

PERTURBATION OF TWO ELEMENTARY WAVES WITH INTERACTION

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ABSTRACT. In this paper, we study the perturbation of two interacting elementary waves for the system of isentropic gas dynamics equations. The entropy solutions for the perturbed problems are obtained by Glimm's scheme. Under the condition that the initial data will issue two elementary waves with interaction, we conclude that the corresponding perturbed solutions exist if the total variation of the perturbation of the initial data is sufficiently small.

1. Introduction

Consider the Cauchy problem for the system of isentropic gas dynamics equations in Lagrangian coordinates:

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v)_x &= 0 \end{aligned} \quad x \in R, \quad t > 0, \quad (\mathbf{E})$$

$$(u, v)(x, 0) = (u_0(x), v_0(x)) \quad x \in R \quad (\mathbf{I})$$

where $p(v) \in C^2(0, +\infty)$, $p'(v) < 0$, $p''(v) > 0$.

We study some perturbation problems for (\mathbf{E}) and (\mathbf{I}) . Suppose the initial data (\mathbf{I}) are perturbed; the perturbed initial data are

$$(\bar{u}, \bar{v})(x, 0) = (\bar{u}_0(x), \bar{v}_0(x)) \quad x \in R \quad (\bar{\mathbf{I}})$$

where $\|(\bar{u}_0(x) - u_0(x), \bar{v}_0(x) - v_0(x))\|$ is sufficiently small; $\|\cdot\|$ is a norm in a certain sense. The function defined by $(\bar{u}_0(x) - u_0(x), \bar{v}_0(x) - v_0(x))$ is called the perturbation of the initial data $(u_0(x), v_0(x))$.

Definition 1. The entropy solutions $(\bar{u}(x, t), \bar{v}(x, t))$ of the perturbed problem (\mathbf{E}) and $(\bar{\mathbf{I}})$ are called perturbed solutions of the corresponding unperturbed problem (\mathbf{E}) and (\mathbf{I}) .

Definition 2. The entropy solution $(u(x, t), v(x, t))$ of the Cauchy problem (\mathbf{E}) and (\mathbf{I}) is called stable if

$$\|(\bar{u}(x, t) - u(x, t), \bar{v}(x, t) - v(x, t))\| \rightarrow 0 \text{ as } \|(\bar{u}_0(x) - u_0(x), \bar{v}_0(x) - v_0(x))\| \rightarrow 0.$$

Obviously, the existence of perturbed solutions is an important prior condition for the study of stability.

In the celebrated paper by Glimm [6], his famous result can be seen as the stability to constant solutions for $n \times n$ nonlinear hyperbolic conservation laws. In [13], Wang

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and Li obtained the perturbed solutions to Riemann problems for the isentropic gas dynamics equations (E). Since there is no interaction between the elementary waves in [13], this core result can be thought of as the perturbation of a single shock. The stability of the single shock for $n \times n$ nonlinear hyperbolic conservation laws was obtained by Chern [4]. Furthermore, under the Chern stability condition [4], Asakura [1] obtained asymptotic stability of solutions with a single shock for $n \times n$ systems. In [2, 3], Asakura studied asymptotic stability of solutions with strong shocks which do not interact with each other for the isothermal gas dynamics equations. If the solutions of the Cauchy problems for 2×2 nonlinear hyperbolic conservation laws only consist of rarefaction waves, then the corresponding perturbation problems have been solved by Liu [9] and Lin and Xie [8].

In this paper, the core result is about the perturbation of two elementary waves for (E) and (I). In other words, we study the existence of entropy solutions for the perturbed problem (E) and ($\bar{\mathbf{I}}$) by Glimm's scheme where the unperturbed initial data (I) will produce two elementary waves with interaction.

2. Properties of shock curves and estimates on interactions

The characteristics of the system (E) are $\lambda_1 = -\sqrt{-p'(y)} = -\lambda_2$. The corresponding Riemann invariants can be taken as $r = u + \int_1^v \sqrt{-p'(y)} dy$ and $s = u - \int_1^v \sqrt{-p'(y)} dy$. The mapping from $(u, v), v > 0$ to $(r, s), s - r > -2 \int_1^v \sqrt{-p'(y)} dy$ is one-to-one and onto, so we may use (u, v) or (r, s) as the variables at our convenience.

To carry out our analysis, we improve the well-known framework given in [6] and [10].

We now briefly describe Glimm's difference scheme [6]. Choose mesh lengths l, h so that l/h is a constant which is greater than $\max\{|\lambda_1|, |\lambda_2|\}$. In each step $nh < t < (n+1)h$ the approximate solution is obtained exactly and consists of elementary waves generated at points of discontinuity $x = ml$ where $n \geq 0, m$ are integers. At time $t = (n+1)h$, the value of the approximate solution $(u^l(x, t), v^l(x, t))$, $(m-1)l < x < (m+1)l$, is set to be the value of the exact solution in the strip $nh < t < (n+1)h$ at $t = (n+1)h$ and $x = (m + e_{n+1})l$. Here $\{e_n\}$ is a prechosen random sequence in $(-1, 1)$. The upper half (x, t) -plane is covered by diamonds $\Delta_{m,n}$ with vertices $a_{m,n-1}, a_{m+1,n}, a_{m,n+1}, a_{m-1,n}$, where the mesh point $a_{i,j} = (i + e_j, j)$, $i + j = \text{even}$. We consider curves consisting of line segments of the form $L_{m,n,m+1,n+1}$ joining $a_{m,n}$ to $a_{m+1,n+1}$ and $L_{m,n,m+1,n-1}$ joining $a_{m,n}$ to $a_{m+1,n-1}$. If the mesh index m increases monotonically on such a curve, we call it an I -curve. Let O denote the unique I -curve passing through the mesh point at $t = 0$ and $t = h$. In order to obtain a uniform bound on the approximate solutions $(u^l(x, t), v^l(x, t))$ in $0 \leq t \leq T, x \in R$, and a uniform bound on the total variation of $(u^l(\cdot, t), v^l(\cdot, t))$ on each line $t = \text{const} > 0$, we consider the interactions of shocks and centered rarefaction waves occurring in the diamond $\Delta_{m,n}$ bounded by the I -curve J_1 and its immediate successor J_2 .

If R_1 (resp., S_1) and R_2 (resp., S_2) are the rarefaction (resp., shock) waves corresponding to the first and second characteristic families respectively, then there are six nontrivial two elementary wave interactions [10]:

- (i) R_2 interacts with R_1 ;
- (ii) S_1 interacts with S_1 (or S_2 interacts with S_2);
- (iii) R_2 interacts with S_1 (or S_2 interacts with R_1);

- (iv) S_2 interacts with S_1 ;
- (v) S_2 interacts with R_2 (or R_1 interacts with S_1);
- (vi) R_2 interacts with S_2 (or S_1 interacts with R_1).

Let β (resp., γ) be the shock wave of the first (resp., second) kind on J . Let b (resp., g) be the rarefaction wave of the first (resp., second) kind on J . We define the major strength, denoted by $|\cdot|$, of the shock, the minor strength, denoted by $|\cdot|_-$, of the shock, and the strength of the rarefaction waves, denoted by $|\cdot|$, as follows: $|\beta| = r_l - r_r$, $|\beta|_- = s_l - s_r$, $|b| = r_r - r_l$, $|\gamma| = s_l - s_r$, $|\gamma|_- = r_l - r_r$, $|g| = s_r - s_l$, where (r_l, s_l) and (r_r, s_r) denote the left state and right state of the waves, respectively.

