RATIONAL INTERPOLATION OF FUNCTIONS ON THE UNIT CIRCLE

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Dedicated to Dick Askey on the occasion of his sixty-fifth birthday

ABSTRACT. We establish various results on the convergence of a sequence of rational functions that interpolate a function on the unit circle. In particular, we extend Walsh's equi-convergence theorem and the Walsh-Sharma theorem on $L_2$ convergence (a special case of a theorem of Lozinski) for the interpolating rational functions.

1. Introduction

Let $T$ be the unit circle in the complex plane $C$, and let $D$ and $\overline{D}$ denote the open and closed unit disks, respectively. Also, let $z_{n,k} = e^{i\theta_{n,k}}$, $0 \leq \theta_{n,0} < \theta_{n,1} < \cdots < \theta_{n,n} < 2\pi$, and for a function $f$ defined on $T$, let $L_n(f; \cdot) \in \mathcal{P}_n$, the set of polynomials of degree at most $n$, such that $L_n(f; z_{n,k}) = f(z_{n,k})$ for $k = 0, 1, \ldots, n$. As early as in 1884, Méray showed that if $\{z_{n,k} : k = 0, 1, \ldots, n\}$ are the $n+1$st roots of unity, i.e., $\theta_{n,k} = 2k\pi/(n+1)$, $k = 0, 1, \ldots, n$, then the polynomials $L_n(f; \cdot)$ do not necessarily converge to $f$. Indeed, Méray took the function $f_M(z) = 1/z$. So, $L_n(f_M; z) = z^n$, $n = 1, 2, 3, \ldots$. These polynomials do not approach the function $f_M(z)$ for all $z \in D$, as $n \to \infty$, but approach the limit zero. Méray's example was examined later by Walsh [24] who pointed out that the singularity of $f_M(z) = 1/z$ at $z = 0$ caused the sequence $\{L_n(f_M; \cdot)\}$ to fail to approach $f_M(z)$ for $z \in D$. Walsh also furnished the following lovely companion to Méray's example.

Walsh's Theorem (1932, [24, Theorem 1]). Let $f$ be continuous (or more generally Riemann integrable) on $T$, and let $\{z_{n,k} : k = 0, 1, \ldots, n\}$ be the $n+1$st roots of unity. Then the sequence $\{L_n(f; \cdot)\}$ converges to the limit

$$f_D(z) = \frac{1}{2\pi i} \int_T \frac{f(t)dt}{t - z}$$

for $z \in D$, uniformly on any compact subset of $D$.

It is worthwhile to recall some steps in the historical development prior to Walsh's Theorem. Runge (1904, [19]) proved that if $\{z_{n,k} : k = 0, 1, \ldots, n\}$ are the $n+1$st roots of unity and if $f$ is analytic on $\overline{D}$, then the sequence $\{L_n(f; z)\}$ converges to $f(z)$ on $D$. After that, Fejér (1918, [9]) showed that $\{L_n(f; z)\}$ converges to $f(z)$ for $z \in D$ if $f(z)$ is assumed only to be continuous on $\overline{D}$ and analytic in $D$. See also [11].
There were further developments after Walsh’s Theorem. When the points of interpolation are the roots of unity, Lozinski (1940, [15]) proved, under Fejér’s assumption, the $L_p$ ($p > 0$) convergence of $\{L_n(f; \cdot)\}$ on the unit circle, i.e.,
\[
\lim_{n \to \infty} \int_{T} |f(z) - L_n(f; z)|^p |dz| = 0, \quad p > 0.
\] (1)
This result, when $p = 2$, was rediscovered later by Walsh and Sharma (1964, [26]). Some more recent results on interpolation at the roots of unity can be found in the work of Saff and Walsh [21], Cavaretta, et al. [6, 7], and Sharma and Vertesi [22].

Walsh gave an example to show that when the points of interpolation $\{z_{n,k}\}$ are not the roots of unity, $\{L_n(f; z)\}$ may fail to converge in $D$ even though $\{z_{n,k}\}$ are sufficiently dense on $T$.

**Walsh’s Example** ([25, pp. 293-294]). Let $z_{n,k}$ be the roots of
\[
\left(\frac{1 - \alpha z}{\alpha - z}\right)^{n+1} = 1, \quad \alpha > 1.
\]

Let $f_W(z) = 1/(\zeta + \beta)$, $0 < \beta < 1$, where $\zeta = (1 - \alpha z)/(\alpha - z)$. Then
\[
L_n(f_W; z) = \frac{1}{\zeta + \beta} + \frac{(\alpha + \beta)^n(\zeta^{n+1} - 1)}{(-1)^n + \beta^{n+1}(\zeta + \beta)(\zeta - \alpha)^n},
\]
which converges for $z$ in only a part of the unit disk $D$.

Walsh’s example shows that the distribution of the points of interpolation $\{z_{n,k}\}$ on $T$ should be taken into account when we study the convergence of interpolating polynomials $\{L_n(f; \cdot)\}$. Closely related to this observation, the following result is now well known (cf. [25, Chapter 7]).

**The Uniform Distribution Theorem.** A necessary and sufficient condition for
\[
\lim_{n \to \infty} L_n(f; z) = f(z),
\]
uniformly for $z \in \overline{D}$, for every $f$ analytic on $\overline{D}$, is that the sequence of points $\{z_{n,k} : k = 0, 1, \ldots, n\}$ is uniformly distributed on $T$, i.e.,
\[
\lim_{n \to \infty} \left| \prod_{k=0}^{n} (z - z_{n,k}) \right|^{1/n} = |z|, \quad |z| > 1.
\]

Now, let us take a closer look at the points of interpolation $\{z_{n,k}\}$ in Walsh’s example, which satisfy, for $|z| > 1$,
\[
\lim_{n \to \infty} \left| \prod_{k=0}^{n} (z - z_{n,k}) \right|^{1/n} = \lim_{n \to \infty} \left| \frac{(1 - \alpha z)^{n+1} - (\alpha - z)^{n+1}}{(-\alpha)^{n+1} - (-1)^{n+1}} \right|^{1/n} = \left| z - \frac{1}{\alpha} \right|.
\]

