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A DISCRETE CURVE-SHORTENING EQUATION

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Dedicated to Martin D. Kruskal on the occasion of his 70th birthday.

ABSTRACT. The usual "curve-shortening equation" describes the planar motion of a smooth curve that moves in a direction normal to itself with a speed proportional to its local curvature. We present here an analogous theory for the planar motion of a discrete (i.e., piecewise linear) curve. In the discrete case, an arbitrary nonintersecting, closed N-sided curve shrinks in on itself, and its enclosed area vanishes in a finite time. We conjecture that the discrete curve tends to an equi-angle N-polygon as it shrinks.

Geometrical models which describe the motion of manifolds (curves and surfaces) in a higher dimensional space have been used successfully in various branches of science [12]. A simple example that has been studied extensively is the so-called curve-shortening equation [4, 6, 7], which describes the planar motion of a smooth curve that moves normal to itself with a speed proportional to its local curvature:

$$\frac{d\mathbf{r}}{dt} = U\mathbf{n}, \qquad U = \kappa, \tag{1}$$

Here $\mathbf{r}$ is the position vector of a point on the curve, $\mathbf{n}$ is the inward-facing unit normal vector of the curve, and κ is its curvature. Two important features of such motion are: (1) that every smooth closed curve shrinks to a point in a finite time; and (2) that the asymptotic shape of such a shrinking curve is always a circle.

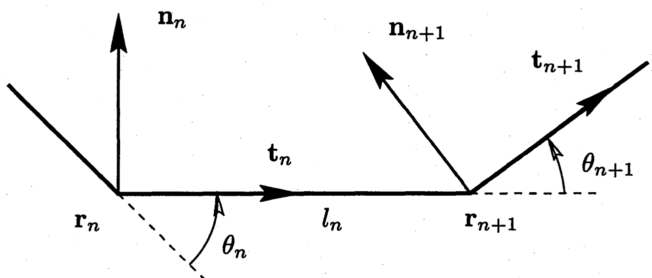

FIGURE 1. A discrete curve

The purpose of this paper is to formulate a discrete analogue of the curve-shortening equation, and to analyze its behavior. This is one of several discrete models of curve-shortening that have been proposed [2, 3, 8, 9, 13, 14]. We discuss some relations

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between these models at the end of this paper. We begin with a description of a discrete curve, (i.e., a piecewise linear curve) in a plane. The curve consists of N points, connected by straight line-segments ("links"). The position vector of the n-th point is denoted by $\mathbf{r}_n$. By definition (Figure 1), we have

$$\mathbf{r}_{n+1} = \mathbf{r}_n + l_n \mathbf{t}_n \tag{2}$$

where $\mathbf{t}_n$ is the unit "tangent" vector and l_n is the length of the n-th link. The unit "normal" vector $\mathbf{n}_n$ is chosen to be a 90° counter-clockwise rotation of $\mathbf{t}_n$:

$$\mathbf{n}_n \cdot \mathbf{t}_n = 0.$$

Each pair of vectors $\{\mathbf{t}_n, \mathbf{n}_n\}$ forms an orthonormal basis for the plane. Thus there exists a set of angles $\{\theta_n\}$ (Figure 1) such that

$$\begin{pmatrix} \mathbf{t}_{n+1} \\ \mathbf{n}_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\theta_{n+1} & \sin\theta_{n+1} \\ -\sin\theta_{n+1} & \cos\theta_{n+1} \end{pmatrix} \begin{pmatrix} \mathbf{t}_n \\ \mathbf{n}_n \end{pmatrix}. \tag{3}$$

Equation (3) is referred to as the discrete Serret-Frenet equation [5, 11].