In what follows, following [10], we denote by $\gamma + \beta \rightarrow \beta' + \gamma'$ the interaction of an S_2 and an S_1 , which produces an S_1 and an S_2 ; the other cases are written in a similar way.

Following [10], the 1-shock curve S_1 and 2-shock curve S_2 with initial point (r_0, s_0) can be written, respectively, as

$$\begin{aligned} s_0 - s &= g_1(r_0 - r, (r_0, s_0)) & r < r_0, \\ r_0 - r &= g_2(s_0 - s, (r_0, s_0)) & s < s_0. \end{aligned}$$

It is easy to prove:

Lemma 2.1. *For the 1-shock curve S_1 in the region $P_1 \subset P_0$ ($P_0 = \{(r, s) \mid v > 0\}$):*

- (i) $g'_1(a, (r_0, s_0)) \in [0, 1]$.
- (ii) *For every $\varepsilon > 0$, there is a $\delta > 0$ only depending on ε and $p(v)$ such that $g'_1(a, (r_0, s_0)) < \varepsilon$, as $a = r_0 - r < \delta$.*
- (iii) $\inf_{P_1} (1 - g'_1(a, (r_0, s_0))) > \bar{C} > 0$, where \bar{C} is a constant only depending on the region P_1 and $p(v)$.
- (iv) $g''_1(a, (r_0, s_0)) \geq 0$.

We have similar conclusions for the 2-shock curve S_2 .

For convenience, we introduce a more refined lemma in [8].

Lemma 2.2. *Suppose that all waves considered below are contained in the region $P_1 \subset P_0$, in the (r, s) -plane. Then the following estimates are valid for the corresponding interactions.*

- (i) $\gamma + \beta \rightarrow \beta' + \gamma'$:
 $||\beta'| - |\beta|| \leq C|\beta||\gamma|^3, \quad ||\gamma'| - |\gamma|| \leq C|\gamma||\beta|^3.$
- (ii) (1) $\gamma + b \rightarrow b' + \gamma'$:
 $||b'| - |b|| \leq C|b||\gamma|^3, \quad |\gamma'| = |\gamma|.$
 (2) $g + \beta \rightarrow \beta' + g'$:
 $||g'| - |g|| \leq C|g||\beta|^3, \quad |\beta'| = |\beta|.$
- (iii) (1) $\gamma_1 + \gamma_2 \rightarrow b' + \gamma'$:
 $|b'| \leq C|\gamma_1||\gamma_2|(|\gamma_1| + |\gamma_2|), \quad |\gamma'| = |\gamma_1| + |\gamma_2|.$
 (2) $\beta_1 + \beta_2 \rightarrow \beta' + g'$:
 $|g'| \leq C|\beta_1||\beta_2|(|\beta_1| + |\beta_2|), \quad |\beta'| = |\beta_1| + |\beta_2|.$

- (iv) (1) $g + \gamma \rightarrow \beta' + \gamma'$:
 $|\gamma'| - |\gamma| \leq -|\beta'| (1 + C|\gamma|^{-2})$, $|\beta'| \leq C|\gamma|^3$.
 (2) $g + \gamma \rightarrow \beta' + g'$:
 $|g'| - |g| \leq |\gamma| (C|\gamma|^8 - 1)$, $|\beta'| \leq C|\gamma|^3$.
 (3) $\beta + b \rightarrow \beta' + \gamma'$:
 $|\beta'| - |\beta| \leq -|\gamma'| (1 + C|\beta|^{-2})$, $|\gamma'| \leq C|\beta|^3$.
 (4) $\beta + b \rightarrow b' + \gamma'$:
 $|b'| - |b| \leq |\beta| (C|\beta|^8 - 1)$, $|\gamma'| \leq C|\beta|^3$.
- (v) (1) $\gamma + g \rightarrow \beta' + \gamma'$:

There exist β_0, γ_0 such that the interaction

$\gamma_0 + \beta_0 \rightarrow \beta' + \gamma'$ is the same as in (i) and

$$|\beta_0| + |\gamma_0| - |\gamma| \leq -C|\beta_0| |\gamma|^{-2}, \quad |\beta_0| \leq C|\gamma|^3.$$

- (2) $\gamma + g \rightarrow \beta' + g'$:
 $|g'| - |g| \leq |\gamma| (C|\gamma|^8 - 1)$, $|\beta'| \leq C|\gamma|^3$.
 (3) $b + \beta \rightarrow \beta' + \gamma'$:
 $|\beta_0| - |\beta| \leq -|\gamma_0| (1 + C|\beta|^{-2})$, $|\gamma_0| \leq C|\beta|^3$
where β_0, γ_0 are defined as in (1).
 (4) $b + \beta \rightarrow b' + \gamma'$:
 $|b'| - |b| \leq |\beta| (C|\beta|^8 - 1)$, $|\gamma'| \leq C|\beta|^3$.

- (vi) $g + b \rightarrow b' + g'$:

$$|g'| = |g|, \quad |b'| = |b|.$$

Here C is a generic constant which only depends on the region P_1 and $p(v)$.

If both waves with interaction are not small, we have the following lemma of estimates on the interaction:

Lemma 2.3. [6, 8] *Under the same assumption as in Lemma 2.2, the following estimates are valid for the corresponding interactions.*

- (1) $\gamma + \beta \rightarrow \beta' + \gamma'$: $|\gamma'| \leq (1 + C_1|\beta|)|\gamma|$.
 (2) $\beta_1 + \beta_2 \rightarrow \beta' + g'$: $|g'| \leq C_1|\beta_1| |\beta_2|$.
 (3) $b + \beta \rightarrow \beta' + \gamma'$: $|\gamma'| \leq C_1|\beta| |b|$.
 (4) $g + \beta \rightarrow \beta' + g'$: $|g'| \leq (1 + C_1|\beta|)|g|$.
 (5) $\beta + b \rightarrow \beta' + \gamma'$: $|\gamma'| + |\beta'| - |\beta| \leq -C_0|\gamma'|$.
 (6) $g + b \rightarrow b' + g'$: $|g'| = |g|, |b'| = |b|$.

Here C_0, C_1 are positive constants only depending on $p(v)$ and the region P_1 .

We can obtain similar estimates for the remaining cases.

Remark 2.1. In the bounded region, the key point about Glimm's interaction estimates is whether they are suitable to small waves. Once the estimates are true for small waves, they also are true for large waves. Since the region considered here is bounded, Lemma 2.3 is obviously true.

In view of (i) and (iii) in Lemma 2.1 and (5) in Lemma 2.3, it follows that there exists a constant $D > 1$ such that

$$\sup_{P_1} g'_i(a, (r_0, s_0)) < \frac{D-1}{D} \quad (i = 1, 2), \quad (2.1)$$

$$\frac{1}{1+C_0} < \frac{D-1}{D}. \quad (2.2)$$

For interactions between a large wave and a small wave belonging to the same characteristic family, we have

Lemma 2.4. *Suppose that all waves considered below are contained in the region P_1 . Then the following estimates are valid for the corresponding interactions:*

- (i) $\beta + \beta_1 \rightarrow \beta' + g' : \quad |g'| < \frac{D-1}{D} |\beta_1|.$
- (ii) $\beta + b \rightarrow \beta' + \gamma' : \quad \text{If } g'_2\left(\frac{|b|}{C_0}, (r_0, s_0)\right) < C_0 - \frac{1}{D-1}, \text{ for } (r_0, s_0) \in P_1,$
then $|\gamma'| < \frac{D-1}{D} |b|.$

There is no difficulty in proving this lemma. We can get similar conclusions for interactions between two waves belonging to the second characteristic family.

