Methods and Applications of Analysis
6 (2) 1999, pp. 175–194

ISSN 1073–2772

On the Andrews-Bowman continued fraction

Dharma P. Gupta and David R. Masson

Dedicated to Dick Askey on the occasion of his 65th birthday

ABSTRACT. Andrews and Bowman have obtained a continued fraction representing a ratio of very-well-poised ${}_8\phi_7$ basic hypergeometric series. They call this a full extension of the Rogers-Ramanujan continued fraction. We rederive their result starting with contiguous relations for a terminative ${}_4\phi_3$ series. We then use Carleson's theorem to prove corresponding contiguous relations for a non-terminating very-well-poised ${}_8\phi_7$ series. Both minimal and dominant solutions to the Andrews-Bowman three-term recurrence then are obtained. The minimal solution yields their continued fraction result via Pincherle's Theorem. A special parameterization of the three-term recurrence yields a polynomial recurrence of the type associated with rational biorthogonality of type R_{II} with the dominant solution giving an explicit expression for the polynomial. An explicit discrete biorthogonality then is stated. The basic beta integral associated with this orthogonality is a summation formula for a pair of ${}_3\phi_2$ s of type I which generalizes the q-Vandermonde summation.

1. Notation

We follow the notation of Gasper and Rahman [4], but we omit the designation 'q' for the base in the q-shifted factorials and the basic hypergeometric functions. Thus, with $|q| < 1$, we have

$$(a)_0 := 1, \quad (a)_n := \prod_{j=1}^{n}(1 - aq^{j-1}), \quad n > 0,$$
$$(a_1, a_2, \ldots, a_k)_n := \prod_{j=1}^{k}(a_j)_n. \tag{1.1}$$

A basic hypergeometric series is

$${}_r\phi_s(a_1, a_2, \ldots, a_r;\ b_1, b_2, \ldots, b_s;\ z) := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_n}{(b_1, b_2, \ldots, b_s)_n} \frac{[(-1)^n q^{n(n-1)/2}]^{1+s-r}}{(q)_n} z^n. \tag{1.2}$$

Received April 21, 1998, revised August 24, 1998.
1991 *Mathematics Subject Classification*: 33045, 40A15, 39A10, 47B39.
Key words and phrases: q-series, contiguous relations, difference equations, minimal solution, continued fraction, biorthogonal rational functions.

For a very-well-poised $_8\phi_7$ series, we make use of the notation

$$\begin{aligned} W &:= {}_8W_7(a;b,c,d,e,f) \\ &:= {}_8\phi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; \frac{a^2q^2}{bcdef}\right). \end{aligned} \tag{1.3}$$

We also denote a balanced $_4\phi_3$ series by

$$\phi := {}_4\phi_3(a,b,c,d;e,f,g) := {}_4\phi_3\left(\begin{matrix} a,b,c,d \\ e,f,g \end{matrix}; q\right)$$

where

$$efg = abcdq.$$

We follow the usual notation for variations of ϕ or W with respect to the parameters. For example, $W(f+)$ represents W with f replaced by fq, $W(e+, f-)$ represents W with e replaced by eq and f replaced by f/q, and ϕ_+ represents ϕ with a, b, c, d, e, f replaced by aq, bq, cq, dq, eq, fq, respectively.

In [8], we obtained a contiguous relation connecting $W(f+)$, W, and $W(f-)$ and obtained the related continued fraction. This was shown to be connected with Watson's q-analogue [14] of Ramanujan's Entry 40 [13]. In [9], we made a more detailed study of the above three-term recurrence. It was shown that, when the related continued fraction terminates, it has a singularity structure of simple poles which is associated with a system of biorthogonal rational functions. We were unable to exhibit the orthogonality explicitly in the non-terminating case. The associated $_8W_7$ case was studied in [10] as a limit of a special linear combination of two balanced very-well-poised $_{10}\phi_9$ series. In another paper [7], we have used the $_8\phi_7$ model to study the associated case of the Askey-Wilson polynomials [4].