The discrete Serret-Frenet equation (3) can be rewritten as

$$\mathbf{t}_{n+1} = \mathbf{t}_n + \tan\frac{\theta_{n+1}}{2}\mathbf{n}_n + \tan\frac{\theta_{n+1}}{2}\mathbf{n}_{n+1}, \tag{4a}$$

$$\mathbf{n}_{n+1} = \mathbf{n}_n - \tan\frac{\theta_{n+1}}{2}\mathbf{t}_n - \tan\frac{\theta_{n+1}}{2}\mathbf{t}_{n+1}. \tag{4b}$$

It is interesting to compare (4) with the Ablowitz-Ladik system for a certain class of integrable differential-difference equations[1, 5]:

$$v_{1,n+1} = z v_{1,n} + Q_n(t) v_{2,n} + S_n v_{2,n+1},$$

$$v_{2,n+1} = \frac{1}{z} v_{2,n} + R_n(t) v_{1,n} + T_n(t) v_{1,n+1}.$$

We observe that the discrete Serret-Frenet equation is the Ablowitz-Ladik system at $z = 1$ with $Q_n = S_n = -R_n = -T_n$. The corresponding statement for smooth curves is known [10].

We now consider a motion of a discrete curve. Let t denote time. We express the velocity of the n-th point by

$$\frac{d\mathbf{r}_n}{dt} = U_n \mathbf{n}_n + W_n \mathbf{t}_n. \tag{5}$$

Here U_n and W_n are the $\mathbf{n}_n$ and $\mathbf{t}_n$ components of the velocity, respectively. Explicit forms of U_n and W_n can be chosen to represent a variety of physical or geometrical phenomena. We require the consistency condition imposed on the dynamical variables,

$$\frac{d}{dt} E = E \frac{d}{dt} \tag{6}$$

where E is the shift operator

$$E f_n = f_{n+1}.$$

Applying (6) to (3) and (5), we get

$$\frac{d}{dt}\begin{pmatrix}\mathbf{t}_n\\ \mathbf{n}_n\end{pmatrix} = \begin{pmatrix}0 & \omega_n\\ -\omega_n & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}_n\\ \mathbf{n}_n\end{pmatrix}, \tag{7}$$

$$\frac{d}{dt} l_n = \gamma_n, \tag{8}$$

$$\frac{d}{dt}\theta_n = \Delta\,\frac{\omega_{n-1}}{l_{n-1}} \tag{9}$$

where

$$\begin{pmatrix}\gamma_n\\ \omega_n\end{pmatrix} = \begin{pmatrix}\cos\theta_{n+1} & -\sin\theta_{n+1}\\ \sin\theta_{n+1} & \cos\theta_{n+1}\end{pmatrix}\begin{pmatrix}W_{n+1}\\ U_{n+1}\end{pmatrix} - \begin{pmatrix}W_n\\ U_n\end{pmatrix} \tag{10}$$

and Δ is the difference operator

$$\Delta\, f_n = (E-1)f_n = f_{n+1} - f_n.$$

It is worth mentioning that all of the formulae for a smooth curve are recovered in the continuum limit

$$l_n \to 0 \text{ and } \theta_n \to 0. \tag{11}$$

In fact, in the limit (11), eqs. (2), (3), (7)–(9) become

$$\mathbf{t} = \frac{\partial \mathbf{r}}{\partial s},$$

$$\frac{\partial}{\partial s}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & \kappa\\ -\kappa & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\end{pmatrix},$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & \omega\\ -\omega & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\end{pmatrix},$$

$$\frac{\partial}{\partial t}\sqrt{g} = \sqrt{g}\gamma,$$

$$\frac{\partial}{\partial t}\kappa = \frac{\partial}{\partial s}\omega$$

where

$$\begin{pmatrix}\gamma\\ \omega\end{pmatrix} = \left[\frac{\partial}{\partial s} + \begin{pmatrix}0 & -\kappa\\ \kappa & 0\end{pmatrix}\right]\begin{pmatrix}W\\ U\end{pmatrix}.$$

Here g is the metric related to the length of the link, l, by $l = \sqrt{g}\sigma$ with σ being a time-independent parameter.

