Methods and Applications of Analysis
4 (3) 1997, pp. 283–310

ISSN 1073–2772

A RATIONAL MOMENT PROBLEM ON THE UNIT CIRCLE

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ABSTRACT. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of not necessarily distinct points on the complex unit circle. We consider the moment problem: find a positive measure on $[-\pi,\pi]$ such that for $\omega_0 = 1$ and $\omega_n(z) = (z-\alpha_1)\cdots(z-\alpha_n)$, $n = 1, 2, \ldots$, we have

$$\int_{-\pi}^{\pi} d\mu(\theta) = 1, \quad \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})} = \mu_n, \quad n = 1, 2, \ldots,$$

for a given sequence of moments $\{\mu_n\}_{n=0}^{\infty}$. This paper gives results which to some extent generalise the limit point – limit circle situation of classical moment problems.

1. Introduction

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the complex unit circle, and let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of not necessarily distinct points on $\mathbb{T} \setminus \{1\}$. Introduction of the "forbidden" point 1 is not a severe restriction because there is only a countable number of α_k's so that there always exists such a point on $\mathbb{T}$, which by a simple rotation can be brought to the position 1. Define $\omega_n(z) = \prod_{k=1}^{n}(z-\alpha_k)$ for $n \geq 1$ and set $\omega_0 = 1$. By Π_n we denote the set of polynomials of degree at most n. We consider the spaces

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\omega_n(z)} : p_n \in \Pi_n \right\}, \quad \mathcal{L} = \bigcup_{k=0}^{\infty} \mathcal{L}_k.$$

For any complex function f, let $f_*(z) = \overline{f(1/\overline{z})}$. It is obvious that $\mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\} = \mathcal{L}_n$ and, similarly, $\mathcal{L}_* = \mathcal{L}$. Finally, we set $\mathcal{R}_n = \mathcal{L}_n \cdot \mathcal{L}_n$ and $\mathcal{R} = \mathcal{L} \cdot \mathcal{L}$. Let M be a linear functional defined on $\mathcal{L} \cdot \mathcal{L}$ which is real and positive, i.e., which satisfies

$$M\{f_*\} = \overline{M\{f\}}, \quad f \in \mathcal{L} \cdot \mathcal{L} \quad \text{and} \quad M\{ff_*\} > 0, \quad 0 \neq f \in \mathcal{L}.$$

This functional defines an inner product by

$$\langle f, g \rangle = M\{fg_*\}.$$

An example of such a linear functional is given by

$$M\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta), \quad \langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\mu(\theta)$$

where μ is a positive measure on $[-\pi,\pi]$. The subject of this paper, just as in Section 8 of [3], is to solve the following moment problem: Given the moments $\{\mu_n\}_{n=0}^{\infty}$ defined

Received October 4, 1996, revised February 13, 1997.
1991 *Mathematics Subject Classification*: 30E05, 42C10, 42A99, 47A57.
Key words and phrases: general moment problem, orthogonal rational functions, nested disks.

by

$$M\{1\} = \mu_0 \quad \text{and} \quad M\left\{\frac{1}{\omega_n}\right\} = \mu_n, \quad n = 1, 2, \ldots$$

(without loss of generality we may assume that $\mu_0 = 1$), does there exist a finite and positive measure μ on $[-\pi, \pi]$ such that

$$M\left\{\frac{1}{\omega_n}\right\} = \mu_n = \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})}, \quad n = 0, 1, \ldots.$$

It was shown in [3] that this moment problem always has at least one solution. This existence result was proved in a constructive way. The solution was obtained as the limit of a converging subsequence of quadrature formulas. Since there could be different subsequences converging to different limits, there might be (infinitely) many solutions. Here we will address the problem whether the moment problem is determinate or indeterminate, i.e., whether a solution μ is unique or not. We shall, to a certain extent, generalise the classical limiting point – limiting disk situation for a sequence of nested disks. This technique was known to Weyl [23] and used by Akhiezer [1] and Shohat and Tamarkin [20] when they studied the classical Hamburger moment problem and the situation where the moment problem has a unique solution. See also the work of Stone [22].

For more on moment problems and nested disks see [10–12, 14, 21]. Uniqueness criteria for the strong Hamburger moment problem can be found in [17]. An extended Hamburger moment problem was discussed in [15]. In the Hamburger problem, the unit circle is replaced by the real line. Similar results about unique solvability were obtained in [16] for the extended Hamburger moment problem where a finite number of points α_k on the real line are cyclically repeated. Multipoint matrix versions of the Hamburger and Stieltjes moment problems are found in e.g., [18, 19]. In the case where the points α_k are located inside the unit disk, results similar to the ones of this paper were given in [4] and [7]. Here we treat the case of the unit circle, and at the same time, we consider a general sequence of α_k's which need not be cyclically repeated. We emphasize that the situation where all the points α_k lie inside the open unit disk is substantially different from when they are on its boundary. In the boundary case, nontrivial new problems arise, for example, due to the fact that functions in $\mathcal{L}$ are not continuous on the unit circle. Analogous results for the real line can be obtained in a similar way. However, there the problem is that the support of the measure is not compact, and special attention has to be paid to the point at infinity which causes some trouble. For simplicity, we discuss here only the case of the unit circle. An essential role will be played by the orthogonal rational functions. They play the role of orthogonal polynomials in the Hamburger case or the orthogonal Laurent polynomials [13] in the case of the strong Hamburger moment problem. For the case of points inside the unit disk, such orthogonal rational functions were first studied in [2, 8, 9]. For points on the boundary, they appear in [3, 5, 6, 16].

2. Orthogonal rational functions and recurrence relation

First we observe that

$$\mathcal{L}_n = \text{Span}\,\{b_0, b_1, \ldots, b_n\}$$

with

$$b_0 = 1, \quad b_n = Z_1 Z_2 \cdots Z_n, \quad n \geq 1,$$

where

$$Z_k(z) = \frac{i(z-1)(\alpha_k - 1)}{z - \alpha_k}, \quad k \geq 1.$$

We use this notation also for $k = 0$, in which case we set $\alpha_0 = -1$. Thus

$$Z_0(z) = 2i\frac{1-z}{1+z}.$$

Note that the basis functions b_k satisfy $b_{k*} = b_k$. By a Gram-Schmidt procedure, these basis functions are orthogonalised to give the orthonormal functions ϕ_n, $n = 0, 1, \ldots$. Let

$$\phi_n(z) = \phi_n(1) + \cdots + \kappa_n' b_{n-1}(z) + \kappa_n b_n(z). \tag{2.1}$$

The orthonormal functions can be fixed uniquely by requiring $\kappa_n > 0$. This is what will always be assumed when we refer to the orthonormal functions. Note that, because $M\{1\} = 1$, we find $\phi_0 = \kappa_0 = 1$. Furthermore,

$$\kappa_n = \left[\frac{\phi_n(z)}{b_n(z)}\right]_{z=\alpha_n} \quad \text{and} \quad \kappa_n + \frac{\kappa_n'}{Z_n(\alpha_{n-1})} = \left[\frac{\phi_n(z)}{b_n(z)}\right]_{z=\alpha_{n-1}}, \quad n \geq 1. \tag{2.2}$$

If we set $\phi_n = p_n/\omega_n$ with $p_n \in \Pi_n$, then we say that ϕ_n (and also its index n) is singular if $p_n(\alpha_{n-1}) = 0$. Otherwise, ϕ_n and n are called regular. The system $\{\phi_n\}$ is called regular if all the indices are regular.

With this normalization, we have the following lemma.

Lemma 2.1. *The orthonormal functions ϕ_n have real coefficients with respect to the basis b_k, and $\phi_{n*} = \phi_n$.*

Proof. Because $b_{k*} = b_k$, it is obvious that if the coefficients are real, then $\phi_{n*} = \phi_n$. The proof that the coefficients are real follows easily by induction as follows. The result is true for $n = 0$. Suppose it is true for $i \leq n-1$. By the Gram-Schmidt procedure,

$$\phi_n = \chi_n / \|\chi_n\| \quad \text{with} \quad \chi_n = b_n - \sum_{i=0}^{n-1} \gamma_i \phi_i, \quad \gamma_i = \langle b_n, \phi_i \rangle.$$

Using $M\{f_*\} = \overline{M\{f\}}$, $\langle f, g \rangle = M\{f g_*\}$, $b_{n*} = b_n$, and $\phi_{i*} = \phi_i$ for $i < n$, it follows that the coefficients $\gamma_i = \langle b_n, \phi_i \rangle = M\{b_n \phi_{i*}\} = \overline{M\{b_{n*}\phi_i\}} = \overline{M\{b_n \phi_{i*}\}} = \overline{\gamma}_i$ are real. Since ϕ_i has real coefficients with respect to the basis b_k, then χ_n and thus also ϕ_n will have real coefficients with respect to the basis b_k. □

The following is a slight modification of Theorem 4.1 in [3].

Theorem 2.2. *Suppose the system $\{\phi_n\}$ is regular. Then the following recurrence holds:*

$$\phi_n(z) = \left(A_n Z_n(z) + B_n \frac{Z_n(z)}{Z_{n-2}(z)}\right) \phi_{n-1}(z) + C_n \frac{Z_n(z)}{Z_{n-2}(z)} \phi_{n-2}(z), \quad n = 2, 3, \ldots, \tag{2.3}$$

with constants A_n, B_n, C_n satisfying the conditions

$$E_n = A_n + B_n/Z_{n-2}(\alpha_{n-1}) \neq 0, \qquad k = 2, 3, \ldots, \tag{2.4}$$

$$C_n \neq 0, \qquad k = 2, 3, \ldots. \tag{2.5}$$

Let us introduce the real numbers

$$D_n = \frac{1}{Z_{n-2}(z)} - \frac{1}{Z_{n-1}(z)} = -i\frac{\alpha_{n-1} - \alpha_{n-2}}{(1-\alpha_{n-1})(1-\alpha_{n-2})} \in \mathbb{R}. \tag{2.6}$$

Since this is a constant not depending on z, and because $1/Z_k(\alpha_k) = 0$ for all k, we can use

$$D_n = \frac{1}{Z_{n-2}(\alpha_{n-1})} = -\frac{1}{Z_{n-1}(\alpha_{n-2})} \in \mathbb{R}$$

for $n \geq 2$ when convenient.

Lemma 2.3. *Let κ_n and κ_n' be as in (2.1). Define*

$$E_n = \frac{1}{\kappa_{n-1}}\left[\kappa_n + \frac{\kappa_n'}{Z_n(\alpha_{n-1})}\right], \qquad n \geq 1. \tag{2.7}$$

Then $E_n \in \mathbb{R}$. If the recurrence relation holds, and hence A_n and B_n are defined, then the E_n of (2.7) coincide with the E_n of (2.4), i.e., $E_n = A_n + B_n D_n$. The latter also holds for $n = 1$ if we set $A_1 = \kappa_1$ and $B_1 = \kappa_1'$.

Proof. Because the coefficients of the orthonormal functions are real and D_n is real, it follows that E_n also is real. To show that $E_n = A_n + B_n D_n$, we divide the recurrence relation (2.3) by $b_n(z)$ and set $z = \alpha_{n-1}$. With (2.2) and $1/Z_k(\alpha_k) = 0$, we get the result. □

Note that it easily follows from this definition of E_n that $E_n = 0$ if and only if n is a singular index.