They are not uniformly distributed unless $\alpha = \infty$. How do we recover a function $f$ analytic on $\overline{D}$ if we are given the values of $f$ at the points $\{z_{n,k}\}$ as in Walsh’s example? According to the Uniform Distribution Theorem, the interpolating polynomials do not necessarily converge to $f$ on $\overline{D}$; we must use interpolating functions that are more general than polynomials. Indeed, in Walsh’s example, we just need to use rational functions of the form
\[
\frac{p_n(z)}{(\alpha - z)^{n+1}}, \quad p_n \in \mathcal{P}_n
\]
that interpolate $f$ at $\{z_{n,k} : k = 0, 1, \ldots, n\}$. Then, by a simple transformation $z = (1 - \alpha\zeta)/(\alpha - \zeta)$, Runge's result mentioned earlier would imply that this sequence of interpolating rational functions converges to $f$ on $\overline{D}$. It is the purpose of this paper to explore this direction further and hence extend Walsh's Theorem to interpolation on the unit circle $T$ by rational functions.

2. More Notation

Let $\alpha_n, n = 0, 1, \ldots$, be given (not necessarily distinct) points in $D$, and let

$$w_n(z) := \prod_{k=0}^{n}(z - \alpha_k) \quad \text{and} \quad w_n^*(z) := \prod_{k=0}^{n}(1 - \overline{\alpha_k}z).$$

We now define the space of rational functions for each $n$. Let

$$\mathcal{R}_n := \left\{ \frac{p(z)}{w_n^*(z)} : p \in \mathcal{P}_n \right\},$$

and let

$$\mathcal{R}_n^* := \left\{ r(1/\bar{z}) : r \in \mathcal{R}_n \right\}.$$  

Then we see that, when we restrict $z$ to $T$, we have $\mathcal{R}_n^* = \{ \bar{r} : r \in \mathcal{R}_n \}$. In general, an element in $\mathcal{R}_n^*$ has the form

$$\frac{zp(z)}{w_n(z)} \quad \text{for some } p \in \mathcal{P}_n.$$  

Define

$$\mathcal{R} = \bigcup_{n=0}^{\infty} \mathcal{R}_n \quad \text{and} \quad \mathcal{R}^* = \bigcup_{n=0}^{\infty} \mathcal{R}_n^*.$$  

We will write $B_n(z) = w_n(z)/w_n^*(z)$. From now on, we will let the points of interpolation $\{z_{n,k} : k = 0, 1, \ldots, n\}$ be given as the roots of

$$w_n(z) = w_n^*(z) \quad \text{or, equivalently,} \quad B_n(z) = 1.$$  

It is easy to see that $z_{n,k} \in T$ for all $k = 0, 1, \ldots, n$ and $n = 1, 2, \ldots$. Moreover, the $z_{n,k}$s are distinct as implied by formula (9) to be given later. For a function $f$ defined on $T$, let $R_n(f; \cdot)$ denote the unique rational function from $\mathcal{R}_n$ that interpolates $f$ at $\{z_{n,k} : k = 0, 1, \ldots, n\}$. Then, it is easy to verify that

$$R_n(f; z) = \frac{\sum_{k=0}^{n} f(z_{n,k})(B_n(z) - 1)}{B_n'(z_{n,k})(z - z_{n,k})}.$$  

We will use $C(T)$ to denote the space of functions continuous on $T$ equipped with the sup-norm on $T$. Let $A(D)$ denote the disk algebra, that is, the set of functions continuous on $\overline{D}$ and analytic in $D$.

Finally, let

$$\sigma_n := \sum_{k=0}^{n}(1 - |\alpha_k|) \quad \text{and} \quad \delta_n := \min_{0 \leq k \leq n}(1 - |\alpha_k|).$$
3. Main results

We will establish various convergence results of the sequence \( \{R_n(f, \cdot)\} \) according to different assumptions on \( f \). First, we give a simple and natural extension of Walsh’s Theorem to rational interpolation.

**Theorem 1.** Assume that \( \lim_{n \to \infty} \sigma_n = \infty \). If \( f \in C(T) \), then

\[
\lim_{n \to \infty} R_n(f; z) = f_D(z)
\]

for \( z \in D \), and the convergence is uniform on any compact subset of \( D \).

We remark that the requirement \( \lim_{n \to \infty} \sigma_n = \infty \) is equivalent to saying \( \lim_{n \to \infty} B_n(z) = 0 \) for \( z \in D \) (see, e.g., [25, §10.1]). It is also a necessary and sufficient condition for the denseness of certain rational functions in \( C(T) \) (cf. [1, 12]); see Lemma 3 for a more precise statement.

Our proof of Theorem 1 depends on the following result of weak-star convergence of a sequence of discrete measures supported at the points of interpolation. Let \( d\delta_z \) denote the unit measure supported at the single point \( z \). For \( n = 0, 1, 2, \ldots \), define

\[
d\nu_n = \sum_{k=0}^{n} \frac{d\delta_{z_{n,k}}}{|B_n'(z_{n,k})|}.
\]

**Theorem 2.** If \( \lim_{n \to \infty} \sigma_n = \infty \), then \( \{d\nu_n\} \) converges in the weak-star topology to the uniform measure on \( T \), \( d\theta/(2\pi) \). More precisely,

\[
\lim_{n \to \infty} \sum_{k=0}^{n} \frac{f(z_{n,k})}{|B_n'(z_{n,k})|} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \, d\theta,
\]

for every \( f \in C(T) \).

Note that when \( \alpha_n = 0 \) for all \( n = 0, 1, 2, \ldots \), we have \( B_n(z) = z^{n+1} \) and \( z_{n,k} \) are the \( n+1 \)st roots of unity, and \( |B_n'(z)| = n+1 \) for \( z \in T \). So, equation (2) becomes the well-known equality

\[
\lim_{n \to \infty} \sum_{k=0}^{n} \frac{f(e^{2i\pi/(n+1)})}{n+1} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \, d\theta.
\]

In general, one should compare Theorem 2 with the quadrature formulas for the Poisson integrals as obtained recently by Bultheel et al. [3]. It is essentially (though not exactly) equivalent to the convergence result in [3] (via suitable transformations and modifications). We will give a proof of Theorem 2 that is independent of [3] to indicate a different and direct approach.