The discussion so far is general. Next, we propose a specific model which leads to a curve-shortening equation. We assume that the velocity of the n-th particle is determined by

$$U_n = \frac{\theta_n}{l_n}, \tag{12a}$$

$$W_n = \frac{\theta_n(l_n - l_{n-1}\cos\theta_n)}{l_n l_{n-1}\sin\theta_n}. \tag{12b}$$

See the Appendix for the derivation. In the continuum limit (11), (5) with (12) yields (1). Making use of (12) in (8) and (9), we obtain

$$\frac{dl_n}{dt} = -\frac{\theta_{n+1}}{l_{n+1}\sin\theta_{n+1}} - \frac{\theta_n}{l_{n-1}\sin\theta_n} + \frac{\theta_{n+1}\cos\theta_{n+1}}{l_n\sin\theta_{n+1}} + \frac{\theta_n\cos\theta_n}{l_n\sin\theta_n}, \tag{13}$$

$$\frac{d\theta_n}{dt} = \frac{\theta_{n+1}-\theta_n}{l_n^2} - \frac{\theta_n-\theta_{n-1}}{l_{n-1}^2}. \tag{14}$$

The model (12) has remarkable features. In what follows, we assume that the discrete curve is closed, and that it does not intersect itself.

Theorem 1. *The area S enclosed by the curve satisfies*

$$\frac{dS}{dt} = -2\pi. \tag{15}$$

Proof. The area S can be expressed as

$$S = \frac{1}{2}\sum_n \mathbf{r}_n \times \mathbf{r}_{n+1}. \tag{16}$$

Making use of (2) and (7), we obtain

$$\frac{dS}{dt} = -\sum_n \theta_n = -2\pi < 0. \tag{17}$$

□

Comment 1. It follows that the area enclosed by the curve vanishes in a finite time, and that the time required is proportional to the initial area enclosed. The second equality in (17) is a special case of the Gauss-Bonnet theorem.

The usual (continuous) curve-shortening equation has property (15); the model (12) was chosen in order to maintain this property in the discrete version. This correspondence is discussed in detail in the Appendix.

Theorem 2. *The total length $L = \sum l_n$ decreases,*

$$\frac{dL}{dt} = -\sum_n \left(\frac{1}{l_n} + \frac{1}{l_{n-1}}\right)\theta_n \tan\frac{\theta_n}{2} < 0. \tag{18}$$

Proof. From the definition of L and (8), it is easy to show (18). Note that $|\theta_n| < \pi$. □

Theorem 3. *A shrinking regular N-polygon is linearly stable to arbitrary small perturbations other than dilatation and rigid-body motion.*

Proof. Let us introduce a new time variable τ by

$$\frac{dt}{d\tau} = S, \quad \tau(t=0) = 0. \tag{19}$$

The area S can be expressed in two ways as

$$S = S_0 - 2\pi t = S_0\exp(-2\pi\tau) \tag{20}$$

where $S_0 = S(t=0)$.

Since the curve is closed, the closing conditions are

$$\sum_{j=0}^{N-1} \theta_j = 2\pi, \quad \sum_{j=0}^{N-1} l_j \mathbf{t}_j = 0. \tag{21}$$

For a regular N-polygon, we have

$$\theta_n = \theta \equiv \frac{2\pi}{N}, \quad l_n = l(t), \quad S(t) = \frac{1}{4} N l(t)^2 \cot\frac{\theta}{2}. \tag{22}$$

Equation (13) can be solved explicitly to yield

$$l(t) = l(0)\sqrt{1 - \frac{t}{t_0}}, \quad l(0)^2 = 4\theta t_0 \tan\frac{\theta}{2}. \tag{23}$$

