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Stability of Localized Structures in Non-Local Reaction-Diffusion Equations

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ABSTRACT. The stability of non-homogeneous, steady state solutions of a scalar, non-local reaction-diffusion equation is considered. Sufficient conditions are provided that guarantee that the relevant linear operator possesses a countable infinity of discrete eigenvalues. These eigenvalues are shown to interlace the eigenvalues of a related local Sturm-Liouville operator. An oscillation theorem for the corresponding non-local eigenfunctions also is established. These results are applied to assess the stability of n-pulse solutions of a model which describes hot spot formation in a microwave heated ceramic fiber. Each n-pulse solution contains n spatially localized regions of elevated temperature. It is shown that the 1-pulse solution is metastable in that the principal eigenvalue of the corresponding linear operator is exponentially small. For $n \geq 2$, all solutions are unstable with corresponding principal eigenvalues bounded away from the origin.

1. Introduction

Spatially non-trivial solutions arise in a variety of contexts. An important property for any physically realizable solution is that it be asymptotically stable to perturbations in the initial data. In this paper, we consider the stability problem for non-homogeneous steady state solutions of non-local reaction-diffusion equations. We apply our results to spatially localized solutions that arise in a model for hot spot formation in a microwave heated ceramic fiber.

The equation of interest is

$$u_t = D^2 u_{xx} + G\Big(u, \int_0^1 f(u)\,dx\Big), \tag{1.1a}$$

$$u_x(0,t) = u_x(1,t) = 0 \tag{1.1b}$$

where G and f are assumed to be sufficiently smooth, $0 \leq x \leq 1$, and D is the diffusion coefficient. A steady state solution, $U_0(x)$, of (1.1) satisfies

$$0 = D^2 U_{0_{xx}} + G(U_0, I_0), \tag{1.2a}$$

$$U_{0_x}(0) = U_{0_x}(1) = 0, \tag{1.2b}$$

$$I_0 = \int_0^1 f(U_0)\,dx. \tag{1.2c}$$

For (1.1), Chafee [3] has shown that linear stability of solutions implies asymptotic stability with respect to the underlying partial differential equation (1.1) in an

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appropriate function space. The linear stability of the solution U_0 is determined in the usual manner by inserting $u = U_0(x) + e^{-\lambda t}\phi(x)$ into (1.1) and linearizing about $U_0(x)$. This yields

$$D^2\phi'' + \Big(\frac{\partial G}{\partial u}(U_0(x), I_0) + \lambda\Big)\phi = -\frac{\partial G}{\partial I}(U_0(x), I_0)\int_0^1 \frac{\partial f}{\partial u}(U_0(x))\phi\, dx, \tag{1.3a}$$

$$\phi'(0) = \phi'(1) = 0. \tag{1.3b}$$

We rewrite this as

$$D^2\phi'' + (A(x) + \lambda)\phi = B(x)\int_0^1 C(x)\phi\, dx, \tag{1.4a}$$

$$\phi'(0) = \phi'(1) = 0 \tag{1.4b}$$

where $A(x) = \frac{\partial G}{\partial u}(U_0(x), I_0)$, $B(x) = -\frac{\partial G}{\partial I}(U_0(x), I_0)$, and $C(x) = \frac{\partial f}{\partial u}(U_0(x))$. The stability of U_0 is assured if solutions of (1.4) satisfy Re $\lambda > 0$.

The non-local eigenvalue problem defined in (1.4) is non-standard due to the presence of the integral operator on the right-hand side. However, its spectrum is intimately linked with the eigenvalues of the related local Sturm-Liouville problem

$$D^2\psi'' + (A(x) + \nu)\psi = 0, \tag{1.5a}$$

$$\psi'(0) = \psi'(1) = 0. \tag{1.5b}$$

It is well known that there exists a countable infinity of discrete and simple local eigenvalues $\{\nu_n\}$ of (1.5) with corresponding eigenfunctions $\{\psi_n\}$, $n = 0, 1, 2\ldots$; see e.g., [4]. Furthermore, there exists an oscillation theorem for the local eigenpair (ψ_n, ν_n), which states that the number of interior zeros of the eigenfunction equals n. For the generic non-local problem (1.4), an eigenvalue λ may be complex-valued. Motivated by the application to microwaves, we provide a condition which assures that the non-local eigenvalue λ will be real. This condition will allow us to show that there exists a countable infinity of discrete and simple non-local eigenvalues $\{\lambda_j\}$ of (1.4). Moreover, the analysis establishes the novel result that the non-local eigenvalues interlace the local ones, i.e., between any two local eigenvalues, there is a non-local eigenvalue. The structure of the non-homogeneous solutions being studied provides a second condition which guarantees that the non-local eigenfunctions obey an oscillation theorem which is analogous to the local oscillation theorem. However, there exists a fundamental difference between the local and non-local cases. Depending on the context, the non-local problem may not possess an eigenfunction of strictly one sign. In the local case, there necessarily exists such an eigenfunction. Thus the non-local sequence $\{\lambda_j\}$ need not start with $j = 0$.

These two results are applied to an example involving formation of spatially localized hot spots in a microwave heated ceramic fiber. This phenomenon occurs when a thin fiber is heated in a highly resonant, single mode cavity. The spot forms along the axis of the sample and begins to propagate outward elevating the temperature of the ceramic sample [10, 12, 13]. In most instances, the spot eventually becomes stationary, thus leaving a localized region of the fiber at a dramatically higher temperature than the rest. These interesting phenomena occur even though the cavity geometry and the polarization of the exciting electric field produce an electric field whose intensity is constant along the axis of the fiber.

In [11], Kriegsmann derives an equation to model this situation. He shows that for different values of the power, the model supports the existence of an S shaped curve of spatially homogeneous, steady state solutions. Analysis on this curve shows that the upper and middle branches lose stability as the value of the diffusion constant is lowered. It is natural to ask, therefore, if non-homogeneous steady state solutions exist and, if so, are they stable? For D sufficiently small, using an asymptotic analysis, Kriegsmann shows that a highly localized, symmetric, pulse-like solution can be formally constructed. He also obtains numerical results from a time-dependent code which suggest that this 1-pulse solution is, in fact, stable in parameter regimes where the spatially homogeneous solution is unstable. In a companion paper, Bose [1] shows how to construct an actual symmetric 1-pulse solution using geometric singular perturbation theory for a variant of the model in [11]. This solution lies in a neighborhood of the asymptotic one and is realized as the transverse intersection of relevant invariant manifolds. It also is proved that n-pulse solutions exist [1]. These solutions are time independent and contain n localized hot spots.