Lemma 2.4. *Suppose the recurrence relation for the orthonormal functions ϕ_n holds with coefficients A_n, B_n, and C_n. Let D_n be defined by (2.6) and $E_n = A_n + B_n D_n$. Then*

$$E_n = -C_n E_{n-1}, \qquad n \geq 2. \tag{2.8}$$

Proof. Because $b_{n-1}/Z_n \in \mathcal{L}_{n-1}$, b_{n-1} is orthogonal to ϕ_n. Using the recurrence relation for ϕ_n, we get

$$\begin{aligned} 0 &= \langle \phi_n, b_{n-1}/Z_n \rangle \\ &= A_n \langle b_{n-1}, \phi_{n-1} \rangle + B_n \left\langle \frac{b_{n-1}}{Z_{n-2}}, \phi_{n-1} \right\rangle + C_n \left\langle \frac{b_{n-1}}{Z_{n-2}}, \phi_{n-2} \right\rangle. \end{aligned}$$

Because $\langle b_{n-2}, \phi_{n-1} \rangle = 0$ and $\langle b_i, \phi_i \rangle = 1/\kappa_i$, we have

$$0 = \frac{A_n}{\kappa_{n-1}} + \frac{B_n D_n}{\kappa_{n-1}} + \frac{C_n}{\kappa_{n-2}} + C_n D_n \langle b_{n-1}, \phi_{n-2} \rangle. \tag{2.9}$$

The remaining inner product can be evaluated when we use

$$b_{n-1}(z) = \frac{\phi_{n-1}(z)}{\kappa_{n-1}} - \frac{\kappa_{n-1}'}{\kappa_{n-1}} b_{n-2}(z) + \cdots,$$

so that

$$\langle b_{n-1}, \phi_{n-2}\rangle = \left\langle \frac{\phi_{n-1}}{\kappa_{n-1}}, \phi_{n-2}\right\rangle - \frac{\kappa'_{n-1}}{\kappa_{n-1}}\langle b_{n-2}, \phi_{n-2}\rangle = -\frac{\kappa'_{n-1}}{\kappa_{n-1}\kappa_{n-2}}. \tag{2.10}$$

When we combine (2.9) and (2.10), we get

$$\begin{aligned} 0 &= \frac{A_n}{\kappa_{n-1}} + \frac{B_n D_n}{\kappa_{n-1}} + \frac{C_n}{\kappa_{n-2}} - \frac{C_n D_n \kappa'_{n-1}}{\kappa_{n-1}\kappa_{n-2}} \\ &= \frac{A_n + B_n D_n}{\kappa_{n-1}} + \frac{C_n}{\kappa_{n-2}}\left[1 - D_n \frac{\kappa'_{n-1}}{\kappa_{n-1}}\right] \\ &= \frac{E_n}{\kappa_{n-1}} + \frac{C_n}{\kappa_{n-1}} \cdot \frac{1}{\kappa_{n-2}}\left[\kappa_{n-1} + \frac{\kappa'_{n-1}}{Z_{n-1}(\alpha_{n-2})}\right] \\ &= \frac{1}{\kappa_{n-1}}[E_n + C_n E_{n-1}], \end{aligned}$$

which gives us the expression we wanted. □

For solutions of the recurrence relation, we can prove a general summation formula. To obtain this formula, we define, for $n \geq 1$,

$$H(z,w) = \frac{1}{Z_{n-1}(z)} - \frac{1}{Z_{n-1}(w)} = -i\frac{z-w}{(w-1)(z-1)}. \tag{2.11}$$

Note that this expression does not depend on n. Furthermore, we set

$$\begin{aligned} H_n(z,w) &= \frac{1}{Z_{n-2}(w)Z_{n-1}(z)} - \frac{1}{Z_{n-2}(z)Z_{n-1}(w)} \\ &= \left[\frac{1}{Z_{n-2}(w)Z_{n-1}(z)} - \frac{1}{Z_{n-2}(w)Z_{n-2}(z)}\right] \\ &\quad + \left[\frac{1}{Z_{n-2}(w)Z_{n-2}(z)} - \frac{1}{Z_{n-2}(z)Z_{n-1}(w)}\right] \\ &= \frac{-D_n}{Z_{n-2}(w)} + \frac{D_n}{Z_{n-2}(z)} = D_n H(z,w). \end{aligned} \tag{2.12}$$

Theorem 2.5. *Let $x_n(z)$ and $y_n(z)$ be two solutions of the recurrence relation* (2.3) *and define*

$$F_n(z,w) = \frac{x_n(w)y_{n-1}(z)}{Z_n(w)Z_{n-1}(z)} - \frac{y_n(z)x_{n-1}(w)}{Z_n(z)Z_{n-1}(w)}.$$

Then, with $H(z,w)$ given by (2.11) *and E_n given by* (2.7),

$$\begin{aligned} F_n(z,w) &= y_{n-1}(z)x_{n-1}(w)H(z,w)E_n - C_n F_{n-1}(z,w) \\ &= \left[\sum_{k=1}^{n-1} y_k(z)x_k(w)\right]H(z,w)E_n + (-1)^n C_n C_{n-1}\cdots C_2 F_1(z,w). \end{aligned}$$

Proof. We use the recurrence relation for x_n and y_n in the definition of $F_n(z,w)$, which gives

$$\begin{aligned}F_n(z,w) = A_n x_{n-1}(w)y_{n-1}(z)&\left[\frac{1}{Z_{n-1}(z)} - \frac{1}{Z_{n-1}(w)}\right]\\ + B_n x_{n-1}(w)y_{n-1}(z)&\left[\frac{1}{Z_{n-1}(z)Z_{n-2}(w)} - \frac{1}{Z_{n-1}(w)Z_{n-2}(z)}\right]\\ - C_n F_{n-1}(z,w).&\end{aligned}$$

Using the expressions (2.11) and (2.12), we find

$$\begin{aligned}F_n(z,w) &= x_{n-1}(w)y_{n-1}(z)H(z,w)\big[A_n + B_n D_n\big] - C_n F_{n-1}(z,w)\\ &= x_{n-1}(w)y_{n-1}(z)H(z,w)E_n - C_n F_{n-1}(z,w).\end{aligned}$$

An induction argument leads to the result. □

From this formula, it is possible to derive the Christoffel-Darboux-type formulas which are given below. However, this would require that the system ϕ_n be regular, since the above formula is based on the existence of the recurrence relation. It is possible, however, to prove the Christoffel-Darboux formulas without using the recurrence relation and only relying on the orthogonality properties of the ϕ_n. That is what we shall do here.

3. Functions of the second kind

With the orthonormal functions ϕ_n, we associate functions of the second kind ψ_n defined by

$$\begin{aligned}\psi_0 &= -\frac{1-z}{1+z} = -\frac{Z_0(z)}{Z_0(0)},\\ \psi_n &= M_t\Big\{D(t,z)\big[\phi_n(t) - \phi_n(z)\big]\Big\}, \quad n \geq 1\end{aligned}$$

where $D(t,z)$ is the Riesz-Herglotz kernel

$$D(t,z) = \frac{t+z}{t-z}.$$

We also introduced the notation M_t to indicate that M operates on its argument as a function of t. We prove that these functions of the second kind are also solutions of the recurrence (2.3).

Theorem 3.1. *Suppose that the system of orthogonal rational functions ϕ_n is regular, and let ψ_n be the associated functions of the second kind. Then these ψ_n satisfy the same recurrence relation (2.3) as the ϕ_n.*

Proof. We use the recurrence relation for $\phi_n(t)$ and $\phi_n(z)$ in the definition of ψ_n. This gives, for $n \geq 2$,

$$\begin{aligned}\psi_n(z) &= A_n M_t\Big\{D(t,z)\big[Z_n(t)\phi_{n-1}(t) - Z_n(z)\phi_{n-1}(z)\big]\Big\} \\ &\quad + B_n M_t\left\{D(t,z)\left[\frac{Z_n(t)}{Z_{n-2}(t)}\phi_{n-1}(t) - \frac{Z_n(z)}{Z_{n-2}(z)}\phi_{n-1}(z)\right]\right\} \\ &\quad + C_n M_t\left\{D(t,z)\left[\frac{Z_n(t)}{Z_{n-2}(t)}\phi_{n-2}(t) - \frac{Z_n(z)}{Z_{n-2}(z)}\phi_{n-2}(z)\right]\right\} \\ &= A_n Z_n(z)\psi_{n-1}(z) + B_n\frac{Z_n(z)}{Z_{n-2}(z)}\psi_{n-1}(z) + C_n\frac{Z_n(z)}{Z_{n-2}(z)}\psi_{n-2}(z) \\ &\quad + M_t\big\{D(t,z)f_n(t,z)\big\} + \delta_{n2}\frac{Z_2(z)}{Z_0(0)}C_2 \qquad (3.1)\end{aligned}$$

with

$$\begin{aligned}f_n(t,z) &= A_n\big[Z_n(t) - Z_n(z)\big]\phi_{n-1}(t) \\ &\quad + \left[\frac{Z_n(t)}{Z_{n-2}(t)} - \frac{Z_n(z)}{Z_{n-2}(z)}\right]\big[B_n\phi_{n-1}(t) + C_n\phi_{n-2}(t)\big].\end{aligned}$$

We note that

$$Z_n(t) - Z_n(z) = -i\frac{(t-z)(\alpha_n - 1)^2}{(t-\alpha_n)(z-\alpha_n)}$$

and

$$\frac{Z_n(t)}{Z_{n-2}(t)} - \frac{Z_n(z)}{Z_{n-2}(z)} = \frac{(t-z)(1-\alpha_n)(\alpha_{n-2}-\alpha_n)}{(t-\alpha_n)(z-\alpha_n)(1-\alpha_{n-2})}.$$

Therefore,

$$\begin{aligned}f_n(t,z) = \frac{(t-z)(1-\alpha_n)}{(t-\alpha_n)(z-\alpha_n)}\Bigg[&A_n i(\alpha_n - 1)\phi_{n-1}(t) \\ &+ \frac{\alpha_{n-2}-\alpha_n}{1-\alpha_{n-2}}\big[B_n\phi_{n-1}(t) + C_n\phi_{n-2}(t)\big]\Bigg].\end{aligned}$$

Next, we split $D(t,z)$ as $D_1(t,z) + D_2(t,z)$:

$$D(t,z) = \frac{t-\alpha_n}{t-z} + \frac{z+\alpha_n}{t-z}.$$

Thus, in the argument of M_t in (3.1), the factor $t-z$ in the numerator of $f_n(t,z)$ cancels the denominator of $D(t,z)$.

Using the orthogonality of the ϕ_k, we find

$$\begin{aligned}M_t\big\{D_1(t,z)f_n(t,z)\big\} &= 0, &&\text{for } n \geq 3, \\ &= C_2\frac{(\alpha_2 - 1)(1+\alpha_2)}{2(z-\alpha_2)}, &&\text{for } n = 2.\end{aligned}$$

For the second term with $D_2(t,z)$, we use again the recurrence relation to write $f_n(t,z)$ as

$$f_n(t,z) = \frac{t-z}{z-\alpha_n}\Big[-\phi_n(t) - i(1-\alpha_n)A_n\phi_{n-1}(t) + \frac{1-\alpha_n}{1-\alpha_{n-2}}\Big(B_n\phi_{n-1}(t) + C_n\phi_{n-2}(t)\Big)\Big].$$

Again, by the orthogonality of the ϕ_k, we get

$$\begin{aligned} M_t\{D_2(t,z)f_n(t,z)\} &= 0, && \text{for } n \geq 3, \\ &= C_2\frac{(z+\alpha_2)(1-\alpha_2)}{2(z-\alpha_2)}, && \text{for } n = 2. \end{aligned}$$

That proves the recurrence relation for $n \geq 3$ directly. For $n = 2$, we can put together all the terms involved, and we find that the recurrence relation is satisfied also because

$$\frac{Z_2(z)}{Z_0(0)}C_2 - C_2\frac{(1-\alpha_2)(1+\alpha_2)}{2(z-\alpha_2)} + C_2\frac{(z+\alpha_2)(1-\alpha_2)}{2(z-\alpha_2)} = 0.$$

□

It is possible to build some redundancy in the definition of the ψ_n. For an analogous proof in the case where the points α_k are inside the unit disk, see [7].