Walsh’s Theorem holds for all \( f \) that are Riemann integrable on \( T \), while in Theorem 1 we only managed to obtain the case when \( f \) is assumed continuous on \( T \). With more restrictive conditions on the rate in which the poles are allowed to approach the unit circle \( T \), we do have an extension of Walsh’s theorem for all Riemann integrable functions as follows.
Theorem 3. Assume that
\[ \lim_{n \to \infty} \sigma_n \delta_n = \infty. \] (3)

If \( f \) is Riemann integrable on \( T \), then
\[ \lim_{n \to \infty} \sum_{k=0}^{n} \frac{f(z_{n,k})}{|B'_n(z_{n,k})|} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \, d\theta \] (4)
and
\[ \lim_{n \to \infty} R_n(f; z) = f_D(z) \] (5)
for \( z \in D \), and the convergence is uniform on any compact subset of \( D \).

Condition (3) still allows \( \alpha_n \) to approach \( T \) but is more restrictive on its rate of convergence. For example, \( \alpha_n = 1 - 1/n^\delta \), \( n = 1, 2, \ldots \), with \( \delta \in [0, 1/2) \) satisfy (3). Since \( \sigma_n > \sigma_n \delta_n \), (3) implies \( \lim_{n \to \infty} \sigma_n = \infty \), which is what we assumed in Theorem 1. It is not clear whether (3) can be replaced by the weaker condition \( \lim_{n \to \infty} \sigma_n = \infty \) in the assumption of Theorem 3.

Next, we present the \( L^2 \) convergence of \( \{R_n(f; \cdot)\} \) for \( f \in A(D) \), which generalizes the result of Walsh and Sharma [26] and that of Lozinski [15] for \( p = 2 \) in (1).

Theorem 4. If \( \lim_{n \to \infty} \sigma_n = \infty \), then
\[ \lim_{n \to \infty} \int_T |R_n(f; z) - f(z)|^2 |dz| = 0, \]
for every \( f \in A(D) \).

Similar to the polynomial case, this implies a rational version of Fejer’s result mentioned in the introduction.

The truths of Theorems 1 and 4 not only reveal a clear analogy between polynomial and rational interpolations on the unit circle \( T \), but also implicitly indicate how the extension from polynomial cases to rational cases can be carried out. For example, the well-known result of Walsh, the equi-convergence theorem [25, Theorem 1, p. 153], is extended for rational interpolation in the following result.

Theorem 5. Assume that the set \( \{\alpha_0, \alpha_1, \ldots\} \) has no limit point on \( T \) and its closure \( \text{cls}(\{\alpha_0, \alpha_1, \ldots\}) \) does not separate the plane \( C \). If
\[ \lim_{n \to \infty} |B_n(z)|^{1/n} = \psi(z) \neq \text{constant} \] (6)
holds locally uniformly on \( C \setminus (D \cup \{1/\alpha_0, 1/\alpha_1, \ldots\}) \), then
\[ \lim_{n \to \infty} [R_n(f; z) - r_n(f; z)] = 0 \quad \text{for } \psi(z) < \rho^2 \]
where \( f \) is assumed analytic for \( \psi(z) < \rho \) (\( \rho > 1 \)) and \( r_n(f; z) \) is the least-square approximation to \( f \) out of \( \mathcal{R}_n \), i.e.,
\[ \int_T |f(z) - r_n(f; z)|^2 |dz| = \min_{r \in \mathcal{R}_n} \int_T |f(z) - r(z)|^2 |dz|. \]
This result is analogous to one proved by Saff and Sharma in [20] for a different rational system in which poles are the roots of $z^n - r^n = 0$ for some $r > 1$ (and so the poles are not from a single sequence as we have assumed in this paper).

In the assumption of Theorem 5, the required range of $z$ implied by (6) can be made smaller. We have used the stronger assumption for simplicity. When $\{\alpha_n\}_{n=0}^\infty$ is cyclic, i.e.,

$$\alpha_{km+r} = \alpha_r, \quad r = 0, 1, \ldots, m - 1, \quad k = 1, 2, \ldots,$$

for some $m \geq 1$, then (6) is satisfied as

$$\lim_{n \to \infty} |B_n(z)|^{1/n} = \left(\prod_{k=0}^{m-1} \frac{z - \alpha_k}{1 - \overline{\alpha_k}z}\right)^{1/m}.$$

We refer to Walsh's discussion in [25, §§9.4, 9.5, in particular, p. 236] for more properties of $\psi(z)$. Here we only mention the following fact that will be needed in our proof: The locus $\gamma(\rho_1) := \{z : \psi(z) = \rho_1\}$ is rectifiable and, for $1 < \rho_1 < \rho$, $\gamma(\rho_1)$, lies in $\{z : \psi(z) < \rho\}$ and is exterior to $T$.

Note that for $f$ analytic on $D$, the least square approximation to $f$ out of $P_n$ is the same as the $n$th Taylor polynomial of $f$ about $z = 0$. As all our theorems imply the corresponding known results for polynomial interpolation, Theorem 5 will be given the following form when we let $\alpha_n = 0$ for all $n = 0, 1, 2, \ldots$.

**Corollary 6.** If $f$ is analytic for $|z| < \rho$ ($\rho > 1$), then

$$\lim_{n \to \infty} |L_n(f; z) - s_n(f; z)| = 0 \quad \text{for } |z| < \rho^2$$

where $L_n(f; z)$ is the polynomial interpolating $f$ at the $n + 1$st roots of unity, and $s_n(f; z)$ is the Taylor polynomial of degree $n$ of $f$ about $z = 0$.

This is the celebrated Walsh's equi-convergence theorem mentioned above. This theorem has been much extended in recent years; for the latest results, see [4, 5, 17, 18] and the bibliographies therein.

### 4. Auxiliary results

We collect some known results and prove some new ones that are needed in the proofs of the theorems.

