are identical, we can infer convergence of each sequence. Finally, because the limits are identical for each choice of sequence $t_n \to \infty$, we obtain convergence of $I(\cdot, t)$ to 0 as $t \to \infty$. The fact that for $a \in [0, \tau + \sigma]$,

$$\lim_{t \to \infty} \|\ell(\cdot, t, a)\|_{\infty,\Omega} = 0 \tag{3.17}$$

follows from the bound on $\|S(\cdot, t)\|_{\infty,\Omega}$, (3.16), and (2.5b). □

We now turn to the proof of (3.15). Integrating equation (2.6b) on $\Omega \times [0, t)$, we have

$$\begin{aligned} \|E(\cdot, t)\|_{1,\Omega} &+ \int_0^t \int_\Omega \ell(x, s, \tau) \, dx \, ds + \lambda \int_0^t \|E(\cdot, s)\|_{1,\Omega} \, ds \\ &= \|E_0(\cdot)\|_{1,\Omega} + \int_0^t \int_\Omega \ell(x, s, 0) \, dx \, ds. \end{aligned} \tag{3.18}$$

We have

$$\begin{aligned} \int_0^\infty \int_\Omega \ell(x, s, 0) \, dx \, ds &= r \int_0^\infty \int_\Omega S(x, s) I(x, s) \, dx \, ds \\ &\leq r \left(\sup_{t \geq 0} \|S(\cdot, t)\|_{\infty,\Omega} \right) \int_0^\infty \|I(\cdot, s)\|_{1,\Omega} \, ds < \infty \end{aligned} \tag{3.19}$$

by Lemma 3.1 and (3.6). Using (3.6) and (3.18), again it follows that

$$\int_0^\infty \int_\Omega \ell(x, s, \tau) \, dx \, ds < \infty. \tag{3.20}$$

Similarly, by integrating equation (2.6c) on $\Omega \times [0, t)$, we find that

$$\int_0^\infty \int_\Omega \ell(x, s, \tau + \sigma) \, dx \, ds < \infty. \tag{3.21}$$

Multiplying equation (2.1a) by S and integrating on $\Omega \times [0, t)$, we obtain

$$\begin{aligned} \frac{1}{2} \|S(\cdot, t)\|_{2,\Omega}^2 &+ d_1 \int_0^t \int_\Omega |\nabla S(x, s)|^2 \, dx \, ds + r \int_0^t \int_\Omega \left(S^2 I\right)(x, s) \, dx \, ds \\ &= \frac{1}{2} \|S_0(\cdot)\|_{2,\Omega}^2 + \int_0^t \int_\Omega S(x, s) \ell(x, s, \tau + \sigma) \, dx \, ds. \end{aligned} \tag{3.22}$$

By the uniform bound on $\|S(\cdot,t)\|_{\infty,\Omega}$, (3.19), and (3.21), we see that

$$\int_0^\infty \left\| \, |\nabla S(\cdot,s)| \, \right\|_{2,\Omega}^2 ds < \infty. \tag{3.23}$$

The *a priori* bounds for $\|S(\cdot,t)\|_{\infty,\Omega}$ and $\|I(\cdot,t)\|_{\infty,\Omega}$ (from Lemma 3.1) will produce uniform estimates for $|\nabla S|$, $|\nabla I|$, $|\partial S/\partial t|$, and $|\partial I/\partial t|$ (cf. [16]). We now can differentiate equation (2.1a) with respect to each spatial variable x_i to produce a uniform estimate for $\left|\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial x_i}\right)\right|$ and thereby develop a uniform estimate for $\left|\frac{\partial}{\partial t} \nabla S\right|$ (cf. Smoller [22, p.226]). Thus, using (3.23), we have

$$\lim_{t\to\infty} \left\| \, |\nabla S(\cdot,t)| \, \right\|_{2,\Omega} = 0.$$

Using the Poincaré-Wittinger inequality [14, p.88], we know that there exists a constant $K > 0$ such that

$$\left\|S(\cdot,t) - \overline{S}(t)\right\|_{2,\Omega} \le K \left\| \, |\nabla S(\cdot,t)| \, \right\|_{2,\Omega}$$

where $S(t) = |\Omega|^{-1} \int_\Omega S(x,t)\,dx$. Using the uniform bounds on $\|S(\cdot,t)\|_{\infty,\Omega}$ and $\left\| \, |\nabla S(\cdot,t)| \, \right\|_{\infty,\Omega}$, we may conclude that for any $p > 1$

$$\lim_{t\to\infty} \left[\left\|S(\cdot,t) - \overline{S}(t)\right\|_{p,\Omega} + \left\| \, |\nabla S(\cdot,t)| \, \right\|_{p,\Omega} \right] = 0.$$