Lemma 3.2. *Let ϕ_n be the orthonormal system and ψ_n the associated functions of the second kind. For $n > 0$ and for any f such that $D(t,z)[f(t) - f(z)] \in \mathcal{L}_{n-1}$ (as a function of t), we have*

$$\psi_n(z)f(z) = M_t\Big\{D(t,z)\big[\phi_n(t)f(t) - \phi_n(z)f(z)\big]\Big\}.$$

Proof. Since

$$M_t\Big\{D(t,z)\big[f(t)\phi_n(t) - f(z)\phi_n(z)\big]\Big\} = f(z)\psi_n(z) + M_t\Big\{D(t,z)\big[f(t) - f(z)\big]\phi_n(t)\Big\},$$

the result follows by the orthogonality of the ϕ_n. □

Note that in particular, we could take $f \in \mathcal{L}_{n-1}$ or $f(t) = g(t)(t-\alpha_n)/(t+z)$ with $g \in \mathcal{L}_n$.

We now shall derive a Liouville-Ostrogradskii-type determinant formula.

Theorem 3.3 (Determinant formula). *Let ϕ_n be the orthonormal functions and ψ_n the functions of the second kind. Then for $n \geq 1$, we have with E_n given by* (2.7)

$$\begin{aligned} \phi_{n-1}(z)\psi_n(z) - \phi_n(z)\psi_{n-1}(z) &= \frac{2iz}{(1-z)^2}E_nZ_{n-1}(z)Z_n(z) \\ &= -\frac{2iz(\alpha_n-1)(\alpha_{n-1}-1)E_n}{(z-\alpha_{n-1})(z-\alpha_n)}. \end{aligned}$$

Proof. We note that

$$\begin{aligned} &\phi_{n-1}(z)\big[\phi_n(t) - \phi_n(z)\big] - \phi_n(z)\big[\phi_{n-1}(t) - \phi_{n-1}(z)\big] \\ &\quad = \phi_{n-1}(t)\big[\phi_n(t) - \phi_n(z)\big] - \phi_n(t)\big[\phi_{n-1}(t) - \phi_{n-1}(z)\big]. \end{aligned}$$

Multiply by $D(t,z)$ and apply M_t to get on the left-hand side

$$\phi_{n-1}(z)\psi_n(z) - \phi_n(z)\psi_{n-1}(z),$$

while for the right-hand side we have

$$M_t\Big\{\phi_{n-1}(t)D(t,z)\big[\phi_n(t)-\phi_n(z)\big]\Big\}-M_t\Big\{\phi_n(t)D(t,z)\big[\phi_{n-1}(t)-\phi_{n-1}(z)\big]\Big\}.$$

Note that in the second term $D(t,z)[\phi_{n-1}(t)-\phi_{n-1}(z)]\in\mathcal{L}_{n-1}$, so that this term is zero by the orthogonality of ϕ_n. To compute the first term, we define

$$h(t)=D(t,z)\big[\phi_n(t)-\phi_n(z)\big]=\gamma_n b_n(t)+\gamma_n' b_{n-1}(t)+\cdots$$

where

$$\gamma_n=\left[\frac{h(t)}{b_n(t)}\right]_{t=\alpha_n}=D(\alpha_n,z)\kappa_n$$

and

$$\begin{aligned}\gamma_n+\frac{\gamma_n'}{Z_n(\alpha_{n-1})}&=\left[\frac{h(t)}{b_n(t)}\right]_{t=\alpha_{n-1}}\\&=D(\alpha_{n-1},z)\left[\kappa_n+\frac{\kappa_n'}{Z_n(\alpha_{n-1})}\right]=D(\alpha_{n-1},z)\kappa_{n-1}E_n.\end{aligned}$$

Thus, by the orthogonality of $\phi_{n-1}=\phi_{(n-1)*}$,

$$M\{\phi_{n-1}h\}=\gamma_n M\{\phi_{n-1}b_n\}+\gamma_n' M\{\phi_{n-1}b_{n-1}\}.$$

Because $\phi_k=\kappa_k b_k+\kappa_k' b_{k-1}+\cdots$, it follows by orthogonality that

$$M\{\phi_{n-1}b_n\}=-\frac{\kappa_n'}{\kappa_n\kappa_{n-1}}\qquad\text{and}\qquad M\{\phi_{n-1}b_{n-1}\}=\frac{1}{\kappa_{n-1}}.$$

So we obtain

$$\begin{aligned}M\{\phi_{n-1}h\}&=-\gamma_n\frac{\kappa_n'}{\kappa_n\kappa_{n-1}}+\frac{\gamma_n'}{\kappa_{n-1}}\\&=-\frac{\gamma_n\kappa_n'}{\kappa_n\kappa_{n-1}}+\big[D(\alpha_{n-1},z)E_n-\frac{\gamma_n}{\kappa_{n-1}}\big]Z_n(\alpha_{n-1})\\&=-D(\alpha_n,z)\frac{\kappa_n'}{\kappa_{n-1}}+D(\alpha_{n-1},z)E_nZ_n(\alpha_{n-1})-D(\alpha_n,z)\kappa_nZ_n(\alpha_{n-1})\\&=-D(\alpha_n,z)E_nZ_n(\alpha_{n-1})+D(\alpha_{n-1},z)E_nZ_n(\alpha_{n-1})\\&=E_nZ_n(\alpha_{n-1})\big[D(\alpha_{n-1},z)-D(\alpha_n,z)\big].\end{aligned}$$

Working this out gives the result. □

4. Christoffel-Darboux relations

We now prove some Chistoffel-Darboux-type relations.

Theorem 4.1 (Christoffel-Darboux relation). *Suppose ϕ_n are orthonormal functions, and let $H(z,w)$ and E_n be defined by* (2.11) *and* (2.7). *Then*

$$\frac{\phi_n(w)\phi_{n-1}(z)}{Z_n(w)Z_{n-1}(z)}-\frac{\phi_n(z)\phi_{n-1}(w)}{Z_n(z)Z_{n-1}(w)}=H(z,w)E_n\sum_{k=0}^{n-1}\phi_k(z)\phi_k(w).$$

Proof. Define

$$g(z,w) = (w-\alpha_n)(z-\alpha_{n-1})\phi_n(w)\phi_{n-1}(z) \quad \text{and} \quad G(w) = g(z,w) - g(w,z). \quad (4.1)$$

Then the Christoffel-Darboux relation, which has to be shown, is equivalent to

$$F(w) = \frac{G(w)}{z-w} = -i(\alpha_n - 1)(\alpha_{n-1}-1)E_n \sum_{k=0}^{n-1} \phi_k(z)\phi_k(w). \quad (4.2)$$

Observe that $F(w) \in \mathcal{L}_{n-1}$, so that it can be written as

$$F(w) = \sum_{k=0}^{n-1} \gamma_k(z)\phi_k(w), \quad \gamma_k(z) = M\{F\phi_{k*}\} = M\{F\phi_k\}.$$

We have

$$\gamma_k(z) = M\{F\phi_k\} = M_w\Big\{F(w)\big[\phi_k(w) - \phi_k(z)\big]\Big\} + \phi_k(z)M\{F\}.$$

The first term is zero. To see this, we write it out as

$$\begin{aligned} M_w&\Big\{F(w)\big[\phi_k(w)-\phi_k(z)\big]\Big\} \\ &= (z-\alpha_{n-1})\phi_{n-1}(z)M_w\left\{\phi_n(w)\frac{w-\alpha_n}{z-w}\left[\phi_k(w)-\phi_k(z)\right]\right\} \\ &\quad - (z-\alpha_n)\phi_n(z)M_w\left\{\phi_{n-1}(w)\frac{w-\alpha_{n-1}}{z-w}\left[\phi_k(w)-\phi_k(z)\right]\right\}. \end{aligned}$$

Because $\phi_n \perp \mathcal{L}_{n-1}$, the first term is zero, and because $\phi_{n-1} \perp \mathcal{L}_{n-2}$, the second term is zero. Thus it remains that $\gamma_k(z) = \phi_k(z)M\{F\}$. We note that

$$\begin{aligned} M\{F\} &= (z-\alpha_{n-1})\phi_{n-1}(z)f_n(z) - (z-\alpha_n)\phi_n(z)f_{n-1}(z), \\ f_i(z) &= M_w\left\{\frac{w-\alpha_i}{z-w}\phi_i(w)\right\}. \end{aligned}$$

By adding and subtracting $(z-\alpha_{n-1})(z-\alpha_n)\phi_{n-1}(z)\phi_n(z)D(z,w)/(2z)$, we can rewrite $M\{F\}$ as

$$M\{F\} = (z-\alpha_{n-1})\phi_{n-1}(z)g_n(z) - (z-\alpha_n)\phi_n(z)g_{n-1}(z)$$

with (for $i = n, n-1 > 0$)

$$g_i(z) = M_w\left\{D(z,w)\left[\frac{w-\alpha_i}{z+w}\phi_i(w) - \frac{z-\alpha_i}{2z}\phi_i(z)\right]\right\} = \frac{z-\alpha_i}{2z}\psi_i(z) \quad (4.3)$$

(the last equality is by Lemma 3.2). For $i=0$, one has to use $\phi_0 = 1$, $\alpha_0 = -1$, and $\psi_0(z) = (z-1)/(z+1)$, and it then follows that the previous relation also holds for $i = 0$. Thus, using the determinant relation of Theorem 3.3, this gives

$$\begin{aligned} M\{F\} &= \frac{(z-\alpha_n)(z-\alpha_{n-1})}{2z}\left[\phi_{n-1}(z)\psi_n(z) - \phi_n(z)\psi_{n-1}(z)\right] \\ &= -i(\alpha_n-1)(\alpha_{n-1}-1)E_n, \end{aligned}$$

so that eventually

$$F(w) = \sum_{k=0}^{n-1}\gamma_k(z)\phi_k(w) = M\{F\}\sum_{k=0}^{n-1}\phi_k(z)\phi_k(w)$$

yields (4.2). □

We also have the following generalization of the determinant formula.

Theorem 4.2. *Let $\{\phi_n\}$ be the orthonormal functions and ψ_n the associated functions of the second kind. Define*

$$F_n(z,w) = \frac{\psi_n(w)\phi_{n-1}(z)}{Z_n(w)Z_{n-1}(z)} - \frac{\phi_n(z)\psi_{n-1}(w)}{Z_n(z)Z_{n-1}(w)}.$$

Let $H(z,w)$ be defined by (2.11), E_n be defined by (2.7), and $D(z,w)$ be the Riesz-Herglotz kernel. Then, for $w \neq z$,

$$F_n(z,w) = \left[\sum_{k=1}^{n-1} \phi_k(z)\psi_k(w) - D(z,w)\right] H(z,w)E_n.$$

For $w = z$, this reduces to the determinant formula.