Now consider small deviations around the regular polygon. Expanding θ_n and l_n as

$$\theta_n = \theta + \delta\theta_n, \quad l_n = l + \delta l_n, \tag{24}$$

and inserting these into (13) and (14), we obtain

$$\frac{d}{d\tau}\delta\theta_n = \frac{N}{4}\cot\frac{\theta}{2}(\delta\theta_{n+1} - 2\delta\theta_n + \delta\theta_{n-1}), \tag{25a}$$

$$\begin{aligned} \frac{d}{d\tau}\delta l_n = {} & \frac{N}{8}\theta \csc^2\frac{\theta}{2}(\delta l_{n+1} - 2\cos\theta\,\delta l_n + \delta l_{n-1}) \\ & - \frac{N}{4} l \left(1 + \frac{\theta}{\sin\theta}\right)(\delta\theta_{n+1} + \delta\theta_n). \end{aligned} \tag{25b}$$

The discrete Fourier transformations,

$$\widetilde{\delta\theta}_k = \sum_{n=0}^{N-1} \delta\theta_n \exp(-ikn\theta), \quad \widetilde{\delta l}_k = \sum_{n=0}^{N-1} \delta l_n \exp(-ikn\theta), \tag{26}$$

recast these into

$$\frac{d}{d\tau}\widetilde{\delta\theta}_k = -N\cot\frac{\theta}{2}\sin^2\frac{k\theta}{2}\cdot\widetilde{\delta\theta}_k, \tag{27a}$$

$$\begin{aligned} \frac{d}{d\tau}\widetilde{\delta l}_k = {} & -\frac{N\theta}{4\sin^2\frac{\theta}{2}}(\cos\theta - \cos k\theta)\cdot\widetilde{\delta l}_k \\ & - \frac{N}{4} l \left(1 + \frac{\theta}{\sin\theta}\right)(\exp(ik\theta) + 1)\cdot\widetilde{\delta\theta}_k \end{aligned} \tag{27b}$$

with the closing condition

$$\widetilde{\delta\theta}_0 = 0, \quad \widetilde{\delta l}_{N-1} + \frac{il}{1 - \exp(i\theta)}\widetilde{\delta\theta}_{N-1} = 0. \tag{28}$$

Note that $\widetilde{\delta l}_1 = \widetilde{\delta l}^*_{N-1}$. Equations (27) can be solved easily. Their solutions show that all of the components go to zero as $\tau \to \infty$, except for $\widetilde{\delta l}_0$. The Fourier mode $\widetilde{\delta l}_0$ represents a dilatation. □

Comment 2. A time variable τ defined by (19) turns out to be convenient also for later discussions.

Moreover, some exact results have been obtained.

Theorem 4. *An arbitrary triangle shrinks to a point in a finite time, and its limiting shape is a regular triangle.*

Proof. The allowed region Ω for the angle variables is given by

$$\Omega = \{\, (\theta_0, \theta_1, \theta_2) \mid \theta_0 + \theta_1 + \theta_2 = 2\pi, \theta_n < \pi \,\}. \tag{29}$$

Because the area S can be expressed as

$$S = \frac{1}{2} l_2 l_0 \sin\theta_0 = \frac{1}{2} l_0 l_1 \sin\theta_1 = \frac{1}{2} l_1 l_2 \sin\theta_2,$$

we have

$$\begin{aligned} \frac{S}{l_0^2} &= \frac{1}{2}\frac{\sin\theta_0 \sin\theta_1}{\sin\theta_2}, \\ \frac{S}{l_2^2} &= \frac{1}{2}\frac{\sin\theta_2 \sin\theta_0}{\sin\theta_1}. \end{aligned}$$