In this paper, we conduct a detailed stability analysis of the n-pulse solutions. We find that for $n \geq 2$, the n-pulse solution is unstable, with principal eigenvalue bounded away from the origin as $D \to 0$. The situation for the 1-pulse solution is much more delicate. Our analysis shows that the 1-pulse solution also is unstable, but that its principal eigenvalue is exponentially small in the diffusion constant D. Thus it, and its translates, persist for exponentially long amounts of time. The unstable 1-pulse and its translates fall into a class of solutions known as 'metastable'. Establishing that the 1-pulse is metastable follows directly from results of Ward [14]. Metastable behavior also has been studied in [2, 9] for scalar, local problems. It is interesting to note that the type of metastability found here is qualitatively different than in those works. For the local problem, pulse solutions are $O(1)$ unstable, and solutions that are constructed by piecing together reduced versions of front and back solutions are metastable. Arbitrarily high numbers of such reduced solutions can be suitably concatenated to form a metastable solution, with each piece contributing an exponentially small eigenvalue [2, 9]. In our case, for the non-local problem, the 1-pulse alone is metastable and the n-pulses are simply unstable, i.e., additional pulses contribute $O(1)$ unstable eigenvalues. While the non-local problem is capable of producing the metastability of a pulse, it cannot duplicate the diversity of patterns available in the local case.

Establishing the oscillation theorem, below, does not specifically rely on the singular nature of the solutions under consideration. In fact, it is quite general. However, utilizing it to assess the stability of the 1-pulse relies on the singular structure of U_0 in two key ways. First, the localized structure is needed to prove the existence of an exponentially small eigenvalue. Second, it is used to establish the non-existence of an eigenfunction of strictly one sign. The singular nature of the n-pulses is used to establish their instability.

This paper is organized as follows. In Section 2, we discuss the non-local eigenvalue problem (1.4). Sufficient conditions are provided to establish the existence and oscillatory behavior of solutions. The oscillation theorem is stated and proved. In Section 3, we analyze the non-local model for ceramic fibers that arises in [11]. We establish the existence of at most two exponentially small eigenvalues and show that the other eigenvalues are strictly positive. We provide arguments which indicate that there is actually only one exponentially small eigenvalue. Section 4 contains numerical simulations and a discussion.

2. The non-local eigenvalue problem

The linear stability analysis for scalar non-local problems has previously been conducted by Freitas [5–7]. He establishes his results by viewing the linear operator of the non-local problem as a perturbation of a local operator and then showing that the perturbation can be continued to large values of the perturbation parameter. Information about the specific problem (1.4) then is obtained as a subcase of the more general results. Freitas proves a number of important and interesting results concerning solutions to generic non-local equations for the case of Dirichlet boundary conditions. In particular, in [6], he proves an oscillation theorem similar to, but less precise, than ours. For Neumann boundary conditions [7], he considers a specific equation in which the non-local term enters in a convenient fashion. For this case, he does not establish any analogous oscillation theorem.

Our approach to the non-local eigenvalue equation will be similar to Freitas in that we will relate the spectra of the local and non-local operators. This will be achieved without using a perturbation argument, however. Rather, we shall directly compare the non-local equation (1.4) with the local equation (1.5). Our results will be less general but, as a result, more focused to the microwave application.

In order to establish the existence of eigenvalues and the oscillation of eigenfunctions, we impose the following two conditions on the functions $A(x)$, $B(x)$, and $C(x)$:

(C1) $B(x) = kC(x)$, where $k \in \Re/\{0\}$, and

(C2) $A(x)$, $B(x)$, and $C(x)$ are symmetric about $x = 1/2$.

Of these two conditions, (C1) is definitely the more restrictive. First, (C1) requires that whenever, and however, the non-local term enters the equation, it must be multiplied by a factor of $f(u)$. Second, $f(u)$ must be equal to $e^{c_1 u}$ where $c_1 > 0$. The exponential form of $f(u)$ is widely used in ceramic applications [8], including the one of interest in this paper. Thus it is natural to exploit its properties. The major simplification (C1) offers is that it implies that the eigenvalues of (1.4) must be real. This is easily demonstrated by showing that the linear operator associated with (1.4) then becomes self-adjoint in L_2. The condition (C2) does not restrict the problem in any substantial way. Note that an n-pulse solution to (1.2) will necessarily be symmetric about the midpoint $x = 1/2$. This can be seen by phase plane arguments, since (1.2) defines a Hamiltonian system. Thus, for these solutions, $A(x)$, $B(x)$, and $C(x)$ are automatically symmetric about $x = 1/2$. (C1) is needed to establish the existence of non-local eigenvalues. Once this is established, (C2) is used to obtain the oscillation result.

2.1. Interspersing of local and non-local eigenvalues. Assume initially that only condition (C1) holds. Let $\{\psi_n\}$ denote the complete set of orthonormal eigenfunctions of (1.5) and $\{\nu_n\}$ the corresponding eigenvalues. Then the solution of (1.4) can be expanded in terms of these functions as the series

$$\phi = \sum_{m=0}^{\infty} a_m(\lambda)\psi_m \tag{2.1}$$

where the unknown coefficients depend upon λ. Multiplying (1.4a) by ψ_n, integrating the ensuing equation from $x = 0$ to $x = 1$, applying integration by parts twice, and using (1.5) and (2.1), we obtain

$$a_n = J\frac{k\beta_n}{\lambda - \nu_n} \tag{2.2}$$

where

$$J = \int_0^1 C(x)\phi\, dx, \tag{2.3a}$$

$$\beta_n = \int_0^1 C(x)\psi_n\, dx. \tag{2.3b}$$

Multiplying (2.1) by $C(x)$, integrating the resulting expression from $x = 0$ to $x = 1$, and using condition (C1) yields

$$J = Jk\sum_{n=0}^{\infty}\frac{\beta_n^2}{\lambda - \nu_n}. \tag{2.4}$$

For each λ, the series converges since ν_n behaves asymptotically like n^2 [4], and β_n remains bounded as $n \to \infty$ since ψ_n is highly oscillatory in this limit. There are two possibilities to consider for equation (2.4). If $J = 0$, then the non-homogeneous term in (1.4) drops out and the eigenvalues of the non-local problem coincide with those of the local problem. If $J \neq 0$, then (2.4) reduces to

$$\frac{1}{k} = \sum_{n=0}^{\infty}\frac{\beta_n^2}{\lambda - \nu_n}. \tag{2.5}$$