**Lemma 1.** For $k = 0, 1, \ldots, n$, we have $z_{n,k}B_n'(z_{n,k}) = |B_n'(z_{n,k})|$.

**Proof.** A straightforward calculation (cf. (i) of [14, Lemma 1]) shows that

$$\frac{zB_n'(z)}{B_n(z)} = |B_n'(z)| \quad \text{for all } z \in T.$$

Letting $z = z_{n,k}$ yields the desired equality. $\square$

**Lemma 2.** For $n = 0, 1, 2, \ldots$, we have

$$\sum_{k=0}^{n} \frac{1}{|B_n'(z_{n,k})|} = \frac{1 - |B_n(0)|^2}{|1 - B_n(0)|^2}.$$
Proof. Let \( f_I(z) := 1 - B_n(0)B_n(z) \). Then \( f_I \in \mathcal{R}_n \) and
\[
f(z_{n,k}) = 1 - B_n(0)B_n(z_{n,k}) = 1 - B_n(0), \quad k = 0, 1, \ldots, n.
\]
So, by the uniqueness of the interpolating rational function,
\[
f_I(z) = R_n(f_I; z) = \sum_{k=0}^{n} \frac{(1 - B_n(0))(B_n(z) - 1)}{B'_n(z_{n,k})(z - z_{n,k})}.
\]
Hence, we have the identity
\[
\sum_{k=0}^{n} \frac{1}{B'_n(z_{n,k})(z - z_{n,k})} = \frac{1 - B_n(0)B_n(z)}{(1 - B_n(0))(B_n(z) - 1)}.
\]
Letting \( z = 0 \) gives us
\[
\sum_{k=0}^{n} \frac{1}{z_{n,k}B'_{n}(z_{n,k})} = \frac{1 - |B_n(0)|^2}{|1 - B_n(0)|^2},
\]
which, by Lemma 1, is the identity we need to verify. \( \square \)

Lemma 3. The linear span of \( \mathcal{R} \cup \mathcal{R}^* \) is dense in \( C(T) \) if and only if \( \lim_{n \to \infty} \sigma_n = \infty \).

Proof. This follows from an application of a result in [1, Addendum A, §2, p. 244]. It can be proved by the same method. \( \square \)

Lemma 4. For \( k = 0, 1, \ldots, n - 1 \), there exist numbers \( \xi_k, \zeta_k \in (\theta_{n,k}, \theta_{n,k+1}) \) such that
\[
\theta_{n,k+1} - \theta_{n,k} = \frac{2\pi}{|B'_n(e^{i\xi_k})|}
\]
and
\[
2\pi - (\theta_{n,k+1} - \theta_{n,k})|B'_{n}(z_{n,k})| = \frac{2\pi\gamma''(\zeta_k)(\theta_{n,k+1} - \zeta_k)}{\gamma'_{n}(\zeta_k)}
\]
where \( \gamma_n(\theta) \) is defined as a continuous function satisfying
\[
B_n(e^{i\theta}) = e^{i\gamma_n(\theta)}, \quad \theta \in [0, 2\pi].
\]

It turns out that, such a function \( \gamma_n \) is uniquely determined (up to an additional constant) and continuously differentiable on \( [0, 2\pi] \). Indeed, we have (see, for example, [13])
\[
\gamma'_n(\theta) = |B'_{n}(e^{i\theta})| = \sum_{k=0}^{n} \frac{1 - |\alpha_k|^2}{|e^{i\theta} - \alpha_k|^2} = \sum_{k=0}^{n} \frac{1 - r_k^2}{1 - 2r_k \cos(\theta - \omega_k) + r_k^2}
\]
where \( \alpha_k = r_ke^{i\omega_k} \) with \( r_k > 0, \ k = 0, 1, \ldots. \) From (9), we can estimate \( \gamma' \) as follows:
\[
\gamma'_{n}(\theta) = |B'_{n}(e^{i\theta})| \geq \sum_{k=0}^{n} \frac{1 - r_k}{1 + r_k} > \frac{1}{2} \sum_{k=0}^{n} (1 - r_k) = \frac{1}{2} \sigma_n.
\]
Proof of Lemma 4. Note that \( \gamma_n(\theta_{n,k+1}) - \gamma_n(\theta_{n,k}) = 2\pi \). Now, by the mean value theorem, we can write, for some \( \xi_k \in (\theta_{n,k}, \theta_{n,k+1}) \),
\[
2\pi = \gamma_n(\theta_{n,k+1}) - \gamma_n(\theta_{n,k}) = \gamma'_n(\xi_k)(\theta_{n,k+1} - \theta_{n,k}),
\]
which, by (9), implies (7).

To verify (8), let
\[
g(\theta) := -\gamma_n(\theta_{n,k+1}) + \gamma_n(\theta) + (\theta_{n,k+1} - \theta)\gamma'_n(\theta).
\]

Then,
\[
g(\theta_{n,k+1}) - g(\theta_{n,k}) = 2\pi - (\theta_{n,k+1} - \theta_{n,k})|B'_n(z_{n,k})|.
\]

and
\[
g'(\theta) = \gamma''_n(\theta)(\theta_{n,k+1} - \theta).
\]

Then, by the generalized Cauchy mean value theorem, there exists a number \( \zeta_k \in (\theta_{n,k}, \theta_{n,k+1}) \) such that
\[
\frac{g(\theta_{n,k+1}) - g(\theta_{n,k})}{\gamma_n(\theta_{n,k+1}) - \gamma_n(\theta_{n,k})} = \frac{g'(\zeta_k)}{\gamma'_n(\zeta_k)}.
\]

Now, equation (8) follows.