3. Existence of perturbed solutions

In this section, we shall study perturbation problems of two elementary waves with interaction for (E). Namely, we shall prove existence theorems for the perturbed solutions of (E) with (I), where the initial data (I) generate two elementary waves with interaction.

In the (r, s) -plane, the initial data corresponding to (I) are

$$(r, s)(x, 0) = (r_0(x), s_0(x)) \quad x \in R \quad (3.1)$$

where $(r_0(x), s_0(x)) = (r(u_0(x), v_0(x)), s(u_0(x), v_0(x)))$. The perturbed initial data of (3.1) are

$$(\bar{r}, \bar{s})(x, 0) = (r_0(x) + r_1(x), s_0(x) + s_1(x)) \quad (3.2)$$

where $(r_1(x), s_1(x))$ is the perturbation of the initial data $(r_0(x), s_0(x))$ in the (r, s) -plane. We assume $(r_1(x), s_1(x))$ satisfies

$$\text{TV}\{r_1(x), s_1(x)\} < \varepsilon, \quad (3.3)$$

$$\lim_{x \rightarrow \pm\infty} (r_1(x), s_1(x)) = (0, 0), \quad (3.4)$$

where TV denotes the total variation and $\varepsilon > 0$ is a sufficiently small constant only dependent on $p(v)$, $(r_0(x), s_0(x))$, and the region P_1 .

First, we study the case in which the unperturbed initial data (I) will generate two shock waves belonging to the same characteristic family. For definiteness, we assume that the two shock waves are 1-shock waves.

Theorem 3.1. *Suppose that the unperturbed initial data $(u_0(x), v_0(x))$ consist of three pieces of constant functions generating two given 1-shock waves $x_1(t)$ and $x_2(t)$, where $x_2(t)$ lies to the right of $x_1(t)$ and the strengths of $x_1(t)$ and $x_2(t)$ are $|r_1^0| = r_- - r_m$ and $|r_2^0| = r_m - r_+$, respectively (here the state $(-)$ is the left state of $x_1(t)$, $(+)$ is the right state of $x_2(t)$, and (m) is the state between $x_1(t)$ and $x_2(t)$). We assume the small perturbation has this form of initial data. If the perturbation $(r_1(x), s_1(x))$*

of the initial data $(r_0(x), s_0(x))$ in the (r, s) -plane satisfies (3.3) and (3.4), then there exists a perturbed solution of the corresponding unperturbed problem (E) with (I).

Remark 3.1. For this unperturbed problem (E) with (I) and the unperturbed problem which will be discussed in the next theorem, the authors in papers [7, 11, 14] proved the existence of unperturbed solutions and gave their structure and properties.

From (3.3), we know that $\text{TV}\{r_1(x), s_1(x)\}$ is sufficiently small. This means that comparing the strengths of $x_1(t)$ and $x_2(t)$, $\text{TV}\{r_1(x), s_1(x)\}$ is small. For definiteness, we refer to $x_1(t)$ and $x_2(t)$ as “large” shock waves. Throughout this paper, the “large” shock waves have similar meanings. From now on, we stipulate that the “small” perturbation is the perturbation of the initial data satisfying (3.3) and (3.4).

Proof of Theorem 3.1. According to Glimm’s scheme [6], we only need to prove that the decreasing variation of the approximate perturbed solutions $(\bar{u}^l(x, t), \bar{v}^l(x, t))$ is uniformly bounded for any $t > 0$. We denote by $x_1^l(t)$ and $x_2^l(t)$ the perturbed approximate large 1-shock waves of $x_1(t)$ and $x_2(t)$, respectively (Figure 1).

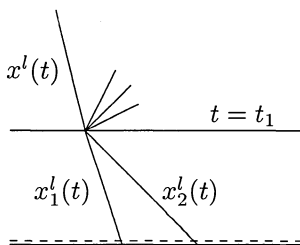


FIGURE 1

Let $t = t_1$ denote the time at which $x_1^l(t)$ interacts with $x_2^l(t)$, $x^l(t)$ denote the approximate large 1-shock wave produced by the interaction of $x_1^l(t)$ and $x_2^l(t)$, and $J_{(t_1-h)}$ (or $J_{(t_1+h)}$), where h is the time mesh, denote the I -curve between $t = t_1$ and $t = t_1 - h$ (or $t = t_1 + h$). Now we design uniformly small functionals on any I -curve J on $0 \leq t \leq t_1$ and $t \geq t_1$, respectively, to dominate the total sum of the strengths of all small shock waves crossing J (here those shock waves except $x_i^l(t)$ ($i = 1, 2$) and $x^l(t)$ are defined as small shock waves). In the following definitions of the functionals, the strengths of the 2-rarefaction waves are not considered because the interaction of $x_1^l(t)$ and $x_2^l(t)$ will generate a large 2-rarefaction wave, whereas functionals with the strengths of 2-rarefaction waves cannot be uniformly small.

(I) When an I -curve J lies on the plane $0 \leq t \leq t_1$, we define the functionals on J :

$$F_1^{(s)}(J) = \sum_{i=1}^3 F_0(J^i),$$

$$F_0(J^i) = M_i L_0(J^i) + Q_0(J^i) \quad (i = 1, 2, 3)$$

where J^1 (resp., J^3) is the part of J lying to the left (resp., right) of $x_1^l(t)$ (resp., $x_2^l(t)$), J^2 is the part of J between $x_1^l(t)$ and $x_2^l(t)$, and

$$L_0(J^i) = \sum \{ |\alpha| : \alpha \in J^i, \alpha \text{ is a shock wave or 1-rarefaction wave} \} \quad (i = 1, 2, 3),$$

$$Q_0(J^i) = \sum \left\{ |\alpha_j| |\alpha_k| : \alpha_j \in J^i, \alpha_k \in \bigcup_{i=1}^3 J^i, \alpha_j, \alpha_k \text{ are not} \right. \\ \left. 2\text{-rarefaction waves and approach each other} \right\} \quad (i = 1, 2, 3),$$

$$M_i = \prod_{j=i}^2 (1 + 4C_1 |r_j^0|) \quad (i = 1, 2), \quad M_3 = 1,$$

where C_1 is a constant appearing in Lemma 2.3 and $\alpha \in J^i$ means that α is a wave crossing J^i ($i = 1, 2, 3$). If there is no wave crossing J^i , then we stipulate $F_0(J^i) = 0$ ($i = 1, 2, 3$).

(II) When any I -curve J is on $t \geq t_1$, we define

$$F_2^{(s)}(J) = F_0(J^1) + F_0(J^3)$$

where J^1 (resp., J^3) is the part of J lying to the left (resp., right) of $x^l(t)$. The definition of $F_0(J^i)$ ($i = 1, 3$) is consistent with that of (I).

If $x_1^l(t)$ does not interact with $x_2^l(t)$ forever, then there is no need to define $F_2^{(s)}(J)$.

We shall prove by induction that

(A) When J lies on $0 \leq t \leq t_1$, $F_1^{(s)}(J) \leq F_1^{(s)}(O) < \eta$ and the total sum of the strengths of all shock waves crossing J is $L_s(J) < |r^0| + 12\varepsilon e^{4C_1|r^0|}$.

