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Nonmonotone waves in a three species reaction-diffusion model

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ABSTRACT. This paper establishes the existence of a nonmonotone travelling wave for a reaction-diffusion system modeling three competing species. General existence results for travelling waves in higher dimensional systems depend on monotonicity and, therefore, do not apply to the result obtained here. The proof demonstrates an application of a homotopy invariant, the *connection index*, to a higher dimensional flow where few explicit results are available. The result is obtained in a singular perturbation regime where the fast-slow structure can be exploited to construct a singular limit solution from the lower dimensional reduced flows. *A priori* estimates show the connecting solution to be uniformly approximated by the singular limit, and these estimates make it possible to construct the higher dimensional isolating neighborhoods necessary for defining the index. The index is computed by continuing the equations to a system containing a lower dimensional invariant manifold.

1. Introduction

Reaction-diffusion equations are used extensively as continuous space-time models for interacting and diffusing chemical and biological species, combustion, phase transitions, and neurophysiology. In mathematical ecology, the interactions between individuals can take the form of competition for limited resources, predator-prey interactions, or mutualistic relationships where the diffusion terms model the migration of a species (see [7, 20]). Though these equations are relatively simple, they can exhibit a variety of interesting spatial and spatio-temporal patterns, including wave fronts and wave pulses. The general system for modeling N species interacting in one space dimension is of the form,

$$\frac{\partial u_i}{\partial t} = d_i \frac{\partial^2 u_i}{\partial x^2} + u_i g_i(u), \qquad i = 1, \dots, N. \tag{1.1}$$

Here $u_i(x,t)$ denotes a population density at time t and spatial position x, and the diffusion coefficients are assumed constant, $d_i \geq 0$.

Travelling waves, solutions of the form $u(x,t) = U(x - \theta t)$, play an important role in the dynamics of (1.1) and are of fundamental interest. A travelling wave represents a segregated spatial pattern propagating through the spatial domain at a constant speed. A homotopy invariant, the *connection index*, has been successful in proving existence results for systems with two species [5, 9, 12], and the *connection matrix* was used in [18] for two mutualist species in competition with a third species.

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General existence results for travelling waves in N species [19,23] ultimately depend on monotonicity and the maximum principle, and do not extend to the results presented here. For singularly perturbed systems, geometric methods can be used to prove existence when certain transversality conditions are satisfied by the limiting solution within a lower dimensional reduced flow [1,13,14].

This paper presents an existence theorem for a heteroclinic travelling wave connecting two stable rest states in a system modeling three competing species. The significance of this result is that it establishes the existence of a *nonmonotone* wave, where the general results for N species do not apply, and the proof demonstrates a method for applying the the connection index to a system with $N > 2$ where very few results have been obtained. The existence results are obtained in a singular perturbation regime where the fast-slow structure in the flow makes it possible to construct a singular limit solution from lower dimensional reduced flows. The singular limit in the fast scaling is a travelling wave for two competing species. The topological proof only requires knowing that the lower dimensional wave exists and that the wave speed is unique, as established in [5] and [8]. The higher dimensional phase space associated with the additional species increases the difficulty of showing that the index is well-defined. The fast-slow structure is used to construct the necessary isolating neighborhoods from sets which are isolating for the lower dimensional reduced flows.

For the result presented here, the governing system of reaction-diffusion equations models the evolution of three competing species:

$$\begin{aligned} u_{1t} &= u_{1xx} + u_1\, g_1(u), \\ u_{2t} &= \varepsilon^2 u_{2xx} + r_2\, u_2\, g_2(u), \\ u_{3t} &= \varepsilon^2 d u_{3xx} + r_3\, u_3\, g_3(u). \end{aligned} \tag{1.2}$$

The travelling wave presented in this paper connects the two rest states P_2 and P_3, as defined by

$$\begin{aligned} &P_2\colon\ g_1(u_1,u_2,0) = g_2(u_1,u_2,0) = 0, \quad \text{with } u_3 = 0, \\ &P_3\colon\ g_1(u_1,0,u_3) = g_3(u_1,0,u_3) = 0, \quad \text{with } u_2 = 0. \end{aligned} \tag{1.3}$$

Travelling waves are steady state solutions in a reference frame moving along with the wave, and so, in one space dimension, they satisfy a system of ordinary differential equations. We look for slow waves with speed $\varepsilon\theta$ where $\theta(\varepsilon)$ remains $O(1)$ as $\varepsilon \to 0$. Transforming the equations to the moving frame $\zeta = x - \varepsilon\theta t$, with $U_i(t,\zeta) = u_i(t,\zeta + \varepsilon\theta t)$, the travelling wave ansatz yields a flow on $\mathbb{R}^6$:

$$\begin{aligned} \dot U_1 &= V_1, \\ \dot V_1 &= -\varepsilon\,\theta\, V_1 - U_1\, g_1(U), \\ \varepsilon\, \dot U_2 &= V_2, \\ \varepsilon\, \dot V_2 &= -\theta\, V_2 - r_2\, U_2\, g_2(U), \\ \varepsilon\, \dot U_3 &= V_3, \\ \varepsilon\, \dot V_3 &= d^{-1}\big[-\theta\, V_3 - r_3\, U_3\, g_3(U)\big]. \end{aligned} \tag{1.4}$$

The singular nature of the limit $\varepsilon \to 0$ results in two distinct time scales, and the ζ parametrization is referred to throughout as the *slow* scaling. To introduce a *fast* scaling, or *stretched* scaling, define $\xi = \zeta/\varepsilon$, $u_i(\xi;\varepsilon) = U_i(\varepsilon\xi;\varepsilon)$, and $v_i(\xi;\varepsilon) = V_i(\varepsilon\xi;\varepsilon)$:

$$\begin{aligned} u_1' &= \varepsilon\, v_1, \\ v_1' &= \varepsilon\big(-\varepsilon\,\theta v_1 - u_1\, g_1(u)\big), \\ u_2' &= v_2, \\ v_2' &= -\theta v_2 - r_2\, u_2\, g_2(u), \\ u_3' &= v_3, \\ v_3' &= d^{-1}\big[-\theta v_3 - r_3\, u_3\, g_3(u)\big]. \end{aligned} \tag{1.5}$$

For $\varepsilon > 0$, equations (1.4) and (1.5) are equivalent, and their solutions will be denoted by $X(\zeta;\varepsilon)$ and $x(\xi;\varepsilon)$, respectively. The variables $y = (u_1, v_1)$ are referred to as the *slow components* and $z = (u_2, v_2, u_3, v_3)$ as the *fast components.* Setting $\varepsilon = 0$ in (1.4) and (1.5) will be called the slow and fast *reduced flows*, respectively.

Proving the existence of the travelling wave for $\varepsilon > 0$ means showing that there is some value of the wave speed parameter $\theta(\varepsilon)$ for which there exists a heteroclinic solution to (1.4) and (1.5) connecting the two rest points P_2 and P_3. The most difficult part of the existence proof is to find appropriate isolating neighborhoods in the six dimensional phase space such that the connection index is well-defined. Moreover, we must show that given a nontrivial index, the only possible solution is the connecting orbit which we are seeking. These properties will follow from *a priori* estimates showing that the solution must lie uniformly close to the singular limit solution as $\varepsilon \to 0$. The final determination of the index is made via a continuation to a product flow consisting of lower dimensional flows for which the indices are already known.

The paper continues with a statement of the hypotheses in Section 1.1. Section 2 presents the reduced equations for $\varepsilon = 0$ and describes the singular limit solution. The main result is stated in Section 3 followed by an overview of the index methods employed in the proof. The remainder of Section 3 is taken up by the details of the existence proof.

1.1. Hypotheses. In the governing equations (1.2), the constants r_2, r_3, and d are strictly positive and two of the species diffuse slowly relative to the third ($\varepsilon \ll 1$). We assume a Lotka-Volterra competition model,

$$g_i(u) = 1 - \sum_{j=1}^{3} \beta_{ij} u_i \tag{1.6}$$

where the competition coefficients β_{ij} are nonnegative and $\beta_{ii} = 1$, for i, $j = 1, 2, 3$. Figure 1 shows the level surfaces $g_i(u) = 0$ and identifies the location of certain crucial intersections, denoted $P_1, \ldots, P_4$. The u_1-components of these four points will be denoted by $p_1, \ldots, p_4$. The rest points P_2 and P_3 already have been defined in (1.3) and are embedded into $\mathbb{R}^6$ by setting $v_i = 0$, for $i = 1, 2, 3$. The remaining two intersection points P_1 and P_4 lie in the planes $u_3 = 0$ and $u_2 = 0$, respectively.

The following hypotheses impose a certain geometric structure on the reduced problems, ensuring that there is an appropriate singular limit solution which will perturb

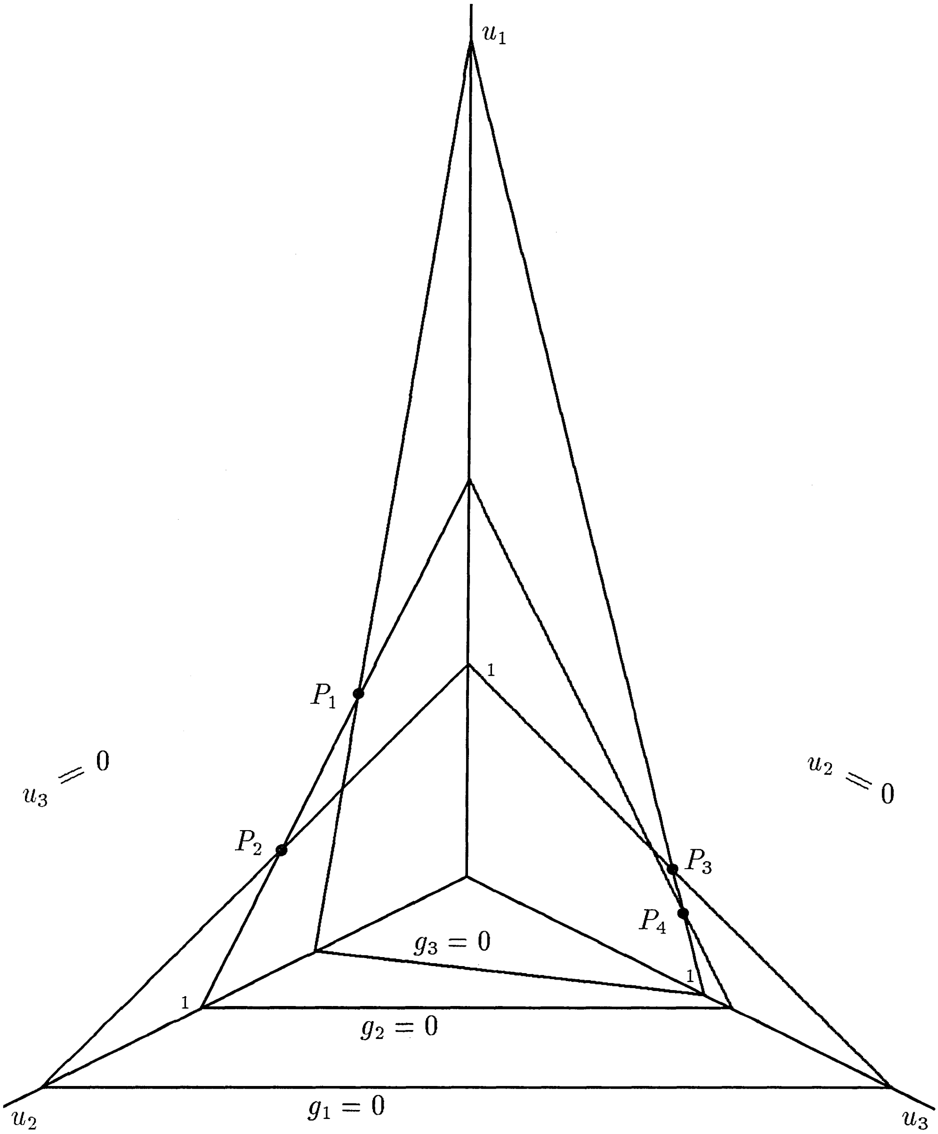

FIGURE 1. The zero sets of the reaction functions.

to a heteroclinic orbit for $\varepsilon > 0$.

$$\begin{aligned} &\text{H1:} && 1 < 1/\beta_{23} < \beta_{32}, \\ &\text{H2:} && \beta_{12} < 1, \\ &\text{H3:} && \beta_{21} < \beta_{31}, \\ &\text{H4:} && p_4 < p_3 < p_2 < p_1. \end{aligned} \tag{1.7}$$

The hypotheses $p_2 < p_1$ and $p_4 < p_3$ ensure that P_2 and P_3 are stable rest points of the reaction flow, whereas the ordering $p_3 < p_2$ is arbitrary. The motivation for

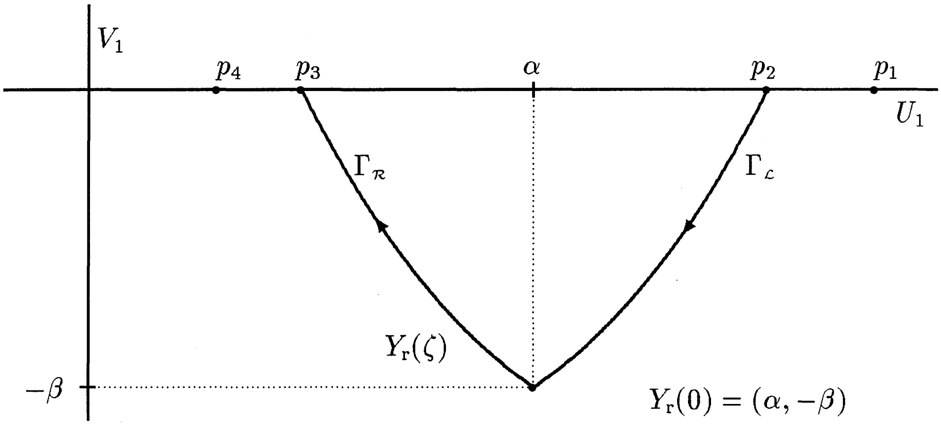

FIGURE 2. The slow singular limit in the U_1-V_1 phase space ($\varepsilon = 0$).

the remaining hypotheses will be made clearer when we define the singular limit solution. Note that these results also will hold for more general nonlinearities with an appropriate reformulation of the hypotheses in (1.7).