Lemma 5. For \( \zeta, z \in \mathbb{C} \), we have
\[
\sum_{j=0}^{n} \frac{B_n(z)}{B'_n(\alpha_j)(z - \alpha_j)(1 - \zeta\alpha_j)} = \begin{cases} 
\frac{zB'_n(z)}{B_n(z)}, & \text{if } z = \frac{1}{\zeta}, \\
\frac{1 - B_n(\zeta)B_n(z)}{1 - \zeta z}, & \text{otherwise}.
\end{cases}
\]

Proof. First, assume \( \zeta \in \mathbb{D} \). Then, note that
\[
1 - \frac{B_n(\zeta)B_n(z)}{1 - \zeta z}
\]
is the rational function from \( \mathcal{R}_n \) that interpolates \( f_\zeta(z) = 1/(1 - \zeta z) \) at \( z = \alpha_0, \alpha_1, \ldots, \alpha_n \). Thus, by Lagrange's interpolation formula, we have
\[
\frac{1 - B_n(\zeta)B_n(z)}{1 - \zeta z} = \sum_{k=0}^{n} \frac{B_n(z)}{B'_n(z_{n,k})(z - \alpha_k)(1 - \zeta\alpha_k)}.
\]

Observe that both sides of (11) are rational functions of the same type in \( z \) and, after taking the complex conjugate of both sides in (11), in \( \zeta \). So, in general, the equality in (11) holds as long as \( \zeta z \neq 1 \).

Finally, the case when \( z = 1/\zeta \) can be handled by first taking the limit as \( z \to 1/\zeta \) in (11) and then write \( \zeta \) as \( 1/z \).

We will need the system of functions from \( \mathcal{R}_n \) given by
\[
\varphi_n(z) = \frac{\sqrt{1 - |\alpha_n|^2}}{z - \alpha_n}B_n(z), \quad n = 0, 1, 2, \ldots.
\]

This system was probably introduced (with a constant multiplier of modulus 1) first by Takenaka in [23] and Malmquist in [16]. See also [8] and [25, p. 224]. It is orthogonal with respect to the uniform measure \( d\theta/(2\pi) \) on the unit circle \( T \) (with \( z = e^{i\theta} \)). We collect some properties of \( \{\varphi_n\} \) in the following lemma.
Lemma 6. (i) For \( n = 0, 1, \ldots \), \( \{\varphi_k(z)\}_{k=0}^n \subseteq \mathcal{R}_n \) and, \( j, k = 0, 1, 2, \ldots \),
\[
\frac{1}{2\pi} \int_{\mathbb{T}} \varphi_j(z) \varphi_k(z) \, dz = \begin{cases} 
0, & j \neq k, \\
1, & j = k.
\end{cases}
\tag{12}
\]

(ii) (Christoffel-Darboux formula for \( \{\varphi_n\} \)) For all \( \zeta, z \in \mathbb{C} \), we have
\[
\sum_{k=0}^n \varphi_k(\zeta) \varphi_k(z) = \frac{1 - B_n(\zeta) B_n(z)}{1 - \zeta z}.
\tag{13}
\]

Proof. A proof of equation (12) can be found in [25, p. 227]. The identity (13) is well known in the literature (see [8]). It is a special case of the more general Christoffel-Darboux formula established for rational functions orthogonal on the unit circle. See, for example, [2]. □

Our next result is interesting in its own right. It generalizes a relation first observed by Walsh and Sharma [26, Formula (16)] on polynomials.

Lemma 7. For \( p, q = 0, 1, \ldots, n \), we have
\[
\frac{1}{2\pi} \int_{\mathbb{T}} \frac{B_n(z) - 1}{z - z_{n,p}} \left( \frac{B_n(z) - 1}{z - z_{n,q}} \right) \, dz = \begin{cases} 
0, & p \neq q, \\
|B'_n(z_{n,p})|, & p = q.
\end{cases}
\]

Proof. Let
\[
I(p, q) := \frac{1}{2\pi} \int_{\mathbb{T}} \frac{B_n(z) - 1}{z - z_{n,p}} \left( \frac{B_n(z) - 1}{z - z_{n,q}} \right) \, dz.
\]

We claim that
\[
I(p, q) = z_{n,q} z_{n,p} \sum_{j=0}^n \frac{B_n(z_{n,p})}{B'_n(\alpha_j)(z_{n,p} - \alpha_j)(1 - \bar{z}_{n,q} \alpha_j)}.
\tag{14}
\]

Assuming the truth of (14), we see that Lemma 5 implies \( I(p, q) = 0 \) when \( p \neq q \), and Lemmas 1 and 5 infer that \( I(p, q) = |B'_n(z_{n,p})| \). Therefore, the lemma follows from (14).

Now, let us verify (14). We need the orthonormal basis \( \{\varphi_k\}_{k=0}^n \) in \( \mathcal{R}_n \) introduced above. Since
\[
\frac{B_n(z) - 1}{z - z_{n,p}}, \quad \frac{B_n(z) - 1}{z - z_{n,q}} \in \mathcal{R}_n,
\]
we can write
\[
\frac{B_n(z) - 1}{z - z_{n,p}} = \sum_{k=0}^n a_k \varphi_k(z) \quad \text{and} \quad \frac{B_n(z) - 1}{z - z_{n,q}} = \sum_{k=0}^n b_k \varphi_k(z)
\]
for some \( \{a_k\}_{k=0}^n \) and \( \{b_k\}_{k=0}^n \). Then, for \( p, q = 0, 1, \ldots, n \), by (12) in Lemma 6, we have \( I(p, q) = \sum_{k=0}^n a_k b_k \). Now,
\[
\frac{1}{2\pi} \int_{\mathbb{T}} \frac{B_n(z) - 1}{z - z_{n,p}} \varphi_k(z) \, dz = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(1 - B_n(z)) \varphi_k(z)}{B_n(z)(1 - \bar{z}_{n,p} z)} \, dz
\]
\[
= \sum_{j=0}^n \frac{\varphi_k(\alpha_j)}{B'_n(\alpha_j)(1 - \bar{z}_{n,p} \alpha_j)},
\]
by the residue theorem. Similarly,
\[
\bar{b}_k = \sum_{j=0}^{n} \frac{\varphi_k(\alpha_j)}{B'_n(\alpha_j)(1 - z_{n,q}\alpha_j)}.
\]