With the proviso of $J \neq 0$, the eigenvalues of (1.4) are given by the solutions of (2.5). Assume for a moment that $\beta_n \neq 0$ for all n. The solutions of (2.5) are real and can be estimated by a graphical analysis of the right-hand side of (2.5). Note that at a solution of (2.5), the right-hand side is decreasing in λ. It is readily seen that since the eigenvalues ν_n of the Sturm-Liouville problem (1.5) are real and distinct that there exist $\{\lambda_n\}$ with the property that if $k > 0$,

$$\nu_0 < \lambda_0 < \nu_1 < \lambda_1 < \nu_2 \cdots ; \tag{2.6a}$$

see Figure 1. Thus, the real non-local eigenvalues interlace the local eigenvalues. That there exists at most one non-local eigenvalue between any two local eigenvalues follows easily by using the monotonicity in λ. If $k < 0$, then there exist $\{\lambda_n\}$ with the property

$$\lambda_0 < \nu_0 < \lambda_1 < \nu_1 < \lambda_2 < \nu_2 \cdots . \tag{2.6b}$$

If it happens that $\beta_i = 0$, $\beta_{i-1} \neq 0$, and $\beta_{i+1} \neq 0$ for some i, then ν_i is a non-local eigenvalue and the graphical analysis shows that there is another distinct non-local eigenvalue between ν_{i-1} and ν_{i+1}. A more thorough examination of this case occurs in the next section.

2.2. Oscillations of the non-local eigenfunctions. The eigenfunctions of a Sturm-Liouville operator are subject to an oscillation theorem. The theorem states that if ψ_n is an eigenfunction, then the number of interior zeros of ψ_n on $(0, 1)$ equals n. Similar to the ordering of the eigenvalues $\{\nu_n\} = \nu_0 < \nu_1 < \nu_2 < \cdots$, the eigenfunctions are ordered by the number of interior zeros they possess. These two

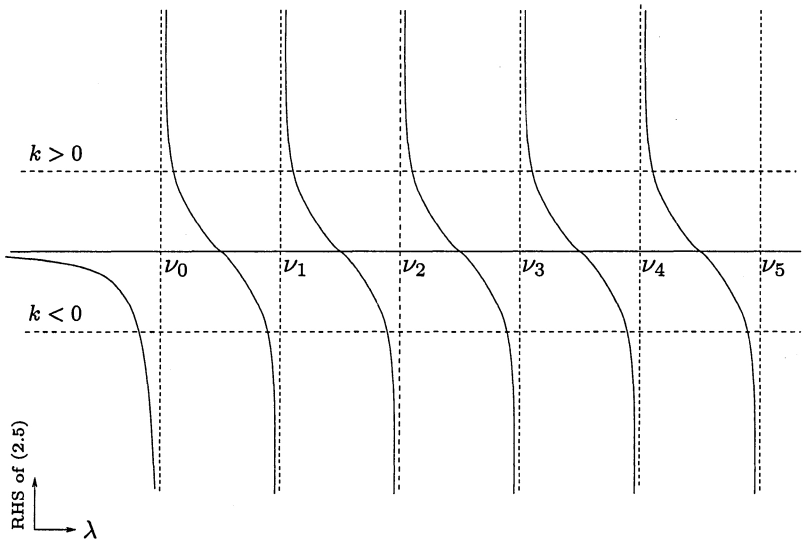

FIGURE 1. The right-hand side of equation (2.5) is plotted as a function of the non-local eigenvalue parameter λ. The cases $k > 0$ and $k < 0$ both are depicted.

ordering results can be obtained simultaneously by using the Prüfer transformation. Let $\tan\theta = \frac{D\psi'}{\psi}$. Using (1.5), this implies

$$D\theta' = \frac{D^2\psi''\psi - (D\psi')^2}{\psi^2 + (D\psi')^2} \tag{2.8a}$$

$$= -(A(x) + \nu)\cos^2\theta - \sin^2\theta \ . \tag{2.8b}$$

Without loss of generality, the boundary condition $\psi'(0) = 0$ transforms to $\theta(0) = 0$. Denote the dependence of the variable θ on ν by $\theta_\nu(x)$. An eigenvalue ν is created if $\theta_\nu(1) = n\pi$ for some integer value of n. It can be shown [4] that the angular variable $\theta_\nu(x)$ has the following properties:

(P1) $\theta_\nu(x) < \pi/2$ for all x and ν,

(P2) $\theta_\nu(x) > 0$ if ν is sufficiently large and negative,

(P3) $\theta_\nu(x)$ is monotone decreasing in ν for each x,

(P4) $\theta_\nu(x) \to -\infty$ as $\nu \to \infty$.

The existence and oscillation results are obtained by an appeal to continuous dependence of solutions on the parameter ν; see Figure 2. Note that an interior zero of an eigenfunction occurs when $\theta_\nu(x) = (2n+1)\pi/2$.