2. Singular limit ($\varepsilon = 0$)

First define a singular limit solution at $\varepsilon = 0$ using a singular perturbation construction (see [17]). The slow and fast scalings give distinctly different reduced flows at $\varepsilon = 0$ and must be considered separately.

2.1. Slow scaling. Setting $\varepsilon = 0$ in (1.4) reduces the equations for the fast components to the algebraic equation $G(Y, Z; \theta) = 0$ where $G(\cdot, \cdot; \theta) : \mathbb{R}^6 \to \mathbb{R}^4$. This determines several two-dimensional manifolds, $Z = h(Y)$, with the reduced flow on the invariant *slow manifold* determined by setting $Z = h(Y)$ in the first two equations for $\dot{Y}$. Consider two particular pieces of the slow manifold, the *left slow manifold* $\mathcal{M}_{\mathcal{L}}$, determined by solving $g_2(u) = 0$ with $u_3 = 0$, and the *right slow manifold* $\mathcal{M}_{\mathcal{R}}$, satisfying $g_3(u) = 0$ with $u_2 = 0$. The rest points P_2 and P_3 lie in the left and right slow manifolds, respectively, and we denote these slow manifolds by the graphs $z = \mathcal{L}(y) = (\mathcal{L}_2(u_1), 0, 0, 0)$ and $z = \mathcal{R}(y) = (0, 0, \mathcal{R}_3(u_1), 0)$. The reduced flows in these slow manifolds are qualitatively the same, each having a center at $(0, 0)$ along with one saddle point. For the flow in $\mathcal{M}_{\mathcal{L}}$, the unstable manifold of $(p_2, 0)$ is denoted by the graph $V_1 = \Gamma_{\mathcal{L}}(U_1)$, and for the flow in $\mathcal{M}_{\mathcal{R}}$, the stable manifold of $(p_3, 0)$ is the graph $V_1 = \Gamma_{\mathcal{R}}(U_1)$. By their monotonicity these two curves intersect uniquely in the lower half-plane at $(U_1, V_1) = (\alpha, -\beta)$ (see Figure 2).

The *slow singular limit* $X_r(\zeta) = (Y_r(\zeta), Z_r(\zeta))$ satisfies $Y_r(0) = (\alpha, -\beta)$. For $\zeta < 0$, $Y_r(\zeta)$ satisfies the reduced flow on $\mathcal{M}_{\mathcal{L}}$ with $Z_r(\zeta) = \mathcal{L}(Y_r(\zeta))$, and for $\zeta > 0$, $Y_r(\zeta)$ satisfies the reduced flow on $\mathcal{M}_{\mathcal{R}}$ with $Z_r(\zeta) = \mathcal{R}(Y_r(\zeta))$. The singular limit $X_r(\zeta)$ is a connection from P_2 to P_3 with a discontinuous transition layer at $\zeta = 0$. Graphs of the U-components in Figure 3 show that the slow component U_1 is continuous at $\zeta = 0$, but the fast components U_2 and U_3 are discontinuous, with U_2 *nonmonotone* in ζ.

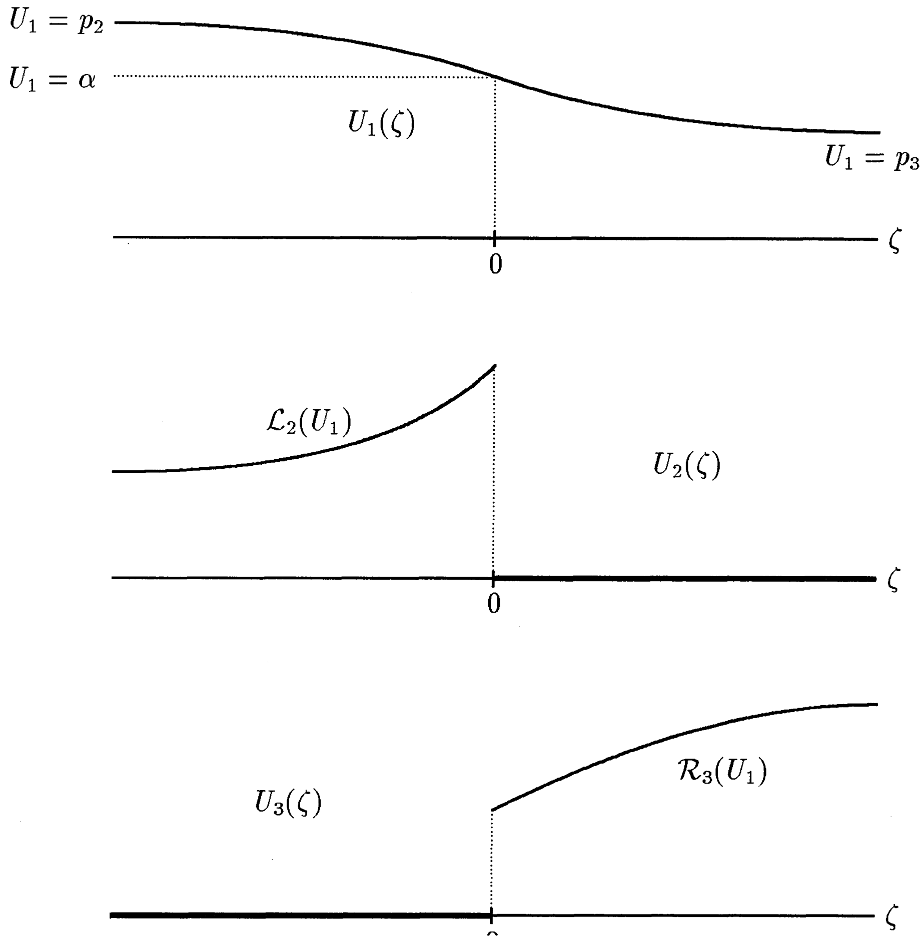

FIGURE 3. The singular limit in the slow scaling ($\varepsilon = 0$).

2.2. Fast scaling. The reduced flow in (1.5) is a flow on $\mathbb{R}^4$ parametrized by u_1. The equations are just the travelling wave equations with wave speed parameter θ, as derived for the two-component reaction-diffusion system,

$$\begin{aligned} u_{2t} &= u_{2xx} + r_2\, u_2\, g_2(u), \\ u_{3t} &= du_{3xx} + r_3\, u_3\, g_3(u). \end{aligned} \tag{2.1}$$

The results in [5] show that for any u_1 in the interval $p_4 < u_1 < p_1$, there exists a bistable travelling wave solution to (2.1) connecting $\mathcal{L}(u_1)$ at $\xi = -\infty$ with $\mathcal{R}(u_1)$ at $\xi = +\infty$. For the existence proof, we also use the result from [8] that the wave speed, denoted by $\Theta(u_1)$, is unique. Hypothesis H4 in (1.7) ensures that the transition point $u_1 = \alpha$ lies in the interval $p_4 < \alpha < p_1$. The *fast singular limit* $x_{\mathrm{r}}(\xi)$ is defined by this heteroclinic orbit for the fast reduced flow connecting $\mathcal{L}(\alpha)$ to $\mathcal{R}(\alpha)$ with wave speed $\Theta(\alpha)$, and $(u_1, v_1) = (\alpha, -\beta)$. The existence of this connecting orbit for the fast reduced flow at $u_1 = \alpha$ is the mechanism by which the solution to the reduced

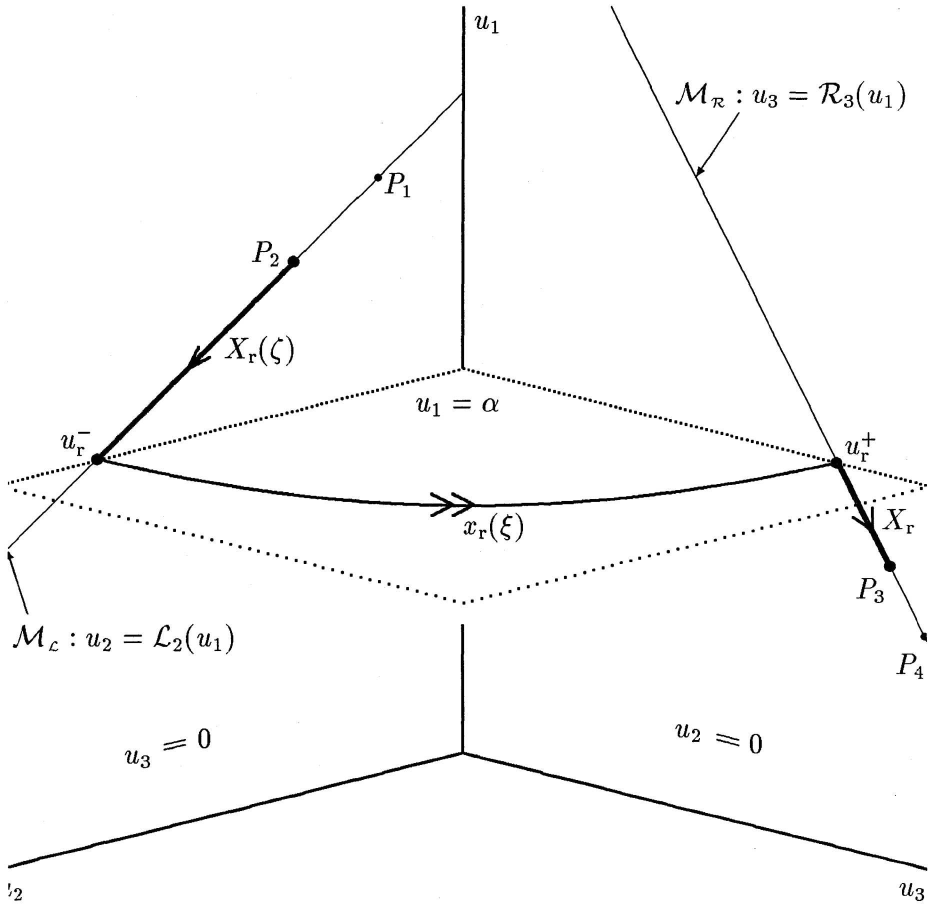

FIGURE 4. The singular limit heteroclinic solution ($\varepsilon = 0$).

equations transitions from the left slow manifold to the right slow manifold. Figure 4 shows both the fast and slow singular limits in phase space as projected onto the u-components.

3. Existence theorem ($\varepsilon > 0$)

We now state the main result, that the singular limit solution described in Section 2 perturbs to a bistable heteroclinic connection for $\varepsilon > 0$.

Theorem 3.1 (Existence). *Suppose the hypotheses* H1–H4 *are satisfied. Then for all sufficiently small* $\varepsilon > 0$, *there exists a solution* $(x(\xi;\varepsilon), \theta(\varepsilon))$ *to* (1.5) *which connects* P_2 *at* $\xi = -\infty$ *with* P_3 *at* $\xi = +\infty$. *The wave speed* $\theta(\varepsilon)$ *tends to* $\Theta(\alpha)$ *as* $\varepsilon \to 0$.

The proof uses the connection index, and we begin with an overview of these methods and some properties needed later in the proof. More complete descriptions of these methods can be found in [3, 5, 9, 22].

3.1. Conley index. Consider a local flow on $\mathbb{R}^n$, $x' = \varphi(x)$. For a compact set $N \subset \mathbb{R}^n$, let $S(N)$ denote the *maximal invariant set* of N relative to the flow, the set of points which stay in N for all time under the action of the flow. If $S(N) \cap \partial N = \emptyset$, N is said to be an *isolating neighborhood* and $S(N)$ an *isolated invariant set.* N is called an *isolating block* if the flow through each point in ∂N leaves N immediately in at least one time direction. The *strict exit set* of N is the set of points in ∂N which leave N immediately in forward time.

The Conley index (or homotopy index) of an isolated invariant set S, denoted by $h(S)$, is the homotopy type $[N/N^+]$ of an *index pair* (N, N^+) where N is an isolating neighborhood for $S = S(N)$ and N/N^+ is the pointed space obtained by collapsing N^+ to a single point. In the simplest situation, where N is an isolating block, the set N^+ can be taken to be the strict exit set of N.