(B) When J lies on $t \geq t_1$, $F_2^{(s)}(J) \leq F_1^{(s)}(O) < \eta$ and the total sum of the strengths of all shock waves crossing J is $L_s(J) < |r^0| + 12\varepsilon e^{4C_1|r^0|}$.

Here $\eta = 4\varepsilon e^{4C_1|r^0|}$, $|r^0| = |r_1^0| + |r_2^0|$, and ε is a sufficiently small constant appearing in (3.3).

The proof will be separated into two steps.

Step I Consider the problem on $0 \leq t \leq t_1$.

$$\begin{aligned} F_1^{(s)}(O) &= \sum_{i=1}^3 M_i L_0(O^i) + \sum_{i=1}^3 Q_0(O^i) \\ &\leq \left(M_1 + \sum_{i=1}^3 L_0(O^i) \right) \sum_{i=1}^3 L_0(O^i) \\ &< 2\varepsilon(M_1 + 2\varepsilon) < 4\varepsilon e^{4C_1|r^0|} = \eta \end{aligned}$$

when $\varepsilon < 1/2$.

Let $(r_-^l(t), s_-^l(t))$ and $(r_m^l(t), s_m^l(t))$ denote the left and right states of $x_1^l(t)$ in the strip $0 < t < h$, respectively, then

$$|r_-^l(t) - r_-| \leq F_0(O^1), \quad |r_m^l(t) - r_m| \leq F_0(O^2).$$

Therefore, when $\eta < |r_1^0|$, in the strip $0 < t < h$, $x_1^l(t)$ is a 1-shock wave and

$$\begin{aligned} 0 &< |r_1^0| - \eta < |r_1^0| - (F_0(O^1) + F_0(O^2)) \leq |x_1^l(t)| \\ &\leq |r_1^0| + F_0(O^1) + F_0(O^2) < |r_1^0| + \eta < 2|r_1^0|. \end{aligned} \quad (3.5)$$

Similarly, we can prove that when $\eta < |r_2^0|$, in the strip $0 < t < h$, $x_2^l(t)$ is a 1-shock wave and

$$\begin{aligned} 0 < |r_2^0| - \eta < |r_2^0| - (F_0(O^2) + F_0(O^3)) &\leq |x_2^l(t)| \\ &\leq |r_2^0| + F_0(O^2) + F_0(O^3) < |r_2^0| + \eta < 2|r_2^0|. \end{aligned} \quad (3.6)$$

Suppose that any I -curve J_1 and its immediate successor J_2 lie on $0 \leq t \leq t_1$, $F_1^{(s)}(J_1) \leq F_1^{(s)}(O) < \eta$, $x_1^l(t)$ and $x_2^l(t)$ crossing J_1 are 1-shock waves, and the strength of $x_i^l(t)$ ($i = 1, 2$) satisfies the following inequality:

$$\begin{aligned} 0 < |r_i^0| - \eta < |r_i^0| - (F_0(J_1^i) + F_0(J_1^{i+1})) &\leq |x_i^l(t)| \\ &\leq |r_i^0| + F_0(J_1^i) + F_0(J_1^{i+1}) < |r_i^0| + \eta < 2|r_i^0|. \end{aligned} \quad (3.7)$$

We shall prove $F_1^{(s)}(J_2) \leq F_1^{(s)}(J_1)$, $x_i^l(t)$ ($i = 1, 2$) crossing J_2 is a 1-shock wave, and (3.7) is valid for J_2 , i.e., if J_2 takes the place of J_1 in (3.7), then the inequality also is valid. For this, we consider all possible types of interactions of waves in the diamond Δ bounded by J_1 and J_2 .

It is well-known that the interaction of three waves can be reduced to a sequence of interactions of two waves for the system of isentropic gas dynamics equations. For example, Wang and Li [13] discussed this problem for the polytropic gas $p = k^2 v^{-\gamma}$ (here $k > 0$ and $\gamma \geq 1$ are constants). Their discussion is applicable to general p -systems and can be generalized with no revision. Therefore, without loss of generality, we may study the interaction of two waves coming into the diamond Δ .

(i) $x_1^l(t)$ and $x_2^l(t)$ are not in Δ . Then the interactions in Δ are among the small waves. Using Lemma 2.2, we obtain $F_1^{(s)}(J_2) \leq F_1^{(s)}(J_1)$ provided

$$0 < \eta < \min \{1, \sqrt[3]{C}, (4Ce^{4C_1|r^0|})^{-1}\}$$

where C is a constant appearing in Lemma 2.2.

(ii) $x_1^l(t)$ is in Δ , but $x_2^l(t)$ is not in Δ .

(1) $\beta + \beta_1 \rightarrow \beta' + g'$ ($\beta = x_1^l(t)$, $\beta_1 \neq x_2^l(t)$, Figure 2).

$$F_0(J_2^1) = F_0(J_1^1),$$

$$F_0(J_2^3) = F_0(J_1^3),$$

$$F_0(J_2^2) - F_0(J_1^2) \leq -M_2|\beta_1|.$$

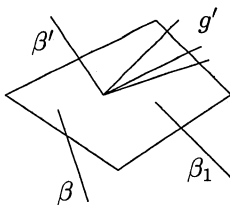


FIGURE 2

Thus $F_1^{(s)}(J_2) - F_1^{(s)}(J_1) \leq 0$.

(2) $\beta + b \rightarrow \beta' + \gamma'$ ($\beta = x_1^l(t)$, Figure 3).

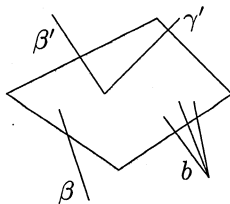


FIGURE 3

By (iv) in Lemma 2.1, we get

$$g_2' \left(\frac{|b|}{C_0}, (r_0, s_0) \right) \leq g_2' \left(\frac{\eta}{C_0}, (r_0, s_0) \right)$$

where $(r_0, s_0) \in P_1$ and C_0 is a constant appearing in Lemma 2.3. From (ii) in Lemma 2.1, there is a constant $d > 0$ only depending on $p(v)$ and $C_0 - \frac{1}{D-1}$ such that

$$g_2' \left(\frac{|b|}{C_0}, (r_0, s_0) \right) < C_0 - \frac{1}{D-1} \quad \text{when} \quad \varepsilon < \frac{dC_0}{4} e^{-4C_1|r^0|}.$$

In view of (ii) in Lemma 2.4, we obtain

$$|\gamma'| < \frac{D-1}{D} |b|.$$

Since $F_0(J_2^1) = F_0(J_1^1)$ and $F_0(J_2^3) = F_0(J_1^3)$,

$$\begin{aligned} F_0(J_2^2) - F_0(J_1^2) &= M_2(|\gamma'| - |b|) + |\gamma'| \sum |\alpha_k| - |b| \sum |\alpha_j| \\ &\leq \frac{|b|}{D} ((D-1)\eta - M_2) \leq 0 \end{aligned}$$

when $(D-1)\eta < 1$. Then $F_1^{(s)}(J_2) - F_1^{(s)}(J_1) \leq 0$.

(3) $\gamma + \beta \rightarrow \beta' + \gamma'$ ($\beta = x_1^l(t)$).