Proof. Define $g(z,w)$ as in (4.1) and set

$$G(z,w) = g(z,w) - g(w,z).$$

Then, by the Chistoffel-Darboux formula,

$$G(z,w) = -i(z-w)(\alpha_n - 1)(\alpha_{n-1} - 1)E_n \sum_{k=0}^{n-1} \phi_k(z)\phi_k(w).$$

Apply M_t to the expression

$$\frac{D(t,w)}{t+w}G(t,z) - \frac{D(t,w)}{2w}G(w,z).$$

This gives, with (4.3),

$$\begin{aligned}(z-\alpha_n)(w-\alpha_{n-1})\phi_n(z)\psi_{n-1}(w) - (w-\alpha_n)(z-\alpha_{n-1})\psi_n(w)\phi_{n-1}(z) \\ = -i(\alpha_n - 1)(\alpha_{n-1} - 1)E_n\left[\sum_{k=1}^{n-1} \phi_k(z)(w-z)\psi_k(w) + (z+w)\right] \\ = i(\alpha_n - 1)(\alpha_{n-1} - 1)(z-w)E_n\left[\sum_{k=1}^{n-1} \phi_k(z)\psi_k(w) - D(z,w)\right].\end{aligned}$$

This can be rewritten in the required form. □

An analog of the Christoffel-Darboux relations for the functions of the second kind also may be obtained.

Theorem 4.3. *Let ψ_n be the functions of the second kind. Then, in analogy with the Christoffel-Darboux relation, we have*

$$\frac{\psi_n(w)\psi_{n-1}(z)}{Z_n(w)Z_{n-1}(z)} - \frac{\psi_n(z)\psi_{n-1}(w)}{Z_n(z)Z_{n-1}(w)} = H(z,w)E_n\left[\sum_{k=1}^{n-1} \psi_k(z)\psi_k(w) - 1\right]$$

with $H(z,w)$ given by (2.11) and E_n given by (2.7).

Proof. We set

$$G(z,w) = (w-\alpha_n)(z-\alpha_{n-1})\phi_n(w)\psi_{n-1}(z) - (z-\alpha_n)(w-\alpha_{n-1})\phi_{n-1}(w)\psi_n(z).$$

Then by the previous theorem

$$G(z,w) = i(\alpha_n - 1)(\alpha_{n-1} - 1)(z-w)E_n\left[\sum_{k=1}^{n-1} \phi_k(w)\psi_k(z) - D(z,w)\right].$$

By applying M_t to

$$\frac{D(t,w)}{t+w}G(z,t) - \frac{D(t,w)}{2w}G(z,w),$$

we get by using (4.3)

$$\begin{aligned}(w-\alpha_n)(z-\alpha_{n-1})\psi_n(w)\psi_{n-1}(z) - (z-\alpha_n)(w-\alpha_{n-1})\psi_{n-1}(w)\psi_n(z)\\ = -i(\alpha_n - 1)(\alpha_{n-1} - 1)(z-w)E_n\left[\sum_{k=1}^{n-1}\psi_k(z)\psi_k(w) - 1\right],\end{aligned}$$

and this is equivalent to the required formula. □

Finally, we give a combination of the previous summation formulas.

Theorem 4.4. *Let ϕ_n be the orthonormal functions, ψ_n the functions of the second kind, and define for an arbitrary complex parameter s*

$$\chi_n(z;s) = \psi_n(z) + s\phi_n(z).$$

Then

$$\begin{aligned}\frac{\chi_n(w;t)\chi_{n-1}(z;s)}{Z_n(w)Z_{n-1}(z)} - \frac{\chi_n(z;s)\chi_{n-1}(w;t)}{Z_n(z)Z_{n-1}(w)}\\ = H(z,w)E_n\left[\sum_{k=1}^{n-1}\chi_k(z;s)\chi_k(w;t) + [st - 1 + D(z,w)(t-s)]\right]\end{aligned}$$

with $H(z,w)$ defined by (2.11) and E_n defined by (2.7).

Proof. This is directly obtained by working out the left-hand side and using the three previous theorems. □

5. Green's formula

We now give complex forms of the previous Christoffel-Darboux-type formulas. Therefore, we introduce

$$\tilde{H}(z,w) = \frac{1}{Z_n(z)} - \frac{1}{\overline{Z_n(w)}} = -\frac{i(1-z\overline{w})}{(1-z)(1-\overline{w})}, \qquad n \geq 0. \tag{5.1}$$

Note that $\tilde{H}(z,w) = H_*(z,w)$ where the substar is with respect to z. For $w=z$, we have $\tilde{H}(z,z) = -iP(1,z)$ with

$$P(t,z) = \frac{1-|z|^2}{|t-z|^2}$$

equal to the Poisson kernel. Furthermore, we define for $n \geq 2$

$$\tilde{H}_n(z,w) = \frac{1}{\overline{Z_{n-2}(w)}Z_{n-1}(z)} - \frac{1}{\overline{Z_{n-1}(w)}Z_{n-2}(z)} = D_n\tilde{H}(z,w). \tag{5.2}$$

We now give without proof the following complex analog of Theorem 2.5. A proof may be given by taking substar conjugates of the relations from the previous section.

Note that we then need the fact that the numbers E_n and C_n are real, which they are by Lemma 2.3 and Lemma 2.4.

Theorem 5.1 (Green's formula). *Let $x_n(z)$ and $y_n(z)$ both be solutions of the recurrence relation* (2.3), *and define*

$$G_n(z,w) = \overline{\left(\frac{x_n(w)}{Z_n(w)}\right)}\left(\frac{y_{n-1}(z)}{Z_{n-1}(z)}\right) - \left(\frac{y_n(z)}{Z_n(z)}\right)\overline{\left(\frac{x_{n-1}(w)}{Z_{n-1}(w)}\right)}.$$

Then, with $\tilde{H}(z,w)$ as in (5.1) *and E_n as in* (2.7),

$$\begin{aligned} G_n(z,w) &= y_{n-1}(z)\overline{x_{n-1}(w)}\tilde{H}(z,w)E_n - C_nG_{n-1}(z,w) \\ &= \left[\sum_{k=1}^{n-1} y_k(z)\overline{x_k(w)}\right]\tilde{H}(z,w)E_n + (-1)^n C_nC_{n-1}\cdots C_2G_1(z,w). \end{aligned}$$

One can use this theorem and take $x_k = y_k = \phi_k$ or $x_k = y_k = \psi_k$ or $x_k = \phi_k$ and $y_k = \psi_k$. One then obtains the formulas in the theorem below which are then proved under the assumption that they satisfy the recurrence relation, in particular that the ϕ_k form a regular system. However, it is equally simple to follow the reasoning used in the previous section to arrive at the Christoffel-Darboux type formulas where orthogonality was the only assumption needed, and again one will arrive at the results of the theorem below. Thus the results hold without assuming that there is a recurrence relation, i.e., without the system ϕ_n being regular. We give the result without proof.

Theorem 5.2. *Let ϕ_n be orthonormal functions and ψ_n functions of the second kind. Let $\tilde{H}(z,w)$ be defined by* (5.1), *E_n defined by* (2.7), *and $D(z,w)$, be the Riesz-Herglotz kernel. Then*

$$\begin{aligned} &\overline{\left(\frac{\phi_n(w)}{Z_n(w)}\right)}\left(\frac{\phi_{n-1}(z)}{Z_{n-1}(z)}\right) - \left(\frac{\phi_n(z)}{Z_n(z)}\right)\overline{\left(\frac{\phi_{n-1}(w)}{Z_{n-1}(w)}\right)} \\ &\qquad = \left[\sum_{k=0}^{n-1} \phi_k(z)\overline{\phi_k(w)}\right]\tilde{H}(z,w)E_n \\ &\overline{\left(\frac{\psi_n(w)}{Z_n(w)}\right)}\left(\frac{\psi_{n-1}(z)}{Z_{n-1}(z)}\right) - \left(\frac{\psi_n(z)}{Z_n(z)}\right)\overline{\left(\frac{\psi_{n-1}(w)}{Z_{n-1}(w)}\right)} \\ &\qquad = \left[\sum_{k=1}^{n-1} \psi_k(z)\overline{\psi_k(w)} + 1\right]\tilde{H}(z,w)E_n \\ &\overline{\left(\frac{\psi_n(w)}{Z_n(w)}\right)}\left(\frac{\phi_{n-1}(z)}{Z_{n-1}(z)}\right) - \left(\frac{\phi_n(z)}{Z_n(z)}\right)\overline{\left(\frac{\psi_{n-1}(w)}{Z_{n-1}(w)}\right)} \\ &\qquad = \left[\sum_{k=1}^{n-1} \phi_k(z)\overline{\psi_k(w)} - D_*(z,w)\right]\tilde{H}(z,w)E_n \end{aligned}$$

where in the last equation, the substar is with respect to z.

Note that this also holds for $w = z$. In that case

$$D_*(z,z) = \frac{1+|z|^2}{1-|z|^2}.$$

The previous relations can be combined to give the following theorem

Theorem 5.3. *Let ϕ_n be the orthonormal functions and ψ_n the functions of the second kind. For an arbitrary complex s, we set*

$$\chi_n(z;s) = \psi_n(z) + s\phi_n(z).$$

Then

$$\begin{aligned}\overline{\left(\frac{\chi_n(w;t)}{Z_n(w)}\right)}\left(\frac{\chi_{n-1}(z;s)}{Z_{n-1}(z)}\right) - \left(\frac{\chi_n(z;s)}{Z_n(z)}\right)\overline{\left(\frac{\chi_{n-1}(w;t)}{Z_{n-1}(w)}\right)}\\ = \left[\sum_{k=1}^{n-1}\chi_k(z;s)\overline{\chi_k(w,t)} + 1 + s\bar{t} - (s+\bar{t})D_*(z,w)\right]\tilde{H}(z,w)E_n.\end{aligned} \tag{5.3}$$

In particular, for $z = w$ and $t = s$, we obtain

$$\begin{aligned}\overline{\left(\frac{\chi_n(z;s)}{Z_n(z)}\right)}\left(\frac{\chi_{n-1}(z;s)}{Z_{n-1}(z)}\right) - \left(\frac{\chi_n(z;s)}{Z_n(z)}\right)\overline{\left(\frac{\chi_{n-1}(z;s)}{Z_{n-1}(z)}\right)}\\ = \left[\sum_{k=1}^{n-1}|\chi_k(z;s)|^2 + |1-s|^2\right]\tilde{H}(z,z)E_n + (s+\bar{s})E_n\frac{2i|z|^2}{|z-1|^2}.\end{aligned} \tag{5.4}$$

Proof. By the previous theorem, we can evaluate the left-hand side and obtain the formula (5.3). For $z = w$ and $t = s$, this equals

$$\left[\sum_{k=1}^{n-1}|\chi_k(z;s)|^2\right]\tilde{H}(z,z)E_n + E_nY(z,s)$$

with

$$\begin{aligned}Y(z,s) &= \tilde{H}(z,z)\big[1 - (s+\bar{s})D_*(z,z) + |s|^2\big]\\ &= \tilde{H}(z,z)|1-s|^2 + (s+\bar{s})\tilde{H}(z,z)\big[1 - D_*(z,z)\big].\end{aligned}$$

Because

$$\tilde{H}(z,z)[1 - D_*(z,z)] = i\frac{1-|z|^2}{|1-z|^2}\left[\frac{1+|z|^2}{1-|z|^2} - 1\right] = i\frac{2|z|^2}{|1-z|^2},$$

we get the result (5.4). □

6. Quasi-orthogonal functions

We define quasi-orthogonal functions as

$$Q_n(z,\tau) = \phi_n(z) + \tau\frac{Z_n(z)}{Z_{n-1}(z)}\phi_{n-1}(z), \quad \tau \in \hat{\mathbb{R}} = \mathbb{R}\cup\{\infty\}, n \ge 1. \tag{6.1}$$