A sum ($\vee$) and product ($\wedge$) are defined on homotopy classes of pointed spaces and the following elementary facts are proved in [3]. If $S_1 \subset \mathbf{X}$ and $S_2 \subset \mathbf{X}$ are disjoint isolated invariant sets, then $h(S_1 \cup S_2) = h(S_1) \vee h(S_2)$. If $S_{\mathrm{X}} \subset \mathbf{X}$ and $S_{\mathrm{Y}} \subset \mathbf{Y}$ are isolated invariant sets, then $S_{\mathrm{X}} \times S_{\mathrm{Y}}$ is an isolated invariant set in the product space $\mathbf{X} \times \mathbf{Y}$ and $h(S_{\mathrm{X}} \times S_{\mathrm{Y}}) = h(S_{\mathrm{X}}) \wedge h(S_{\mathrm{Y}})$. The additive identity is the homotopy class of the empty set, the pointed one-point space denoted by $\bar{0}$. The multiplicative identity is the homotopy class of an isolated attracting rest point, the pointed two-point space or pointed zero-sphere denoted by $\bar{1}$. An isolated hyperbolic rest point with k unstable eigenvalues (Morse index k) has the homotopy class of the pointed k-sphere denoted by Σ^k.

3.2. Connection index. The connection index extends the Conley index to the existence of heteroclinic solutions to parametrized flows, $x' = \varphi(x;\theta)$ where $x \in \mathbb{R}^n$, and $\theta \in \mathbb{R}$. The flow is augmented with the parameter equation $\theta' = 0$, and for S an invariant set relative to the augmented flow on $\mathbb{R}^{n+1}$, let $S_\theta \subset \mathbb{R}^n$ denote the subset of S with parameter value θ.

Definition. Suppose (S', S'', S) is a triple of invariant sets relative to the augmented flow on $\mathbb{R}^n \times [\theta_1, \theta_2]$. Then (S', S'', S) is called a *connection triple* if S'_θ, S''_θ, and S_θ are isolated invariant sets for each $\theta \in [\theta_1, \theta_2]$ and the triple satisfies

1. $S' \cup S'' \subset S$,
2. $S' \cap S'' = \emptyset$,
3. $S_\theta = S'_\theta \cup S''_\theta \quad$ for $\theta = \theta_1, \theta_2$.

The connection index for a connection triple (S', S'', S) is rigorously defined via the Conley index applied to an appropriately perturbed flow [4, 5]. In many applications S' and S'' are just curves of rest points, and we seek a connecting orbit from S'_θ to S''_θ for some value of the parameter θ. To visualize the calculation of the connection index, suppose (N, N^+) is an index pair for the isolated invariant set S_θ for all $\theta \in [\theta_1, \theta_2]$. For $S'_\theta \subset N$, define $\mathcal{W}^{\mathcal{U}}(S'_\theta, N)$ as the set of all points in N which tend to S'_θ in backwards time, the unstable manifold of S'_θ. In particular, define $\mathcal{W}_1^{\mathcal{U}} = \mathcal{W}^{\mathcal{U}}(S'_{\theta_1}, N)$ and $\mathcal{W}_2^{\mathcal{U}} = \mathcal{W}^{\mathcal{U}}(S'_{\theta_2}, N)$, and define the compact sets in $\mathbb{R}^n \times [\theta_1, \theta_2]$,

$$\bar{N} = N \times [\theta_1, \theta_2],$$
$$\bar{N}^+ = \left(N^+ \times [\theta_1, \theta_2]\right) \cup \mathcal{W}_1^{\mathcal{U}} \cup \mathcal{W}_2^{\mathcal{U}}.$$

Then the connection index is defined heuristically as $\bar{h}(S', S'', S) = [\bar{N}/\bar{N}^+]$. The existence of a connecting orbit follows from the result

$$\bar{h}(S', S'', S) \neq (\Sigma^1 \wedge h(S')) \vee h(S'') \rightarrow S \neq S' \cup S'' \tag{3.1}$$

where Σ^1 is the pointed one-sphere and $h(S')$ and $h(S'')$ are just $h(S'_\theta)$ and $h(S''_\theta)$ for any $\theta \in [\theta_1, \theta_2]$. If the computed index satisfies the inequality on the left-hand side of (3.1), additional knowledge of S is needed to conclude that $S \setminus (S' \cup S'')$ contains a connecting solution between S' and S''.

Now suppose that the flow $x' = \varphi(x; \theta; \mu)$ is parametrized continuously by $\mu \in [0, 1]$. Let N'_μ, N''_μ, and N_μ be isolating neighborhoods in $\mathbb{R}^n$ depending continuously on μ and determining a connection triple (S'_μ, S''_μ, S_μ) on $\mathbb{R}^n \times [\theta_1, \theta_2]$, for all $\mu \in [0, 1]$. Then the triples (S'_0, S''_0, S_0) and (S'_1, S''_1, S_1) are related by continuation, and $\bar{h}(S'_0, S''_0, S_0) = \bar{h}(S'_1, S''_1, S_1)$. In our application, the index will be computed by continuing the equations to a product flow,

$$\begin{aligned} x' &= Ax \qquad x \in \mathbb{R}^p, \\ y' &= g(y; \theta) \qquad y \in \mathbb{R}^q, \quad \theta \in \mathbb{R} \end{aligned} \tag{3.2}$$

where A is a hyperbolic constant matrix with k unstable eigenvalues. If $(\tilde{S}', \tilde{S}'', \tilde{S})$ is a connection triple for the flow on $\mathbb{R}^q \times [\theta_1, \theta_2]$ and (S', S'', S) is the inclusion of $(\tilde{S}', \tilde{S}'', \tilde{S})$ into $\mathbb{R}^{p+q}$ with the x-coordinates set to zero, then (S', S'', S) is a connection triple for the product flow on $\mathbb{R}^{p+q} \times [\theta_1, \theta_2]$ with index

$$\bar{h}(S', S'', S) = \Sigma^k \wedge \bar{h}(\tilde{S}', \tilde{S}'', \tilde{S}). \tag{3.3}$$

3.3. Continuing the equations. We first describe the deformation of the equations to be used later in computing the index. The continuation is done in three steps. (1) β_{32} increases to a value $\bar{\beta}_{32}$ sufficiently large to make $p_1 > 1$. (2) β_{12} goes to 0 making $p_2 = 1$. (3) β_{13} goes to 0 making $p_3 = 1$. It is clear that the hypotheses H1–H4 remain satisfied throughout the homotopy.

Notation. To simplify the notation, we use the parameter $\mu \in [0, 1]$ to denote the deformation of the flow where $\mu = 1$ refers to the original equations and $\mu = 0$ denotes the flow with $\beta_{32} = \bar{\beta}_{32}$, $\beta_{12} = 0$, and $\beta_{13} = 0$.

At $\mu = 0$, the $g_1 = 0$ surface coincides with $u_1 = 1$, so that $p_3 = p_2 = 1$, and the fast components have been decoupled from the flow for (u_1, v_1):

$$\begin{aligned} u_1' &= \varepsilon v_1, \\ v_1' &= \varepsilon(-\varepsilon\theta v_1 - u_1(1 - u_1)), \\ u_2' &= v_2, \\ v_2' &= -\theta v_2 - r_2\, u_2\, g_2(u_1, u_2, u_3), \\ u_3' &= v_3, \\ v_3' &= d^{-1}[-\theta v_3 - r_3\, u_3\, g_3(u_1, u_2, u_3)]. \end{aligned} \tag{3.4}$$

The four-dimensional manifold $\mathcal{M}_0 = \{\, x \in \mathbb{R}^6 : u_1 = 1, v_1 = 0 \,\}$ contains the rest points P_2 and P_3 and is invariant under the flow in (3.4). The flow in $\mathcal{M}_0$ is governed by the fast reduced flow with $u_1 = 1$, and a connection exists from P_2 with P_3. In Section 3.8, the system in (3.4) is continued to the product of a hyperbolic linear flow on $\mathbb{R}^2$ and a parametrized flow on $\mathbb{R}^4$ where we then can apply the result in (3.3).

3.4. The connection triple. In this section, we define a connection triple relative to the flow in (1.4) and (1.5) with $\varepsilon > 0$ and for all $\mu \in [0,1]$. First choose θ_1 and θ_2 such that $\Theta(\alpha) \in (\theta_1, \theta_2)$. Using the uniqueness of the wave speed for the fast singular limit as established in [8], it is shown in Lemma 3.6 that $\theta(\varepsilon)$ converges to $\Theta(\alpha)$ as $\varepsilon \to 0$. It follows that for ε sufficiently small, a connection from P_2 to P_3 cannot exist at $\theta = \theta_1$ and $\theta = \theta_2$.

The goal is to define an appropriate neighborhood $N \subset \mathbb{R}^6$ for which the total invariant set consists only of connections from P_2 to P_3, and which is isolating relative to (1.4) and (1.5) for all $\theta \in [\theta_1, \theta_2]$. The invariant sets in $\mathbb{R}^6 \times [\theta_1, \theta_2]$ which make up the triple (S', S'', S) are defined by $S'_\theta = P_2$, $S''_\theta = P_3$, and $S_\theta = S_\theta(N; \varepsilon)$. We construct two neighborhoods, $N_0(\delta)$ and $N_1(\delta)$, with N_0 isolating for $\mu \in [0, \bar{\mu}]$ and N_1 isolating for $\mu \in [\bar{\mu}, 1]$ where $\bar{\mu}$ is to be specified later. Here $\delta = (\delta_f, \delta_0, \delta_1, \delta_2, \delta_3)$ is a vector of small parameters used in defining the neighborhoods, and a key feature of the construction is that these parameters must be set in relation to one another, with

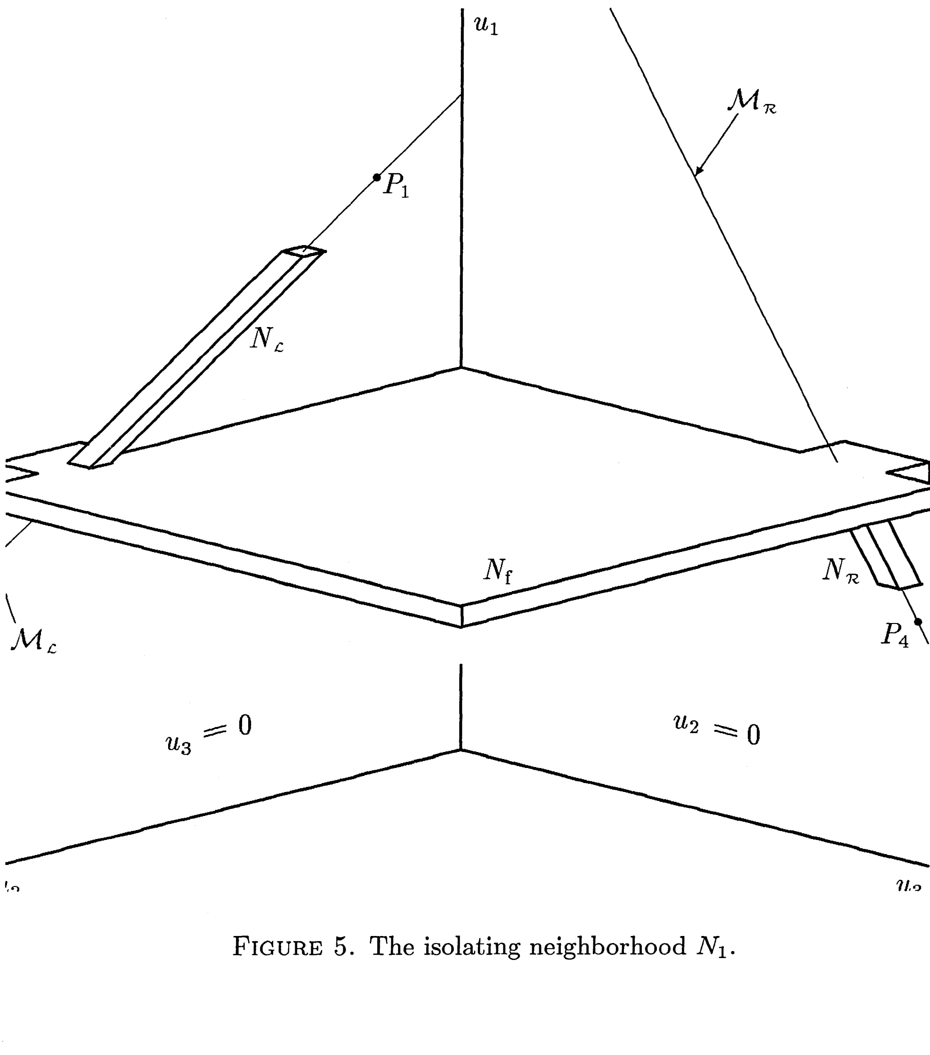

FIGURE 5. The isolating neighborhood N_1.

the order prescribed as follows:

$$\text{fix } \delta_\text{f} \to \text{ fix } \delta_1 \to \text{ fix } \bar{\mu} \to \text{ fix } \delta_1 \to \text{ fix } \delta_2 \to \text{ fix } \delta_3. \tag{3.5}$$

The critical construction is the neighborhood N_1, which consists of three pieces, a *fast block* N_f containing the fast singular limit, and two *slow tubes*, $N_\mathcal{L}$ and $N_\mathcal{R}$, containing the slow singular limit for $\zeta < 0$ and $\zeta > 0$, respectively. The projection of N_1 onto the three-dimensional u-space is shown in Figure 5.

