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STRONG ASYMPTOTICS ON THE SUPPORT OF THE MEASURE OF ORTHOGONALITY FOR POLYNOMIALS ORTHOGONAL WITH RESPECT TO A DISCRETE SOBOLEV INNER PRODUCT

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ABSTRACT. In [6, 8] relative asymptotics for Sobolev orthogonal polynomials were studied. When the continuous part of the measure satisfies the Szegő condition, the strong asymptotics outside the support of the measure follow from [6, 8]. In this paper, we analyze this problem on the support of the measure.

1. Introduction and statements of results

Some problems arising from approximation theory, mathematical physics, and number theory have motivated different generalizations of the standard notion of orthogonality. If μ is a finite positive Borel measure supported on the real line, then $\{p_n\}$ is the sequence of standard orthonormal polynomials (OP) with respect to μ when

$$\int p_n(x)p_m(x)d\mu(x) = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

and $p_n(x) = \alpha_n x^n +$ lower degree terms with $\alpha_n > 0$ for $n \in \mathbb{N}$.

Some of these generalizations are the so-called Hermite-Padé polynomials, [3, 10], matrix polynomials,[2, 4, 5], and orthogonal polynomials with respect to a linear homogeneous differential operator (OPDO) [1]. This last notion is important because we can apply the analytical methods used in the theory of Hermite-Padé approximants to OPDO. An important class among these notions are the Sobolev orthogonal polynomials (SOP) that are included in the class of OPDO. In this paper, we deal with the case of SOP with respect to a discrete Sobolev inner product [6, 8].

Definition 1. Let μ be a finite positive Borel measure, whose support S_μ contains an infinite set of real points ($S_\mu \subset \mathbb{R}$). A *discrete Sobolev inner product* is given by

$$\langle h, g\rangle = \int h(x)g(x)d\mu(x) + \sum_{j=1}^{m}\sum_{i=0}^{N_j} M_{j,i}h^{(i)}(c_j)g^{(i)}(c_j)$$

where $c_j \in \mathbb{R}$, $M_{j,i} \geq 0$, $m, N_j > 0$.

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In [6], a generalization of the above discrete Sobolev inner product in the following sense was considered

$$\langle h, g \rangle = \int h(x)g(x)d\mu(x) + \sum_{j=1}^{m}\sum_{i=0}^{N_j} h^{(i)}(c_j)\mathcal{L}_{j,i}(g; c_j) \tag{1}$$

where $\mathcal{L}_{j,i}(g; c_j)$ is the evaluation at $c_j \in \mathbb{C}$ of a linear differential operator with constant coefficients $\mathcal{L}_{j,i}$ acting on g and $\mathcal{L}_{j,N_j} \neq 0$, $j = 1, \dots, m$. The motivation of this definition comes from the following orthogonality relation

$$0 = \int p(x)Q_n(x)d\mu(x) + \sum_{j=1}^{m}\sum_{i=0}^{N_j} A_{j,i}(p(z)Q_n(z))^{(i)}|_{z=c_j}, \quad p \in \mathbb{P}_{n-1},$$

where $A_{j,i}$ may be complex numbers. The polynomials of the sequence $\{Q_n\}_{n\geq 0}$ are the denominators of the main diagonal sequence of the Padé approximants of Stieltjes-type meromorphic functions

$$f(z) = \int \frac{d\mu(x)}{z-x} + \sum_{j=1}^{m}\sum_{i=0}^{N_j} A_{j,i}\frac{i!}{(z-c_j)^{i+1}}, \quad A_{j,N_j} \neq 0.$$

Given $j = 1, \dots, m$, let J_j be the maximum order of the differential operators $\mathcal{L}_{j,i}$, $i = 0, \dots, N_j$. Then

$$\mathcal{L}_{j,i}(g; x) = \sum_{k=0}^{J_j} \gamma_{i,k}^{j} g^{(k)}(x).$$

For a fixed j, let $\mu_j = (\gamma_{i,k}^{j})$, $i = 0, \dots, N_j$, $k = 0, \dots, J_j$, be the matrix whose elements are the coefficients of the operator $\mathcal{L}_{j,i}$. Denote by $\mu_j{}^*$ the matrix obtained from μ_j, deleting all the zero rows and columns. Following [6], we say that (1) is regular if, for each $j = 1, \dots, m$, the matrix μ_j^* is a square matrix with non-zero determinant. We denote the dimension of μ_j^* by I_j, and $I = \sum_{j=1}^{m} I_j$.

The aim of our paper is the analysis of the strong asymptotics for the polynomials orthogonal with respect to (1) in the regular case when $\{c_j\} \in \mathbb{C} \setminus \mathcal{S}_\mu$, $j = 1, \dots, m$. So we denote by $S_n, n \in \mathbb{Z}_+$, the monic polynomial of least degree such that

$$\langle p, S_n \rangle = 0 \qquad p \in \mathbb{P}_{n-1}.$$

If the inner product is positive definite, then $\deg S_n$ is n, and thus all the S_n 's are distinct. In general, this is not true, and for different values of n, we can have the same S_n.

Despite an extensive effort to investigate the properties of SOP, not much progress has been attained regarding their asymptotics for general conditions on the inner products. The first essential contribution to this topic was [6]. There a new class of measures was introduced.