Finally, by the Sobolev imbedding theorem, it follows that

$$\lim_{t\to\infty} \left\|S(\cdot,t) - \overline{S}(t)\right\|_{\infty,\Omega} = 0. \tag{3.24}$$

If we let $Z(t) = \frac{1}{|\Omega|}\left[\, \|S(\cdot,t)\|_{1,\Omega} + \|E(\cdot,t)\|_{1,\Omega} + \|I(\cdot,t)\|_{1,\Omega} \right]$, then we see by adding equations (2.6a–c) and integrating over Ω, that

$$\frac{dZ(t)}{dt} \le 0,$$

which implies that $\lim_{t\to\infty} Z(t)$ exists and is nonnegative. Since $\lim_{t\to\infty} \|I(\cdot,t)\|_{1,\Omega} = 0$ by (3.16) and $\lim_{t\to\infty} \|E(\cdot,t)\|_{1,\Omega} = 0$ by (3.17) and (1.1), it follows that there is a constant $S_\infty \ge 0$ such that

$$\lim_{t\to\infty} \overline{S}\,(t) = S_\infty.$$

Finally, it remains to show that $S_\infty > 0$ if $S_0(x) > 0$ on a set of positive measure.

By integrating (2.1a) over Ω, we find that

$$\begin{aligned} \frac{d}{dt} \|S(\cdot,t)\|_{1,\Omega} &\ge -r \int_\Omega S(x,t) I(x,t)\,dx \\ &\ge -r \|I(\cdot,t)\|_{\infty,\Omega} \|S(\cdot,t)\|_{1,\Omega}. \end{aligned}$$

Thus,

$$\begin{aligned} S_\infty = \lim_{t\to\infty} S\,(t) &= \lim_{t\to\infty} \frac{1}{|\Omega|} \|S\,(\cdot,t)\|_{1,\Omega} \\ &\ge \frac{1}{|\Omega|} \exp\left[-r \int_0^\infty \|I\,(\cdot,s)\|_{\infty,\Omega}\, ds\right] \|S_0\|_{1,\Omega} > 0. \end{aligned}$$

4. Concluding remarks

The authors feel that currently there is an extensive literature for the use of systems of ordinary differential equations in modeling the spread of infectious disease. The situation is very different concerning distributed parameter models. Although the concept of age-structured populations was introduced first by McKendrick [20] in the 1920s, the idea was not studied widely until after the appearance of the seminal papers of Hoppensteadt [13] and Gurtin and MacCamy [11]. To paraphrase Murray [21], the geographic or spatial spread of disease is far from understood despite the obvious need for such an understanding. The authors maintain an ongoing effort of enhancing this understanding.

For the present discussion, we again wish to make particular note to reference [9] where we established the asymptotic extinction of infection for a diffusive age-structured SEIR model. In this model, the only two possible outcomes of infection were death or its epidemiological equivalent, permanent immunity. However, our analysis showed that this extinction resulted from the reduction of susceptibles to a level below the threshold needed to sustain an epidemic, and the fact that the infectives, after they passed through the age boundary at $\tau + \sigma$ and entered the removed class, no longer affected the dynamics of the evolution of the disease. This extinction initially was thought to be attributable to the mortality due to infection; however, we explicitly demonstrated the persistence of infection extinction even when the mortality rate, λ, was identically zero. On the other hand, the SEIRS model considered here allows for recovery which is clearly represented by (2.6a–c) as a feedback from the infective class to the susceptible class. In this case, we are led to conclude that the extinction of the disease is driven by the mortality rate. Future work will examine diffusive age-dependent epidemic SEIRS models with zero mortality and models with spatially dependent and age-dependent mortality. We shall pay particular attention to cases featuring age-and spatially dependent mortality rates compactly supported on proper subregions of age and space.