Notation. In describing the isolating neighborhoods, N is used to denote neighborhoods in $\mathbb{R}^6$, $\mathcal{U}$ denotes sets in (u_2, u_3) space, $\mathcal{V}$ denotes sets in (v_2, v_3) space, $\mathcal{Z}$ denotes sets in $\mathbb{R}^4$ ($\mathcal{Z} = \mathcal{U} \times \mathcal{V}$), and $\mathcal{Y}$ denotes sets in (u_1, v_1) space. $S(N; \varepsilon)$ denotes the maximal invariant set in N relative to the flow in (1.4) and (1.5), parametrized by θ and ε. The notation $S_\mu(N;\varepsilon)$ is used whenever it is necessary to be explicit about the value of the homotopy parameter, with $S_1(N;\varepsilon)$ denoting the original flow in (1.5) and $S_0(N;\varepsilon)$ denoting the flow at the end of the homotopy in (3.4). The notation $S_\text{f}(\mathcal{Z}; u_1)$ represents the maximal invariant set of $\mathcal{Z} \subset \mathbb{R}^4$ relative to the fast reduced flow parametrized by u_1.

3.4.1. *Isolating neighborhoods for P_2 and P_3.* It is easy to verify that for all sufficiently small $\varepsilon > 0$, P_2 and P_3 are hyperbolic rest points of the flow in (1.5), each having three stable and three unstable eigenvalues. From singular perturbation theory [6], the slow manifold perturbs to an invariant manifold for $\varepsilon > 0$, and, by the hyperbolicity of this manifold, any solution which remains in a small neighborhood of P_2 (resp. P_3) for all time must lie in this invariant manifold. Since the flow on the invariant manifold is an $O(\varepsilon)$ perturbation of the flow on the slow manifold for the reduced equations, it follows that for all sufficiently small neighborhoods N' and N'' and $\varepsilon > 0$, $S(N';\varepsilon) = P_2$ and $S(N'';\varepsilon) = P_3$. Moreover, the isolating properties of N' and N'' do not depend on the smallness of v_2 and v_3, which follows from v_2 and v_3 being the derivatives of u_2 and u_3. We summarize in the following lemma.

Lemma 3.2. *Suppose $N' \subset \mathbb{R}^6$ is a compact neighborhood of P_2 satisfying*

$$|u_1 - p_2|,\ |v_1|,\ |u_2 - \mathcal{L}_2(p_2)|,\ |u_3| \le \delta_c, \qquad |v_2|,\ |v_3| \le K,$$

and N'' is a neighborhood of P_3 satisfying

$$|u_1 - p_3|,\ |v_1|,\ |u_3 - \mathcal{R}_3(p_3)|,\ |u_2| \le \delta_c, \qquad |v_2|, |v_3| \le K.$$

Then for all δ_c and ε sufficiently small, $S(N';\varepsilon) = P_2$ and $S(N'';\varepsilon) = P_3$.

3.4.2. *The fast block.* The *fast block* is given by $N_\text{f} = \mathcal{Y}_\text{f} \times \mathcal{Z}_\text{f}$ where $\mathcal{Z}_\text{f}$ is isolating relative to the fast reduced flow with $u_1 = \alpha$, and $\mathcal{Y}_\text{f}$ is a rectangle in (u_1, v_1) space centered at the transition point $(\alpha, -\beta)$,

$$\mathcal{Y}_\text{f} = \mathcal{Y}_\text{f}(\delta_1) = \{\ (u_1, v_1) : |u_1 - \alpha| \le \delta_1,\ |v_1 + \beta| \le b\ \}. \tag{3.6}$$

For K fixed and assumed large, define the sets

$$\mathcal{U}_K = \{\ (u_2, u_3) : 0 \le u_2,\ u_3 \le K\ \},$$
$$\mathcal{V}_K = \{\ (v_2, v_3) : 0 \le -v_2,\ v_3 \le K\ \},$$
$$\mathcal{V}_\text{f} = \{\ (v_2, v_3) : |v_2|,\ |v_3| \le K\ \}.$$

Define the sets $\mathcal{U}_{\text{f}\mathcal{L}}$ and $\mathcal{U}_{\text{f}\mathcal{R}}$ to be rectangles in (u_2, u_3) space centered at the slow manifolds with $u_1 = \alpha$ (see Figure 6).

$$\mathcal{U}_{\text{f}\mathcal{L}}(\delta_\text{f}) = \{\ (u_2, u_3) : |u_2 - \mathcal{L}_2(\alpha)| \le a_\mathcal{L}\delta_\text{f};\ |u_3| \le \delta_\text{f}\ \},$$

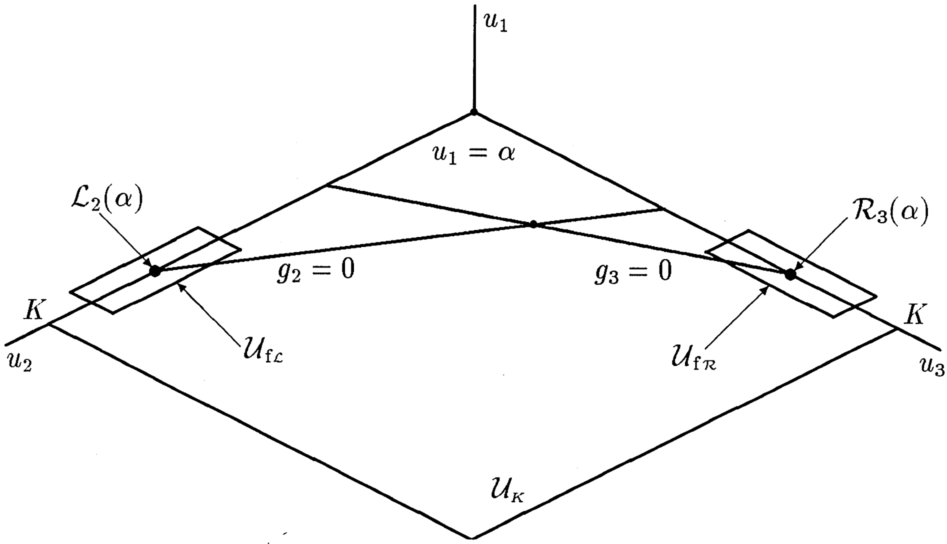

FIGURE 6. $\mathcal{U}_{\mathcal{K}}$, $\mathcal{U}_{f\mathcal{L}}$, $\mathcal{U}_{f\mathcal{R}}$: isolating neighborhood for the fast components.

$$\mathcal{U}_{f\mathcal{R}}(\delta_f) = \{ (u_2, u_3) : |u_2| \le \delta_f;\ |u_3 - \mathcal{R}_3(\alpha)| \le a_{\mathcal{R}}\delta_f \}.$$

Here the positive constants $a_{\mathcal{L}}$ and $a_{\mathcal{R}}$ denote the aspect ratios of the respective rectangles, δ_f controls the size of the neighborhoods, and we assume that $\max\{\delta_f, a_{\mathcal{L}}\delta_f, a_{\mathcal{R}}\delta_f\}$ is less than δ_c as defined in Lemma 3.2. Now define the following sets in $\mathbb{R}^4$, $\mathcal{Z}_{\mathcal{K}} = \mathcal{U}_{\mathcal{K}} \times \mathcal{V}_{\mathcal{K}}$, $\mathcal{Z}_{f\mathcal{L}}(\delta_f) = \mathcal{U}_{f\mathcal{L}}(\delta_f) \times \mathcal{V}_f$, and $\mathcal{Z}_{f\mathcal{R}}(\delta_f) = \mathcal{U}_{f\mathcal{R}}(\delta_f) \times \mathcal{V}_f$. Excise neighborhoods $\mathcal{Z}'$ and $\mathcal{Z}''$ about the rest point at the origin in $\mathbb{R}^4$ and the rest point at $g_2 = g_3 = 0$. The fast block in $\mathbb{R}^4$ is defined as

$$\mathcal{Z}_f = \mathcal{Z}_f(\delta_f) = \left(\mathcal{Z}_{f\mathcal{L}}(\delta_f) \cup \mathcal{Z}_{\mathcal{K}} \cup \mathcal{Z}_{f\mathcal{R}}(\delta_f)\right) \setminus (\mathcal{Z}' \cup \mathcal{Z}''). \tag{3.7}$$

The following lemma is proved by Conley and Gardner in [5] where it is shown that the travelling wave exists for the fast reduced system.

Lemma 3.3. *There exists $a_{\mathcal{L}}$, $a_{\mathcal{R}} > 0$ such that for all K sufficiently large and all $\mathcal{Z}'$, $\mathcal{Z}''$, δ_f sufficiently small, $\mathcal{Z}_{f\mathcal{L}}(\delta_f)$ isolates the rest point $\mathcal{L}(\alpha)$, $\mathcal{Z}_{f\mathcal{R}}(\delta_f)$ isolates the rest point $\mathcal{R}(\alpha)$, and $\mathcal{Z}_f(\delta_f)$ is an isolating neighborhood in $\mathcal{R}^4$, relative to the fast reduced system with $u_1 = \alpha$. By the monotonicity built into $\mathcal{Z}_{\mathcal{K}}$, it follows that $S_f(\mathcal{Z}_f; \alpha)$ consists only of connections from $\mathcal{L}(\alpha)$ to $\mathcal{R}(\alpha)$.*

3.4.3. *The slow tubes.* First define tubes in (u_1, v_1) space, $\tilde{\mathcal{Y}}_{\mathcal{L}}$ and $\tilde{\mathcal{Y}}_{\mathcal{R}}$, which extend beyond the slow singular limit, as shown in Figure 7:

$$\tilde{\mathcal{Y}}_{\mathcal{L}}(\delta_2) = \bigcup_{u_1 \in [p_4, p_1]} \{u_1\} \times \mathcal{I}_{\mathcal{L}}(\delta_2; u_1),$$

$$\tilde{\mathcal{Y}}_{\mathcal{R}}(\delta_2) = \bigcup_{u_1 \in [p_4, p_1]} \{u_1\} \times \mathcal{I}_{\mathcal{R}}(\delta_2; u_1).$$

Here $\mathcal{I}_{\mathcal{L}}(\delta_2; u_1)$ denotes an interval in v_1 of length $2\delta_2$ centered at the point $(u_1, \Gamma_{\mathcal{L}}(u_1))$ on the unstable manifold of $(p_2, 0)$, and $\mathcal{I}_{\mathcal{R}}$ is an interval centered at $(u_1, \Gamma_{\mathcal{R}}(u_1))$ on the stable manifold of $(p_3, 0)$.

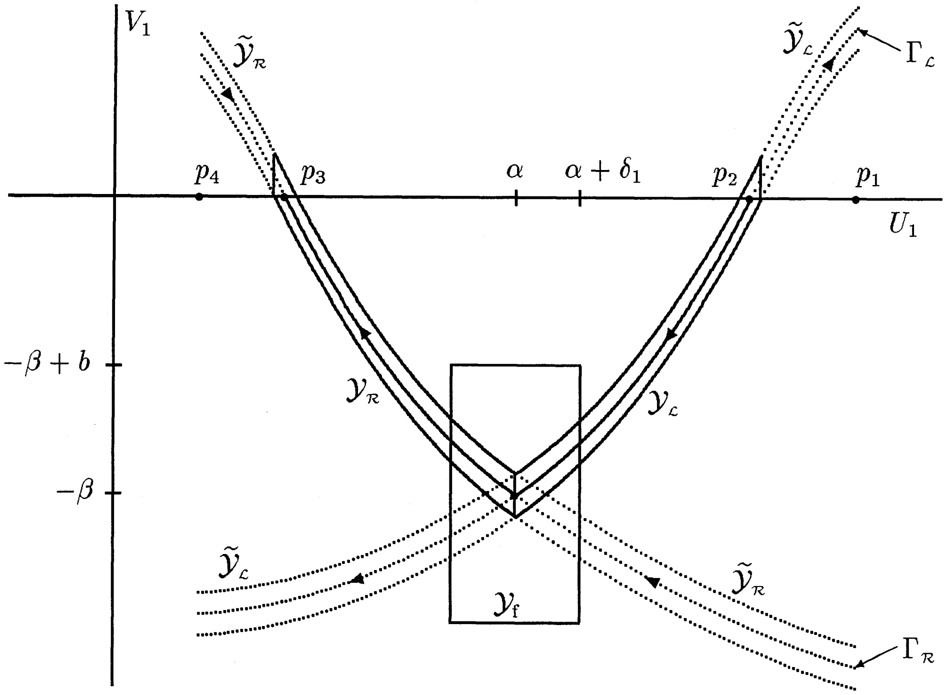

FIGURE 7. $\mathcal{Y}_{\mathrm{f}}$, $\mathcal{Y}_{\mathcal{L}}$, $\mathcal{Y}_{\mathcal{R}}$: isolating neighborhoods for the slow components.