Definition 2. Let μ be a complex measure. We say that $\mu \in M_{\mathbb{C}}(0,1)$ if the corresponding orthonormal polynomials $\{q_n\}$ satisfy

$$xq_n(x) = \alpha_{n+1}q_{n+1}(x) + \beta_n q_n(x) + \alpha_n q_{n-1}(x), \quad n \geq n_0, \quad \alpha_n, \beta_n \in \mathbb{C}$$

with

$$\lim_{n\to\infty} \alpha_n = \frac{1}{2}, \quad \lim_{n\to\infty} \beta_n = 0,$$
$$\int |q_n(x)q_{n+m}(x)|\,|d\mu(x)| \leq C < \infty \tag{2}$$

where the support of μ is equal to $[-1,1]\cup E$, and E is at most a denumerable set of isolated points in $\mathbb{C}\setminus[-1,1]$, with $E'\subset[-1,1]$.

The following asymptotic behavior for the sequence $\{q_n\}$ holds

$$\frac{q_{n+1}^{(\nu)}(z)}{q_n^{(\nu)}(z)} \rightrightarrows \varphi(z), \quad K\subset\mathbb{C}\setminus S_\mu \tag{3}$$

where

$$\varphi(z) = z+\sqrt{z^2-1}, \quad |\varphi(z)|>1,$$

and

$$\frac{1}{n}\frac{q_n^{(\nu+1)}(z)}{q_n^{(\nu)}(z)} \rightrightarrows \frac{1}{\sqrt{z^2-1}}, \quad K\subset\mathbb{C}\setminus S_\mu.$$

Notation. The symbol $\rightrightarrows$ means, in this work, uniform convergence on compact subsets of the indicated region.

For this class of measures, a method for the analysis of relative asymptotic properties of SOP with respect to OP has been developed. In particular, the following theorem was proved [6, Th. 4].

Theorem 3. *Consider a regular inner product of type* (1) *such that* $\mu\in M_{\mathbb{C}}(0,1)$ *and* $c_1,\dots,c_m\in\mathbb{C}\setminus S_\mu$. *Let* $\{L_n\}$, $n\in\mathbb{Z}_+$ *be the sequence of monic orthogonal polynomials with respect to* μ, *and let* $\{S_n\}$, $n\in\mathbb{Z}_+$ *be the monic orthogonal polynomials with respect to the above inner product. Then for all sufficiently large* n, $\deg S_n$ *is* n *and each point* c_j *attracts exactly* I_j *zeros of* S_n, *while the other zeros concentrate on* S_μ. *Also for each fixed* $\nu\in\mathbb{Z}_+$,

$$\frac{S_n{}^{(\nu)}(z)}{L_n{}^{(\nu)}(z)} \rightrightarrows \prod_{j=1}^{m}\left(\frac{(\varphi(z)-\varphi(c_j))^2}{2\varphi(z)(z-c_j)}\right)^{I_j}, \quad K\subset\overline{\mathbb{C}}\setminus S_\mu, \tag{4}$$

We use the following corollary [6, Cor.2] later.

Corollary 4. *Under the hypothesis of the above theorem, for all sufficiently large* n, $\langle S_n,S_n\rangle\neq 0$. *Let* η_n *be the leading coefficient of* l_n, *the* n*-th orthonormal polynomial with respect to* μ. *Then* $\gamma_n=\langle S_n,S_n\rangle^{-\frac{1}{2}}$ *may be taken so that*

$$\lim_{n\to\infty}\frac{\gamma_n}{\eta_n} = \prod_{j=1}^{m}\frac{1}{\varphi(c_j)^{I_j}},$$

and, in particular,

$$\lim_{n\to\infty}\frac{\gamma_{n+1}}{\gamma_n} = 2.$$

The advantage of such a theorem is the following. If we know the asymptotic behavior of the OP, then we can deduce the corresponding behavior for the SOP. To illustrate this with an example, we can consider $d\mu(x) = \omega(x)dx$ where $\omega(x)$ is a positive and integrable function in $[-1,1]$ satisfying the Szegő condition

$$\int_{-1}^{1} \frac{\ln \omega(x)}{\sqrt{1-x^2}} dx > -\infty.$$

We denote by $\{l_n = \eta_n x^n + \cdots\}$ the set of polynomials orthonormal with respect to the function $\omega(x)$

$$\int_{-1}^{1} l_n(x) l_m(x) \omega(x) dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}.$$

It is well known [11] that the function

$$D_\omega(z) = \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{+\pi} \frac{1+ze^{-it}}{1-ze^{-it}} \ln\left(\omega(\cos t)|\sin t|\right) dt \right\}$$

satisfies

(a) $D_\omega(z) \in H_2$,

(b) $D_\omega(e^{i\theta}) = \lim_{r\to 1^-} D_\omega(re^{i\theta})$ exists a.e. and $|D_\omega(e^{i\theta})|^2 = \omega(\cos\theta)|\sin\theta|$,

(c) $D_\omega(z) \neq 0$, $|z| < 1$, and $D_\omega(0) > 0$.