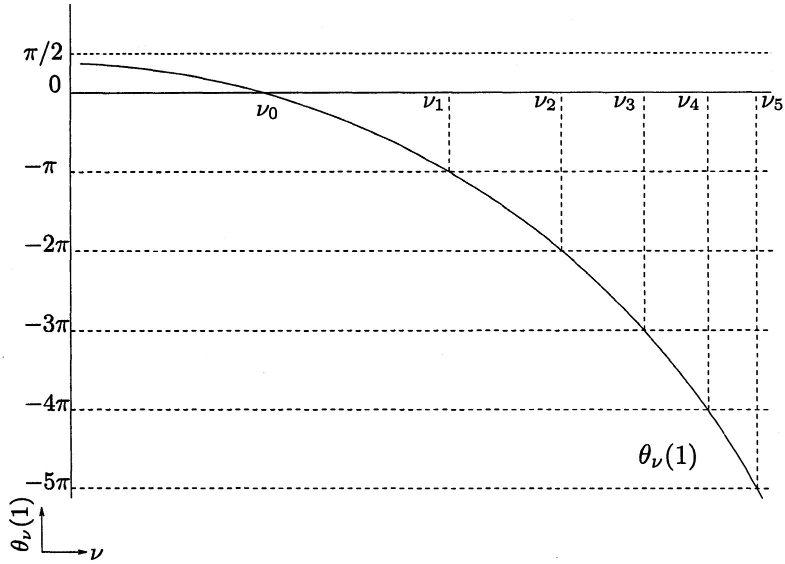

FIGURE 2. The angular variable θ_ν evaluated at $x = 1$ is plotted as a function of the local eigenvalue parameter ν. Local eigenvalues are created at those values of ν where the graph crosses $\theta_\nu(1) = n\pi$.

For the non-local problem, the Prüfer transformation also is applicable. Now let $\tan\theta = \frac{D\phi'}{\phi}$. As before, we derive

$$D\theta' = \frac{D^2\phi''\phi - (D\phi')^2}{\phi^2 + (D\phi')^2} \tag{2.9a}$$

$$= -(A(x) + \lambda)\cos^2\theta - \sin^2\theta + \frac{kC(x)J\phi}{\phi^2 + (D\phi')^2}. \tag{2.9b}$$

Here, the dependence of θ on λ is denoted $\theta_\lambda(x)$. As before, the left boundary condition transforms to $\theta_\lambda(0) = 0$, and an eigenvalue is created when $\theta_\lambda(1) = n\pi$ for an integer value of n. The property (P1) above also applies to the non-local angular variable $\theta_\lambda(x)$. However, properties (P2)–(P4) cannot, in general, be established. This is because the non-local term $kC(x)J\phi/(\phi^2 + (D\phi')^2)$ can change signs not only as λ is varied, but also for different values of x along a given trajectory of θ. In some sense, (2.8b) can be analyzed independently of (1.5), whereas (2.9b) cannot be disassociated from (1.4). Thus, it is not immediately clear how to use the angular variable to obtain an oscillation result. This difficulty is overcome by imposing condition (C2) and using it in conjunction with (2.6).

Condition (C2) requires $A(x)$, $B(x)$, and $C(x)$ to be symmetric, or even, about $x = 1/2$. By checking to see what equation $\psi(1-x)$ satisfies, it is not difficult to show that if $A(x)$ is even, then the local eigenfunctions break up into two subsets: $\{\psi_{2n}\}$ which are even and $\{\psi_{2n+1}\}$ which are odd about $x = 1/2$. The evenness requirement on $C(x)$ has profound consequences. It implies that

$$\beta_{2n+1} = \int_0^1 C(x)\psi_{2n+1}\,dx = 0 \tag{2.10}$$

since $C(x)\psi_{2n+1}$ is odd. Thus, the local eigenfunctions $\{\psi_{2n+1}\}$ also must be non-local eigenfunctions and the local eigenvalues $\{\nu_{2n+1}\}$ also must be non-local eigenvalues, i.e., $\phi_{2n+1} = \psi_{2n+1}$ and $\lambda_{2n+1} = \nu_{2n+1}$. Investigating conditions under which $\phi(1-x)$ satisfies (1.4) and using the evenness of $C(x)$ implies that the set of non-local eigenfunctions ϕ_n also consists of subsets of even and odd eigenfunctions. The odd, local eigenfunctions (and thus also non-local eigenfunctions), together with (2.6), serve as a backbone on which to build the following oscillation theorem.

Oscillation Theorem. *Let λ be a non-local eigenvalue with corresponding eigenfunction ϕ. Then, for $n \geq 1$:*

(a) *$\lambda = \nu_{2n-1}$ if and only if $\phi = \psi_{2n-1}$ has $2n-1$ interior zeros.*

(b) *$\nu_{2n-1} < \lambda < \nu_{2n+1}$ if and only if ϕ has $2n$ interior zeros.*

(c) *There exists at most one interval (ν_{2n-1}, ν_{2n+1}) that contains two non-local eigenvalues. All other such intervals contain exactly one non-local eigenvalue.*

Proof. Part (a) of the proof is obvious. To prove (b), first consider $k > 0$. Since the odd local eigenfunctions drop out of the sequence, (2.6) now becomes

$$\nu_0 < \lambda_{G_1} < \nu_2 < \lambda_{G_2} < \nu_4 < \cdots \tag{2.11}$$

where we introduce the subscript G_i to denote the ith non-local eigenvalue obtained from the graphical argument. Next, observe from (2.9a) that $\phi(x) = 0$ implies $\cos\theta = 0$, $D\theta' = -1$, and $\theta = (2n+1)\pi/2$. Thus, the angular variable is strictly decreasing at a zero of the eigenfunction. For $n = 1, 2, \ldots$, consider next $\theta_{\nu_{2n-1}}(x)$ and $\theta_{\nu_{2n+1}}(x)$. From the local result, it follows that $\theta_{\nu_{2n-1}}(1) = -(2n-1)\pi$ and $\theta_{\nu_{2n+1}}(1) = -(2n+1)\pi$. Thus, by continuous dependence on parameters, there exists at least one value λ in the open interval (ν_{2n-1}, ν_{2n+1}) such that $\theta_\lambda(1) = -2n\pi$. This value of λ must be a non-local eigenvalue. See Figure 3. From (2.11), since there uniquely exists $\lambda_{G_n} \in (\nu_{2n-2}, \nu_{2n})$, and $\lambda_{G_{n+1}} \in (\nu_{2n}, \nu_{2n+2})$, there can exist at most two graphical eigenvalues in the interval (ν_{2n-1}, ν_{2n+1}). Crucially, note that any associated eigenfunction must have exactly $2n$ interior zeros. If the eigenfunction had either $2n-1$ or $2n+1$ interior zeros, then its associated eigenvalue also would need to be a local eigenvalue, which would violate (P3). If the eigenfunction had j interior zeros where $j \neq 2n-1$, $2n$, $2n+1$, this would imply the existence of yet another eigenvalue which would need to be a local one and thus violate (P3).