For each fixed u_1, define the sets in $\mathbb{R}^4$, $\mathcal{Z}_{\mathcal{L}} = \mathcal{U}_{\mathcal{L}} \times \mathcal{V}_{\mathrm{s}}$ and $\mathcal{Z}_{\mathcal{R}} = \mathcal{U}_{\mathcal{R}} \times \mathcal{V}_{\mathrm{s}}$, such that the set $\{u_1\} \times \mathcal{I}_{\mathcal{L}} \times \mathcal{Z}_{\mathcal{L}}$ is a neighborhood in $\mathbb{R}^5$ centered on the slow singular limit in $\mathcal{M}_{\mathcal{L}}$, and $\{u_1\} \times \mathcal{I}_{\mathcal{R}} \times \mathcal{Z}_{\mathcal{R}}$ is a neighborhood centered on the slow singular limit in $\mathcal{M}_{\mathcal{R}}$. Here $\mathcal{V}_{\mathrm{s}}(\delta_3)$ is a square in (v_2, v_3) space centered at the origin with sides of length $2\delta_3$, and $\mathcal{U}_{\mathcal{L}}$ and $\mathcal{U}_{\mathcal{R}}$ are defined by

$$\mathcal{U}_{\mathcal{L}} = \mathcal{U}_{\mathcal{L}}(\delta_3; u_1) = \{\ (u_2, u_3) : |u_2 - \mathcal{L}_2(u_1)| \le a_{\mathcal{L}}\delta_3;\ |u_3| \le \delta_3\ \},$$
$$\mathcal{U}_{\mathcal{R}} = \mathcal{U}_{\mathcal{R}}(\delta_3; u_1) = \{\ (u_2, u_3) : |u_2| \le \delta_3;\ |u_3 - \mathcal{R}_3(u_1)| \le a_{\mathcal{R}}\delta_3\ \}.$$

The parameter δ_3 is chosen small relative to the parameter δ_{f}, so that $\mathcal{U}_{\mathcal{L}}(\delta_3; u_1)$ lies interior to $\mathcal{U}_{\mathrm{f}\mathcal{L}}(\delta_{\mathrm{f}})$, and $\mathcal{U}_{\mathcal{R}}$ lies interior to $\mathcal{U}_{\mathrm{f}\mathcal{R}}$.

Define two *slow tubes* in $\mathbb{R}^6$ containing the slow singular limit by letting u_1 range over an appropriately chosen interval:

$$N_{\mathcal{L}} = N_{\mathcal{L}}(\delta) = \bigcup_{u_1 \in [\alpha, \sigma_{\mathcal{L}}]} \{u_1\} \times \mathcal{I}_{\mathcal{L}}(\delta_2; u_1) \times \mathcal{Z}_{\mathcal{L}}(\delta_3; u_1),$$
$$N_{\mathcal{R}} = N_{\mathcal{R}}(\delta) = \bigcup_{u_1 \in [\sigma_{\mathcal{R}}, \alpha]} \{u_1\} \times \mathcal{I}_{\mathcal{R}}(\delta_2; u_1) \times \mathcal{Z}_{\mathcal{R}}(\delta_3; u_1) \tag{3.8}$$

where $\sigma_{\mathcal{L}}$ and $\sigma_{\mathcal{R}}$ are defined in Figure 8. The projection of these tubes in (u_1, v_1) space are denoted by $\mathcal{Y}_{\mathcal{L}}(\delta_2)$ and $\mathcal{Y}_{\mathcal{R}}(\delta_2)$, respectively, with $\mathcal{Y}_{\mathcal{L}} \subset \tilde{\mathcal{Y}}_{\mathcal{L}}$ and $\mathcal{Y}_{\mathcal{R}} \subset \tilde{\mathcal{Y}}_{\mathcal{R}}$. The ends of the tubes containing the rest points P_2 and P_3 are denoted by $N_{\mathcal{L}}^-$ and $N_{\mathcal{R}}^+$, respectively, and their projections in (u_1, v_1) space are denoted by $\mathcal{Y}_{\mathcal{L}}^-$ and $\mathcal{Y}_{\mathcal{R}}^+$ and appear as the shaded regions in Figure 8. The monotonicity of u_1 ensures that

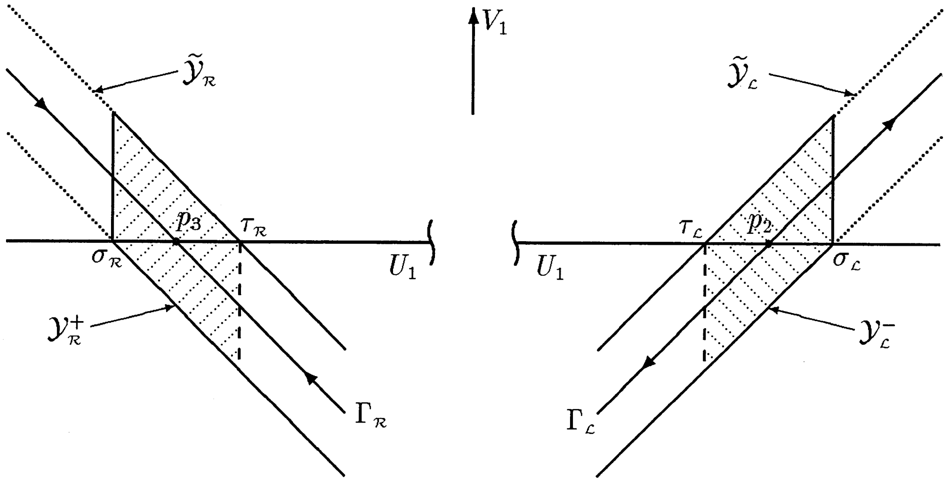

FIGURE 8. $\mathcal{Y}^-_{\mathcal{L}}$, $\mathcal{Y}^+_{\mathcal{R}}$: the slow tubes near the rest points.

solutions in $S(N_1;\varepsilon)$ end up in $N^-_{\mathcal{L}}$ as $\xi \to -\infty$ and in $N^+_{\mathcal{R}}$ as $\xi \to +\infty$. The neighborhoods $N^-_{\mathcal{L}}$ and $N^+_{\mathcal{R}}$ will be chosen small enough to isolate the rest points P_2 and P_3 for all sufficiently small $\varepsilon > 0$.

3.4.4. *The neighborhood* $N_0(\delta)$. Let $\mathcal{Y}_0$ be a square in (u_1, v_1) space centered at $(\alpha, 0)$, with sides of length $2\delta_0$, and define $N_0(\delta) = \mathcal{Y}_0(\delta_0) \times \mathcal{Z}_{\mathrm{f}}$. Here $\mathcal{Z}_{\mathrm{f}}$ is the isolating region defined in section 3.4.2, relative to the fast reduced flow with $u_1 = \alpha$. For the flow at $\mu = 0$ in (3.4), we have $\alpha = 1$ and a four-dimensional invariant manifold $\mathcal{M}_0$ where the flow in $\mathcal{M}_0$ is the fast reduced flow with $u_1 = 1$. Recall that $S_0(N;\varepsilon)$ denotes the maximal invariant set in $N \subset \mathbb{R}^6$ with respect to the flow in (3.4) and $S_{\mathrm{f}}(\mathcal{Z};1)$ denotes the maximal invariant set in $\mathcal{Z} \subset \mathbb{R}^4$ relative to the fast reduced flow with $u_1 = 1$. The following lemma shows that for δ_0 sufficiently small, $\mathcal{M}_0$ contains the maximal invariant set of $N_0(\delta)$, relative to the flow in (3.4).

Lemma 3.4. *Let $\mathcal{Z}$ be any compact set in $\mathbb{R}^4$ and $S_f(\mathcal{Z};1)$ the maximal invariant set in $\mathcal{Z}$ relative to the fast reduced flow with $u_1 = 1$. Then for all $\delta_0 < 1/2$ and all $\varepsilon \in (0,1]$, the total invariant set of $\mathcal{Y}_0(\delta_0) \times \mathcal{Z}$ relative to* (3.4) *is given by,*

$$S_0(\mathcal{Y}_0 \times \mathcal{Z};\varepsilon) = \{(1,0)\} \times S_f(\mathcal{Z};1).$$

Proof. Let $\varphi(\xi)$ be a solution in $S_0(\mathcal{Y}_0 \times \mathcal{Z};\varepsilon)$. By the invariance of the manifold $\mathcal{M}_0$ it is clear that

$$S_0(\mathcal{Y}_0 \times \mathcal{Z};\varepsilon) \cap \mathcal{M}_0 = \{(1,0)\} \times S_{\mathrm{f}}(\mathcal{Z};1), \tag{3.9}$$

and if $(u_1, v_1) = (1,0)$ at any point on the solution φ, then $\varphi(\xi)$ must lie in $\mathcal{M}_0$ for all ξ. Therefore, fix $\delta_0 > 0$ and assume there is a curve $\varphi(\xi) \in S_0(\mathcal{Y}_0 \times \mathcal{Z};\varepsilon)$ with $\varphi(\xi) \notin \mathcal{M}_0$ for all ξ. There is a largest $\bar{\delta} \le \delta_0$ such that the curve φ touches the boundary $\partial\mathcal{Y}_0(\bar{\delta}) \times \mathcal{Z}$. If there is such a curve φ, the flow at this point must necessarily have an *internal tangency* relative to the neighborhood $\mathcal{Y}_0(\bar{\delta}) \times \mathcal{Z}$. The idea is to show that for $\bar{\delta} < 1/2$, the differential equations imply that such a point

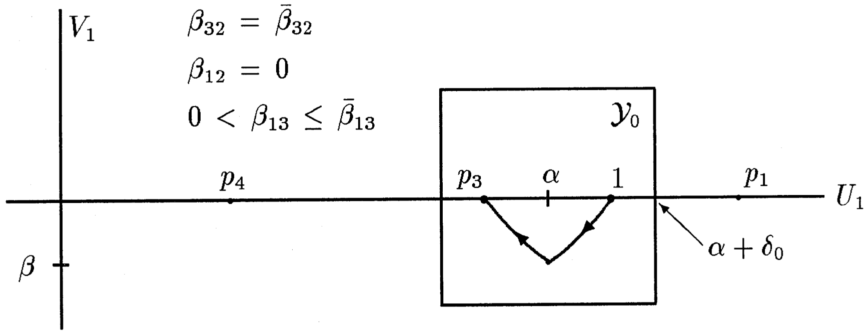

FIGURE 9. $\mathcal{Y}_0$: the isolating neighborhood at $\mu = \bar\mu$.

has an *external tangency.* Then, for $\delta_0 < 1/2$, $S_0(\mathcal{Y}_0 \times \mathcal{Z};\varepsilon)$ must lie in $\mathcal{M}_0$, and it follows from (3.9) that $S_0(\mathcal{Y}_0 \times \mathcal{Z};\varepsilon) = \{(1,0)\} \times S_{\mathrm{f}}(\mathcal{Z};1)$.

To add some generalization to the lemma for later use, express the equation for v_1' as a perturbation about $u_1 = 1$ and multiply the v_1 and $(u_1-1)^2$ terms by a parameter λ:

$$v_1' = \varepsilon[-\lambda\varepsilon\theta v_1 + (u_1 - 1) + \lambda(u_1-1)^2].$$

Later, in completing the continuation, we cite the fact that this lemma holds for all $\lambda \in [0,1]$. To prove the lemma, two possibilities must be considered.

1. Suppose that $v_1 = \pm\bar\delta$ and $v_1' = 0$. Then $v_1'' = \varepsilon^2 v_1[1 + 2\lambda(u_1-1)]$ and for all $\bar\delta < 1/2$ and $\lambda \in [0,1]$, $\mathrm{sgn}(v_1'') = \mathrm{sgn}(v_1)$. It follows that the flow at such a point has an external tangency.

2. Suppose $u_1 = 1 \pm \bar\delta$ and $u_1' = 0$. Then $u_1'' = \varepsilon v_1' = \varepsilon^2(\pm\bar\delta)(1 \pm \lambda\bar\delta)$. If $\bar\delta < 1$, then $u_1'' < 0$ for $u_1 = 1 - \bar\delta$, and $u_1'' > 0$ for $u_1 = 1 + \bar\delta$, implying that the flow at these points has an external tangency. □

It follows that for the flow with $\mu = 0$, $N_0^- = \mathcal{Y}_0 \times \mathcal{Z}_{\mathrm{f}\mathcal{L}}$ and $N_0^+ = \mathcal{Y}_0 \times \mathcal{Z}_{\mathrm{f}\mathcal{R}}$ isolate the rest points at P_2 and P_3, respectively, and $N_0 = \mathcal{Y}_0 \times \mathcal{Z}_{\mathrm{f}}$ is an isolating neighborhood in $\mathbb{R}^6$ with $S_0(N_0;\varepsilon)$ consisting only of connections from P_2 to P_3.