Hence,
\[
I(p, q) = \sum_{k=0}^{n} \sum_{j_1=0}^{n} \frac{\varphi_k(\alpha_{j_1})}{B'_n(\alpha_{j_1})(1 - z_{n,p}\alpha_{j_1})} \sum_{j_2=0}^{n} \frac{\varphi_k(\alpha_{j_2})}{B'_n(\alpha_{j_2})(1 - z_{n,q}\alpha_{j_2})} = \sum_{j_1,j_2=0}^{n} \frac{1}{B'_n(\alpha_{j_1})B'_n(\alpha_{j_2})(1 - z_{n,p}\alpha_{j_1})(1 - z_{n,q}\alpha_{j_2})} \frac{1}{1 - \alpha_{j_1}\alpha_{j_2}} \sum_{k=0}^{n} \varphi_k(\alpha_{j_1})\varphi_k(\alpha_{j_2})
\]
\[
= \sum_{j_1,j_2=0}^{n} \frac{1}{B'_n(\alpha_{j_1})B'_n(\alpha_{j_2})(1 - z_{n,p}\alpha_{j_1})(1 - z_{n,q}\alpha_{j_2})} \frac{1}{1 - \alpha_{j_1}\alpha_{j_2}} \sum_{j_2=0}^{n} \frac{1}{B'_n(\alpha_{j_2})(1 - z_{n,q}\alpha_{j_2})(1 - \alpha_{j_1}\alpha_{j_2})}.
\]

by (13) in Lemma 6. So,
\[
I(p, q) = \sum_{j_1,j_2=0}^{n} \frac{1}{B'_n(\alpha_{j_1})B'_n(\alpha_{j_2})(1 - z_{n,p}\alpha_{j_1})(1 - z_{n,q}\alpha_{j_2})} \frac{1}{1 - \alpha_{j_1}\alpha_{j_2}} \sum_{j_2=0}^{n} \frac{1}{B'_n(\alpha_{j_2})(1 - z_{n,q}\alpha_{j_2})(1 - \alpha_{j_1}\alpha_{j_2})}.
\]

Now, note that
\[
\sum_{j_2=0}^{n} \frac{1}{B'_n(\alpha_{j_2})(1 - z_{n,q}\alpha_{j_2})(1 - \alpha_{j_1}\alpha_{j_2})} = z_{n,q} \sum_{j_2=0}^{n} \frac{B_n(z_{n,q})}{B'_n(\alpha_{j_2})(z_{n,q} - \alpha_{j_2})(1 - \alpha_{j_1}\alpha_{j_2})},
\]
which, by using Lemma 5 with \( z = z_{n,q} \) and \( \zeta = \alpha_{j_1} \), is equal to
\[
z_{n,q} \frac{1 - B_n(\alpha_{j_1})B_n(z_{n,q})}{1 - \alpha_{j_1}z_{n,q}} = \frac{z_{n,q}}{1 - \alpha_{j_1}z_{n,q}}.
\]

Thus,
\[
I(p, q) = \sum_{j_1=0}^{n} \frac{1}{B'_n(\alpha_{j_1})(1 - z_{n,p}\alpha_{j_1})} \frac{z_{n,q}}{1 - \alpha_{j_1}z_{n,q}} = z_{n,q}z_{n,p} \sum_{j=0}^{n} \frac{B_n(z_{n,p})}{B'_n(\alpha_j)(z_{n,p} - \alpha_j)(1 - z_{n,q}\alpha_j)},
\]
which is (14). This completes the proof. \(\square\)
5. Proofs of the main theorems

We prove Theorem 2 first since our proof of Theorem 1 is based on it.

Proof of Theorem 2. We first claim that

$$\lim_{n \to \infty} \int_T r \, d\nu_n = \frac{1}{2\pi} \int_T r(z) \, |dz|$$

(15)

for every \( r \in \mathcal{R} \cup \mathcal{R}^* \). This is a consequence of the following identity.

$$\sum_{k=0}^{n} \frac{f(z_{n,k})}{|B'_n(z_{n,k})|} = \frac{1}{2\pi(1-B_n(0))} \int_T f(z) \, |dz|$$

(16)

for every \( f \in \mathcal{R}_n \). Indeed, for \( f \in \mathcal{R}_n \), we have \( f(z) = R_n(f; z) \). So, by Lemma 1,

$$f(0) = R_n(f; 0) = \sum_{k=0}^{n} \frac{f(z_{n,k})(1-B_n(0))}{z_{n,k}B'_n(z_{n,k})} = \sum_{k=0}^{n} \frac{f(z_{n,k})(1-B_n(0))}{|B'_n(z_{n,k})|}.$$

This, on writing \( f(0) = \int_T f(z) \, |dz| \), implies equation (16). Since \( \mathcal{R}_n \subset \mathcal{R}_{n+1} \), we let \( n \to \infty \) in (16) to obtain (15) for \( r \in \mathcal{R} \) by using the fact that \( \lim_{n \to \infty} B_n(0) = 0 \).

Next, by taking the complex conjugate of both sides of (15), we see that (15) holds for \( f \in \mathcal{R}^*_n \) as well.

Now, from Lemma 2, it follows that

$$\int_T d\nu_n = \frac{1-|B_n(0)|^2}{|1-B_n(0)|^2} \leq \frac{2}{1-|\alpha_0|},$$

for \( n = 0, 1, 2, \ldots \). Thus \( \{d\nu_n\} \) is compact in the weak-star topology. Let \( d\nu \) be any weak-star limit of \( \{d\nu_n\} \) and

$$\lim_{n \to \infty} \int_T f \, d\nu_n = \int_T f \, d\nu,$$

for every \( f \in C(\mathbb{T}) \). In view of (15), this means

$$\int_T r \, d\nu = \frac{1}{2\pi} \int_T r(z) \, |dz|$$

for every \( r \in \mathcal{R} \cup \mathcal{R}^* \). Since, by Lemma 3, the linear span of \( \mathcal{R} \cup \mathcal{R}^* \) is dense in \( C(\mathbb{T}) \), then \( d\nu(\theta) = d\theta/(2\pi) \). Therefore, the whole sequence converges to \( d\theta/(2\pi) \).