To prove (c), we must show that if the interval (ν_{2i-1}, ν_{2i+1}) contains two non-local eigenvalues for some i, then for all $j \neq i$, (ν_{2j-1}, ν_{2j+1}) contains exactly one non-local eigenvalue. To clarify this, consider first the intervals (ν_0, ν_2) and (ν_1, ν_3). From the graphical argument, we know there exists a unique non-local eigenvalue in (ν_0, ν_2). From the Prüfer argument, there exists at least one, not necessarily unique, non-local eigenvalue in (ν_1, ν_3). Suppose that there exists a non-local eigenvalue in (ν_0, ν_1), then this must be the one obtained from the graphical argument. Thus, the non-local eigenvalue obtained from the Prüfer argument must lie in (ν_2, ν_3). But then this must be the unique non-local eigenvalue that lies in (ν_2, ν_4). So (ν_1, ν_3) contains exactly one non-local eigenvalue. By repeating the argument, we now see that (ν_3, ν_5) also contains exactly one non-local eigenvalue and so on. Next, suppose that there does not exist a non-local eigenvalue in (ν_0, ν_1). Then the graphical argument necessitates that there be non-local eigenvalue in (ν_1, ν_2). A second (Prüfer) eigenvalue now may lie in the interval (ν_2, ν_3). Thus (ν_1, ν_3) may contain two non-local eigenvalues. If it does, then the argument presented directly above shows that every subsequent interval

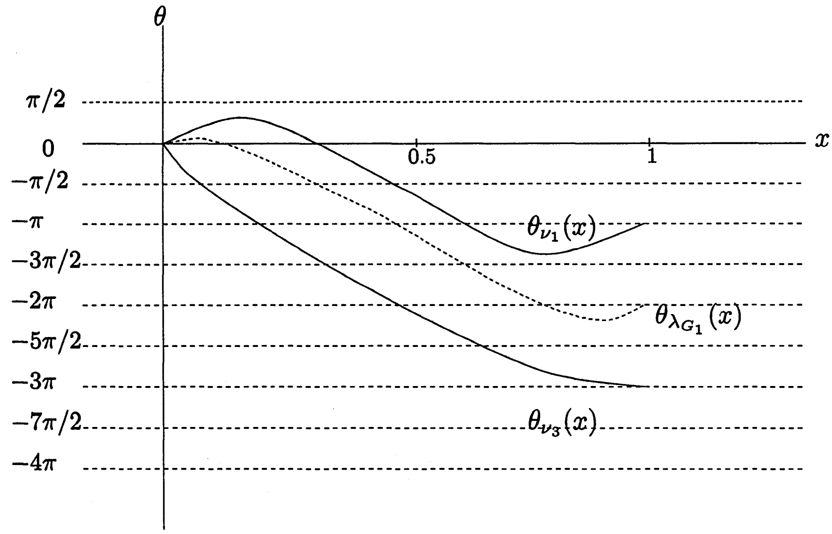

FIGURE 3. A non-local eigenvalue is shown to exist in the interval (ν_1, ν_3). The corresponding eigenfunction has exactly two interior zeros.

(ν_{2n+1}, ν_{2n+3}) for $n = 1, 2, \ldots$ contains exactly one non-local eigenvalue. If (ν_1, ν_3) does not contain two non-local eigenvalues, then check (ν_3, ν_5) and proceed as above.

Our argument does not guarantee that an interval of the form (ν_{2j-1}, ν_{2j+1}) exists which contains two-non local eigenvalues. Nor does it say exactly where this interval is, should it exist. It does show, however, that there exists at most one pair of non-local eigenfunctions that have the same number of interior zeros. □

Remark 1. The oscillation theorem gives a nearly complete description of the location of the non-local eigenvalues in terms of the local ones. It also states that the non-local eigenvalues can be ordered by comparison to the number of interior zeros of their corresponding eigenfunctions. Statement (c) relates information obtained from the graphical argument to that obtained from the Prüfer argument.

Remark 2. Although the theorem does not make explicit mention of the case $n = 0$, information about this case is implicitly present. For example, if a non-local eigenfunction of strictly one sign were to exist, then its associated eigenvalue would necessarily be less than ν_1. What the oscillation theorem does not say is whether such an eigenfunction exists. In fact, this must be resolved on a case-by-case basis.

Remark 3. For $k > 0$, the non-local eigenvalue problem (1.4) need not have a solution of strictly one sign. In the microwave problem, we show that the lack of existence of an eigenfunction of one sign is the precise reason that the 1-pulse solution is metastable, and not $O(1)$ unstable.

Remark 4. For $k < 0$, (1.4) must have a solution of strictly one sign. This is because $\lambda_{G_1} < \nu_0$, which in this case implies $\theta_{\lambda_{G_1}}(1) = 0$ and that ϕ_{G_1} has no interior zeros. Establishing the oscillation theorem for $k < 0$ then proceeds as above.

Remark 5. If condition (C1) were not enforced, then a modified version of the oscillation theorem would still hold. Points (a) and (b) would continue to hold, but (c) would not necessarily. In this case, the exact count of eigenvalues in (ν_{2n-1}, ν_{2n+1}) cannot be determined by our methods. Without (C1), complex eigenvalues also may exist. If (C1) were retained, but (C2) were discarded, then relationship (2.6) would continue to hold. However, the oscillation theorem would not follow by our methods. It is entirely likely that an oscillation theorem for this case exists. We, however, do not pursue it here. See [1] for further details.

Remark 6. The oscillation theorem is not directly tied to the fact that we have imposed Neumann boundary conditions. Suitable modifications of the theorem hold for Dirichlet and mixed boundary conditions as well. Thus, our results also can be used to sharpen those in Freitas [6].