By (1) in Lemma 2.3, $|\gamma'| \leq (1 + C_1|\beta|)|\gamma|$,

$$\begin{aligned} F_1^{(s)}(J_2) - F_1^{(s)}(J_1) &= M_2|\gamma'| + (|\gamma'| - |\gamma|) \sum |\alpha_k| - M_1|\gamma| \\ &\leq M_2(1 + C_1|\beta|)|\gamma| + C_1|\beta| |\gamma| F_1^{(s)}(O) - M_1|\gamma| \\ &\leq (M_2(1 + 2C_1|\beta|) - M_1)|\gamma| \quad \text{when } \eta < 1. \end{aligned}$$

By (3.7), $|\beta| < 2|r_1^0|$; then

$$F_1^{(s)}(J_2) - F_1^{(s)}(J_1) \leq (M_2(1 + 4C_1|r_1^0|) - M_1)|\gamma| = 0.$$

(4) $g + \beta \rightarrow \beta' + g'$ ($\beta = x_1^l(t)$).

(5) $b + \beta \rightarrow \beta' + \gamma'$ ($\beta = x_1^l(t)$).

(6) $\beta_1 + \beta \rightarrow \beta' + g'$ ($\beta = x_1^l(t)$).

The proofs of (4)–(6) are similar to that of (3).

In view of (1)–(6), we have $F_1^{(s)}(J_2) \leq F_1^{(s)}(J_1)$.

Similar to the proof of (3.5), we can prove (3.7) is valid for J_2 , and $x_1^l(t)$ crossing J_2 is a 1-shock wave.

(iii) $x_2^l(t)$ is in Δ , but $x_1^l(t)$ is not in Δ . The proof is similar to that of (ii).

From (i)–(iii), we have $F_1^{(s)}(J) \leq F_1^{(s)}(O) < \eta$ for any I -curve J on $0 \leq t \leq t_1$, and (3.7) is valid for J (in particular, we have $F_1^{(s)}(J_{(t_1-h)}) < \eta$). Thus by (3.7), the total sum of the strengths of all shock waves crossing J is

$$\begin{aligned} L_s(J) &\leq |x_1^l(t)| + |x_2^l(t)| + F_1^{(s)}(J) \\ &\leq |r^0| + 3F_1^{(s)}(J) < |r^0| + 12\epsilon e^{4C_1|r^0|}. \end{aligned}$$

Step II Consider the problem on $t \geq t_1$.

First, we can show by using the techniques similar to the ones in Step I:

$$\begin{aligned} F_2^{(s)}(J_{(t_1+h)}) &= F_0(J_{(t_1+h)}^1) + F_0(J_{(t_1+h)}^3) \\ &\leq F_0(J_{(t_1-h)}^1) + F_0(J_{(t_1-h)}^3) \\ &= F_1^{(s)}(J_{(t_1-h)}) < \eta. \end{aligned} \quad (3.8)$$

Let $t = t_1$ be the initial line and $J_{(t_1+h)}$ be the initial I -curve. Similar to Step I, using (3.8), we can obtain $F_2^{(s)}(J) \leq F_2^{(s)}(J_{(t_1+h)}) < \eta$ for any I -curve J on $t \geq t_1$; moreover, $x^l(t)$ crossing J is a 1-shock wave and

$$\begin{aligned} 0 &< |r^0| - \eta < |r^0| - (F_0(J^1) + F_0(J^3)) \leq |x^l(t)| \\ &\leq |r^0| + F_0(J^1) + F_0(J^3) < |r^0| + \eta < 2|r^0|. \end{aligned} \quad (3.9)$$

Thus, it follows from (3.9) that the total sum of the strengths of all shock waves crossing J is

$$\begin{aligned} L_s(J) &\leq |x^l(t)| + F_2^{(s)}(J) \leq |r^0| + 2F_2^{(s)}(J) \\ &< |r^0| + 12\epsilon e^{4C_1|r^0|}. \end{aligned} \quad (3.10)$$

In view of Step I and Step II, according to [6, 12], the decreasing variation of the approximate perturbed solutions $(\bar{u}^l(\cdot, t), \bar{v}^l(\cdot, t))$ is bounded uniformly for any $t \geq 0$ when

$$0 < \epsilon < \min \left\{ \frac{|r_1^0|}{B}, \frac{|r_2^0|}{B}, \frac{1}{B}, \frac{1}{CB^2}, \frac{\sqrt[3]{C}}{B}, \frac{dC_0}{B}, \frac{1}{(D-1)B} \right\},$$

where $B = 4e^{4C_1|r^0|}$. □

For the case in which the unperturbed data (I) generates a 2-centered rarefaction wave (or 2-shock wave) and a 1-shock wave (or 1-centered rarefaction wave) which will interact, we state the existence theorem for the perturbed solution and give the definition of a uniformly small functional. The proof of the theorem is similar to that for Theorem 3.1.

Theorem 3.2. *Suppose that the unperturbed initial data $(u_0(x), v_0(x))$ consists of three pieces of constant functions generating a 2-centered rarefaction wave (or 2-shock wave) and a 1-shock wave $x(t)$ (or 1-centered rarefaction wave) which will interact. If we give a small perturbation to $(u_0(x), v_0(x))$, then there exists a perturbed solution of the corresponding unperturbed problem (E) and (I).*

Let $x^l(t)$ denote the perturbed approximate large 1-shock wave of $x(t)$. We design a uniformly small functional $F_1^{(s)}(J)$ on any I -curve J on the upper half-plane to control the total sum of the strengths of all small shock waves crossing J . Let

$$\begin{aligned} F_1^{(s)}(J) &= F_0(J^1) + F_0(J^2), \\ F_0(J^i) &= M_i L_0(J^i) + Q_0(J^i) \quad i = 1, 2, \\ M_1 &= 1 + 4C_1|r^0|, \quad M_2 = 0, \end{aligned}$$

where J^1 (resp., J^2) is the part of J lying to the left (resp., right) of $x^l(t)$ and $|r^0|$ is the strength of $x^l(t)$. $L_0(J^i)$ and $Q_0(J^i)$, $i = 1, 2$, are defined as in Theorem 3.1.

Now we turn to the case in which the unperturbed data (I) generates a 2-shock wave and a 1-shock wave which will interact.

Theorem 3.3. *Suppose that the unperturbed initial data $(u_0(x), v_0(x))$ consist of three pieces of constant functions generating a 2-shock wave $x_2(t)$ and a 1-shock wave $x_1(t)$ where $x_2(t)$ lies to the left of $x_1(t)$. If we give a small perturbation to $(u_0(x), v_0(x))$, then there exists a perturbed solution of the corresponding unperturbed problem (E) and (I)*

Remark 3.2. For this unperturbed problem and the unperturbed problem which will be discussed in the next theorem, Ding et al. [5] gave the existence, structure, and properties of the unperturbed solutions.

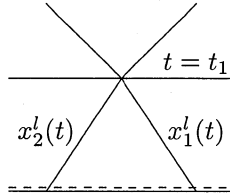


FIGURE 4

Proof of Theorem 3.3. Let $x_2^l(t)$ (resp., $x_1^l(t)$) denote the perturbed approximate large shock wave of $x_2(t)$ (resp., $x_1(t)$) (Figure 4). We assume $x_2^l(t)$ interacts with $x_1^l(t)$ at time $t = t_1$. We still denote by $x_2^l(t)$ (resp., $x_1^l(t)$) the penetrated approximate large 2-shock wave (resp., 1-shock wave) (but $x_1^l(t)$ lies to the left of $x_2^l(t)$).