Now consider the continuation near $\mu = 0$ with $0 \le \beta_{13} \le \bar\beta_{13}$, $\beta_{12} = 0$, and $\beta_{32} = \bar\beta_{32}$. This is just an $O(\beta_{13})$ regular perturbation of the flow in (3.4). For $\bar\beta_{13}$ sufficiently small, the neighborhoods N_0^-, N_0^+, and N_0 remain isolating relative to the flow in (1.5) for all $\varepsilon \in (0,1]$. Moreover, we take δ_0 and $\bar\beta_{13}$ sufficiently small that $S(N_0^-;\varepsilon) = P_2$, $S(N_0^+;\varepsilon) = P_3$, and for all $(u_1,v_1) \in \mathcal{Y}_0$, $\mathcal{Z}_{\mathrm{f}}$ is isolating for the fast reduced flow, with $S_{\mathrm{f}}(\mathcal{Z}_{\mathrm{f}};u_1)$ consisting only of connections from $\mathcal{L}(u_1)$ to $\mathcal{R}(u_1)$. In particular, we can assume that the point $g_2(u) = g_3(u) = 0$ remains excised from $\mathcal{Z}_{\mathrm{f}}$ for all $y \in \mathcal{Y}_0$. Figure 9 shows the slow singular limit in (u_1,v_1) space relative to the neighborhood $\mathcal{Y}_0(\delta_0)$ at $\mu = \bar\mu$, where the notation $\bar\mu$ is used to denote the point in the homotopy where $\beta_{32} = \bar\beta_{32}$, $\beta_{12} = 0$, and $\beta_{13} = \bar\beta_{13}$.

3.4.5. *Specifying parameters.* Fix δ_c, δ_{f}, and δ_0 from the previous lemmas and take $0 < \varepsilon \le \bar\varepsilon$ satisfying Lemma 3.2. We now restrict the remaining parameters, and it will be clear that the following conditions can be satisfied by first fixing δ_1, then δ_2, and finally δ_3. These conditions will be cited later in proving the *a priori* estimates and isolating properties of N_1 and N_0.

(C1) First restrict b, δ_1, δ_2, and δ_3 so that $N_1(\delta) \subset N_0(\delta)$ at $\mu = \bar{\mu}$. For b, δ_1, and δ_2 sufficiently small, we have $\mathcal{Y}_{\mathrm{f}}(\delta_1) \cup \mathcal{Y}_{\mathcal{L}}(\delta_2) \cup \mathcal{Y}_{\mathcal{R}}(\delta_2) \subset \mathcal{Y}_0(\delta_0)$. For δ_3 sufficiently small, $\mathcal{Z}_{\mathcal{L}}(\delta_3; u_1) \subset \mathcal{Z}_{\mathrm{f}\mathcal{L}}$ and $\mathcal{Z}_{\mathcal{R}}(\delta_3; u_1) \subset \mathcal{Z}_{\mathrm{f}\mathcal{R}}$ for all $(u_1, v_1) \in \mathcal{Y}_0$, and it follows that $N_1 \subset N_0$ at $\mu = \bar{\mu}$.

(C2) We use δ_1 to control the approximation by the fast reduced flow. In the fast scaling (1.5), the flow in the fast variables is an $O(\varepsilon|u_1 - \alpha|)$ regular perturbation of the fast reduced flow at $u_1 = \alpha$. Let ξ_{f} be the maximal exit time for all points in the boundary $\partial \mathcal{Z}_{\mathrm{f}}$ relative to this fast reduced flow. Choose δ_1 sufficiently small so that if $z(0; \varepsilon) \in \partial \mathcal{Z}_{\mathrm{f}}$ and $|u_1(\xi; \varepsilon) - \alpha| \leq 2\delta_1$ for all $|\xi| \leq \xi_{\mathrm{f}}$, then $z(\xi; \varepsilon)$ lies outside $\mathcal{Z}_{\mathrm{f}}$ relative to the flow in (1.5) for some $|\xi| \leq \xi_{\mathrm{f}}$.

(C3) Use δ_1 and δ_2 to ensure that the slow tubes $\tilde{\mathcal{Y}}_{\mathcal{L}} \setminus \mathcal{Y}_{\mathrm{f}}$ and $\tilde{\mathcal{Y}}_{\mathcal{R}} \setminus \mathcal{Y}_{\mathrm{f}}$ are uniformly bounded away from each other in (u_1, v_1) space (see Figure 7). For δ_1 sufficiently small, the curves $\Gamma_{\mathcal{L}}$ and $\Gamma_{\mathcal{R}}$ intersect the boundary of $\mathcal{Y}_{\mathrm{f}}(\delta_1)$ along the sides $u_1 = \alpha \pm \delta_1$. Now choose δ_2 sufficiently small so that if y is not in $\mathcal{Y}_{\mathrm{f}}$ then y must be bounded away from at least one of the slow tubes, that is, $\mathrm{dist}(y, \tilde{\mathcal{Y}}_{\mathcal{L}}) > \delta_2$ or $\mathrm{dist}(y, \tilde{\mathcal{Y}}_{\mathcal{R}}) > \delta_2$.

(C4) Use δ_1 and δ_3 to minimize the type of boundary points where the slow tubes join the fast block. Make δ_1 sufficiently small so that for $|u_1 - \alpha| \leq 2\delta_1$, $\mathcal{L}(u_1)$ is interior to $\mathcal{Z}_{\mathrm{f}\mathcal{L}}$ and $\mathcal{R}(u_1)$ is interior to $\mathcal{Z}_{\mathrm{f}\mathcal{R}}$. Then for δ_3 sufficiently small and $(u_1, v_1) \in \mathcal{Y}_{\mathrm{f}}$, the set $\mathcal{Z}_{\mathcal{L}}(\delta_3; u_1)$ lies interior to $\mathcal{Z}_{\mathrm{f}\mathcal{L}}$ and $\mathcal{Z}_{\mathcal{R}}(\delta_3; u_1)$ remains interior to $\mathcal{Z}_{\mathrm{f}\mathcal{R}}$.

(C5) For δ_2 and δ_3 sufficiently small and $0 < \varepsilon \leq \bar{\varepsilon}$, it follows from Lemma 3.2 that $S(N_{\mathcal{L}}^-; \varepsilon) = P_2$ and $S(N_{\mathcal{R}}^+; \varepsilon) = P_3$.

(C6) We use δ_3 and $\bar{\varepsilon}$ to control the approximation to the reduced flow on the slow manifolds. Let $\mathcal{Y}_m$ denote a compact set in (u_1, v_1) space large enough to contain the sets $\mathcal{Y}_0$ and $\tilde{Y}_{\mathcal{L}} \cup \mathcal{Y}_{\mathrm{f}} \cup \tilde{Y}_{\mathcal{R}}$ (see Figure 7). For the reduced flow on the left slow manifold, points in $\mathcal{Y}_m \setminus \tilde{\mathcal{Y}}_{\mathcal{L}}$ exit $\mathcal{Y}_m$ in both time directions, points in $\tilde{\mathcal{Y}}_{\mathcal{L}} \setminus \mathcal{Y}_{\mathcal{L}}$ exit $\mathcal{Y}_m$ in forward time, and by compactness, we can define a maximal exit time $\zeta_{\mathrm{s}}(\delta_2)$. In the slow scaling (1.4), the slow components $Y(\zeta; \varepsilon)$ are uniformly approximated by the reduced flow on the left slow manifold, as long as the fast components $Z(\zeta; \varepsilon)$ stay uniformly close to $\mathcal{L}(Y(\zeta; \varepsilon)$ and ε is sufficiently small:

$$\dot{Y} = F\left(Y, \mathcal{L}(Y)\right) + O(|Z - \mathcal{L}(Y)|) - \varepsilon\theta V_1. \tag{3.10}$$

Choose δ_3 and $\bar{\varepsilon}$ sufficiently small such that if $Y(0; \varepsilon) \in \mathcal{Y}_m \setminus \tilde{\mathcal{Y}}_{\mathcal{L}}$ and $Z(\zeta; \varepsilon)$ stays in $\mathcal{Z}_{\mathcal{L}}(\delta_3; U_1(\zeta; \varepsilon))$ for $-\zeta_{\mathrm{s}} \leq \zeta \leq 0$, then $Y(\zeta; \varepsilon)$ exits $\mathcal{Y}_m$ for some ζ in the interval $[-\zeta_{\mathrm{s}}, 0]$. Similarly, if $Z(\zeta; \varepsilon)$ stays in $\mathcal{Z}_{\mathcal{L}}$ for $0 \leq \zeta \leq \zeta_{\mathrm{s}}$, then points in both $\mathcal{Y}_m \setminus \tilde{\mathcal{Y}}_{\mathcal{L}}$ and $\tilde{\mathcal{Y}}_{\mathcal{L}} \setminus \mathcal{Y}_{\mathcal{L}}$ should exit $\mathcal{Y}_m$ in the interval $[0, \zeta_{\mathrm{s}}]$. A similar approximation holds for the case where $Z(\zeta; \varepsilon)$ is in the neighborhood $\mathcal{Z}_{\mathcal{R}}(\delta_3; U_1(\zeta; \varepsilon))$ using approximation by the reduced flow on the right slow manifold.

3.5. *A priori* estimates. In this section, we establish *a priori* estimates for non-constant solutions in $S(N_0; \varepsilon)$ and $S(N_1; \varepsilon)$, relative to the flow in (1.4) and (1.5). Lemma 3.5 shows that the total invariant sets consist only of connecting solutions from P_2 and P_3, and Lemma 3.6 shows that any such connecting solution must be uniformly close to a singular limit solution.

Lemma 3.5. *For all ε sufficiently small, $S(N_1;\varepsilon)$ and $S(N_0;\varepsilon)$ consist only of the rest points P_2 and P_3 and connections from P_2 to P_3.*

Proof. For the neighborhood N_1, the lemma follows easily from the monotonicity of u_1 and the fact that $S(N_{\mathcal{L}}^-;\varepsilon) = P_2$ and $S(N_{\mathcal{R}}^+;\varepsilon) = P_3$. To prove the lemma for N_0 observe that solutions entering $\mathcal{U}_{f\mathcal{L}}$ in backward time remain there for all $\xi \to -\infty$ and solutions entering $\mathcal{U}_{f\mathcal{R}}$ in forward time remain there for all $\xi \to +\infty$. If a solution in $S(N_0;\varepsilon)$ stays outside $(\mathcal{U}_{f\mathcal{L}} \cup \mathcal{U}_{f\mathcal{R}})$ for all $\xi \to -\infty$, monotonicity in u_2 and u_3 implies that the fast variables approach a limit $(\bar{u}_2, 0, \bar{u}_3, 0)$. For β_{13} and ε sufficiently small, the flow for the slow variables will be asymptotic to a saddle point at $(1-\beta_{13}\bar{u}_3, 0)$, and, therefore, $u(\xi;\varepsilon)$ must be asymptotic to a rest point of the reaction flow. Two of these rest points, P_2 and P_3, lie interior to $\mathcal{U}_{f\mathcal{L}} \cup \mathcal{U}_{f\mathcal{R}}$ and the remaining two have been excised from N_0 for $\beta_{13} \in [0, \bar{\beta}_{13}]$. A similar argument in forward time shows that a solution in $S(N_0;\varepsilon)$ cannot stay outside of $\mathcal{U}_{f\mathcal{L}} \cup \mathcal{U}_{f\mathcal{R}}$ for all $\xi \to +\infty$. The lemma for N_0 follows since $\mathcal{Y}_0 \times \mathcal{Z}_{f\mathcal{L}}$ isolates P_2 and $\mathcal{Y}_0 \times \mathcal{Z}_{f\mathcal{R}}$ isolates P_3. □

From the previous lemma, we can parametrize nonconstant solutions in $S(N_1;\varepsilon)$ and $S(N_0;\varepsilon)$ to be bounded away from both slow manifolds at $\xi = 0$, for example, $u_2(0;\varepsilon) = \mathcal{L}_2(\alpha)/2$. The next lemma states that any such connecting solution lies close to the slow manifolds outside a compact ξ-interval where the ξ-interval can be chosen uniformly for $\varepsilon \to 0$. The notation $\mathcal{Z}^0$ denotes the interior of the set $\mathcal{Z}$.

Lemma 3.6. *Suppose $(x(\xi;\varepsilon), \theta(\varepsilon))$ is a nonconstant solution in the total invariant set $S(N_1;\varepsilon)$ or $S(N_0;\varepsilon)$, with $u_2(0;\varepsilon) = \mathcal{L}_2(\alpha)/2$. Given any $\delta_3 > 0$, there exists $\bar{\varepsilon} > 0$ and $\bar{\xi} > 0$ such that if $0 < \varepsilon \le \bar{\varepsilon}$, then $x(\xi;\varepsilon) = (y(\xi;\varepsilon), z(\xi;\varepsilon))$ satisfies*

$$\begin{aligned} z(\xi;\varepsilon) &\in \mathcal{Z}^0_{\mathcal{L}}(\delta_3; u_1(\xi;\varepsilon)) \text{ for all } \xi \le -\bar{\xi}, \\ z(\xi;\varepsilon) &\in \mathcal{Z}^0_{\mathcal{R}}(\delta_3; u_1(\xi;\varepsilon)) \text{ for all } \xi \ge +\bar{\xi}. \end{aligned} \tag{3.11}$$

Furthermore, the wave speed $\theta(\varepsilon)$ tends to $\Theta(\alpha)$ as $\varepsilon \to 0$.