3. Stability of hot spots

We now apply our results to the equation derived by Kriegsmann [11] which models hot spot formation in a ceramic fiber. The equation of interest is

$$u_t = D^2 u_{xx} - L(u) + \frac{pf(u)}{1 + \chi^2 (\int_0^1 f(u)\,dx)^2}, \tag{3.1a}$$

$$L(u) = 2(u + \beta[(u+1)^4 - 1]), \quad f(u) = e^{c_1 u}, \quad c_1 > 0, \tag{3.1b}$$

$$u_x(0,t) = u_x(1,t) = 0 \tag{3.1c}$$

where u denotes a dimensionless temperature along the axis, $L(u)$ models heat loss at the surface of the fiber due to convection and radiation, and the exponential function $f(u)$ represents the effective electrical conductivity of a low-loss ceramic, such as alumina [8]. The parameters D, β, and χ are assumed to be sufficiently small, which is true for fibers, and p is the dimensionless power which is proportional to the square of the amplitude of the mode which excites the cavity. The nonlocal term in (3.1a) models the detuning effect the heated fiber has upon the cavity. There exists a curve of homogeneous solutions which is determined by the solutions of

$$pf(u) = L(u)[1 + \chi^2 f^2(u)]. \tag{3.2}$$

The graph of this curve is S shaped when p is plotted on the horizontal axis, and u is plotted on the vertical axis. Kriegsmann [11] shows that the lower and middle branches of this curve lose stability as D is lowered. Naturally, one would like to know what other types of steady state solutions exist. Kriegsmann constructs a locally unique 1-pulse using the method of matched asymptotic expansions. Locally unique n-pulse solutions of (3.1) also exist [1]. These solutions are symmetric about $x = 1/2$ and have n interior layers on which a hot spot forms. We show that for $n \geq 2$, the n-pulse solutions are unstable with $O(\ln(1/D))$ instabilities. The 1-pulse solution, however, is metastable. That is, its principal eigenvalue is $O(e^{-a/D})$ where $a > 0$. Perturbations of the 1-pulse in the translational direction persist for exponentially long amounts of time. Other perturbations decay quickly to the original 1-pulse or to one of its translates. In Section 4, we discuss the implications for asymmetric hot spot formation.

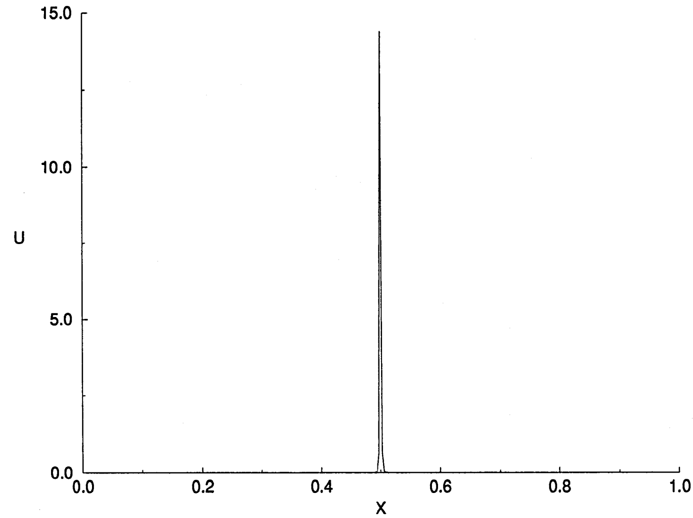

FIGURE 4. The 1-pulse solution $U_0(x)$ is depicted.

3.1. Metastability of the 1-pulse solution. We use much of the notation developed earlier in the paper. Let $U_0(x)$ denote the 1-pulse steady state solution of (3.1) pictured in Figure 4. Let $I_0 = \int_0^1 f(U_0)dx$. The non-local eigenvalue problem associated with (3.1) satisfies conditions (C1) and (C2). It can be written as follows:

$$D^2\phi'' + (A(U_0) + \lambda)\phi = kf(U_0)\int_0^1 f(U_0)\phi\, dx, \tag{3.3a}$$

$$\phi'(0) = \phi'(1) = 0. \tag{3.3b}$$

Here we have explicitly included the dependence of the potential A on the underlying 1-pulse solution and suppressed dependence on x. Now $A(U_0)$ and k are given by

$$A(U_0) = -2(1 + 4\beta(U_0 + 1)^3) + \frac{pc_1 f(U_0)}{1 + \chi^2 I_0^2}, \tag{3.3c}$$

$$k = \frac{2p\chi^2 I_0 c_1}{(1 + \chi^2 I_0^2)^2}. \tag{3.3d}$$

Note that $k > 0$, thus ν_0 will serve as a lower bound for the principal non-local eigenvalue. Also, note that this implies that there need not be an eigenfunction of strictly one sign.

As before, the symmetry of $U_0(x)$ is crucial as it implies

$$\beta_{2n+1} = \int_0^1 f(u_0)\psi_{2n+1} = 0, \tag{3.4}$$

from which it follows that the odd subscripted eigenvalues and eigenfunctions also will be non-local ones. Thus, a necessary first step in assessing the stability of $U_0(x)$ is to locate ν_1, which is the first local eigenvalue that coincides with a non-local one, and λ_{G_1}, which is the first eigenvalue obtained from the graphical analysis. Apriori, we know that the eigenfunction associated with ν_1 has one interior zero. We do not know the structure of the eigenfunction associated with λ_{G_1}. After normalization, it will be either strictly positive, or it will contain two interior zeros. Using the oscillation theorem, this will determine the location of λ_{G_1} relative to ν_1.