Using these relations in (14), we obtain time evolution equations for the angles,

$$\frac{d\theta_0}{d\tau} = \frac{1}{2}\sin\theta_0 \sin\theta_1 \sin\theta_2 \left[\frac{\theta_1 - \theta_0}{\sin^2\theta_2} - \frac{\theta_0 - \theta_2}{\sin^2\theta_1}\right], \tag{30}$$

and cyclic permutations. Let us introduce a function

$$V = -(\pi - \theta_0)(\pi - \theta_1)(\pi - \theta_2). \tag{31}$$

The function V is negative in Ω, zero on $\partial\Omega$, and has a unique minimum at $P_* = (2\pi/3, 2\pi/3, 2\pi/3)$. Its time derivative is given by

$$\frac{dV}{d\tau} = -\frac{1}{2}\sin\theta_0 \sin\theta_1 \sin\theta_2 \left[\frac{(\pi - \theta_2)(\theta_1 - \theta_0)^2}{\sin^2\theta_2} + \text{cyclic permutations}\right]. \tag{32}$$

The right-hand side is negative on $\Omega \setminus \{P_*\}$ and zero at P_*. Thus V can be considered as a Liapunov function, and therefore, P_* is asymptotically stable. □

Theorem 5. *An arbitrary parallelogram shrinks to a point in a finite time, and its limiting shape is a rectangle.*

Proof. Let $a = l_0$, $b = l_1$, $\theta = \theta_0$, and $\phi = \theta_1 = \pi - \theta_0$. The evolution equations for the lengths and the angles are given by

$$\frac{da}{d\tau} = -\pi a + b(\phi - \theta)\cos\phi, \tag{33a}$$

$$\frac{db}{d\tau} = -\pi b + a(\phi - \theta)\cos\phi, \tag{33b}$$

$$\frac{d\phi}{d\tau} = -(\phi - \theta)S\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \tag{33c}$$

$$\frac{d\theta}{d\tau} = +(\phi - \theta)S\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \tag{33d}$$

where $S = ab\sin\theta = ab\sin\phi = S_0\exp(-2\pi\tau)$. Define $\Theta = (\phi - \theta)/2$, $A = a\exp(\pi\tau)$, and $B = b\exp(\pi\tau)$. Since $0 < \theta < \pi$, we have $-\pi/2 < \Theta < \pi/2$. The evolution equations (33) can be recast into

$$\frac{dA}{d\tau} = -2B\Theta\sin\Theta, \tag{34a}$$

$$\frac{dB}{d\tau} = -2A\Theta\sin\Theta, \tag{34b}$$

$$\frac{d\Theta}{d\tau} = -2S_0\Theta\left(\frac{1}{A^2}+\frac{1}{B^2}\right). \tag{34c}$$

Note that $\frac{d}{d\tau}(A^2 - B^2) = 0$. There are two cases.

(i) If $A = B$ initially, then the figure is initially equilateral, and it remains so. From (34c),

$$\frac{d\left(\Theta^2\right)}{d\tau} = -\frac{4S_0\Theta^2}{A^2} \leq -\frac{4S_0\Theta^2}{A_0^2}$$

where $A_0 = A(\tau = 0)$. So $\Theta \to 0$ as $\tau \to \infty$, and the equilateral parallelogram approaches a square as it shrinks to a point.

(ii) If $A \neq B$ initially, then we may assume without loss of generality that $A > B$, and that

$$A = C\cosh\frac{u}{2}, \quad B = C\sinh\frac{u}{2}, \quad 0 < u < \infty \tag{35}$$

where C is a positive constant. Now the evolution equations read

$$\frac{du}{d\tau} = -4\Theta\sin\Theta, \tag{36a}$$

$$\frac{d\Theta}{d\tau} = -\frac{8S_0}{C^2}\Theta\frac{\cosh u}{\sinh^2 u}. \tag{36b}$$

Integrating these equations, we have

$$\cos\Theta - \frac{2S_0}{C^2}\frac{1}{\sinh u} = D = \text{constant}. \tag{37}$$

Eliminating u in (36b), we obtain

$$\frac{d}{d\tau}\left(\Theta^2\right) = -\Theta^2 f(\Theta) \tag{38}$$

where

$$f(\Theta) = \frac{4C^2}{S_0}\left(\cos\Theta - D\right)^2\sqrt{1+\frac{4S_0^2}{C^4\left(\cos\Theta - D\right)^2}}. \tag{39}$$