For $\tau = \infty$, we set

$$Q_n(z,\infty) = \frac{Z_n(z)}{Z_{n-1}(z)}\phi_{n-1}(z).$$

It was proved in [3] that if ϕ_n is regular, then there always exists infinitely many so-called regular values $\tau = \tau_n \in \hat{\mathbb{R}}$ such that $Q_n(z,\tau_n)$ has n simple zeros, all lying on $\mathbb{T} \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. In fact, all except finitely many $\tau \in \hat{\mathbb{R}}$ are regular values. Suppose we have chosen a regular value τ_n, and suppose that the zeros of $Q_n(z,\tau_n)$ are given by $\xi_k = \xi_{nk}(\tau)$, $k = 1, \ldots, n$. They can be used to construct quadrature formulas. These formulas use the ξ_k for their abscissas, and their weights $\lambda_{nk}(\tau)$ are given by

$$\lambda_{nk}(\tau) = M\{L_{nk}\} = M\{L_{nk}(z,\tau)L_{nk*}(z,\tau)\} > 0$$

with

$$L_{nk}(z,\tau) = \frac{(z-\xi_1)\cdots(z-\xi_{k-1})(z-\xi_{k+1})\cdots(z-\xi_n)\omega_{n-1}(\xi_k)}{(\xi_k-\xi_1)\cdots(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})\cdots(\xi_k-\xi_n)\omega_{n-1}(z)}.$$

It also was proved that these quadrature formulas are exact in $\mathcal{R}_{n-1} = \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}$.

Theorem 6.1. *With the notation just introduced, we have that*

$$M\{f\} = \sum_{k=1}^{n} \lambda_{nk}(\tau) f(\xi_{nk}(\tau))$$

for all $f \in \mathcal{R}_{n-1} = \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}$.

Alternative expressions for the weights can be obtained as follows. First introduce quasi-orthogonal functions of the second kind by

$$P_n(z) = P_n(z,\tau) = \psi_n(z) + \tau\frac{Z_n(z)}{Z_{n-1}(z)}\psi_{n-1}(z), \qquad \tau \in \hat{\mathbb{R}}, n \geq 1.$$

We then have

Theorem 6.2. *The weights of the quadrature formula are given by*

$$\lambda_{nk} = \frac{1}{2\xi_k}\frac{P_n(\xi_k)}{Q_n'(\xi_k)}$$

where the prime means derivative.

Proof. From the partial fraction decomposition, we see that

$$R_n(z) = -\frac{P_n(z)}{Q_n(z)} = \sum_{j=1}^{n} \lambda_{nj} D(\xi_j, z)$$

with $D(t,z)$ the Riesz-Herglotz kernel. Therefore,

$$(z-\xi_k)R_n(z) = \sum_{j\neq k} \lambda_{nj} D(\xi_j,z)(z-\xi_k) + \lambda_{nk} D(\xi_k,z)(z-\xi_k).$$

Taking the limit $z \to \xi_k$ cancels the sum, and we get

$$-\frac{P_n(\xi_k)}{Q_n'(\xi_k)} = \lambda_{nk} \lim_{z\to\xi_k} D(\xi_k,z)(z-\xi_k),$$

and this gives the result. □

Another expression, which clearly shows the positivity of the weights, is

Theorem 6.3. *The weights of the quadrature formula are given by*

$$\lambda_{nk} = \left[\sum_{j=0}^{n-1} |\phi_j(\xi_k)|^2\right]^{-1}.$$

Proof. Since $Q_n(\xi_k) = 0$, the first formula in Theorem 5.2 with $w = \xi_k$ and $z = t$ given the equality

$$\frac{\overline{\phi_{n-1}(\xi_k)}}{\overline{Z_{n-1}(\xi_k)}Z_n(t)}\left[Q_n(\xi_k) - Q_n(t)\right] = \left[\sum_{j=0}^{n-1} \phi_j(t)\overline{\phi_j(\xi_k)}\right] \tilde{H}(t, \xi_k)E_n.$$

Dividing by $\xi_k - t$ and letting t tend to ξ_k, we get

$$\frac{\overline{\phi_{n-1}(\xi_k)}}{\overline{Z_{n-1}(\xi_k)}Z_n(\xi_k)}Q_n'(\xi_k) = k_{n-1}(\xi_k, \xi_k)E_n \lim_{t \to \xi_k} \frac{\tilde{H}(t, \xi_k)}{\xi_k - t}$$

where $k_{n-1}(z, z) = \sum_{j=0}^{n-1} |\phi_j(z)|^2$, and the limit on the right-hand side is $i/(1 - \xi_k)^2$, provided that $\xi_k \neq 1$.

Similarly, using the third formula in Theorem 5.2 with $z = w = \xi_k$ and taking into account that $Q_n(\xi_k) = 0$, we get

$$\frac{\overline{\phi_{n-1}(\xi_k)}}{\overline{Z_{n-1}(\xi_k)}Z_n(\xi_k)}P_n(\xi_k) = -E_n \lim_{t \to \xi_k} \overline{D_*(t, \xi_k)}\,\overline{\tilde{H}(t, \xi_k)}$$

where the limit is (again for $\xi_k \neq 1$)

$$-i\frac{2}{|1 - \xi_k|^2}.$$

We now substitute the values of $Q_n'(\xi_k)$ and $P_n(\xi_k)$ into the expression for λ_{nk} in Theorem 6.3, and we get the result for $\xi_k \neq 1$.

For $\xi_k = 1$, we use a perturbation technique. First we choose a bicontinuous perturbation $\tau(\epsilon) \in \hat{\mathbb{R}}$ of the parameter τ in Q_n. Assume that $\lim_{\epsilon \to 0} \tau(\epsilon) = \tau$. The coefficients of Q_n depend bicontinuously on ϵ and hence also on its zero ξ_k. Therefore, we can always choose $\epsilon \neq 0$ so that $\xi_k(\epsilon) \neq 1$. Then the relation we want holds true by our previous arguments. Since $\xi_k(\epsilon) \to 1$ as $\epsilon \to 0$ and since both P_n and Q_n' as well as their ratio depend bicontinuously on ϵ in the neighbourhood of $\epsilon = 0$, we finally obtain the result for $\xi_k = 1$:

$$\frac{1}{k_{n-1}(1, 1)} = \lim_{\epsilon \to 0} \frac{1}{k_{n-1}(\xi_k(\epsilon), \xi_k(\epsilon))} = \lim_{\epsilon \to 0} \frac{1}{2\xi_k(\epsilon)} \frac{P_n(\xi_k(\epsilon))}{Q_n'(\xi_k(\epsilon))} = \frac{1}{2}\frac{P_n(1)}{Q_n'(1)} = \lambda_{nk}.$$

That concludes the proof. □

The discrete measure which takes weights $\lambda_{nk}(\tau)$ at the points $\xi_{nk}(\tau)$, $k = 1, \ldots, n$ will be denoted by $\mu_n = \mu_n(\,\cdot\,, \tau)$. The set of regular τ values for which such a measure exists is dense in $\mathbb{R}$.

7. Nested disks

In this section, we use the notation

$$s = s_n(z) = R_n(z, \tau) = -\frac{P_n(z, \tau)}{Q_n(z, \tau)}$$

where Q_n are the quasi-orthogonal functions and P_n are the quasi-orthogonal functions of the second kind. Obviously when n is a regular index, $\tau \to s_n(z)$ maps (for a fixed $z \in \hat{\mathbb{C}} \setminus \mathbb{T}$ with $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) the extended real line $\hat{\mathbb{R}}$ onto a circle $K_n(z)$. The closed disk with boundary $K_n(z)$ is denoted by $\Delta_n(z)$.

For $z \in \mathbb{T}$, the circle $K_n(z)$ degenerates to a line (the imaginary axis).

When n is a singular index, the transformation is degenerate. In that case, the whole plane is mapped to a point. Indeed, since for a singular n we have $E_n = 0$, it follows from the Christoffel-Darboux relation that

$$\frac{\phi_n(w)}{Z_n(w)}\frac{\phi_{n-1}(z)}{Z_{n-1}(z)} = \frac{\phi_n(z)}{Z_n(z)}\frac{\phi_{n-1}(w)}{Z_{n-1}(w)}$$

for all z and w. Choose w such that $\phi_{n-1}(w)/Z_{n-1}(w) \neq 0$, then there is a constant c_n such that

$$\phi_n(z) = c_n \frac{Z_n(z)}{Z_{n-1}(z)}\phi_{n-1}(z).$$

It follows similarly from Theorem 4.3 that the same holds for the functions of the second kind ψ_n, i.e., we may replace ϕ by ψ in the previous relation and the constant c_n is the same for ϕ and ψ as follows from the determinant formula of Theorem 3.3. Hence we get

$$R_n(z,\tau) = -\frac{P_n(z,\tau)}{Q_n(z,\tau)} = -\frac{(c_n Z_n/Z_{n-1}+\tau)\psi_{n-1}(z)}{(c_n Z_n/Z_{n-1}+\tau)\phi_{n-1}(z)} = -\frac{\psi_{n-1}(z)}{\phi_{n-1}(z)}.$$

This is independent of τ for any $\tau \in \hat{\mathbb{C}}$ (not only the real ones), and thus the whole Riemann sphere is mapped to a point, which we denote by $K_n(z)$.

We have the following theorem.

Theorem 7.1. *Suppose that the index n is regular. Then for $z \in \hat{\mathbb{C}} \setminus \mathbb{T}$*

1. *The equation of the circle $K_n(z)$ is given by*

$$\sum_{k=1}^{n-1} |\psi_k(z) + s\phi_k(z)|^2 + |1-s|^2 = (s+\overline{s})\frac{2|z|^2}{1-|z|^2}.$$

2. *The (closed) disk $\Delta_n(z)$ is obtained by replacing the equality sign by $\leq$.*
3. *The center c_n and the radius r_n of the disk are*

$$c_n = -\frac{\left(\frac{\psi_n}{Z_n}\right)\overline{\left(\frac{\phi_{n-1}}{Z_{n-1}}\right)} - \left(\frac{\psi_{n-1}}{Z_{n-1}}\right)\overline{\left(\frac{\phi_n}{Z_n}\right)}}{\left(\frac{\phi_n}{Z_n}\right)\overline{\left(\frac{\phi_{n-1}}{Z_{n-1}}\right)} - \left(\frac{\phi_{n-1}}{Z_{n-1}}\right)\overline{\left(\frac{\phi_n}{Z_n}\right)}},$$

$$r_n = \left|\frac{\left(\frac{\psi_n}{Z_n}\right)\left(\frac{\phi_{n-1}}{Z_{n-1}}\right) - \left(\frac{\psi_{n-1}}{Z_{n-1}}\right)\left(\frac{\phi_n}{Z_n}\right)}{\left(\frac{\phi_n}{Z_n}\right)\overline{\left(\frac{\phi_{n-1}}{Z_{n-1}}\right)} - \left(\frac{\phi_{n-1}}{Z_{n-1}}\right)\overline{\left(\frac{\phi_n}{Z_n}\right)}}\right| = \frac{1}{k_{n-1}(z,z)}\left|\frac{2z}{1-|z|^2}\right|$$

   where $k_{n-1}(z,z) = \sum_{k=0}^{n-1} |\phi_k(z)|^2$.
4. *The circle $K_n(z)$ is in the right (left) half plane iff $|z| < 1$ ($|z| > 1$).*
5. *If m is also a regular index and $m > n$, then $\Delta_m(z) \subset \Delta_n(z)$.*
6. *The circles $K_n(z)$ and $K_{n-1}(z)$ touch.*