Proof. First, work in the fast scaling where solutions to (1.5) can be approximated by solutions to the fast reduced flow, uniformly on compact ξ-intervals as $\varepsilon \to 0$. Consider a sequence of heteroclinic orbits $(x(\hat{\xi};\varepsilon_n), \theta(\varepsilon_n))$ in $S(N_1;\varepsilon_n)$ (resp. $S(N_0;\varepsilon_n)$), parameterized such that $u_2(0;\varepsilon_n) = \mathcal{L}_2(\alpha)/2$ where $\hat{\xi}$ is fixed and $\varepsilon_n \to 0$. Let $(\hat{x}, \hat{\theta})$ be a limit point and $x_r(\xi) = (\hat{y}, z_r(\xi))$ the solution through $\hat{x}$ for the fast reduced flow parametrized by $\hat{u}_1$ and $\hat{\theta}$. We first show that $(\hat{x}, \hat{\theta})$ lies on a connecting solution for the fast reduced problem with $\hat{\theta} = \Theta(\hat{u}_1)$. By uniform approximation on the interval $[0, \hat{\xi}]$, the point $\hat{x}$ satisfies $|\hat{u}_1 - \alpha| \le \delta_1$, and it follows from (C4) that $\hat{z} \in \mathcal{Z}_f$. If $(\hat{x}, \hat{\theta})$ does not lie on a connecting orbit for the fast reduced problem, $z_r(\xi^*)$ is outside $\mathcal{Z}_f$ at some finite ξ^*, and for ε_n sufficiently small, $z(\xi^*;\varepsilon_n)$ also lies outside $\mathcal{Z}_f$ with $|u_1(\xi^*;\varepsilon_n) - \alpha| \le 2\delta_1$. By condition (C4), $x(\xi^*;\varepsilon_n)$ must lie outside N_1 as $\varepsilon_n \to 0$. The same result holds for the neighborhood $N_0 = \mathcal{Y}_0 \times \mathcal{Z}_f$.

Let $z_r(\xi)$ be a solution to the fast reduced problem connecting $\mathcal{L}(u_1)$ to $\mathcal{R}(u_1)$ with wave speed $\Theta(u_1)$ and $u_{2r}(0) = \mathcal{L}_2(\alpha)/2$. Given $\delta_3 > 0$, there exists $\bar{\xi} > 0$ such that

$$\begin{aligned} z_r(\xi) &\in \mathcal{Z}^0_{\mathcal{L}}(\delta_3; u_1) \quad \text{for all } \xi \le -\bar{\xi}, \\ z_r(\xi) &\in \mathcal{Z}^0_{\mathcal{R}}(\delta_3; u_1) \quad \text{for all } \xi \ge \bar{\xi}. \end{aligned} \tag{3.12}$$

It follows that given any fixed $\xi \geq \bar{\xi}$ and ε sufficiently small, a connecting orbit $(x(\xi;\varepsilon), \theta(\varepsilon))$ must satisfy

$$z(-\xi;\varepsilon) \in \mathcal{Z}^0_{\mathcal{L}}(\delta_3; u_1(-\xi;\varepsilon)) \quad \text{and} \quad z(\xi;\varepsilon) \in \mathcal{Z}^0_{\mathcal{R}}(\delta_3; u_1(\xi;\varepsilon)). \tag{3.13}$$

It remains to show that there exists an $\bar{\varepsilon} > 0$ such that if $0 < \varepsilon \leq \bar{\varepsilon}$, then (3.13) holds for all $|\xi| \geq \bar{\xi}$. Suppose that the choice of $\bar{\varepsilon}$ does not hold uniformly for $\xi \to -\infty$. Then there exists a sequence $(x(\xi_n;\varepsilon_n), \theta(\varepsilon_n))$ of connections in $S(N_1;\varepsilon)$ (resp. $S(N_0;\varepsilon)$), with $\xi_n \to -\infty$, $\varepsilon_n \to 0$, and $z(\xi_n;\varepsilon_n) \notin \mathcal{Z}^0_{\mathcal{L}}(\delta_3; u_1(\xi_n;\varepsilon_n))$. Let $(\hat{x}, \hat{\theta})$ denote a limit point and $(x_r(\xi), \hat{\theta})$ the solution to the fast reduced flow with $x_r(0) = \hat{x}$. From the hyperbolicity of the slow manifold, $z_r(\xi^*)$ must lie outside $\mathcal{Z}_{\mathcal{L}}(\delta_3; \hat{u}_1)$ for some finite ξ^*, as must $z(\xi^*;\varepsilon_n)$ for ε_n sufficiently small. It follows that the limit point $\hat{x}$ is in the fast block $N_f = \mathcal{Y}_f \times \mathcal{Z}_f$ (resp. $N_0 = \mathcal{Y}_0 \times \mathcal{Z}_f$), and therefore, $\hat{x}$ lies on a connecting orbit to the fast reduced flow with wave speed $\hat{\theta} = \Theta(\hat{u}_1)$. It follows from (3.12) that this solution must lie in $\mathcal{Z}^0_{\mathcal{R}}(\delta_3, \hat{u}_1)$ for all $\xi \geq 2\bar{\xi}$ and, as $\varepsilon_n \to 0$, the connecting solutions to (1.5) also must lie near the right slow manifold at $\xi = \xi_n + 2\bar{\xi}$. From the choice of parametrization and the monotonicity away from the slow manifolds, $x(\xi;\varepsilon_n)$ can be near the right slow manifold only for $\xi > 0$. However, as $\varepsilon_n \to 0$, $\xi_n + 2\bar{\xi} \to -\infty$, yielding the necessary contradiction.

Having established the estimate for the location of the connecting solutions, we now show that the wave speed tends to $\Theta(\alpha)$ as $\varepsilon \to 0$. Consider a sequence of connecting solutions $(x(0;\varepsilon_n); \theta(\varepsilon_n))$ with $u_2(0;\varepsilon_n) = \mathcal{L}_2(\alpha)/2$, converging to $(\hat{x}, \hat{\theta})$ with $\hat{\theta} = \Theta(\hat{u}_1)$. Suppose $\hat{y} \neq (\alpha, -\beta)$ and, for definiteness, assume that $\hat{y}$ does not lie on the slow singular limit on $\mathcal{M}_{\mathcal{L}}$. The solution to the reduced flow on $\mathcal{M}_{\mathcal{L}}$ with $Y(0) = \hat{Y}$ will lie outside $\mathcal{Y}_{\mathcal{L}} \cup \mathcal{Y}_f \cup \mathcal{Y}_{\mathcal{R}}$ (resp. $\mathcal{Y}_0$) in finite time $-\zeta^* < 0$. By the previous estimate, $Y(\zeta;\varepsilon_n)$ is uniformly approximated by the reduced flow on $\mathcal{L}_{\mathcal{M}}$ on the half-line $\zeta \leq -\varepsilon_n\bar{\xi}$. As $\varepsilon_n \to 0$, $Y(-\varepsilon_n\bar{\xi};\varepsilon_n)$ tends to $\hat{Y}$, and by approximation to the slow reduced flow, $Y(-\varepsilon_n\bar{\xi} - \zeta^*;\varepsilon_n)$ lies outside $\mathcal{Y}_{\mathcal{L}} \cup \mathcal{Y}_f \cup \mathcal{Y}_{\mathcal{R}}$ (resp. $\mathcal{Y}_0$), contradicting $X(\zeta;\varepsilon_n) \in S(N_1;\varepsilon_n)$ (resp. $S(N_0;\varepsilon_n)$). □

Remark 1. In the following sections, we apply Lemma 3.6 as follows: if $x(\xi;\varepsilon)$ is any nonconstant solution in $S(N_1;\varepsilon)$ or $S(N_0;\varepsilon)$, then $x(\xi;\varepsilon)$ cannot stay uniformly bounded away from both of the slow manifolds for a time greater than $2\bar{\xi}$.

3.6. Isolating property of N_1 and N_0. It follows from Lemma 3.4 that the neighborhood $N_0(\delta)$ is known to be isolating for $0 \leq \mu \leq \bar{\mu}$ and $0 < \varepsilon \leq 1$. The following lemma establishes the necessary isolating properties for $N_1(\delta)$, relative to (1.4) and (1.5). The proof cites the restrictions on δ_1, δ_2, and δ_3 as described in Section 3.4.5 and the *a priori* estimates from Section 3.5.

Lemma 3.7. *Fix δ_c, δ_f, and δ_0 as specified in Sections* 3.4.1, 3.4.2, *and* 3.4.4, *and fix δ_1, δ_2, and δ_3 to satisfy conditions* (C1)–(C6) *as described in Section* 3.4.5. *Then there exists $\bar{\varepsilon} > 0$ such that for $0 < \varepsilon \leq \bar{\varepsilon}$, $N_1(\delta)$ is isolating relative to the flow in* (1.4) *and* (1.5). *That is, $S(N_1;\varepsilon) \cap \partial N_1 = \emptyset$, for all $\theta \in [\theta_1, \theta_2]$.*

Proof. Let $x^0 = (y^0, z^0)$ denote an arbitrary point in the boundary of the set N_1, and let $x(\xi;\varepsilon) = (y(\xi;\varepsilon), z(\xi;\varepsilon))$ be the solution to (1.5) satisfying $x(0;\varepsilon) = x^0$. We must show that for all sufficiently small $\varepsilon > 0$, the solution $x(\xi;\varepsilon)$ exits N_1 in at least one time direction.

1. $x^0 \in \partial N_{\mathcal{L}}$, $x^0 \notin N_f$

There are two cases to be considered, $y^0 \in \partial\mathcal{Y}_{\mathcal{L}}$ and $z^0 \in \partial\mathcal{Z}_{\mathcal{L}}$.

(a) Suppose $y^0 \in \partial\mathcal{Y}_{\mathcal{L}}(\delta_2)$ with $u_1^0 > \alpha + \delta_1$. If $u_1^0 = \sigma_{\mathcal{L}}$, the solution leaves immediately in forward time. If $v_1^0 = \Gamma_{\mathcal{L}}(u_1^0) \pm \delta_2$, then for $\xi < 0$, $z(\xi;\varepsilon) \in \mathcal{Z}_{\mathcal{L}}(\delta_3; u_1)$ as long as $x(\xi;\varepsilon)$ stays in N_1. Using the approximation by the slow reduced flow on $\mathcal{M}_{\mathcal{L}}$, it follows that $X(\zeta;\varepsilon)$ exits N_1 in finite negative time.

(b) Now suppose $z^0 \in \partial\mathcal{Z}_{\mathcal{L}}(\delta_3, u_1^0)$. From the estimate in Lemma 3.6, $z(\xi;\varepsilon)$ must stay in $\mathcal{Z}_{\mathcal{R}}(\delta_3, u_1(\xi;\varepsilon))$ for all $\xi \geq 2\bar{\xi}$. Because $y^0 \in \mathcal{Y}_{\mathcal{L}} \setminus \mathcal{Y}_{\mathrm{f}}$, it follows from (C3) that $y(2\bar{\xi};\varepsilon) \notin \tilde{\mathcal{Y}}_{\mathcal{R}}$ for all ε sufficiently small. From the approximation by the slow reduced flow on $\mathcal{M}_{\mathcal{R}}$, $X(\zeta;\varepsilon)$ exits N_1 in some time $\zeta \leq \varepsilon 2\bar{\xi} + \zeta_{\mathrm{s}}$.

2. $x^0 \in \partial N_{\mathcal{R}}$, $x^0 \notin N_{\mathrm{f}}$
The argument is similar to part 1, with a time reversal.

3. $x^0 \in \partial N_{\mathrm{f}}$, $x^0 \notin (N_{\mathcal{L}} \cup N_{\mathcal{R}})$
Again consider separately the cases $y^0 \in \partial\mathcal{Y}_{\mathrm{f}}$ and $z^0 \in \partial\mathcal{Z}_{\mathrm{f}}$.

(a) Suppose $y^0 \in \partial\mathcal{Y}_{\mathrm{f}}$. If $u_1^0 = \alpha \pm \delta_1$, then the solution $y(\xi;\varepsilon)$ exits $\mathcal{Y}_{\mathrm{f}}$ immediately in one time direction. Because x^0 does not lie in either of the slow tubes, the solution $x(\xi;\varepsilon)$ exits N_1 immediately. If $v_1^0 = -\beta \pm b$, then we can use the approximation by the slow reduced flow. The parameters b, δ_1, and δ_2 have been chosen such that y^0 is bounded away from both slow tubes (see Figure 7). If x^0 is in $S(N_1;\varepsilon)$, the estimate in Lemma 3.6 implies that at least one of the following statements will be true:

$$\text{(i)} \quad z(\xi;\varepsilon) \in \mathcal{Z}_{\mathcal{R}}(\delta_3; u_1(\xi;\varepsilon)), \qquad \text{for all } \xi \geq \bar{\xi},$$
$$\text{(ii)} \quad z(\xi;\varepsilon) \in \mathcal{Z}_{\mathcal{L}}(\delta_3; u_1(\xi;\varepsilon)), \qquad \text{for all } \xi \leq -\bar{\xi}.$$

In either case, we can choose ε sufficiently small so that $y(\pm\bar{\xi};\varepsilon)$ lies outside both $\tilde{\mathcal{Y}}_{\mathcal{L}}$ and $\tilde{\mathcal{Y}}_{\mathcal{R}}$. Using approximation by the slow reduced flow, the solution $X(\zeta;\varepsilon)$ must exit N_1 in some uniform time ζ_{s} contradicting $x^0 \in S(N_1;\varepsilon)$.