Let $J_{(t_1+h)}$ denote the I -curve between $t = t_1$ and $t = t_1 + h$ (h is time mesh). For any I -curve J on $0 \leq t \leq t_1$ and $t \geq t_1$, respectively, we design uniformly small functionals to control the total sum of the strengths of all small waves (i.e., all waves except $x_i^l(t)$ ($i = 1, 2$)) crossing J .

(I) When J is on $0 \leq t \leq t_1 + h$, we define on J :

$$\begin{aligned} F_1^{(s)}(J) &= \sum_{i=1}^3 F(J^i), \\ F(J^i) &= L(J^i) + Q(J^i) \quad i = 1, 3, \\ F(J^2) &= ML(J^2) + Q(J^2), \end{aligned}$$

where J^1 (or J^3) denotes the part of J lying to the left (or right) of $x_2^l(t)$ (or $x_1^l(t)$) and J^2 the part of J between $x_2^l(t)$ and $x_1^l(t)$. Moreover,

$$\begin{aligned} L(J^i) &= \sum \{|\alpha| : \alpha \in J^i, \text{ and } \alpha \neq x_1^l(t), x_2^l(t)\} \quad i = 1, 3, \\ L(J^1) &= \sum \{|\alpha| : \alpha \in J^1\}, \\ Q(J^i) &= \sum \{|\alpha_j| |\alpha_k| : \alpha_j \in J^i, \alpha_k \in J, \alpha_j, \alpha_k \text{ approach each other} \\ &\quad \text{and } \alpha_j, \alpha_k \neq x_1^l(t), x_2^l(t)\} \quad i = 1, 2, 3, \\ M &= \max\{1 + 4C_1|r^0|, 1 + 4C_1|s^0|\} \end{aligned}$$

(here $|r^0|$ and $|s^0|$ are the strengths of $x_1(t)$ and $x_2(t)$, respectively, and C_1 is a constant appearing in Lemma 2.3).

In $L(J^i)$ and $Q(J^i)$, $i = 1, 3$, $\alpha \in J^i$ seems to imply $\alpha \neq x_j^l(t)$, $j = 1, 2$. In fact, when $J = J_{(t_1+h)}$, $\alpha \in J^1$ (or J^3) implies $\alpha = x_1^l(t)$ (or $\alpha = x_2^l(t)$). We stipulate that $F(J^2) = 0$ when there is no wave crossing J^2 .

(II) When J is on $t \geq t_1$, we define on J :

$$\begin{aligned} F_2^{(s)}(J) &= \sum_{i=1}^3 F(J^i), \\ F(J^i) &= M_i L(J^i) + Q(J^i) \quad i = 1, 3, \\ F(J^2) &= L(J^2) + Q(J^2), \end{aligned}$$

where J^1 (or J^3) is the part of J lying to the left (or right) of $x_1^l(t)$ (or $x_2^l(t)$), J^2 is the part of J between $x_1^l(t)$ and $x_2^l(t)$, $M_1 = 1 + 4C_1|B_1^0|$, $M_2 = 1 + 4C_1|B_2^0|$, $|B_1^0| = 2(1 + 2C_1|s^0|)|r^0|$, $|B_2^0| = 2(1 + 2C_1|r^0|)|s^0|$, and the rest of the definition is as in (I).

If $x_2^l(t)$ does not interact with $x_1^l(t)$ forever, then there is no need to define $F_2^{(s)}(J)$.

We now sketch the proof of the conclusion that the total variation of the approximate perturbed solutions $(\bar{u}^l(\cdot, t), \bar{v}^l(\cdot, t))$ is uniformly bounded for any $t \geq 0$. The details are omitted.

(I) Using the techniques similar to the ones for Theorem 3.1, for any I -curve J on $0 \leq t \leq t_1 + h$, we can inductively prove

$$F_1^{(s)}(J) < \eta \quad (\eta = 4M\varepsilon),$$

and the total sum of the strengths of all waves crossing J is

$$\begin{aligned} L_s(J) &\leq |r^0| + |s^0| + F(J^1) + 2F(J^2) + F(J^3) + F_1^{(s)}(J) \\ &< |r^0| + |s^0| + 3\eta. \end{aligned}$$

(II) There is no difficulty in proving

$$F_2^{(s)}(J_{(t_1+h)}) < \max\{M_1, M_2\}\eta.$$

(III) Let $t = t_1$ be the initial line and $J_{(t_1+h)}$ be the initial I -curve. Similar to Theorem 3.1, we have that the total sum of the strengths of all waves crossing J is uniformly bounded for any I -curve J on $t \geq t_1$. \square

Now, we shall discuss the case in which the unperturbed initial data $(u_0(x), v_0(x))$ will generate a 2-shock wave and a 2-centered rarefaction wave lying to its right. For the case in which the unperturbed initial data will generate a 2-centered rarefaction wave and a 2-shock wave lying to its right, we can study it similarly.

Theorem 3.4. *Suppose that the unperturbed initial data $(u_0(x), v_0(x))$ consist of three pieces of constant functions generating a 2-shock wave $x_2(t)$ and a 2-centered rarefaction wave, where $x_2(t)$ lies to the left of the 2-centered rarefaction wave and the strengths of the 2-shock wave $x_2(t)$ and 2-centered rarefaction wave are $|s^0|$ and $|g^0|$, respectively. If we give a small perturbation to $(u_0(x), v_0(x))$, then there exists a perturbed solution of the corresponding unperturbed problem (E) and (I).*

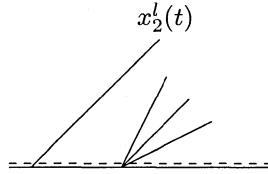


FIGURE 5

Proof. Let $x_2^l(t)$ denote the perturbed approximate shock wave of $x_2(t)$ (Figure 5). The perturbed 2-rarefaction wave which originated from $t = 0^+$ will be scattered in the time evolution. Because of the perturbation, some small 2-rarefaction waves will appear at $t = 0^+$. The interaction of 2-rarefaction waves and $x_2^l(t)$ from the left or right will produce some new approximate 1-shock waves. We denote by \mathbf{S} the set of $x_2^l(t)$, these new approximate 1-shock waves and those approximate 1-shock waves produced by interactions among these new approximate 1-shock waves. Let $t = t_i$, $i = 1, 2, \dots, N$ (N may be an arbitrary positive integer) be the time at which 2-rarefaction waves orderly interact with $x_2^l(t)$ from the left or right. Let $|r_i^0|$ be the strength of the 1-shock wave produced by the interaction of the 2-rarefaction wave and $x_2^l(t)$ from the left or right at time $t = t_i$, $i = 1, 2, \dots, N$, whose corresponding approximate 1-shock wave is denoted by $x_1^i(t)$, $i = 1, 2, \dots, N$.

We now inductively design functional sequences $\{F_i^{(s)}\}_{i=1}^{N+1}$ and $\{F_i^{(L)}\}_{i=1}^{N+1}$ such that for any I -curve J on $t_{i-1} \leq t \leq t_i$ where $1 \leq i \leq N+1$, $t_0 = 0$, the functional $F_i^{(s)}(J)$ dominates the total sum of the strengths of all shock waves crossing J except those belonging to \mathbf{S} and is uniformly small for i and J , and $F_i^{(L)}(J)$ dominates

the total sum of the strengths of the shock waves crossing J belonging to \mathbf{S} and is uniformly bounded for i and J .