We have the following strong asymptotics for $\{l_n\}$ (see [11], Theorems 12.1.2 and 12.1.4):

(a) *Outside the support*:

$$l_n\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right) = \frac{1}{\sqrt{2\pi}} \frac{1}{z^n} \frac{1}{D_\omega(z)} [1 + o(1)], \quad |z| < 1.$$

(b) *On the support*:

$$l_n(\cos\theta) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{D_\omega(e^{i\theta})} e^{-in\theta} + \frac{1}{\overline{D_\omega(e^{i\theta})}} e^{in\theta} \right] + o(1)$$

where $o(1)$ may be understood in the sense of convergence in $\mathcal{L}_{2,\omega}$.

(c) *The leading coefficient*:

$$\eta_n = \frac{1}{\sqrt{2\pi}} 2^n \frac{1}{D_\omega(0)} [1 + o(1)].$$

Combining these results with [6, Th. 4], we obtain as an immediate consequence the strong asymptotics of SOP outside the support of the measure of orthogonality.

Corollary 5. *Using Corollary* 4, *we can define for sufficiently large* n, $s_n = \gamma_n S_n$. *Then, for* $|z| < 1$

$$s_n\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right) = \frac{1}{\sqrt{2\pi}} \frac{1}{z^n} \frac{\prod_{j=1}^{m} \beta_j(z)^{I_j}}{D_\omega(z)} [1 + o(1)]$$

uniformly for $z \in K \subset \{z : |z| < 1\} \setminus \{z_j\}_{j=1}^m$, *where we have used the notation*

$$\beta_j(z) := \frac{z_j - z}{(1 - zz_j)},$$
$$\varphi(x) = x + \sqrt{x^2 - 1}, \quad |\varphi(x)| > 1,$$
$$\frac{1}{2}(z + 1/z) = x, \qquad \varphi(x) = \frac{1}{z},$$
$$\frac{1}{2}(z_j + 1/z_j) = c_j, \qquad \varphi(c_j) = \frac{1}{z_j}.$$

At the same time, we should mention that the theorem from [6, Th. 4] does not allow us to make a conclusion about asymptotics of SOP on the support of the measure of orthogonality. This is not a surprise because the values of the entries of the matrix μ_j do not appear in [6, Th. 4], but as we show later, the asymptotics at the mass points strictly depend on them. So an investigation of the strong asymptotic properties of the polynomials on the support of the measure requires a more delicate technique to be developed. The main aim of the present paper is to fill this gap. We prove the following.

Theorem 6. *Let* ω *be a positive and integrable function which satisfies the Szegő condition* $\int_{-1}^{1} \ln \omega(x)/\sqrt{1-x^2}dx > -\infty$. *Let us consider an inner product* (1) *in the regular case and* $\{c_j\}_{j=1}^m \in \mathbb{C} \setminus \mathcal{S}_\mu$ *where* $d\mu(x) = \omega(x)dx$. *For sufficiently large* $N \in \mathbb{N}$, *let* $\{S_n = x^n + \cdots\}_{n \geq N}$ *be the sequence of monic orthogonal polynomials with respect to* (1), *and* $\gamma_n = \langle S_n, S_n \rangle^{-1/2}$. *Let* $s_n = \gamma_n S_n$. *Then we have the following asymptotics for* $\{s_n\}_{n \geq N}$:

(a) *On the interval* $[-1, 1]$,

$$\left\| s_n(\cos\theta) - \frac{1}{\sqrt{2\pi}} \left[\frac{1}{D_\omega(e^{i\theta})/\beta(e^{i\theta})} e^{-in\theta} + \frac{1}{\left[\overline{D_\omega(e^{i\theta})/\beta(e^{-i\theta})}\right]} e^{in\theta} \right] \right\| = o(1)$$

where $\|\cdot\|$ *means the norm in* $\mathcal{L}^2(\omega(\cos\theta)|\sin\theta|d\theta, [-\pi, \pi])$ *and*

$$\beta(z) := \prod_{j=1}^{m} \left[\frac{z_j - z}{(1 - zz_j)} \right]^{I_j}.$$

(b) *At the mass points* $\{c_j\}_{j=1}^m$, *in the case that* $J_j = N_j = I_j - 1$,

$$\begin{pmatrix} s_n(c_j) \\ s_n'(c_j) \\ \vdots \\ s_n^{(J_j)}(c_j) \end{pmatrix} = (\mu_j)^{-1} (G_j)^{-1} \begin{pmatrix} h_n(c_j) \\ h_n'(c_j) \\ \vdots \\ \frac{1}{J_j!} h_n^{(J_j)}(c_j) \end{pmatrix} [1 + o(1)]$$

where

$$h_n(x) := \tilde{h}_n\left(\frac{1}{\varphi(x)}\right), \quad \varphi(x) = x + \sqrt{x^2 - 1},$$

$$\tilde{h}_n(z) := -\frac{\sqrt{2\pi}}{2^{M-1}} D_\omega(z) \frac{1}{z - 1/z} z^{n-M} \prod_{k=1}^{m} \frac{(zz_k - 1)^{2(J_k+1)}}{z_k^{J_k+1}},$$

and G_j, $j=1,\dots,m$, are the following matrices:

$$G_j(k,l)=\begin{cases}0 & J_j+2-k>l,\\ \binom{l-1}{J_j+1-k}(J_j+1-k)!\,g_j^{(l-(J_j+2-k))}(c_j) & J_j+2-k\le l,\end{cases}$$

and $g_j(x)=\prod_{i=1,\,i\neq j}^{m}(x-c_i)^{J_i+1}$.