Since $f(\Theta) > 0$ for $u < \infty$, we have

$$\frac{d}{d\tau}\left(\Theta^2\right) = -\Theta^2 f(\Theta) \leq -\Theta^2 f(\Theta_0) \tag{40}$$

where $\Theta_0 = \Theta(\tau = 0)$. So $\Theta \to 0$ as $\tau \to \infty$, and the figure approaches a rectangle as it shrinks. Moreover, from (36a), $\frac{du}{d\tau} \leq 0$, so $u \not\to \infty$ as $\tau \to \infty$. Thus $B/A = b/a \not\to 1$ as $\tau \to \infty$. In other words, the limiting figure is not a square. □

Theorem 6. *An arbitrary, pairwise symmetric (*i.e., *with opposite sides parallel and of equal length, as in Figure 2), equi-angle hexagon shrinks to a point in a finite time and its limiting shape is a regular hexagon.*

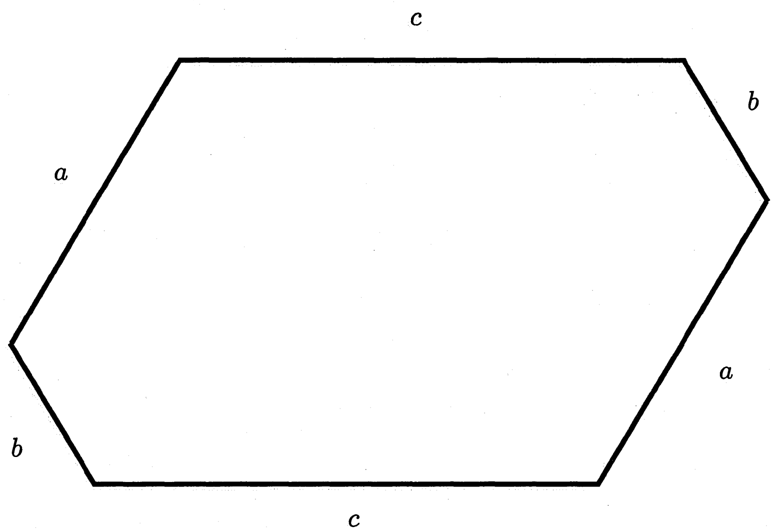

FIGURE 2. A pairwise symmetric, equi-angle hexagon

Proof. Let $a = l_0 = l_3$, $b = l_1 = l_4$, and $c = l_2 = l_5$. Time evolution equations for these length variables are given by

$$\frac{1}{\alpha}\frac{da}{dt} = \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \tag{41}$$

and cyclic permutations, where $\alpha = 2\pi/3\sqrt{3}$. Define

$$\begin{gathered} L = a + b + c, \quad w = -\log L, \\ A = \frac{a}{L}, \quad B = \frac{b}{L}, \quad C = \frac{c}{L}, \\ Q = \frac{1}{A} + \frac{1}{B} + \frac{1}{C}. \end{gathered} \tag{42}$$

From (41), we get the following equations

$$\frac{dw}{dt} = \alpha \exp(2w) Q, \tag{43}$$

$$\frac{dA}{dw} = \frac{1}{Q}\left(\frac{1}{A} - \frac{1}{B} - \frac{1}{C} + AQ\right), \tag{44}$$

and cyclic permutations. For A, B, and C, the allowed region Ω is given by

$$\Omega = \{(A, B, C) \mid A + B + C = 1, \ A, B, C > 0\}.$$

First let us show that $a, b, c \to 0$ simultaneously as $S \to 0$.

(1) Suppose $a \to 0$ and b, c are finite as $S \to 0$. But (41) shows that if $0 < a \ll b, c$, then $da/dt > 0$, so $\{a \to 0$ with b, c finite$\}$ is impossible.