(I) When any I -curve J is on $0 \leq t \leq t_1 + h$ (h is time mesh), we define on J :

$$\begin{aligned} F_1^{(s)}(J) &= F_0(J^1) + F_0(J^{N+2}), \\ F_0(J^i) &= L_0(J^i) + Q_0(J^i), \quad i = 1, N+2, \\ L_0(J^i) &= \sum \{|\alpha| : \alpha \in J^i, \alpha \notin \mathbf{S}, \alpha \text{ is not a 2-rarefaction wave} \}, \quad i = 1, N+2, \\ Q_0(J^i) &= \sum \{|\alpha_j| |\alpha_k| : \alpha_j \in J^i, \alpha_k \in J, \alpha_j, \alpha_k \notin \mathbf{S}, \\ &\quad \alpha_j, \alpha_k \text{ are not 2-rarefaction waves} \\ &\quad \text{and approach each other} \}, \quad i = 1, N+2, \\ F_1^{(L)}(J) &= L_s^{(L)}(J) + F_1^{(s)}(J) + L_{2s}^{(s)}(J), \end{aligned}$$

$$L_s^{(L)}(J) = \sum \{|\alpha| : \alpha \in J \cap \mathbf{S},$$

$$L_{2s}^{(s)}(J) = \sum \{|\alpha| : \alpha \in J, \alpha \text{ is a 2-shock wave, and } \alpha \neq x_2^l(t) \}$$

$$M = 2 + 4C_1|s^0|,$$

where J^1 (or J^{N+2}) is the part of J lying to the left (or right) of $x_2^l(t)$, and C_1 is a constant appearing in Lemma 2.3,

(II) When J is on $t_n \leq t \leq t_n + h$, $n \leq N$ is arbitrary. It does no harm to suppose that there are no interactions among the waves belonging to \mathbf{S} below J and that each of these waves is not eliminated. Then the approximate shock waves $x_1^{l_1}(t), \dots, x_1^{l_n}(t)$ belonging to \mathbf{S} are all crossing J . Otherwise, our problem will be simpler. We define functionals on J :

$$\begin{aligned} F_{n+1}^{(s)}(J) &= \sum_{i=1}^{n+1} F_0(J^i) + F_0(J^{N+2}), \\ F_0(J^i) &= M_i L_0(J^i) + Q_0(J^i), \quad i = 1, 2, \dots, n+1, \end{aligned}$$

where J^1 is the part of J lying to the left of $x_1^{l_1}(t)$, J^i , $i = 2, 3, \dots, n$, is the part of J between $x_1^{l_{i-1}}(t)$ and $x_1^{l_i}(t)$, J^{n+1} is the part of J between $x_1^{l_n}(t)$ and $x_2^l(t)$, and J^{N+2} is the part of J lying to the right of $x_2^l(t)$. The definition of $F_0(J^{N+2})$ is as in (I).

$$\begin{aligned} L_0(J^i) &= \sum \{|\alpha| : \alpha \in J^i, \alpha \notin \mathbf{S}, \text{ and } \alpha \text{ is not a 2-rarefaction wave} \} \\ &\quad i = 1, \dots, n+1, \end{aligned}$$

$$\begin{aligned} Q_0(J^i) &= \sum \{|\alpha_j| |\alpha_k| : \alpha_j \in J^i, \alpha_k \in J, \alpha_j, \alpha_k \notin \mathbf{S}, \\ &\quad \alpha_j, \alpha_k \text{ are not 2-rarefaction waves and approach each other} \} \\ &\quad i = 1, \dots, n+1, \end{aligned}$$

$$M_i = \prod_{j=i}^n (1 + 6C_1|r_j^0|) \quad i = 1, 2, \dots, n, \quad M_{n+1} = 1,$$

$$F_{n+1}^{(L)}(J) = L_s^{(L)}(J) + F_{n+1}^{(s)}(J) + L_{2s}^{(s)}(J).$$

The definitions of $L_s^{(L)}(J)$ and $L_{2s}^{(s)}(J)$ are as in (I).

We now define another functional $F^+(J)$ to dominate the total sum of the strengths of those waves which interact with $x_2^l(t)$ from the right below any I -curve J on the upper half plane. Let

$$F^+(J) = L(J^{N+2}) + Q(J^{N+2}),$$

$$L(J^{N+2}) = \sum \{|\alpha| : \alpha \in J^{N+2}\},$$

$$Q(J^{N+2}) = \sum \{|\alpha_j| |\alpha_k| : \alpha_j, \alpha_k \in J^{N+2}, \alpha_j, \alpha_k \text{ approach each other} \}$$

where J^{N+2} is the part of J lying to the right of $x_2^l(t)$, whose definition is consistent with the above.

Since a large 2-rarefaction wave issued from the initial line $t = 0^+$ and newly generated 2-rarefaction waves are not small, in order to guarantee that $F_i^{(s)}(J)$ is decreasing on $t_{i-1} \leq t \leq t_i$ and uniformly small for i and J , we did not consider the strengths of 2-rarefaction waves in the definition of $F_i^{(s)}(J)$, $1 \leq i \leq N+1$. When a 2-shock wave not belonging to \mathbf{S} interacts with $x_2^l(t)$ from the left, $L_s^{(L)}(J) + F_i^{(s)}(J)$ is not decreasing; thus we add a term $L_{2s}^{(s)}(J)$ in $F_i^{(L)}(J)$ to ensure that $F_i^{(L)}(J)$ is decreasing on $t_{i-1} \leq t \leq t_i$, $i = 1, 2, \dots, N+1$. Here $F_i^{(s)}(J)$ and $F_i^{(L)}(J)$ decreasing on $t_{i-1} \leq t \leq t_i$ means that for any I -curve J_I on $t_{i-1} \leq t \leq t_i$ and its immediate successor J_{II} on $t_{i-1} \leq t \leq t_i$, we have $F_i^{(s)}(J_{II}) \leq F_i^{(s)}(J_I)$ and $F_i^{(L)}(J_{II}) \leq F_i^{(L)}(J_I)$.

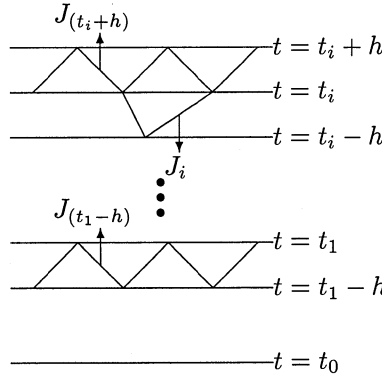


FIGURE 6

Let $J_{(t_0+h)}, J_{(t_1-h)}, J_{(t_1+h)}, \dots, J_{(t_N-h)}, J_{(t_N+h)}$ denote the I -curves lying in $t_0 \leq t \leq t_0 + h, t_1 - h \leq t \leq t_1, t_1 \leq t \leq t_1 + h, \dots, t_N - h \leq t \leq t_N, t_N \leq t \leq t_N + h$, respectively, where $t_0 = 0$ and $J_{(t_0+h)} = O$ (Figure 6). Let $J_i, i = 1, 2, \dots, N$, denote the I -curve satisfying the condition that $J_{(t_i+h)}$ is an immediate successor to it, and in the diamond Δ bounded by it and $J_{(t_i+h)}$, a 2-rarefaction wave interacts with $x_2^l(t)$ from the left or right. At each time $t = t_n, 1 \leq n \leq N$, the interaction of $x_2^l(t)$ and the elementary wave lying to its left or right is either a 2-rarefaction wave overtaking $x_2^l(t)$

or $x_2^l(t)$ overtaking a 2-rarefaction wave, namely, a part of $|r_i^0|$, $i = 1, 2, \dots, n$, coming from the 2-rarefaction waves which interact with $x_2^l(t)$ from the left, and another part coming from the 2-rarefaction waves which interact with $x_2^l(t)$ from the right. We relabel them. Those from the left are $|r_{l_1}^0|, \dots, |r_{l_{p(n)}}^0|$; those from the right are $|r_{r_1}^0|, \dots, |r_{r_{q(n)}}^0|$. The 2-rarefaction waves which interact with $x_2^l(t)$ from the right are denoted by $g_{r_1}, \dots, g_{r_{q(n)}}$, respectively, where $p(n) + q(n) = n$.