For a better understanding of the result, we give some corollaries.

Corollary 7. *We first consider the simplest case $N_j=0$, and*

$$\langle h,g\rangle=\int_{-1}^{1}h(x)g(x)\omega(x)dx+\gamma h(c)g(c)\,,\quad c\in\mathbb{R},\quad \gamma>0.$$

In this case, if we denote $\frac{1}{2}(z_1+1/z_1)=c$ with $|z_1|<1$, then

$$\gamma_n=\frac{2^n}{D_\omega(0)}\frac{|z_1|}{\sqrt{2\pi}}\left[1+o(1)\right],\quad \gamma_n>0,$$

$$\gamma s_n(c)=\frac{|z_1|}{2}h_n(z_1)\left[1+o(1)\right],$$

$$h_n(z)=\sqrt{2\pi}\frac{D_\omega(z_1)}{1/2}z_1^{n-1}\Big(z_1-\frac{1}{z_1}\Big).$$

Denoting $F(z)=\frac{D_\omega(z)(zz_1-1)|z_1|}{(z-z_1)z_1}$, we get that

$$s_n(c)=\frac{D_\omega(z_1)}{\frac{1}{z_1-1/z_1}}|z_1|\frac{\sqrt{2\pi}}{\gamma}z_1^{n-1}\left[1+o(1)\right]=\frac{z_1^{n-1}}{\gamma}\sqrt{2\pi}\,\mathrm{Res}_{z=z_1}F(z)\left[1+o(1)\right]$$

holds.

This result agrees with [9, (17), p.2691].

Corollary 8. *Now if we consider*

$$\langle h,g\rangle=\int_{-1}^{1}h(x)g(x)\omega(x)dx+\gamma_{00}h(c)g(c)+\gamma_{11}h'(c)g'(c)\,,\quad c\in\mathbb{C}\setminus[-1,1],$$

then in this case $F(z)=\frac{D_\omega(z)}{(z-z_1)^2}(zz_1-1)^2$, and

$$\begin{pmatrix}s_n(c)\\ s_n'(c)\end{pmatrix}=\frac{\sqrt{2\pi}z_1^n}{\gamma_{00}\gamma_{11}}\begin{pmatrix}0 & \gamma_{11}\\ \gamma_{00} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\frac{z_1^2-1}{z_1^3}\rho_1\\ \frac{1}{z_1}\rho_2+\left[\frac{1}{z_1^3}(n-1)-\frac{2}{z_1^2-1}\right]\rho_1\end{pmatrix}\left[1+o(1)\right],$$

where $\rho_1=\mathrm{Res}_{z=z_1}((z-z_1)F(z))$ and $\rho_2=\mathrm{Res}_{z=z_1}F(z)$.

The method of proof of Theorem 6 is based on the investigation of the coefficients of a polynomial modification of SOP in terms of OP

$$w_M(x)S_n(x)=L_{n+M}(x)+\sum_{j=1}^{2M}a_{n,j}L_{n+M-j}(x)$$

where $w_M(x)=\prod_{j=1}^{m}(x-c_j)^{N_j+1}$ and $\deg w_M$ is equal to M.

As an intermediate step, which at the same time is of independent interest, we prove

Lemma 9. *Consider a regular inner product of type* (1) *such that* $\mu \in M_{\mathbb{C}}(0,1)$ *and* $c_1, \dots, c_m \in \mathbb{C} \setminus S_\mu$. *For sufficiently large* n, *let* $S_n(x)$ *be the monic Sobolev orthogonal polynomial of degree* n *with respect to this inner product. Let* $L_n(x) = x^n + \cdots$ *be the monic orthogonal polynomial of degree* n *with respect to* $d\mu(x)$. *Then*

$$w_M(x) S_n(x) = L_{n+M}(x) + \sum_{j=1}^{2M} a_{n,j} L_{n+M-j}(x) \tag{5}$$

where $w_M(x) = \prod_{j=1}^{m} (x - c_j)^{N_j+1}$, $\deg w_M(x) = M$. *Furthermore,* $\lim_{n\to\infty} a_{n,j}$ *exists for* $1 \le j \le m$.

This theorem gives us an analog of condition (2) of $M_{\mathbb{C}}(0,1)$ in the sense that the limits of the coefficients in a recurrence relation exist. The special case in the above theorems when $J_j = N_j = 0$ was proved by Nikishin [9]. The next section is devoted to the proofs of Lemma 9 and of Theorem 6.

2. Proof of the main results

Proof of Lemma 9. The proof is given in two steps.