Proof. We solve the relation $s = R_n(z,\tau)$ for τ, and we then get

$$\begin{aligned}\tau &= -\frac{Z_{n-1}(z)}{Z_n(z)}\frac{\psi_n(z)+s\phi_n(z)}{\psi_{n-1}(z)+s\phi_{n-1}(z)}\\ &= -\frac{|Z_{n-1}|^2}{Z_n\overline{Z}_{n-1}}\frac{[\psi_n+s\phi_n]\overline{[\psi_{n-1}+s\phi_{n-1}]}}{|\psi_{n-1}+s\phi_{n-1}|^2}.\end{aligned}$$

Taking the imaginary part gives

$$\Im\tau = \frac{1}{2i}\frac{|Z_{n-1}|^2}{|\psi_{n-1}+s\phi_{n-1}|^2}g_n(z,s)$$

with

$$g_n(z,s) = \frac{(\overline{\psi_n+s\phi_n})(\psi_{n-1}+s\phi_{n-1})}{\overline{Z}_n Z_{n-1}} - \frac{(\psi_n+s\phi_n)(\overline{\psi_{n-1}+s\phi_{n-1}})}{Z_n\overline{Z}_{n-1}}.$$

Now suppose that the index n is regular. Then we can write the equation of the circle $K_n(z)$, which corresponds to $\Im\tau = 0$ since $g_n(z,s) = 0$. With the help of Green's formula (5.4) (since n is a regular index and hence $E_n \neq 0$), we also may write the equation for the circle as

$$\sum_{k=1}^{n-1} |\psi_k(z) + s\phi_k(z)|^2 + |1-s|^2 = (s+\overline{s})\frac{X(z)}{\tilde{H}(z,z)}$$

with $X(z) = 2i|z|^2/|z-1|^2$. This is (1) of the theorem.

Since the denominator $1-|z|^2$ is positive, negative, or zero as $|z|<1$, $|z|=1$, or $|z|>1$, the circle will be in the right- or left-half plane depending on whether $|z|$ is less than 1 or greater than 1, respectively. This is (4).

The disk $\Delta_n(z)$ with boundary $K_n(z)$ is given by putting a $\leq$ sign instead of equality. Since

$$(s+\overline{s})\frac{2|z|^2}{1-|z|^2} - \sum_{k=1}^{n-1}|\psi_k(z)+s\phi_k(z)|^2 - |1-s|^2 = A|s|^2 + Bs + \overline{Bs} + D$$

with $A = -\sum_{k=0}^{n-1}|\phi_k|^2 < 0$, it follows that this will become negative for $|s|$ sufficiently large. Thus this expression is negative outside the disk. Therefore, the closed disk is described by

$$\sum_{k=1}^{n-1} |\psi_k(z) + s\phi_k(z)|^2 + |1-s|^2 \leq (s+\overline{s})\frac{2|z|^2}{1-|z|^2}. \tag{7.1}$$

This is (2).

Since the sum in the left-hand side of (7.1) is non-decreasing with n, it follows that we have nested disks for regular indices, i.e., if $m > n$ and m and n regular indices, then $\Delta_m(z) \subset \Delta_n(z)$. This is (5). We have to exclude the singular indices because when n is singular, Δ_n collapses to a point, and a subsequent disk with positive radius cannot be a subset of Δ_n.

Since $R_n(z,\infty) = R_{n-1}(z,0)$, the circles will touch, even if the index n is singular. In the latter case, $K_n(z)$ is a point from $K_{n-1}(z)$.

The expressions for center and radius for a general linear fractional transform $(\tau \in \hat{\mathbb{R}})$

$$s = -\frac{a-\tau b}{c-\tau d} = -\frac{a\overline{d}-b\overline{c}}{c\overline{d}-d\overline{c}} + \frac{ad-bc}{c\overline{d}-d\overline{c}}\cdot\frac{\overline{c}-\tau\overline{d}}{c-\tau d}$$

are obviously

$$c = -\frac{a\overline{d}-b\overline{c}}{c\overline{d}-d\overline{c}} \quad \text{and} \quad r = \left|\frac{ad-bc}{c\overline{d}-d\overline{c}}\right|.$$

Using the Green and Christoffel-Darboux formulas, the expressions for c_n and r_n as in (3) will follow. □

Corollary 7.2. *For $z = 0$, all the circles $K_n(z)$ $n = 1, 2, \dots$ reduce to the same point $s(z) \equiv 1$. For $z = \infty$, they all reduce to $s(z) \equiv -1$.*

Proof. When n is regular, it follows from the expression for the radius that for $z \in \{0, \infty\}$, it is zero. From the equation of the circle, it follows that $s = 1$ or $s = -1$.

Since successive circles touch also for singular indices, we get the same point for all n. □

Assume from now on that there are infinitely many regular indices. Because the disks are nested, it follows that when $n(\nu)$ is the sequence of regular indices, then $\Delta_\infty = \cap_{n(\nu)} \Delta_{n(\nu)}$ is a disk with radius

$$r(z) = \lim_{\nu \to \infty} r_{n(\nu)}(z),$$

which reduces to a point when this radius is zero. We have the following lemma.

Lemma 7.3. *Suppose $z \in \mathbb{C}_0 = \mathbb{C} \setminus (\mathbb{T} \cup \{0\})$. If $\Delta_\infty(z)$ is a disk (with positive radius), then*

$$\sum_{k=0}^{\infty} |\phi_k(z)|^2 < \infty \qquad \text{and} \qquad \sum_{k=0}^{\infty} |\psi_k(z)|^2 < \infty.$$

If $\Delta_\infty(z)$ is a point, then

$$\sum_{k=0}^{\infty} |\phi_k(z)|^2 = \infty \qquad \text{and} \qquad \sum_{k=0}^{\infty} |\psi_k(z)|^2 = \infty.$$

Proof. It follows from the expression for the radii that $r(z)$ is positive (zero) iff $k_\infty(z, z)$ is finite (infinite), i.e., iff $\sum_{k=0}^{\infty} |\phi_k(z)|^2$ is finite (infinite).

Let s be a point from the disk $\Delta_\infty(z)$. Then it follows from (7.1) that in any case (disk or point)

$$\sum_{k=1}^{\infty} |\psi_k(z) + s(z)\phi_k(z)|^2 < \infty.$$

Thus $\sum_{k=1}^{\infty} |\phi_k(z)|^2 < \infty$ iff $\sum_{k=1}^{\infty} |\psi_k(z)|^2 < \infty$. □

Before we can prove an invariance theorem, we also need the following lemma.

Lemma 7.4. *If ϕ_n are the orthonormal functions and ψ_n the functions of the second kind, then for any $w \in \mathbb{C}_0 = \mathbb{C} \setminus (\mathbb{T} \cup \{0\})$*

$$\phi_n(z) = \phi_n(w) + \frac{z - w}{\alpha_n - z} \Big[A_{0n} + \sum_{k=1}^{n-1} \phi_k(z) A_{kn}(w) \Big],$$

$$\psi_n(z) = \psi_n(w) + \frac{z - w}{\alpha_n - z} \Big[B_{0n} + \sum_{k=1}^{n-1} \psi_k(z) A_{kn}(w) \Big],$$

where

$$A_{0n}(w) = V_n(w)\phi_n(w) - Y_n(w)\psi_n(w),$$
$$B_{0n}(w) = V_n(w)\psi_n(w) - Y_n(w)\phi_n(w),$$

$$A_{kn}(w) = Y_n(w)a_{kn}(w), \quad a_{kn}(w) = \psi_k(w)\phi_n(w) - \phi_k(w)\psi_n(w), \quad k = 1, \ldots, n-1,$$

$$Y_n(w) = \frac{w - \alpha_n}{2w}, \quad \textit{and} \quad V_n(w) = \frac{w + \alpha_n}{2w}.$$

Proof. From the Christoffel-Darboux formula in Theorem 4.1 and the mixed formula in Theorem 4.2, we get

$$\frac{\phi_n(z)\phi_{n-1}(w)}{Z_n(z)Z_{n-1}(w)} - \frac{\phi_n(w)\phi_{n-1}(z)}{Z_n(w)Z_{n-1}(z)} = -H(z,w)E_n \sum_{k=0}^{n-1} \phi_k(z)\phi_k(w),$$

$$\frac{\phi_n(z)\psi_{n-1}(w)}{Z_n(z)Z_{n-1}(w)} - \frac{\psi_n(w)\phi_{n-1}(z)}{Z_n(w)Z_{n-1}(z)} = -H(z,w)E_n \left[\sum_{k=1}^{n-1} \phi_k(z)\psi_k(w) - D(z,w) \right].$$

Elimination of $\phi_{n-1}(z)$ gives

$$\frac{\phi_n(z)}{Z_n(z)Z_{n-1}(w)} a_{n-1,n}(w)$$

$$= -H(z,w)E_n \left[\sum_{k=1}^{n-1} \phi_k(z)a_{k,n}(w) - \left[\psi_n(w) + D(z,w)\phi_n(w)\right] \right].$$

From the determinant formula of Theorem 4.2, we find

$$a_{n-1,n}(w) = -E_n Z_{n-1}(w)Z_n(w)\frac{2iw}{(1-w)^2}.$$

Hence

$$-\frac{H(z,w)E_n Z_n(z)Z_{n-1}(w)}{a_{n-1,n}(w)} = \frac{z-w}{\alpha_n - z}\left[\frac{w-\alpha_n}{2w}\right] = \frac{z-w}{\alpha_n - z}Y_n(w).$$

Thus

$$\phi_n(z) = \frac{z-w}{\alpha_n - z}Y_n(w)\left[\sum_{k=1}^{n-1} \phi_k(z)a_{kn}(w) - \left[\psi_n(w) + D(z,w)\phi_n(w)\right]\right].$$

Next we compute

$$-\frac{z-w}{\alpha_n - z}Y_n(w)D(z,w) = \frac{(z+w)(\alpha_n - w)}{2w(\alpha_n - z)} = 1 + \frac{(z-w)(w+\alpha_n)}{(\alpha_n - z)2w}.$$

Thus

$$\phi_n(z) = \phi_n(w) + \frac{z-w}{\alpha_n - z}\left[V_n(w)\phi_n(w) - Y_n(w)\psi_n(w)\right]$$

$$+ \frac{z-w}{\alpha_n - z}\left[\sum_{k=1}^{n-1} \phi_k(z)\left[Y_n(w)a_{kn}(w)\right] \right].$$

This is the first formula required. The second formula is proved similarly. □

We now can prove the invariance theorem.

Theorem 7.5 (Invariance). *Suppose* $w \in \mathbb{C}_0 = \mathbb{C}\backslash(\mathbb{T}\cup\{0\})$*, and suppose that* $\Delta_\infty(w)$ *is a disk with positive radius. Then* $\Delta_\infty(z)$ *is a disk with positive radius for every* $z \in \mathbb{C}_0$*, and*

$$\sum_{k=0}^{n} |\phi_k(z)|^2 \quad \textit{and} \quad \sum_{k=0}^{n} |\psi_k(z)|^2$$

converge locally uniformly in $\mathbb{C}_0$ *as* $n \to \infty$.