In order to prove this theorem, we only need to prove for all $k \in Z^+$ that:

(A) When any I -curve J is on $t_{k-1} \leq t \leq t_k$,

$$\begin{aligned} F_k^{(s)}(J) &\leq F_k^{(s)}(J_{(t_{k-1}+h)}) < \eta, \\ F_k^{(L)}(J) &\leq F_k^{(L)}(J_{(t_{k-1}+h)}) < G, \\ F^+(J) &\leq F^+(J_{(t_{k-1}+h)}) \leq F^+(O) < W. \end{aligned}$$

(B) When J is on $t_k \leq t \leq t_k + h$, i.e., $J = J_{(t_k+h)}$,

$$\begin{aligned} F_{k+1}^{(s)}(J) &\leq F_1^{(s)}(O) \prod_{j=1}^k (1 + 6C_1|r_j^0|) < \eta, \\ F_{k+1}^{(L)}(J) &\leq F_1^{(L)}(O) - \frac{C_0}{2} \prod_{j=1}^{p(k)} |r_{l_j}^0| + 4C_1|s^0| \sum_{j=1}^{q(k)} |g_{r_j}| < G, \\ \sum_{j=1}^{q(k)} |r_{r_j}^0| &\leq 2C_1|s^0| \sum_{j=1}^{q(k)} |g_{r_j}|, \\ \sum_{j=1}^{q(k)} |g_{r_j}| &\leq F^+(O) - F^+(J) \leq F^+(O) < W \end{aligned}$$

where $W = (|g^0| + 2\varepsilon)^2 + |g^0| + 2\varepsilon$, $G = (4C_1W + 1)|s^0| + 10M\varepsilon$,

$$\eta = 4M\varepsilon \exp \left\{ \frac{12C_1}{C_0} ((1 + 4C_1W + C_0C_1W)|s^0| + 10M\varepsilon) \right\},$$

C_0, C_1 are constants appearing in Lemma 2.3, and ε is a sufficiently small constant appearing in (3.3).

We shall prove conclusions (A) and (B) by induction. We only give the main steps below. The details are omitted here.

(I) We prove (A) and (B) are valid for $k = 1$.

(i) By considering all possible types of interactions of waves, it is not difficult to show that for any I -curve J on $0 \leq t \leq t_1$, $F_1^{(s)}(J)$ and $F^+(J)$ are decreasing and the strength of $x_2^l(t)$ crossing J satisfies

$$\begin{aligned} |x_2^l(t)| &\leq |s^0| + 2\varepsilon + \sum_{i=1}^r (|\beta| + |\gamma|) + \sum_{i=1}^{\ell} |\gamma| \\ &< |s^0| + 2\varepsilon + 2D(F_1^{(s)}(O) - F_1^{(s)}(J)) \\ &< 2|s^0| \end{aligned}$$

where $\sum^r (|\beta| + |\gamma|)$ denotes the total sum of the strengths of 1-shock waves and 2-shock waves which interact with $x_2^l(t)$ from the right below J and $\sum^\ell |\gamma|$ denotes the total sum of the strengths of 2-shock waves which interact with $x_2^l(t)$ from the left below J .

Therefore, we can prove (A) is valid for $k = 1$.

(ii) When the I -curve $J = J_{(t_1+h)}$, then at time $t = t_1$, the interaction is either $g + \gamma \rightarrow \beta' + \gamma'$ ($\gamma = x_2^l(t)$) or $\gamma + g \rightarrow \beta' + \gamma'$ ($\gamma = x_2^l(t)$) in the diamond Δ bounded by $J_{(t_1+h)}$ and J_1 . If the interaction is $g + \gamma \rightarrow \beta' + \gamma'$ ($\gamma = x_2^l(t)$), then $|r_{l_1}^0| = |r_1^0| = |\beta'|$, $|r_{r_1}^0| = 0$, $|g_{r_1}| = 0$, and $|x_2^l(t)$ crossing $J| < 2|s^0|$. If the interaction is $\gamma + g \rightarrow \beta' + \gamma'$ ($\gamma = x_2^l(t)$), then $|r_{l_1}^0| = 0$, $|r_{r_1}^0| = |r_1^0| = |\beta'|$, $g_{r_1} = g$, and $|x_2^l(t)$ crossing $J| < 2|s^0|$. Thus we can prove the conclusion (B) is valid for $k = 1$.

(II) Suppose that the conclusions (A) and (B) are valid for $k = n$. We need to prove that (A) and (B) are valid for $k = n + 1$.

Let $t = t_n$ be the initial line and $J_{(t_n+h)}$ is the initial I -curve.

(i) When any I -curve J is on $t_n \leq t \leq t_{n+1}$, by considering all possible types of interactions of waves, we have that $F_{n+1}^{(s)}(J)$, $F_{n+1}^{(L)}(J)$, and $F^+(J)$ are decreasing and the strength of $x_2^l(t)$ crossing J satisfies:

$$\begin{aligned} |x_2^l(t)| &\leq |s^0| + 2\varepsilon + \sum^r (|\beta| + |\gamma|) + \sum^\ell |\gamma| \\ &< |s^0| + 2\varepsilon + 2D(F_1^{(s)}(O) + 6C_1\eta \sum_{i=1}^n |r_i^0| - F_{n+1}^{(s)}(J)) \\ &< 2|s^0|. \end{aligned}$$

Therefore, using the results that (A) and (B) are valid for $k = n$, we can prove (A) is valid for $k = n + 1$.

(ii) When the I -curve $J = J_{(t_{n+1}+h)}$, then at time $t = t_{n+1}$, the interaction is either $g + \gamma \rightarrow \beta' + \gamma'$ ($\gamma = x_2^l(t)$) or $\gamma + g \rightarrow \beta' + \gamma'$ ($\gamma = x_2^l(t)$) in the diamond Δ bounded by $J_{(t_{n+1}+h)}$ and J_{n+1} .

If the interaction is $g + \gamma \rightarrow \beta' + \gamma'$ ($\gamma = x_2^l(t)$), then $|r_{l_{p(n+1)}}^0| = |r_{n+1}^0| = |\beta'|$, $|r_{q(n+1)}^0| = 0$, $|g_{r_{q(n+1)}}| = 0$, and the strength of $x_2^l(t)$ crossing J is less than $2|s^0|$.

If the interaction is $\gamma + g \rightarrow \beta' + \gamma'$ ($\gamma = x_2^l(t)$), then $|r_{l_{p(n+1)}}^0| = 0$, $|r_{q(n+1)}^0| = |r_{n+1}^0| = |\beta'|$, $g_{r_{q(n+1)}} = g$, and the strength of $x_2^l(t)$ crossing J is less than $2|s^0|$.

Consequently, using the results obtained in (i) and the result that (B) is valid for $k = n$, we also can prove (B) is valid for $k = n + 1$. \square

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