1. We consider $p(z) = w_M(z) p_1(z)$ where p_1 is an arbitrary polynomial of degree at most $n - M - 1$. We have

$$0 = \langle w_M(z) p_1(z), S_n(z) \rangle = \int p_1(z) w_M(z) S_n(z) d\mu(z). \tag{6}$$

By Corollary 5, we know that for $n \ge n_0$

$$\int L_n^2(x) d\mu(x) \neq 0.$$

We can conclude that any polynomial, for sufficiently large n, satisfying (6), must be of degree at least $n - M$. If we try to write

$$w_M(z) S_n(z) = L_{n+M}(z) + \sum_{j=1}^{2M} a_{n,j} L_{n+M-j}(z),$$

then to do this we require that

$$p_n(z) = \frac{a_{n,0} L_{n+M}(z) + \sum_{j=1}^{2M} a_{n,j} L_{n+M-j}(z)}{w_M(z)}$$

be a polynomial. For doing this, we have M homogeneous linear equations in the unknowns $\{a_{n,j}\}$, $j = 0, \dots, 2M$.

With the above condition, one has

$$0 = \langle w_M(z) p_1(z), p_n(z) \rangle \qquad 0 \le \deg p_1 \le n - M - 1. \tag{7}$$

Now, if we also impose

$$0 = \langle z^k, p_n(z) \rangle \qquad k = 0, \dots, M - 1, \tag{8}$$

then we can guarantee a non-trivial solution for (7) and (8) because we have $2M$ homogeneous linear equations and $2M+1$ unknowns. Thus this polynomial p_n satisfies the same orthogonality condition as S_n and, for sufficiently large n, $\deg S_n = n$ so $a_{n,0} \neq 0$. Taking the normalization $a_{n,0} = 1$, we can say $p_n = S_n$ and (5) holds.

2. To prove the existence of the limit of $a_{n,j}$ for $j = 1, \ldots, 2M$, we follow the technique used in [6]. Let

$$w_M(z)S_n(z) = L_{n+M}(z) + \sum_{j=1}^{2M} a_{n,j} L_{n+M-j}(z).$$

We consider

$$\Omega_n(z) = \frac{w_M(z)S_n(z)}{L_{n-M}(z)} \quad \text{and} \quad a_n^* = \frac{1}{1+\sum_{j=1}^{2M} |a_{n,j}|}.$$

Set

$$\Omega_n^* = a_n^* \Omega_n = a_n^* \frac{L_{n+M}(z)}{L_{n-M}(z)} + \sum_{j=1}^{2M} a_{n,j}^* \frac{L_{n+M-j}(z)}{L_{n-M}(z)} \tag{9}$$

where $a_{n,j}^* = a_n^* a_{n,j}$. The new sequence $\{a_{n,j}^*\}$ is uniformly bounded. Let Λ be a subset of $\mathbb{N}$ such that for $j = 1, \ldots, 2M$

$$\lim_{n\to\infty, n\in\Lambda} a_{n,j}^* = a_j^* \quad \text{and} \quad \lim_{n\to\infty, n\in\Lambda} a_n^* = a_0^*.$$

Taking the limit in (9), when $n \to \infty$, $n \in \Lambda$, and making use of (3) and (4),

$$a_0^* \prod_{j=1}^{m} \frac{(z-c_j)^{N_j+1-I_j} (\varphi(z) - \varphi(c_j))^{2I_j}}{(2\varphi(z))^{I_j}} \left(\frac{\varphi(z)}{2}\right)^M$$
$$= a_0^* \left(\frac{\varphi(z)}{2}\right)^{2M} + \sum_{j=1}^{2M} a_j^* \left(\frac{\varphi(z)}{2}\right)^{2M-j}$$

for $z \in \mathbb{C} \setminus S_\mu$. If we notice that $|a_n^*| + \sum_{j=1}^{2M} |a_{n,j}^*| = 1$, then it is clear that $|a_0^*| + \sum_{j=1}^{2M} |a_j^*| = 1$.

Now if we use the fact that $\{\varphi^k\}$ for $k = 0, \ldots, 2M$ is a set of linearly independent functions because φ is a one-to-one holomorphic function, then it holds that $a_0^* \neq 0$.

In particular, this implies that $\{a_{n,j}\}$ is uniformly bounded, so to prove the existence of the limits, we only need to prove the uniqueness of the limit points for every $j = 1, \ldots, 2M$.

Let Λ be a subset of $\mathbb{N}$ such that for $j = 1, \ldots, 2M$, the limits $\lim_{n\to\infty,\, n\in\Lambda} a_{n,j} = a_j$ exist. Taking $\lim_{n\to\infty,\, n\in\Lambda} \Omega_n(z)$, one obtains

$$\prod_{j=1}^{m} \frac{(z-c_j)^{N_j+1-I_j} (\varphi(z) - \varphi(c_j))^{2I_j}}{(2\varphi(z))^{I_j}} \left(\frac{\varphi(z)}{2}\right)^M = \left(\frac{\varphi(z)}{2}\right)^{2M} + \sum_{j=1}^{2M} a_j \left(\frac{\varphi(z)}{2}\right)^{2M-j}$$

for $z \in \mathbb{C} \setminus S_\mu$.

From this last equation, we have the uniqueness of a_j. □

Proof of Theorem 6. First we are going to deduce the strong asymptotics on $[-1, 1]$.