(2) Suppose $a, b \to 0$ and c remains finite as $S \to 0$. Thus da/dt and db/dt should be negative for sufficiently small S. However, time evolution equations show that $(da/dt)(db/dt) < 0$. Then this also never happens.

(3) Thus we conclude that a, b, and c simultaneously become zero as $S \to 0$. In particular, $w \to \infty$ as $S \to 0$.

Next we show that $a/b, a/c \to 1$ as $w \to \infty$. For this purpose, let us introduce a function,

$$V = -ABC. \tag{45}$$

The function V is zero on $\partial\Omega$, negative in Ω, and has a unique minimum at $P_* = (1/3, 1/3, 1/3)$. Its derivative dV/dw is negative in $\Omega \setminus \{P_*\}$ and zero at P_*. Thus V is a Liapunov function, and P_* is asymptotically stable. □

We conjecture that an arbitrary, pairwise symmetric, equi-angle $2N$-polygon for $N > 2$ shrinks to a point and its limiting shape is a regular one. The case $N = 2$ is exceptional.

These properties ensure that the model (12) is appropriate for a discrete curve-shortening equation. However, other models for discrete curve-shortening also have been proposed, as we now discuss. The first of these is due to Taylor, who set forth a theory for the motion of curves by crystalline curvature [2, 14]. A similar model was developed independently by Angenent and Gurtin [3], while Girão and Kohn [8, 9] establish the convergence of a numerical scheme (i.e., discrete in both space and time) for such motions. Their scheme can be compared to Roberts' [13] numerical scheme for the original curve-shortening equation.

Comparing our model (12) with Taylor's [2, 14], we note that in her model, line-segments translate in their normal directions, keeping their orientations. Her model is simpler than (12) because only l_n varies in time, not θ_n.

As we discuss in the Appendix, property (15) is an intrinsic property of the continuous curve-shortening equation, which is inherited by our model (12). Girão [8] shows that his model acquires this property in the continuum limit. A referee pointed out to us that Taylor's model exhibits this property (15) under some circumstances; whether that is always the case seems to be unknown.

Appendix: A derivation of the model (12)

In this Appendix we shall derive a discrete curve-shortening equation (5) with (12) from the continuous curve-shortening equation (1).

First of all, we note that the equation (1) is the one-dimensional reduction of a diffusion equation

$$\frac{d\mathbf{r}}{dt} = \Delta \mathbf{r} \tag{46}$$

where $\Delta = (\det g)^{-1/2}\partial_\mu(\det g)^{1/2} g^{\mu\nu}\partial_\nu$ is the Laplacian on the hypersurface $\mathbf{r}(u^1, \ldots, u^k)$ in $\mathbb{R}^{k+1}$ with respect to the induced metric g, and that the area A enclosed by a simple closed curve driven by the flow (1) satisfies

$$\frac{dA}{dt} = -\oint \kappa\sqrt{g}\, du = -2\pi. \tag{47}$$

Equation (46) indicates that a natural discretization of (1) is

$$\frac{d\mathbf{r}_n}{dt} = \frac{1}{g_n}(\mathbf{r}_{n+1} - 2\mathbf{r}_n + \mathbf{r}_{n-1}). \tag{48}$$

To determine g_n, we impose the condition (47) which also should hold for discrete curves since it represents a topological property of embedded curves. A little calculation shows that

$$\frac{dA}{dt} = -\sum_n \frac{l_n l_{n-1} \sin\theta_n}{g_n}, \tag{49}$$

and the metric is identified as

$$g_n = \frac{l_n l_{n-1} \sin\theta_n}{\theta_n}. \tag{50}$$

This choice of the metric gives a discrete curve-shortening equation (5) with (12). We remark that g_n in (50) consists of geometric quantities associated only with the n-th vertex $\mathbf{r}_n$ (see Figure 1).

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