We first derive an estimate for ν_1 since, as we show below, it turns out to be the principal non-local eigenvalue. Let $\zeta = U_0'$ and note that ζ satisfies

$$D^2\zeta'' + A(U_0)\zeta = 0, \tag{3.5a}$$

$$\zeta(0) = \zeta(1) = 0. \tag{3.5b}$$

Also $\zeta(1/2) = 0$ since $U_0(1/2)$ is a maximum for the 1-pulse. Note that (3.5) is the equation satisfied by the eigenfunction associated with a zero eigenvalue of the local Dirichlet problem. From the ordering property of Dirichlet and Neumann eigenvalues, we can immediately conclude that $\nu_1 < 0$. To estimate the magnitude of ν_1, consider (3.3) for the eigenfunction ψ_1 and the eigenvalue ν_1 (keep in mind that $J = 0$ in this case). Using a standard trick, multiply (3.3a) by ζ and (3.5a) by ψ_1, and subtract the two ensuing equations to obtain

$$D^2[\psi_1''\zeta - \psi_1\zeta''] + \nu_1\psi_1\zeta = 0. \tag{3.6}$$

Integrating (3.6) on the interval $[1/2, 1]$, using integration by parts where necessary, and inserting the appropriate boundary conditions, we obtain

$$D^2[\psi_1'(1)\zeta(1) - \psi_1'(1/2)\zeta(1/2) - \psi_1(1)\zeta'(1) + \psi_1(1/2)\zeta'(1/2)] + \nu_1 \int_{1/2}^{1} \psi_1\zeta \, dx = 0. \tag{3.7}$$

The first, second, and fourth boundary terms are zero. Thus

$$\nu_1 = \frac{D^2\psi_1(1)\zeta'(1)}{\int_{1/2}^{1} \psi_1\zeta \, dx}. \tag{3.8}$$

Without loss of generality, the value of $\psi_1(1)$ can be chosen negative, and thus the integral in the denominator is positive. The derivative $\zeta'(1)$ is also positive. All of these imply that ν_1 is negative. Again, without loss of generality, the integral term and $\psi_1(1)$ in (3.8) can be chosen as $O(1)$ terms. Thus, ν_1 is $O(\zeta'(1))$, which is the order of the second derivative of the solution U_0 at the $x = 1$ boundary. Ward [14] shows that $\zeta'(1)$ is exponentially small. If (3.3) is rescaled by $\xi = (x - 1/2)/D$, then the derivative of the 1-pulse solution will be an eigenfunction of the ensuing equations. Due to translational invariance on $\Re$, the corresponding eigenvalue is 0. Ward shows that when the domain then is rescaled back to [0,1], this 0 eigenvalue is perturbed by an exponentially small amount. Ward's results [14] apply directly to the present situation, and we obtain $\nu_1 \sim O(e^{-a/D})$ and negative. The value of the positive constant a can be found by determining the eigenvalues of the critical points of the scaled version of (3.1). Thus, the non-local equation possesses an exponentially small, unstable eigenvalue.

We next locate λ_{G_1}. Since $\nu_1 < 0$, by Sturm-Liouville theory, $\nu_0 < 0$ also. Using the above trick with ψ_2, it is easy to show that $\nu_2 > 0$. The graphical argument implies the existence of $\lambda_{G_1} \in (\nu_0, \nu_2)$. Since $\nu_0 < 0$, we are not guaranteed that λ_{G_1} is actually a stable eigenvalue. We prove that the eigenfunction associated with λ_{G_1} must have exactly two interior zeros. Thus, an appeal to the oscillation theorem implies that $\lambda_{G_1} > \nu_1$. This will finally show that the principal non-local eigenvalue is the exponentially small ν_1.

Integrating (3.3) from $x = 0$ to $x = 1$ with $\Gamma = pc_1/(1+\chi^2 I_0^2)$ yields

$$(\lambda - 2)\int_0^1 \phi\, dx - 8\beta \int_0^1 (U_0+1)^3\phi\, dx + \Gamma J = kI_0 J. \tag{3.10}$$

Assume that there exists a strictly positive eigenfunction ϕ. Then $J > 0$. Since β, U_0, and ϕ are all positive, after rearrangement, we obtain the following estimate:

$$\begin{aligned}(\lambda - 2)\int_0^1 \phi\, dx &> J[kI_0 - \Gamma] \\ &= \frac{Jpc_1}{(1+\chi^2 I_0^2)^2}[\chi^2 I_0^2 - 1]. \end{aligned} \tag{3.11}$$

In [11], Kriegsmann derives a relationship between the parameters D, β, χ, p, and c_1 and the value of I_0. He shows that $\chi I_0 \gg 1$ for D sufficiently small. Since p and c_1 are positive by definition and $J > 0$ by assumption, the right-hand side of (3.11) is strictly positive. This implies $\lambda - 2 > 0$ since $\int_0^1 \phi\, dx > 0$. Thus, $\lambda > 2$. This result violates the oscillation theorem as $\nu_1 < \lambda$ contradicts ϕ having no interior zeros. Thus, there cannot exist a strictly positive eigenfunction.

In summary, we have shown that $\nu_1 \sim O(e^{-a/D})$ is the principal non-local eigenvalue. This eigenvalue governs the metastability of the 1-pulse solution in the translational direction along the x-axis. The first graphical eigenvalue $\lambda_{G_1} > \nu_1$. We have not ruled out that λ_{G_1} may itself be exponentially small. Heuristically, if λ_{G_1} were exponentially small, though, then there would exist another nearly invariant translational direction. The physical setup of the problem precludes this possibility. We further discuss these and related issues below in Section 4.

3.2. Instability of the n-pulse solutions. We have just shown that the 1-pulse solution is metastable. We show that if $n \geq 2$, then the n-pulse solution is unstable with principal eigenvalue bounded away from the origin as $D \to 0$.

We present the argument for a 2-pulse solution. Denote the 2-pulse solution by $\Phi_2(x)$ and its derivative by $\Upsilon(x)$. By symmetry considerations, the interior zeros of $\Upsilon(x)$ occur at $x = 1/4$, $1/2$, and $3/4$. The relevant equations for $\Upsilon(x)$ and for the related Sturm-Liouville problem are

$$D^2\Upsilon'' + A(\Phi_2)\Upsilon = 0, \tag{3.12a}$$

$$\Upsilon(0) = \Upsilon(1) = 0, \tag{3.12b}$$

and

$$D^2\psi'' + (A(\Phi_2)+\nu)\psi = 0, \tag{3.13a}$$

$$\psi'(0) = \psi'(1) = 0. \tag{3.13b}$$

Let ψ_1 be the eigenfunction of (3.13) with exactly one interior zero. Using the same procedure as the previous section, since $\beta_1 = \int_0^1 f(\Phi_2)\psi_1\, dx = 0$, it is not hard to show that $\lambda_1 = \nu_1$ and

$$\nu_1 = \frac{D^2\psi(3/4)\Upsilon'(3/4)}{\int_{1/2}^{3/4} \psi\Upsilon\, dx}. \tag{3.14}$$

Note that ν_1 is again negative. However, the value of ν_1 is now $O(D^2U_0''(1/2))$, instead of $O(D^2U_0''(1))$ for the 1-pulse. Using Kriegsmann's asymptotic analysis [11], it is straightforward to show that $|D^2U_0''(1/2))|$ grows like $\ln(1/D)$ as $D \to 0$. Thus, ν_1

remains bounded away from the origin in this limit. This implies that perturbations of the 2-pulse solution grow quickly in time. Finally, using estimates as above, it is not hard to show that ν_3 is negative and $O(e^{-a/D})$.