Let

$$w_M(z) = \prod_{j=1}^{m} (z - c_j)^{N_j+1}.$$

We expand $S_n w_M$ in terms of $\{L_n\}$.

$$S_n(z) w_M(z) = L_{n+M}(z) + \sum_{j=1}^{2M} a_{n,j} L_{n+M-j}(z). \tag{10}$$

Dividing by S_n,

$$\begin{aligned} w_M(z) &= \frac{L_{n+M}(z)}{S_n(z)} + \sum_{j=1}^{2M} a_{n,j} \frac{L_{n+M-j}(z)}{S_n(z)} \\ &= \frac{L_{n+M}(z)}{L_n(z)} \frac{L_n(z)}{S_n(z)} + \sum_{j=1}^{2M} a_{n,j} \frac{L_{n+M-j}(z)}{L_n(z)} \frac{L_n(z)}{S_n(z)}. \end{aligned}$$

Now taking the limit in a compact set off supp μ, using (4) as well as the existence of $\lim a_{n,j} = a_j$, (Lemma 9)

$$w_M(z) = \prod_{j=1}^{m} \left(\frac{2\varphi(z)(z - c_j)}{(\varphi(z) - \varphi(c_j))^2} \right)^{I_j} \left[\left(\frac{\varphi(z)}{2} \right)^M + \sum_{j=1}^{2M} a_j \left(\frac{\varphi(z)}{2} \right)^{M-j} \right],$$

$$\begin{aligned} &\prod_{j=1}^{m} (\varphi(z) - \varphi(c_j))^{2I_j} (z - c_j)^{N_j+1-I_j} \\ &\qquad = 2^I \varphi^I(z) \left[\left(\frac{\varphi(z)}{2} \right)^M + \sum_{j=1}^{2M} a_j \left(\frac{\varphi(z)}{2} \right)^{M-j} \right]. \end{aligned}$$

Now, denoting $\varphi(z) = t$, we arrive at the equation,

$$\begin{aligned} &\prod_{j=1}^{m} (t - \varphi(c_j))^{2I_j} \left(\frac{1}{2} \left(t + \frac{1}{t} \right) - c_j \right)^{N_j+1-I_j} \\ &\qquad = 2^I t^I \left[\left(\frac{t}{2} \right)^M + \sum_{j=1}^{2M} a_j \left(\frac{t}{2} \right)^{M-j} \right]. \end{aligned} \tag{11}$$

that these $a_j, j = 1, \ldots, 2M$ verify. We come back to (10), and, putting $z = \cos\theta$, we have

$$S_n(\cos\theta) w_M(\cos\theta) = L_{n+M}(\cos\theta) + \sum_{j=1}^{2M} a_{n,j} L_{n+M-j}(\cos\theta).$$

Now using the strong asymptotics for L_n on $[-1, 1]$, we deduce

$$S_n(\cos\theta)w_M(\cos\theta)$$
$$= \frac{\frac{1}{\sqrt{2\pi}}\left[\frac{1}{D_\omega(e^{i\theta})}e^{-i(n+M)\theta} + \frac{1}{\overline{D_\omega(e^{i\theta})}}e^{i(n+M)\theta}\right]}{\frac{1}{\sqrt{2\pi}}2^{n+M}\frac{1}{D_\omega(0)}}$$
$$+\sum_{j=1}^{2M} a_j \frac{D_\omega(0)}{2^{n+M-j}}\left[\frac{1}{D_\omega(e^{i\theta})}e^{-i(n+M-j)\theta} + \frac{1}{\overline{D_\omega(e^{i\theta})}}e^{i(n+M-j)\theta}\right] + o(1)$$
$$= \frac{D_\omega(0)}{2^{n+M}}\left[\frac{1}{D_\omega(e^{i\theta})}e^{-i(n+M)\theta} + \frac{1}{\overline{D_\omega(e^{i\theta})}}e^{i(n+M)\theta}\right]$$
$$+\sum_{j=1}^{2M} a_j \frac{D_\omega(0)}{2^{n+M-j}}\left[\frac{1}{D_\omega(e^{i\theta})}e^{-i(n+M-j)\theta} + \frac{1}{\overline{D_\omega(e^{i\theta})}}e^{i(n+M-j)\theta}\right] + o(1)$$
$$= \frac{D_\omega(0)}{D_\omega(e^{i\theta})}\left[\frac{e^{-in\theta}}{2^n}\left[\frac{1}{2^M}e^{-iM\theta} + \sum_{j=1}^{2M} a_j \frac{1}{2^{M-j}}e^{-i(M-j)\theta}\right]\right]$$
$$+ \frac{D_\omega(0)}{\overline{D_\omega(e^{i\theta})}}\left[\frac{e^{in\theta}}{2^n}\left[\frac{1}{2^M}e^{iM\theta} + \sum_{j=1}^{2M} a_j \frac{1}{2^{M-j}}e^{i(M-j)\theta}\right]\right] + o(1).$$