Proof. From Lemma 7.4, we know that with $x_n = \phi_n$ and $y_n = \psi_n$ or vice versa, we have, with the notation as in Lemma 7.4,

$$\begin{aligned} x_n(z) = x_n(w) &+ \frac{z-w}{\alpha_n - z} Y_n(w) \sum_{k=1}^{n-1} x_k(z) a_{kn}(w) \\ &+ \frac{z-w}{\alpha_n - z} \big[V_n(w) x_n(w) - Y_n(w) y_n(w)\big]. \end{aligned}$$

From the definition of $a_{kn}(w)$, it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} |a_{kn}(w)|^2 &\le 2 \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \left(|\psi_k(w)|^2 |\phi_n(w)|^2 + |\phi_k(w)|^2 |\psi_n(w)|^2 \right) \\ &\le 4 \Big(\sum_{n=1}^{\infty} |\psi_n(w)|^2 \Big) \Big(\sum_{n=1}^{\infty} |\phi_n(w)|^2 \Big) < \infty \end{aligned}$$

by Lemma 7.3. Now suppose that C is a compact subset of $\mathbb{C}_0$. Then for arbitrary $z \in C$ and $w \in \mathbb{C}_0$ fixed,

$$\frac{z-w}{\alpha_n - z} Y_n(w) \quad \text{and} \quad \frac{z-w}{\alpha_n - z} V_n(w)$$

are uniformly bounded, say

$$\left| \frac{z-w}{\alpha_n - z} Y_n(w) \right| \le R_1 \quad \text{and} \quad \left| \frac{z-w}{\alpha_n - z} V_n(w) \right| \le R_2.$$

Thus,

$$\begin{aligned} \Big(\sum_{n=m}^{N} |x_n(z)|^2 \Big)^{1/2} &\le \Big(\sum_{n=m}^{N} |x_n(w)|^2 \Big)^{1/2} + R_1 \Big(\sum_{n=m}^{N} |y_n(w)|^2 \Big)^{1/2} \\ &+ R_2 \Big(\sum_{n=m}^{N} |x_n(w)|^2 \Big)^{1/2} + R_1 \Big(\sum_{n=m}^{N} \Big| \sum_{k=1}^{n-1} a_{kn}(w) x_k(z) \Big|^2 \Big)^{1/2}. \end{aligned}$$

For any $0 < \epsilon < 1$, choose $1 < m = m(\epsilon, R_1, R_2)$ such that

$$\Big(\sum_{n=m}^{\infty} |y_n(w)|^2 \Big)^{1/2} < \frac{\epsilon}{R_1}, \qquad \Big(\sum_{n=m}^{\infty} |x_n(w)|^2 \Big)^{1/2} < \frac{\epsilon}{1+R_2},$$

and

$$\Big(\sum_{n=m}^{\infty} \sum_{k=1}^{n-1} |a_{kn}(w)|^2 \Big)^{1/2} < \frac{\epsilon}{R_1}.$$

Then

$$\begin{aligned}\left(\sum_{n=m}^{N}|x_n(z)|^2\right)^{1/2} &\le \epsilon+\epsilon+R_1\left[\sum_{n=m}^{N}\left(\sum_{k=1}^{n-1}|a_{kn}(w)|^2\right)\left(\sum_{k=1}^{n-1}|x_k(z)|^2\right)\right]^{1/2}\\ &\le 2\epsilon+R_1\left(\sum_{k=1}^{N}|x_k(z)|^2\right)^{1/2}\left(\sum_{n=m}^{\infty}\sum_{k=1}^{n-1}|a_{kn}(w)|^2\right)^{1/2}\\ &\le 2\epsilon+\epsilon\left(\sum_{k=1}^{N}|x_k(z)|^2\right)^{1/2}\\ &\le 2\epsilon+\epsilon\left(\sum_{k=m}^{N}|x_k(z)|^2\right)^{1/2}+\epsilon\left(\sum_{k=1}^{m-1}|x_k(z)|^2\right)^{1/2}.\end{aligned}$$

Hence,

$$(1-\epsilon)\left(\sum_{n=m}^{N}|x_n(z)|^2\right)^{1/2}\le 2\epsilon+\epsilon\left(\sum_{k=1}^{m-1}|x_k(z)|^2\right)^{1/2}.$$

Because $\sum_{k=1}^{m-1}|x_k(z)|^2$ is a continuous function in C, there is an $M_m>0$ such that

$$\left(\sum_{k=1}^{m-1}|x_k(z)|^2\right)^{1/2}\le M_m$$

holds uniformly in C. Thus,

$$\left(\sum_{k=m}^{N}|x_k(z)|^2\right)^{1/2}\le\frac{(2+M_m)\epsilon}{1-\epsilon},\quad N\ge m.$$

From this inequality can be deduced first uniform boundedness and then uniform convergence of $\sum_{n=0}^{\infty}|x_k(z)|^2$ in C. □

We also can prove the analyticity theorem.

Theorem 7.6 (Analyticity). *Let $\Delta_\infty(z)$ be a point. Then the limit*

$$s(z)=\lim_{n\to\infty}R_n(z,\tau),\qquad z\in\mathbb{C}\setminus\mathbb{T},$$

is an analytic function of z not depending on τ. Moreover,

$$\frac{\Re s(z)}{1-|z|^2}>0,\quad z\in\mathbb{C}\setminus\mathbb{T}.$$

Proof. By Theorem 7.1 (6), it follows that if $\Delta_\infty(z)$ is a point, then

$$\lim_{n\to\infty}R_n(z,\tau)=\lim_{n\to\infty}s_n(z)=s(z)$$

exists and is independent of τ.

Obviously, $s_n(z)=R_n(z,\tau)$ is analytic for $z\in\mathbb{C}\setminus\mathbb{T}$. Thus the analyticity of $s(z)$ will follow if the functions $s_n(z)$ are uniformly bounded in compact subsets of $\mathbb{C}\setminus\mathbb{T}$. This is shown as follows. We know that

$$|1-s_n(z)|^2+\sum_{k=1}^{n-1}|\psi_k(z)+s_n(z)\phi_k(z)|^2=(s_n(z)+\overline{s}_n(z))\frac{2|z|^2}{1-|z|^2}.$$

Thus,

$$1+|s_n|^2+\sum_{k=1}^{n-1}|\psi_k+s_n\phi_k|^2=(s_n+\overline{s}_n)\frac{1+|z|^2}{1-|z|^2},$$

or, using $|s_n+\overline{s}_n|\leq 2|s_n|$,

$$|s_n|^2<1+|s_n|^2+\sum_{k=1}^{n-1}|\psi_k+s_n\phi_k|^2\leq 2|s_n|\left|\frac{1+|z|^2}{1-|z|^2}\right|.$$

Therefore,

$$|s_n(z)|\leq\left|\frac{1+|z|^2}{1-|z|^2}\right|.$$

Since the right-hand side is uniformly bounded on compact subsets of $\mathbb{C}\setminus\mathbb{T}$, the analyticity of $s(z)$ follows.

The last inequality is a direct consequence of Theorem 7.1 (4). □

8. The moment problem

We now are ready to approach the moment problem as was described in the introduction. Thus, given the linear functional M, real and positive on $\mathcal{R}$, can we find the measure μ, finite and positive on $[-\pi,\pi]$, such that

$$M\left\{\frac{1}{\omega_n}\right\}=\mu_n=\int_{-\pi}^{\pi}\frac{d\mu(\theta)}{\omega_n(e^{i\theta})},\quad n=0,1,\ldots?$$

Note that if μ is a solution of this moment problem, then it is easily seen that $M\{f\}=\int_{-\pi}^{\pi}f(e^{i\theta})d\mu(\theta)$ for all $f\in\mathcal{L}$. Thus μ defines the linear functional M on the space $\mathcal{L}$. The functional M, however, was defined on the larger space $\mathcal{R}=\mathcal{L}\cdot\mathcal{L}$. Thus, strictly speaking, to solve the moment problem, it would have been sufficient to define M on $\mathcal{L}$ only. As we have seen however, our approach has relied heavily on orthogonality properties, i.e., on the fact that we have a real and positive inner product, and that is why we needed the functional M to be defined on $\mathcal{R}$.

Another, more general moment problem, which could have been considered, is to write the functional M as an integral against a positive measure μ for all functions in $\mathcal{R}=\mathcal{L}\cdot\mathcal{L}$ (and not just for functions in $\mathcal{L}$). Thus, $M\{f\}=\int_{-\pi}^{\pi}f(e^{i\theta})d\mu(\theta)$ for all $f\in\mathcal{R}$. Then we should consider the problem of finding a positive measure μ such that the moment relations

$$M\left\{\frac{1}{\omega_n\omega_{m*}}\right\}=\mu_{nm}=\int_{-\pi}^{\pi}\frac{d\mu(\theta)}{\omega_n(e^{i\theta})\omega_{m*}(e^{i\theta})},\quad n,m=0,1,\ldots \tag{8.1}$$

hold. This moment problem indeed is equivalent to finding a positive measure μ such that

$$\langle f,g\rangle=M\{fg_*\}=\int_{-\pi}^{\pi}f(e^{i\theta})\overline{g(e^{i\theta})}d\mu(\theta),\quad\forall f,g\in\mathcal{L}. \tag{8.2}$$

We shall call this the moment problem in $\mathcal{R}=\mathcal{L}\cdot\mathcal{L}$, while the previously explained simpler form is called the moment problem in $\mathcal{L}$. Note that the moment problem in $\mathcal{R}$ prescribes also the moments $\mu_{n0}=\mu_n$, which implies that a solution of the moment problem in $\mathcal{R}$ will automatically be a solution of the moment problem in $\mathcal{L}$. It is not clear whether the converse is true in general.

It was shown in [3] that the moment problem in $\mathcal{L}$ has at least one solution. Since the proof given in [3] is rather concise, we include it here in more detail. The arguments which are made explicit in this proof also show why the same arguments do not lead to a solution of the moment problem in $\mathcal{R}$.

Theorem 8.1. *Let M be a real positive functional defined on $\mathcal{R} = \mathcal{L} \cdot \mathcal{L}$ and assume that the sequence of orthogonal rational functions ϕ_n has infinitely many regular indices. Then there exists at least one measure μ on $\mathbb{T}$ which solves the moment problem in $\mathcal{L}$, i.e., which satisfies* (8.1).