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References

1. N. Alikakos, *An application of the invariance principle to reaction diffusion equations*, J. Diff. Eqns. **33** (1979), 201–225.
2. S. Busenberg, K. Cooke, and M. Iannelli, *Endemic thresholds and stability in a class of age-structured epidemics*, SIAM J. Appl. Math. **48** (1988), 1379–1395.
3. S. Busenberg and K. Cooke, *Vertically Transmitted Diseases, Models in Dynamics*, Biomath. Series, Vol. 23, Springer-Verlag, New York, 1993.
4. V. Capasso, *Global solution for a diffusive nonlinear deterministic epidemic model*, SIAM J. Appl. Math. *35* (1978), 274–294.
5. P. DeMottoni, E. Orlandi, and A. Tesei, *Asymptotic behavior for a system describing epidemics with migration and spatial spread of infection*, Nonlinear Anal. **3** (1979), 663–675.
6. W. Fitzgibbon, C. Martin, and J. Morgan, *A diffusive epidemic model with criss-cross dynamics*, J. Math. Anal. Appl. **184** (1994), 399–414.
7. W. Fitzgibbon and J. Morgan, *A diffusive epidemic model on a bounded domain of arbitrary dimension*, Diff. and Int. Eqns. **1** (1988), 125–132.
8. W. Fitzgibbon, J. Morgan, and S. Waggoner, *A quasilinear system modeling the spread of infectious disease*, Rocky Mtn. J. Math. **22** (1992), 579–592.
9. W. Fitzgibbon, M. Parrott, and G. Webb, *Diffusive epidemic models with spatial and age-dependent heterogeneity*, Discrete and Cont. Dyn. Sys. **1** (1995), 35–37.
10. ———, *Diffusion epidemic models with incubation and crisscross dynamics*, Math. Biosci., to appear.

11. M. Gurtin and R. MacCamy, *Non-linear age-dependent population dynamics*, Arch. Rational Mech. Anal. **54** (1974), 281–300.
12. A. Haraux and M. Kirane, *Estimations C^1 pour des problèmes paraboliques semi-lineaires*, Ann. Fac. Sci. Toulouse **5** (1983), 265–280.
13. F. Hoppensteadt, *An age dependent epidemic model*, J. Franklin Inst. **197** (1974), 325-333.
14. S. Kesavan, *Topics in Functional Analysis and Applications*, John Wiley & Sons, New York, 1989.
15. M. Kubo and M. Langlais, *Periodic solutions for a population dynamics problem with age dependence and spatial structure*, J. Math. Biol. **29** (1991), 363–378.
16. O. Ladyzenskaja, V. Solonnikov, and N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monographs, Vol. 23, Amer. Math. Soc., Providence, RI, 1968.
17. M. Langlais, *Large time behavior in a non-linear age dependent population dynamics problem with diffusion*, J. Math. Biol. **26** (1988), 319–346.
18. ———, *Remarks on an epidemic model with age structure*, in Mathematical Population Dynamics, Lecture Notes in Pure and Applied Mathematics, Vol. 131, Marcel Dekker, New York, 1991, pp.75–92.
19. R. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley-Interscience, New York, 1976.
20. A. McKendrick, *Applications of mathematics to medical problems*, Proc. Edin. Math. Soc. **44** (1926), 98–130.
21. J. Murray, *Mathematical Biology*, Biomathematics Texts 19, Springer-Verlag, Berlin, 1989.
22. J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.
23. G. Webb, *An age-dependent epidemic model with spatial diffusion*, Arch. Rat. Mech. Anal. **75** (1980), 91–102.
24. ———, *A reaction-diffusion model for a deterministic diffusive epidemic*, J. Math. Anal. Appl. **84** (1981), 150–161.
25. ———, *Theory of Nonlinear Age-Dependent Population Dynamics*, Marcel Dekker, New York, 1985.

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