Using the notation

$$e^{i\theta} = z, \qquad \frac{1}{\varphi(c_j)} = z_j, \qquad \frac{1}{2}(z_j + \frac{1}{z_j}) = c_j,$$

we have, by (11),

$$S_n(\cos\theta) = \frac{\frac{D_\omega(0)}{D_\omega(z)}\left[\frac{1}{2^n z^n}\prod_{j=1}^{m}\left[\frac{1}{z} - \frac{1}{z_j}\right]^{2I_j}\frac{z^I}{2^I}\right]}{\prod_{j=1}^{m}\left[\frac{1}{2}(z+\frac{1}{z}) - \frac{1}{2}(z_j + \frac{1}{z_j})\right]^{I_j}}$$
$$+ \frac{\frac{D_\omega(0)}{\overline{D_\omega(z)}}\left[\frac{z^n}{2^n}\prod_{j=1}^{m}\left[z - \frac{1}{z_j}\right]^{2I_j}\frac{1}{2^I z^I}\right]}{\prod_{j=1}^{m}\left[\frac{1}{2}(z+\frac{1}{z}) - \frac{1}{2}(z_j + \frac{1}{z_j})\right]^{I_j}} + o(1)$$
$$= \frac{D_\omega(0)}{D_\omega(z)}\left[\frac{1}{2^n z^n}\prod_{j=1}^{m}\left[\frac{z_j - z}{z_j(1 - zz_j)}\right]^{I_j}\right]$$
$$+ \frac{D_\omega(0)}{\overline{D_\omega(z)}}\left[\frac{z^n}{2^n}\prod_{j=1}^{m}\left[\frac{zz_j - 1}{z_j(z - z_j)}\right]^{I_j}\right] + o(1).$$

If we denote

$$\alpha(z)=\prod_{j=1}^{m}\left[\frac{z_j-z}{z_j(1-zz_j)}\right]^{I_j},$$

then

$$S_n(\cos\theta)=\frac{D_\omega(0)}{\frac{D_\omega(e^{i\theta})}{\alpha(e^{i\theta})}}\frac{1}{2^n}e^{-in\theta}+\frac{D_\omega(0)}{\left[\overline{\frac{D_\omega(e^{i\theta})}{\alpha(e^{-i\theta})}}\right]}\frac{1}{2^n}e^{in\theta}+o(1) \tag{12}$$

For sufficiently large n the leading coefficient of s_n is denoted by γ_n. From Corollary 4

$$\gamma_n=\frac{1}{\sqrt{2\pi}}2^n\frac{1}{D_\omega(0)}\frac{1}{\prod_{j=1}^{m}\varphi(c_j)^{I_j}}+o(1)\ \text{ holds.}$$

Using this and (12), we have

$$s_n(\cos\theta)=\frac{1}{\sqrt{2\pi}}\prod_{j=1}^{m}{z_j}^{I_j}\left[\frac{1}{\frac{D_\omega(e^{i\theta})}{\alpha(e^{i\theta})}}e^{-in\theta}+\frac{1}{\left[\overline{\frac{D_\omega(e^{i\theta})}{\alpha(e^{-i\theta})}}\right]}e^{in\theta}\right]+o(1).$$

If we denote

$$\beta(z):=\prod_{j=1}^{m}\left[\frac{z_j-z}{(1-zz_j)}\right]^{I_j},$$

then

$$s_n(\cos\theta)=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{\frac{D_\omega(e^{i\theta})}{\beta(e^{i\theta})}}e^{-in\theta}+\frac{1}{\left[\overline{\frac{D_\omega(e^{i\theta})}{\beta(e^{-i\theta})}}\right]}e^{in\theta}\right]+o(1)$$

where $o(1)$ is in the sense of the $\mathcal{L}_{2,\omega}$ norm. As a last step, we derive the asymptotic behavior for $s_n^{(k)}(c_j)$, $0\le k\le J_j$, $1\le j\le m$. We consider

$$S_n(x)w_M(x)=L_{n+M}(x)+\sum_{j=1}^{2M}a_{n,j}L_{n+M-j}(x)$$

where, using the notation introduced in Lemma 9, we denote

$$g_j(x):=\frac{w_M(x)}{(x-c_j)^{J_j+1}}=\prod_{\substack{i=1\\ i\neq j}}^{m}(x-c_i)^{J_i+1}.$$

From orthogonality for the Sobolev inner product, for $n>M-1$, we get

$$\begin{aligned}0&=\left\langle g_j(x)(x-c_j)^{J_j},S_n(x)\right\rangle\\ &=\int_{-1}^{1}\frac{w_M(x)S_n(x)}{x-c_j}d\mu(x)+\left(0,\cdots,0,\ J_j!g_j(c_j)\right)\mu_j\begin{pmatrix}S_n(c_j)\\ \vdots\\ S_n^{(J_j)}(c_j)\end{pmatrix},\end{aligned}$$

$$
\begin{aligned}
0 &= \langle g_j(x)(x-c_j)^{J_j-1}, S_n(x)\rangle \\
&= \int_{-1}^{1} \frac{w_M(x)S_n(x)}{(x-c_j)^2}\,d\mu(x) \\
&\qquad + \left(0, \cdots, 0, (J_j-1)!g_j(c_j), \tbinom{J_j}{J_j-1}(J_j-1)!g'(c_j)\right)\mu_j \begin{pmatrix} S_n(c_j) \\ \vdots \\ S_n^{(J_j)}(c_j) \end{pmatrix},
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
0 &= \langle g_j(x), S_n(x)\rangle \\
&= \int_{-1}^{1} \frac{w_M(x)S_n(x)}{(x-c_j)^{J_j+1}}\,d\mu(x) + \left(g_j(c_j), g_j'(c_j), \cdots, g_j^{(J_j)}(c_j)\right)\mu_j \begin{pmatrix} S_n(c_j) \\ \vdots \\ S_n^{(J_j)}(c_j) \end{pmatrix}.
\end{aligned}
$$