The stability analysis of n-pulse solutions for $n \geq 3$ is identical to that just presented, so we omit it.

4. Numerical experiments and discussion

In this section, we present a few numerical simulations to support our work. We used an implicit Crank-Nicholson scheme as in Kriegsmann [11]. The following parameter values were used for all simulations: $p = 1.0$, $D = 0.001$, $c_1 = 1$, $\beta = .01$, $\chi = .01$. All simulations were run for the same number of time steps, which is equivalent to six seconds, clearly an $O(1)$ amount of time. Figure 5a shows the evolution of a symmetric perturbation of the homogeneous steady state solution with several local maxima, one of which occurs at $x = 1/2$. Three time snapshots are superimposed. The one with nine equal local maxima corresponds to the initial condition. The snapshot with nine unequal local maxima occurs next. Finally, a snapshot of the final attracting steady state 1-pulse is provided. Notice that this 1-pulse solution is centered at $x = 1/2$.

Figure 5b shows the evolution of an initial condition that does not have a local maximum at $x = 1/2$. Four snapshots are provided. The one with twelve roughly equal local maxima corresponds to the initial condition. The next snapshot has four local maxima at about $U = 12$. The third snapshot shows that the spot centers at $x = 0.125$ and decays at the other three spatial locations. The final snapshot shows the spot centered at $x = 0.125$ and nothing else. This spot is a translate of the 1-pulse solution. We do not yet fully understand why the spot forms at $x = 0.125$ as opposed to some other value of x. By taking non-symmetric perturbations, one can shift the position of this attracting 1-pulse to seemingly any point in the domain (simulations not shown). However, when a perturbation with a sufficiently large local maximum at $x = 1/2$ is evolved, then the symmetric 1-pulse solution is the attracting solution. The asymmetric 1-pulses that are not centered at $1/2$ do not represent steady state solutions. Instead they are moving exponentially slowly towards one of the boundaries. They are manifestations of the metastability afforded by the equation.

Finally, we address the possibility of there existing two exponentially small eigenvalues. The numerical simulations above do not suggest the existence of a second exponentially small eigenvalue. A second exponentially small eigenvalue would imply that a 2-pulse solution is, in fact, metastable. Numerically, we would then expect to see this. That we did not find such a solution gives strong support to the instability of the n-pulse solutions and the metastability of the 1-pulse solution.

In conclusion, using the sufficient conditions (C1) and (C2), we have shown how to locate the spectrum of a non-local linear operator. These results then were employed to assess the stability of solutions of Kriegsmann's model [11]. In microwave heating experiments on ceramic fibers, the hot spot forms in the middle of the sample. The experimental apparatus is such that there is a symmetry about the midpoint of the fiber. Thus, any thermal perturbations to the spatially homogeneous temperature distribution will have this symmetry, too. It is evident, from the above discussions, that the hot spot corresponds to a metastable 1-pulse.

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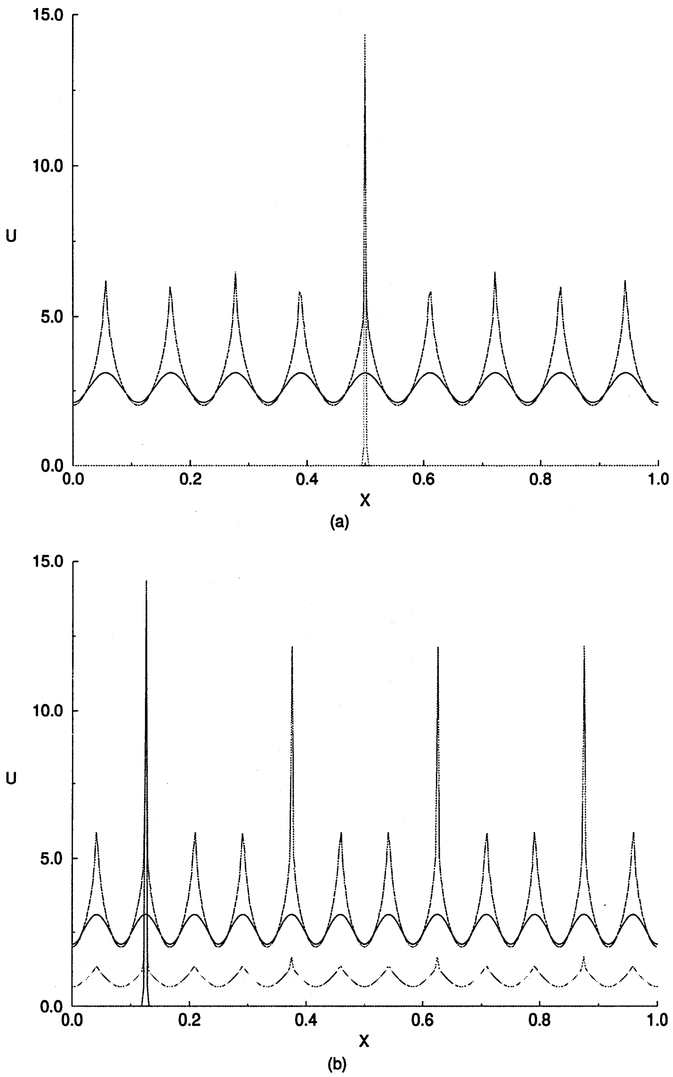

FIGURE 5. (a) The evolution of a symmetric perturbation with local maximum at $x = 1/2$ is shown to converge to the 1-pulse solution; (b) The evolution of a symmetric perturbation without a local maximum at $x = 1/2$ is shown to converge to a translate of the 1-pulse solution, after several spatial locations compete for the hot spot.

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