Proof. The proof is based on the fact that the discrete measures μ_k representing the quadrature formulas

$$\sum_{i=1}^{k} \lambda_{ki}(\tau) f(\xi_{ki}(\tau)) = \int_{-\pi}^{\pi} f(e^{i\theta}) d\mu_k(\theta)$$

which are in $\mathcal{R}_{k-1}$, and hence also in $\mathcal{L}_{k-1}$. They are uniformly bounded by $M\{1\} = 1$, and hence, by Helly's selection principle, the sequence $\{\mu_k\}$ will have a convergent subsequence $\mu_{k(j)} \to \mu$. Next we prove that such a μ solves the moment problem in $\mathcal{L}$, i.e., that

$$M\left\{\frac{1}{\omega_n}\right\} = \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})}, \quad n = 0, 1, \ldots.$$

For $n = 0$, we can apply Helly's convergence theorem since 1 is a continuous function on the interval $[-\pi, \pi]$. Thus it remains to prove this for $n > 0$. Fix $n > 0$. We denote the elements of the subsequence $k(j)$ by k for simplicity. Note that for $n > 0$, we cannot simply apply Helly's convergence theorem because the ω_n^{-1} are not continuous on the interval $I = [-\pi, \pi]$. However, consider any subset $J \subset I$ which does not contain the $\alpha_1, \ldots, \alpha_n$, then ω_n^{-1} indeed is continuous on J, and this means that we can find a $k = k(j)$ large enough such that for a given $\epsilon > 0$,

$$\left| \int_J \frac{d\mu(\theta) - d\mu_k(\theta)}{\omega_n(e^{i\theta})} \right| < \frac{\epsilon}{3}. \tag{8.3}$$

On the other hand, for $k = k(j) > n$,

$$\left| \int_{I-J} \frac{d\mu_k(\theta)}{\omega_n(e^{i\theta})} \right| = \left| \int_{I-J} \frac{\omega_{n*}(e^{i\theta}) d\mu_k(\theta)}{|\omega_n(e^{i\theta})|^2} \right| \le \max_{I-J} |\omega_{n*}(e^{i\theta})| \int_{I-J} \frac{d\mu_k(\theta)}{|\omega_n(e^{i\theta})|^2}$$

$$\le \max_{I-J} |\omega_{n*}(e^{i\theta})| M\left\{\frac{1}{\omega_n \omega_{n*}}\right\}.$$

We always can choose J large enough such that the maximum of $|\omega_{n*}|$ in $I - J$ is arbitrarily small. Because $M\{[\omega_n \omega_{n*}]^{-1}\}$ is finite, it follows that for any $\epsilon > 0$, we can make J large enough to satisfy

$$\left| \int_{I-J} \frac{d\mu_k(\theta)}{\omega_n(e^{i\theta})} \right| < \frac{\epsilon}{3}. \tag{8.4}$$

Note that this holds for any $k > n$, i.e., J is independent of k. Next we want to show that J also can be made large enough to satisfy

$$\left| \int_{I-J} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})} \right| < \frac{\epsilon}{3}. \tag{8.5}$$

To obtain this, we consider sets J_p, none of which contain $\alpha_1, \ldots, \alpha_n$, and such that $J_q \subset J_p$ for $p > q$ and $\bigcup_p J_p = I \setminus \{\alpha_1, \ldots, \alpha_n\}$. Note that for $k = k(j) > n$ and for any p

$$\int_{J_p} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})} = \int_{J_p} \frac{d\mu(\theta) - d\mu_k(\theta)}{\omega_n(e^{i\theta})} + \int_{J_p} \frac{d\mu_k(\theta)}{\omega_n(e^{i\theta})}.$$

Thus, if $p > q$ and $k > n$, then

$$\left| \int_{J_p - J_q} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})} \right| \leq \left| \int_{J_p - J_q} \frac{d\mu(\theta) - d\mu_k(\theta)}{\omega_n(e^{i\theta})} \right| + \left| \int_{J_p - J_q} \frac{d\mu_k(\theta)}{\omega_n(e^{i\theta})} \right|. \tag{8.6}$$

By (8.4), there are always p and q large enough such that for any $\eta > 0$,

$$\left| \int_{J_p - J_q} \frac{d\mu_k(\theta)}{\omega_n(e^{i\theta})} \right| < \frac{\eta}{2} \tag{8.7}$$

for all large k. By (8.3), we can make k so large that for any $\eta > 0$,

$$\left| \int_{J_p - J_q} \frac{d\mu(\theta) - d\mu_k(\theta)}{\omega_n(e^{i\theta})} \right| < \frac{\eta}{2}. \tag{8.8}$$

By combining (8.6)–(8.8), we see that it is possible to make

$$\left| \int_{J_p} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})} - \int_{J_q} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})} \right| < \eta$$

for any $\eta > 0$. This means that

$$\left\{ \int_{J_p} \frac{d\mu(\theta)}{\omega(e^{i\theta})} \right\}_p$$

is a Cauchy sequence so that the limit for $p \to \infty$ (which is easily seen to be equal to the integral over I) exists, thus for any $\epsilon > 0$, there exists a p large enough such that

$$\left| \int_{J_p} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})} - \int_I \frac{d\mu(\theta)}{\omega_n(e^{i\theta})} \right| < \frac{\epsilon}{3},$$

which proves (8.5). Finally, by (8.3)–(8.5),

$$\left| \int_I \frac{d\mu_k(\theta) - d\mu(\theta)}{\omega_n(e^{i\theta})} \right| \leq \left| \int_{I-J} \frac{d\mu_k(\theta)}{\omega_n(e^{i\theta})} \right| + \left| \int_{I-J} \frac{d\mu(\theta)}{\omega_n(e^{i\theta})} \right| + \left| \int_J \frac{d\mu_k(\theta) - d\mu(\theta)}{\omega_n(e^{i\theta})} \right| < \epsilon$$

because each of the terms on the right-hand side can be bounded by $\epsilon/3$. Thus,

$$\lim_{k(j)\to\infty} \int_I \frac{d\mu_{k(j)}(\theta)}{\omega_n(e^{i\theta})} = M\left\{\frac{1}{\omega_n}\right\} = \int_I \frac{d\mu(\theta)}{\omega_n(e^{i\theta})}.$$

This proves the theorem. □

We now use our framework of nested disks to obtain information about the uniqueness of the solution. Let us denote the set of solutions of the moment problem in $\mathcal{L}$ by $\mathcal{M}^{\mathcal{L}}$, and the set of solutions of the moment problem in $\mathcal{R}$ by $\mathcal{M}^{\mathcal{R}}$. Then we have

Theorem 8.2. *Assume that the sequence of orthonormal functions ϕ_n has infinitely many regular indices. Hence, $\mathcal{M}^{\mathcal{L}} \neq \varnothing$. Fix $z \in \mathbb{C} \setminus \mathbb{T}$, and define for $\mu \in \mathcal{M}^{\mathcal{L}}$ the Riesz-Herglotz transform*

$$\Omega_\mu(z) = \int_{-\pi}^{\pi} D(e^{i\theta}, z) d\mu(\theta).$$

Then

$$\{\Omega_\mu(z) : \mu \in \mathcal{M}^{\mathcal{R}}\} \subset \Delta_\infty(z) \subset \{\Omega_\mu(z) : \mu \in \mathcal{M}^{\mathcal{L}}\}.$$

Proof. Let $s = \Omega_\mu(z)$ for some $\mu \in \mathcal{M}^{\mathcal{R}}$. Note that the system $\{\phi_n\}$ is then orthonormal with respect to the inner product defined by the measure μ. Let $f(z) = \overline{D(t,z)}$, $t \in \mathbb{T}$. Writing the generalized Fourier series of $f(z)$ as

$$f(z) \sim \sum_{k=0}^{\infty} \gamma_k \phi_k(z), \quad \gamma_k = \langle f, \phi_k \rangle$$

and then using

$$\gamma_k = \int_{-\pi}^{\pi} \overline{D(t,z)\phi_k(t)} d\mu(\theta), \qquad \text{and}$$

it turns out that $\overline{\gamma}_0 = s$ and $\overline{\gamma}_k = \psi_k(z) + s\phi_k(z)$ for $k \geq 1$ because

$$\begin{aligned}\overline{\gamma}_k &= \int_{-\pi}^{\pi} D(t,z)\phi_k(t) d\mu(\theta) \\ &= \int_{-\pi}^{\pi} D(t,z)[\phi_k(t) - \phi_k(z)] d\mu(\theta) + \phi_k(z)\Omega_\mu(z), \qquad t = e^{i\theta}.\end{aligned}$$

On the other hand, it can be shown that for $t \in \mathbb{T}$

$$-1 + \frac{1+|z|^2}{1-|z|^2}\left[D(t,z) + \overline{D(t,z)}\right] = \frac{|t+z|^2}{|t-z|^2} = |D(t,z)|^2.$$

Using Bessel's inequality, it then follows that

$$\sum_{k=0}^{\infty} |\gamma_k|^2 = |s|^2 + \sum_{k=1}^{\infty} |\psi_k + s\phi_k|^2 \leq -1 + \frac{1+|z|^2}{1-|z|^2}(s + \overline{s}),$$

which can be rearranged as

$$|1-s|^2 + \sum_{k=1}^{\infty} |\psi_k + s\phi_k|^2 \leq \frac{2|z|^2}{1-|z|^2}(s+\overline{s}).$$

This means that $s \in \Delta_\infty(z)$.

Next, it is shown that if $s \in \Delta_\infty(z)$, then s is the Riesz-Herglotz transform of some $\mu \in \mathcal{M}^{\mathcal{L}}$. This is readily shown by using the quadrature formulas we have discussed. We consider the limiting point and the limiting disk cases separately.

If $\Delta_\infty(z)$ is a point, then since $s \in \Delta_\infty(z)$, there must exist $s_n \in K_n(z)$ such that $s_n \to s$. Since there is for each n, some τ_n such that

$$s_n = R_n(z, \tau_n) = \int_{-\pi}^{\pi} D(e^{i\theta}, z) d\mu_n(\theta).$$

Helly's selection criterion then yields that there is a subsequence $\mu_{n(j)} \to \mu$. By the proof of the previous theorem, $\mu \in \mathcal{M}^{\mathcal{L}}$, and by Helly's convergence theorem

$$\lim_{j\to\infty} \int_{-\pi}^{\pi} D(t,z) d\mu_{n(j)}(\theta) = \int_{-\pi}^{\pi} D(t,z) d\mu(\theta) = s.$$

Thus s is the Riesz-Herglotz transform of a $\mu \in \mathcal{M}^{\mathcal{L}}$.

If $\Delta_\infty(z)$ is a disk, let s be a point on the boundary $K_\infty(z)$. Recall that for a fixed n, we can, except for finitely many values of τ, associate a quadrature formula with $R_n(z,\tau)$. Let us denote the discrete measure that is associated with this quadrature by $\mu_n(\,\cdot\,,\tau)$. It depends on n as well as the choice of τ. For every regular index n, we then can choose an $s_n \in K_n(z)$ such that s_n tends to s and such that $s_n = \Omega_{\mu_n}(z)$, where $\mu_n = \mu_n(\,\cdot\,,\tau_n)$ and where τ_n is chosen such that $s_n = R_n(z,\tau_n)$. By Helly's theorems and the proof of the previous theorem, there exists a $\mu \in \mathcal{M}^{\mathcal{L}}$ such that $\Omega_\mu(z) = s$.

Thus every s on the boundary $K_\infty(z)$ is of the form $\Omega_\mu(z)$ with $\mu \in \mathcal{M}^{\mathcal{L}}$. Now let s be an interior point of $\Delta_\infty(z)$. Then s is a convex combination $s = \lambda s_1 + (1-\lambda)s_2$ $(0 < \lambda < 1)$ of points s_1, s_2 on the boundary $K_\infty(z)$. Thus there exist $\mu_1, \mu_2 \in \mathcal{M}^{\mathcal{L}}$ such that

$$s_j = \int_{-\pi}^{\pi} D(t,z)d\mu_j(\theta), \quad t = e^{i\theta}.$$

Thus $\mu = \lambda\mu_1 + (1-\lambda)\mu_2 \in \mathcal{M}^{\mathcal{L}}$ and $s = \Omega_\mu(z)$. □

Now the following corollary is obvious.

Corollary 8.3. *In the case of a limiting disk, for each $s \in \Delta_\infty(z)$, $z \in \mathbb{C} \setminus \mathbb{T}$, the moment problem in $\mathcal{L}$ has infinitely many solutions.*

In the case of a limiting point, a solution of the moment problem in $\mathcal{R}$ is unique.

Acknowledgement. This research was performed as part of the European project ROLLS under contract CHRX-CT93-0416.

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