Taking into account the matrix G_j which appears in the statement of the theorem, we can express, for every j, $1 \le j \le m$, the above equations in matrix form.

$$
G_j\mu_j \begin{pmatrix} s_n(c_j) \\ \vdots \\ s_n^{(J_j)}(c_j) \end{pmatrix} = -\gamma_n \begin{pmatrix} \int_{-1}^{1} \frac{w_M(x)S_n(x)}{(x-c_j)}\omega(x)dx \\ \vdots \\ \int_{-1}^{1} \frac{w_M(x)S_n(x)}{(x-c_j)^{J_j+1}}\omega(x)dx \end{pmatrix}. \tag{13}
$$

To obtain an asymptotic formula, we must study the asymptotic behavior of the right-hand side of (13). Using the change the variable $\frac{1}{2}(z+1/z) = \lambda$, the asymptotics of L_n on $[-1,1]$ as well as (11), we have

$$
\begin{aligned}
&\int_{-1}^{1} \frac{w_M(x)S_n(x)}{(x-c_j)}\omega(x)dx \\
&\quad = \left[\int_{-1}^{1} \frac{L_{n+M}(x)}{(x-\lambda)}\omega(x)dx + \sum_{j=1}^{2M} a_{n,j} \int_{-1}^{1} \frac{L_{n+M-j}(x)}{(x-\lambda)}\omega(x)dx\right]_{\lambda=c_j} \\
&\quad = \left[\frac{z^n}{2^n}\left[\left(\frac{z}{2}\right)^M + \sum_{j=1}^{2M} a_j \left(\frac{z}{2}\right)^{M-j}\right] \int_{-\pi}^{\pi} \frac{D_\omega(z)D_\omega(0)d\theta}{(\cos\theta-\lambda)}\right]_{\substack{\lambda=c_j \\ z=z_j}} \left[1+o(1)\right] \\
&\quad = \left[\frac{4\pi}{z-1/z} D_\omega(z)D_\omega(0)\frac{z^{n-M}}{2^{n+M}} \prod_{k=1}^{m}\left(z-\frac{1}{z_k}\right)^{2(J_k+1)}\right]_{z=z_j} \left[1+o(1)\right]
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\int_{-1}^{1} \frac{L_n(x)}{x-\lambda}\omega(x)dx &= \frac{1}{L_n(\lambda)} \int_{-1}^{1} \frac{L_n^2(x)}{x-\lambda}\omega(x)dx \\
&= \frac{D_\omega(0)}{2^n} D_\omega(z) z^n \int_{-\pi}^{\pi} \frac{1}{|D_\omega(e^{i\theta})|^2} \frac{\omega(\cos\theta)\,|\sin\theta|}{\cos\theta-\lambda}\,d\theta \left[1+o(1)\right].
\end{aligned}
$$

If we denote by

$$h_n(z) = -\frac{\sqrt{2\pi}}{2^{M-1}} D_\omega(z) \frac{1}{z - 1/z} z^{n-M} \prod_{k=1}^{m} \frac{(zz_k - 1)^{2(J_k+1)}}{z_k^{J_k+1}},$$

then

$$-\gamma_n \int_{-1}^{1} \frac{w_M(x) S_n(x)}{(x - c_j)} \omega(x) dx = h_n(z_j)\Big[1 + o(1)\Big].$$

For the other coordinates on the right-hand side of (13), we use again the fact that if $f_n = g_n[1 + o(1)]$ where f_n, g_n are analytic functions and g_n is uniformly bounded, then $f'_n = g'_n[1 + o(1)]$ and

$$-\int_{-1}^{1} \frac{w_M(x) s_n(x)}{(x - \lambda)^{k+1}} \omega(x) dx = \frac{1}{k!} \frac{d^k}{d\lambda^k} \Big[h_n(z)\Big]_{\lambda = c_j} \Big[1 + o(1)\Big].$$

Taking this into account, we obtain

$$G_j \mu_j \begin{pmatrix} s_n(c_j) \\ s'_n(c_j) \\ \vdots \\ s_n^{(J_j)}(c_j) \end{pmatrix} = \begin{pmatrix} h_n(z) \,|_{\lambda = c_j} \\ \frac{d}{d\lambda} h_n(z) \,|_{\lambda = c_j} \\ \vdots \\ \frac{1}{J_j!} \frac{d^{J_j}}{d\lambda^{J_j}} h_n(z) \,|_{\lambda = c_j} \end{pmatrix} \Big[1 + o(1)\Big].$$

□

Remark 1. Notice that on obtaining the strong asymptotics on $[-1, 1]$, we have not used the condition $I_j - 1 = J_j = N_j$. At the points $\{c_j\}$, $j = 1, \ldots, m$, this condition is not necessary either but the derivation of the corresponding formula is quite complicated.

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