(b) Suppose $z^0 \in \partial\mathcal{Z}_{\mathrm{f}}$. Choose ε sufficiently small so that $|u_1(\xi;\varepsilon) - \alpha| \leq 2\delta_1$ for all $|\xi| \leq \xi_{\mathrm{f}}$, where ξ_{f} is the maximal exit time for points in $\partial\mathcal{Z}_{\mathrm{f}}$ relative to the fast reduced flow. From approximation by the fast reduced flow, $z(\xi;\varepsilon)$ also exits $\mathcal{Z}_{\mathrm{f}}$ in time $|\xi| \leq \xi_{\mathrm{f}}$. By condition (C4), we have $\mathcal{Z}_{\mathcal{L}}(\delta_3, u_1(\xi;\varepsilon)) \subset \mathcal{Z}_{\mathrm{f}}$ and $\mathcal{Z}_{\mathcal{R}}(\delta_3, u_1) \subset \mathcal{Z}_{\mathrm{f}}$ for all $|\xi| \leq \xi_{\mathrm{f}}$. It follows that when $z(\xi;\varepsilon)$ exits $\mathcal{Z}_{\mathrm{f}}$, the solution $x(\xi;\varepsilon)$ must lie outside the slow tubes and hence, outside N_1.

4. $x^0 \in \partial N_{\mathcal{L}} \cap \partial N_{\mathrm{f}}$
From condition (C4) all such points must have $y^0 \in \partial\mathcal{Y}_{\mathrm{f}}$ since for $(u_1, v_1) \in \mathcal{Y}_{\mathrm{f}}$, $\partial\mathcal{Z}_{\mathcal{L}}(\delta_3; u_1) \cap \partial\mathcal{Z}_{\mathrm{f}} = \emptyset$. That leaves just two cases to consider.

(a) Suppose $y^0 \in \partial\mathcal{Y}_{\mathrm{f}} \cap \partial\mathcal{Y}_{\mathcal{L}}$. This is the same as part 1-a with $v_1^0 = \Gamma_{\mathcal{L}}(u_1^0) \pm \delta_2$, and the solution exits N_1 in backward time.

(b) Suppose $y^0 \in \partial\mathcal{Y}_{\mathrm{f}} \cap \mathcal{Y}_{\mathcal{L}}$ and $z^0 \in \partial\mathcal{Z}_{\mathcal{L}}(\delta_3; u_1^0)$. Because y^0 is bounded away from $\tilde{\mathcal{Y}}_{\mathcal{R}}$, the argument proceeds as in part 1(b).

5. $x^0 \in \partial N_{\mathcal{R}} \cap \partial N_{\mathrm{f}}$
The argument is similar to part 4, with a time reversal. □

3.7. Continuation from N_1 to N_0. For $0 \le \mu \le \bar{\mu}$, the connection triples associated with N_0 are related by continuation, and similarly, for $\bar{\mu} \le \mu \le 1$, the connection triples associated with N_1 are related by continuation. It remains to be shown that for $\mu = \bar{\mu}$, the sets N_0 and N_1 determine the same connection triple.

Lemma 3.8. *Given $\mu = \bar{\mu}$ as specified following Lemma 3.4, there exists $\bar{\varepsilon} > 0$ such that for all ε satisfying $0 < \varepsilon \le \bar{\varepsilon}$, $S_{\bar{\mu}}(N_0; \varepsilon) = S_{\bar{\mu}}(N_1; \varepsilon)$.*

Proof. From condition (C1), we have $N_1 \subset N_0$, and hence, $S(N_1;\varepsilon) \subset S(N_0;\varepsilon)$ for all $\varepsilon > 0$. To get the inclusion in the other direction, we must show that for points $x^0 \in N_0 \setminus N_1$, the solution through x^0 exits N_0 in at least one time direction. First observe that because $N_0 = \mathcal{Y}_0 \times \mathcal{Z}_{\mathrm{f}}$, it follows that $(y,z) \in N_0 \setminus N_1$ implies $y \in \mathcal{Y}_0 \setminus \mathcal{Y}_{\mathrm{f}}$. Again we let $x(\xi;\varepsilon) = (y(\xi;\varepsilon), z(\xi;\varepsilon))$ denote the solution to (1.5) satisfying $x(0;\varepsilon) = x^0$. We assume $x^0 \in S(N_0;\varepsilon)$ and show that this leads to a contradiction. There are several cases to consider depending on the location of $y^0 \in \mathcal{Y}_0 \setminus \mathcal{Y}_{\mathrm{f}}$.

1. $y^0 \in \mathcal{Y}_0 \setminus (\tilde{\mathcal{Y}}_{\mathcal{L}} \cup \mathcal{Y}_{\mathrm{f}} \cup \tilde{\mathcal{Y}}_{\mathcal{R}})$

From condition (C3), y^0 be must bounded away from at least one of the slow tubes $\tilde{\mathcal{Y}}_{\mathcal{L}}$ or $\tilde{\mathcal{Y}}_{\mathcal{R}}$. For definiteness, suppose $\mathrm{dist}(y^0, \tilde{\mathcal{Y}}_{\mathcal{L}}) > \delta_2$. From the estimate in Lemma 3.6, at least one of the following will be true:

$$\text{(i)}\quad z(\xi;\varepsilon) \in \mathcal{Z}_{\mathcal{L}}(\delta_3; u_1(\xi;\varepsilon)), \qquad \text{for all } \xi \le -2\,\bar{\xi},$$

$$\text{(ii)}\quad z(\xi;\varepsilon) \in \mathcal{Z}_{\mathcal{R}}(\delta_3; u_1(\xi;\varepsilon)), \qquad \text{for all } \xi \ge 0.$$

If (i) is true, then $y(-2\,\bar{\xi};\varepsilon) \notin \tilde{\mathcal{Y}}_{\mathcal{L}}$ for all sufficiently small $\varepsilon > 0$, and the solution exits N_0 in backwards time by approximation to the slow reduced flow on $\mathcal{M}_{\mathcal{L}}$, contradicting $x^0 \in S(N_0;\varepsilon)$. If (ii) is true, then the solution exits N_0 in forward time by approximation to the slow reduced flow on $\mathcal{M}_{\mathcal{R}}$. A similar argument holds for the case where $\mathrm{dist}(y^0, \tilde{\mathcal{Y}}_{\mathcal{R}}) > \delta_2$.

2. $y^0 \in \mathcal{Y}_{\mathcal{L}} \setminus \mathcal{Y}_{\mathrm{f}}$

Since (y^0, z^0) lies outside N_1, it follows that $z^0 \notin \mathcal{Z}_{\mathcal{L}}(\delta_3; u_1^0)$, and therefore, $z(\xi;\varepsilon)$ must be in $\mathcal{Z}_{\mathcal{R}}(\delta_3, u_1(\xi;\varepsilon))$ for all $\xi \ge 2\,\bar{\xi}$. For all ε sufficiently small, $y(2\bar{\xi};\varepsilon)$ lies outside $\tilde{\mathcal{Y}}_{\mathcal{R}}$, and from the approximation by the slow reduced flow on $\mathcal{M}_{\mathcal{R}}$, this solution must exit N_0 in time $\zeta \le \varepsilon 2\bar{\xi} + \zeta\mathrm{s}$.

3. $y^0 \in \mathcal{Y}_{\mathcal{R}} \setminus \mathcal{Y}_{\mathrm{f}}$

The argument is similar to part 2, with a time reversal.

4. $y^0 \in \tilde{\mathcal{Y}}_{\mathcal{L}} \setminus (\mathcal{Y}_{\mathcal{L}} \cup \mathcal{Y}_{\mathrm{f}})$

Applying the estimate in Lemma 3.6, at least one of the following will be true,

$$\text{(i)}\quad z(\xi;\varepsilon) \in \mathcal{Z}_{\mathcal{L}}(\delta_3; u_1(\xi;\varepsilon)), \qquad \text{for all } \xi \le 0,$$

$$\text{(ii)}\quad z(\xi;\varepsilon) \in \mathcal{Z}_{\mathcal{R}}(\delta_3; u_1(\xi;\varepsilon)), \qquad \text{for all } \xi \ge 2\,\bar{\xi}.$$

If (ii) is true, then $y(2\,\bar{\xi};\varepsilon) \notin \tilde{\mathcal{Y}}_{\mathcal{R}}$ for all sufficiently small ε, and the solution exits N_0 in forward time by approximation to the slow reduced flow on $\mathcal{M}_{\mathcal{R}}$. If (i) is true, let $\zeta_* \ge 0$ denote the first time at which $Z(\zeta_*;\varepsilon) \in \partial\mathcal{Z}_{\mathcal{L}}$. Because the flow is well approximated by the slow reduced flow on $\mathcal{M}_{\mathcal{L}}$ for $0 \le \zeta \le \zeta_*$, we can assume $Y(\zeta_*;\varepsilon) \notin \mathcal{Y}_{\mathcal{L}} \cup \mathcal{Y}_{\mathrm{f}}$. If $Y(\zeta_*;\varepsilon) \notin \tilde{\mathcal{Y}}_{\mathcal{L}}$, then the solution must exit N_0 at some time $\zeta \le \zeta_*$ because of the approximation by the slow reduced flow on $\mathcal{M}_{\mathcal{L}}$ for $\zeta \le \zeta_*$. If $Y(\zeta_*;\varepsilon) \in \tilde{\mathcal{Y}}_{\mathcal{L}}$, then the solution must satisfy $Z(\zeta;\varepsilon) \in \mathcal{Z}_{\mathcal{R}}(\delta_3; U_1(\zeta;\varepsilon))$ for all $\zeta \ge$

$\zeta_* + \varepsilon 2\bar{\xi}$ with $Y(\zeta_* + \varepsilon 2\bar{\xi}; \varepsilon) \notin \tilde{\mathcal{Y}}_{\mathcal{R}}$. From the approximation by the slow reduced flow on $\mathcal{M}_{\mathcal{R}}$ for $\zeta \geq \zeta_* + \varepsilon 2\bar{\xi}$, $X(\zeta; \varepsilon)$ exits N_0 in forward time, contradicting $x^0 \in S(N_0; \varepsilon)$.

5. $y^0 \in \tilde{\mathcal{Y}}_{\mathcal{R}} \setminus (\mathcal{Y}_{\mathcal{R}} \cup \mathcal{Y}_{\mathrm{f}})$

The argument is similar to part 4, with a time reversal. □

3.8. Computing the index. We have shown that N_1 determines a connection triple (S_1', S_1'', S_1) relative to the flow at $\mu = 1$, N_0 determines a connection triple (S_0', S_0'', S_0) relative to the flow at $\mu = 0$, and their indices are the same. Introduce a continuation parameter $\lambda \in [0,1]$ into the flow at $\mu = 0$, with $\lambda = 1$ corresponding to the flow in (3.4):

$$\begin{aligned}
u_1' &= \varepsilon v_1, \\
v_1' &= \varepsilon[-\lambda\varepsilon\theta v_1 + (u_1 - 1) + \lambda(u_1 - 1)^2], \\
u_2' &= v_2, \\
v_2' &= -\theta v_2 - r_2 u_2 g_2(1, u_2, u_3) + \lambda O(\varepsilon|u_1 - 1|), \\
u_3' &= v_3, \\
v_3' &= d^{-1}[-\theta v_3 - r_3 u_3 g_3(1, u_2, u_3) + \lambda O(\varepsilon|u_1 - 1|).
\end{aligned} \tag{3.14}$$

It was shown in Lemma 3.4 that N_0 is isolating relative to the flow in (3.14) for all $\lambda \in [0,1]$. At $\lambda = 0$, the system decouples into a product flow, a constant coefficient linear flow on $\mathbb{R}^2$ and a parametrized flow on $\mathbb{R}^4 \times [\theta_1, \theta_2]$. Consider the triple of invariant sets in $\mathcal{R}^4$, $\tilde{S}_\theta' = \mathcal{L}(1)$, $\tilde{S}_\theta'' = \mathcal{R}(1)$, and $\tilde{S}_\theta = S_{\mathrm{f}}(\mathcal{Z}_{\mathrm{f}}; 1)$. These define a connection triple for the flow on $\mathbb{R}^4 \times [\theta_1, \theta_2]$ with index $\bar{h}(\tilde{S}', \tilde{S}'', \tilde{S}) = \bar{0}$, as shown in [5]. For the flow in the slow variables, $\mathcal{Y}_0$ isolates the saddle point at $(1,0)$ with index Σ^1. Applying the formula in (3.3) to compute the index,

$$\bar{h}(S_1', S_1'', S_1) = \Sigma^1 \wedge \bar{h}(\tilde{S}', \tilde{S}'', \tilde{S}) = \Sigma^1 \wedge \bar{0} = \bar{0}.$$

The indices for the isolated rest points are $h(P_2) = h(P_3) = \Sigma^3$, and since

$$\big(h(S_1') \wedge \Sigma^1\big) \vee h(S_1'') = \Sigma^4 \vee \Sigma^3 \neq \bar{0} = \bar{h}(S_1', S_1'', S_1),$$

it follows from (3.1) that $S_1 = S_1(N_1 \times [\theta_1, \theta_2]) \neq S_1' \cup S_1''$. By the *a priori* estimate in Lemma 3.5, there must exist a connection from P_2 to P_3 for some wave speed $\theta(\varepsilon) \in (\theta_1, \theta_2)$.

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