Proof of Theorem 1. Let \( f \in C(\mathbb{T}) \). By using Lemma 1, we can write

$$R_n(f; z) = \sum_{k=0}^{n} \frac{f(z_{n,k})(B_n(z)-1)}{B'_n(z_{n,k})(z-z_{n,k})} = \left(1-B_n(z)\right) \sum_{k=0}^{n} \frac{z_{n,k}f(z_{n,k})}{z_{n,k} - z} \frac{1}{|B'_n(z_{n,k})|}.$$

For each fixed \( z \in \mathbb{D} \), we have \( \lim_{n \to \infty} B_n(z) = 0 \) and that

$$\frac{tf(t)}{t-z}$$

is continuous for \( t \in [0, 2\pi] \). So, using Theorem 2, we obtain

$$\lim_{n \to \infty} R_n(f; z) = \frac{1}{2\pi} \int_T \frac{tf(t)}{t-z} \, |dt| = \frac{1}{2\pi} \int_T \frac{f(t)}{t-z} \, dt,$$

which is \( f_D(z) \).
Proof of Theorem 3. We show that
\[
\left| \frac{1}{|B'_n(z_n,k)|} \frac{\theta_{n,k+1} - \theta_{n,k}}{2\pi} \right| \leq \frac{8\pi}{|B'_n(z_n,k)| \sigma_n \delta_n}. \tag{17}
\]

Dividing both sides of (8) by \(2\pi|B'_n(z_n,k)|\), we have
\[
\left| \frac{1}{|B'_n(z_n,k)|} \frac{\theta_{n,k+1} - \theta_{n,k}}{2\pi} \right| = \left| \frac{\gamma''(\zeta_k)(\theta_{n,k+1} - \zeta_k)}{|B'_n(z_n,k)| \gamma'(\zeta_k)} \right|. \tag{18}
\]

Now, note that
\[
\left| \frac{\gamma''(\zeta_k)}{\gamma'(\zeta_k)} \right| \leq \frac{\sum_{j=0}^n c_j \frac{2r_j |\sin(\zeta_k - \omega_j)|}{1 - 2r_j \cos(\zeta_k - \omega_j) + r_j^2}}{\sum_{j=0}^n c_j},
\]
with
\[
c_j := \frac{1 - r_j^2}{1 - 2r_j \cos(\zeta_k - \omega_j) + r_j^2}, \quad j = 0, 1, \ldots, n.
\]

Note that, \(c_j > 0, j = 0, 1, \ldots, n, \) and, for \(0 \leq r < 1, \)
\[
\max_{\theta \in [0, 2\pi]} \frac{2r |\sin(\theta)|}{1 - 2r \cos(\theta) + r^2} = \frac{2r}{1 - r^2}.
\]

So, we have
\[
\left| \frac{\gamma''(\zeta_k)}{\gamma'(\zeta_k)} \right| \leq \max_{0 \leq j \leq n} \frac{2r_j}{1 - r_j^2} < \frac{2}{\delta_n}. \tag{19}
\]

On the other hand, by (7) in Lemma 4 and (10), we have
\[
|\theta_{n,k+1} - \zeta_k| \leq |\theta_{n,k+1} - \theta_{n,k}| = \frac{2\pi}{|B'_n(e^{i\xi_k})|} \leq \frac{2\pi}{\frac{1}{2} \sigma_n} = \frac{4\pi}{\sigma_n}. \tag{20}
\]

Using (19) and (20) in (18), we obtain (17).

To prove (4), we use (17) and compare the sum
\[
\sum_{k=0}^n \frac{f(z_n,k)}{|B'_n(z_n,k)|}
\]
with
\[
\frac{1}{2\pi} \sum_{k=0}^n f(e^{i\theta_{n,k}})(\theta_{n,k+1} - \theta_{n,k}),
\]
a Riemann sum of $f$ on $T$. We have, with $M_f$ denoting an upper bound of $f$ on $T$, by using (17) and Lemma 2,
\[
\left| \sum_{k=0}^{n} \frac{f(z_{n,k})}{|B'_n(z_{n,k})|} - \frac{1}{2\pi} \sum_{k=0}^{n} f(e^{i\theta_{n,k}})(\theta_{n,k+1} - \theta_{n,k}) \right| \\
\leq \sum_{k=0}^{n} \left| \frac{f(z_{n,k})}{|B'_n(z_{n,k})|} \right| \frac{1}{2\pi} \left| \theta_{n,k+1} - \theta_{n,k} \right| \\
\leq M_f \sum_{k=0}^{n} \frac{8\pi}{|B'_n(z_{n,k})|} \frac{\delta_n}{r_n} = \frac{8\pi M_f (1 - |B_n(0)|^2)}{\delta_n (1 - B_n(0))^2},
\]
which goes to the limit zero by the assumption (3). This proves (4).

The proof of (5) is obtained from (4) in the same way as that of Theorem 1 is obtained from Theorem 2.

\textit{Proof of Theorem 4.} Since $f \in A(D)$, by Mergelyan's theorem ([10, Theorem 1, p. 97]), there exists a sequence of polynomials (the arithmetic means of the partial sums of the Taylor series of $f$ about $z = 0$) that converges uniformly to $f$ on $\overline{D}$. By using the argument in [1, Addendum A, pp. 243–246], we can see that each monomial $z^k$ ($k \geq 0$) can be approximated by rational functions from $R$ as closely as we please. Therefore, for $\varepsilon > 0$, there exists a function $r \in R$ such that
\[
\max_{z \in \overline{D}} |f(z) - r(z)| < \varepsilon.
\]
Note that $R_n(r; z) = r(z)$ for all $n$ large enough. Now we have, for $n$ large enough,
\[
\int_T |R_n(f; z) - f(z)|^2 |dz| = \int_T |R_n(f - r; z) + r(z) - f(z)|^2 |dz| \\
\leq 2 \left\{ \int_T |R_n(f - r; z)|^2 |dz| + \int_T |r(z) - f(z)|^2 |dz| \right\} \\
\leq 2 \int_T |R_n(f - r; z)|^2 |dz| + 4\pi \varepsilon^2.
\]
The integral in the last expression can be expanded as
\[
\sum_{p,q=0}^{n} \frac{f(z_{n,p}) - r(z_{n,p})}{B'_n(z_{n,p})} \left[ \frac{f(z_{n,q}) - r(z_{n,q})}{B'_n(z_{n,q})} \right] \int_T \frac{B_n(z) - 1}{z - z_{n,p}} \left( \frac{B_n(z) - 1}{z - z_{n,q}} \right) |dz|,
\]
which, according to Lemma 7, is equal to
\[
\sum_{k=0}^{n} 2\pi |f(z_{n,k}) - r(z_{n,k})|^2 |B'_n(z_{n,k})|. \\
\]
Now, combining the above estimates and using Lemma 2, we have
\[
\int_T |R_n(f; z) - f(z)|^2 |dz| \leq \frac{4\pi \varepsilon^2 (1 - |B_n(0)|^2)}{|1 - B_n(0)|^2} + 4\pi \varepsilon^2 \leq \frac{12\pi \varepsilon^2}{1 - |B_n(0)|^2}.
\]
Thus, by letting $n \to \infty$, we obtain
\[
0 \leq \limsup_{n \to \infty} \int_T |R_n(f; z) - f(z)|^2 |dz| \leq 12\pi \varepsilon^2.
\]
This implies the desired result. \qed
Proof of Theorem 5. By Hermite’s formula, for each $n = 1, 2, \ldots,$

$$R_n(f; z) = \frac{1}{2\pi} \int_{\gamma} \frac{f(\zeta)(B_n(\zeta) - B_n(z))}{B_n(\zeta)(\zeta - z)} d\zeta$$

(21)

for all $z \in \text{Int}(\gamma)$, the interior of $\gamma$ where $\gamma$ is a Jordan curve in $\{z : \psi(z) < \rho\}$ that surrounds $\{z_0, z_1, \ldots, z_n\} \subseteq T$. On the other hand, by [25, Lemmas I and II on p. 225, Theorem 2 on p. 227], the least square approximation $r_n(f; z)$ is the same as the rational function from $R_n$ that interpolates $f$ at $\alpha_0, \alpha_1, \ldots, \alpha_n$. (For repeated points, $r_n$ interpolates $f$ and its derivatives.) Again, by Hermite’s formula,

$$r_n(f; z) = \frac{1}{2\pi} \int_{\gamma} \frac{f(\zeta)(B_n(\zeta) - B_n(z))}{B_n(\zeta)(\zeta - z)} d\zeta, \quad z \in \text{Int}(\gamma),$$

(22)

where $\gamma$ is as in (21). Therefore, by (21) and (22), we obtain, for $z \in \text{Int}(\gamma),$

$$R_n(f; z) - r_n(f; z) = \frac{1}{2\pi} \int_{\gamma} \frac{f(\zeta)(B_n(\zeta) - B_n(z))}{(\zeta - z)(B_n(\zeta) - 1)B_n(\zeta)} d\zeta.$$

Assume $1 < \rho_1 < \rho_2 < \rho$. Let $\gamma(r) = \{z : \psi(z) = r\}, r > 1$. (For the properties of $\gamma(r)$, see the remarks after the statement of Theorem 5 in Section 3.) Then, for $z \in \text{Int}(\gamma(\rho_1))$ and $\epsilon \in (0, \rho - 1)$, when $n$ is large enough, we have

$$|R_n(f; z) - r_n(f; z)| \leq \frac{1}{2\pi} \frac{\max_{\zeta \in \gamma(\rho_2)} |f(\zeta)|((\rho_2 + \epsilon)^n + (\rho_1 + \epsilon)^n)l(\gamma(\rho_2))}{\text{dist}(\gamma(\rho_1), \gamma(\rho_2))((\rho_2 - \epsilon)^n - 1)(\rho_2 - \epsilon)^n}$$

where $l(\gamma(\rho_2))$ denotes the length of $\gamma(\rho_2)$ and dist$(\gamma(\rho_1), \gamma(\rho_2))$ is the distance between $\gamma(\rho_1)$ and $\gamma(\rho_2)$. It follows that

$$\limsup_{n \to \infty} \max_{z \in \gamma(\rho_1)} |R_n(f; z) - r_n(f; z)|^{1/n} \leq \frac{1}{\rho_2}.$$

Letting $\rho_2 \nearrow \rho$ first and then letting $\rho_1 \nearrow \rho$ yield

$$\limsup_{n \to \infty} \max_{z \in \gamma(\rho)} |R_n(f; z) - r_n(f; z)|^{1/n} \leq \frac{1}{\rho}. \quad (23)$$

Now, from the fact that

$$\frac{R_n(f; z) - r_n(f; z)}{B_n(z)}$$

is analytic in $C \setminus D$ (including $\infty$), by using the maximum modulus principle, we have

$$\max_{z \in \gamma(\rho_2)} \left| \frac{R_n(f; z) - r_n(f; z)}{B_n(z)} \right| \leq \max_{z \in \gamma(\rho_1)} \left| \frac{R_n(f; z) - r_n(f; z)}{B_n(z)} \right|.$$
and so,
\[
\limsup_{n \to \infty} \max_{z \in \gamma(\rho^2)} |R_n(f; z) - r_n(f; z)|^{1/n} \\
\leq \frac{\rho^2}{\rho_1} \limsup_{n \to \infty} \max_{z \in \gamma(\rho_1)} |R_n(f; z) - r_n(f; z)|^{1/n}.
\]

This and (23) imply that, for \(1 < \rho_* < \rho\),
\[
\limsup_{n \to \infty} \max_{z \in \gamma(\rho^2)} |R_n(f; z) - r_n(f; z)|^{1/n} \\
\leq \rho_* \limsup_{n \to \infty} \max_{z \in \gamma(\rho_*)} |R_n(f; z) - r_n(f; z)|^{1/n} \\
\leq \rho_* \limsup_{n \to \infty} \max_{z \in \gamma(\rho)} |R_n(f; z) - r_n(f; z)|^{1/n} \\
\leq \frac{\rho_*}{\rho} < 1,
\]
which implies that \(R_n(f; z) - r_n(f; z)\) converges to the limit zero locally and uniformly in \(\text{Int}(\gamma(\rho^2))\) at a geometric rate. This finishes the proof of Theorem 5. \(\square\)

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