In a recent paper [2], Andrews and Bowman have obtained the second-order q-difference equation for a non-terminating $W(a;b,c,d,e,f)$ using variations of the parameter 'a'. The connected continued fraction and its limiting cases also are given. The proof depends on the properties of some auxiliary functions defined and studied by Andrews [1] in his earlier work on q-difference equations for the very-well-poised basic hypergeometric series.

We feel obliged to study the above problem for several reasons. Firstly, the three-term recurrence satisfied by a hypergeometric function is essentially a three term contiguous relation, and we should be able to obtain the Andrews-Bowman equation from three-term contiguous relations. Starting with some contiguous relations for $_4\phi_3$ series, we obtain a contiguous relation for $_8W_7$ which gives the result obtained by Andrews and Bowman. Secondly, as a bonus, we are able to find a second solution to the second-order q-difference equation which was not given in [2]. From the asymptotics of the two solutions, we show that one of the solutions is a minimal solution. We then apply Pincherle's theorem [12] to obtain the value of the associated continued fraction. Finally, we discuss a rational biorthogonality resulting from this study of $_8W_7$ functions. As a byproduct, we have a summation formula for $_3\phi_2$ s of type I.

Andrews and Bowman have proved the following:

Theorem A (Andrews & Bowman [2]). *If*

$$H(x) = \frac{(xq/b, xq/c, xq/d, xq/e, xq/f)_\infty}{(xq)_\infty}\ {}_8W_7(x; b, c, d, e, f), \tag{1.4}$$

then

$$Q(x)H(x) = P(x)H(xq) + R(x)H(xq^2) \tag{1.5}$$

where

$$\begin{aligned} Q(x) &= (1 - x^2q^2\sigma_5)(1 - x^2q^3\sigma_5)p(xq), && (1.6)\\ P(x) &= -xq(1 - x^2q^3\sigma_5)\big(\sigma_1 - xq\sigma_3 + x^3q^3\sigma_2\sigma_5 - x^4q^4\sigma_4\sigma_5\big)p(xq)\\ &\quad - (-1 - xq\sigma_1 + x^3q^4\sigma_5)p(x)p(xq)\\ &\quad - p(x)\Big(- x^6q^{11}\sigma_2\sigma_5^2 + x^5q^9\sigma_1\sigma_4\sigma_5 - x^5q^9\sigma_4\sigma_5 + x^4q^7\sigma_3\sigma_5 + x^4q^7\sigma_2\sigma_5\\ &\qquad - x^4q^7\sigma_4^2 - x^3q^5\sigma_1\sigma_5 + x^2q^3\sigma_4 - x^2q^3\sigma_3 + xq\sigma_1\Big), && (1.7)\\ R(x) &= xqp(x)\left(1 - \frac{xq^2}{bc}\right)\left(1 - \frac{xq^2}{bd}\right)\cdots\left(1 - \frac{xq^2}{ef}\right), && (1.8)\\ p(x) &= 1 - x^2q^2\sigma_4 + x^3q^3\sigma_1\sigma_5 - x^5q^5\sigma_5^2, && (1.9) \end{aligned}$$

and $\sigma_j := \sigma_j\big(\frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}, \frac{1}{f}\big)$ *denotes the* j^{th} *elementary symmetric function of* $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, $\frac{1}{e}$, $\frac{1}{f}$ *with* $\sigma_1 = \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f}, \ldots, \sigma_5 = 1/bcdef$.

The resulting continued fraction is given by

$$\frac{H(x)}{H(xq)} = \frac{P(x)}{Q(x)} + \cfrac{R(x)/Q(x)}{P(xq)/Q(xq) + \cfrac{R(xq)/Q(xq)}{P(xq^2)/Q(xq^2) + \cfrac{P(xq^2)/Q(xq^2)}{P(xq^3)/Q(xq^3) + \ddots}}}. \tag{1.10}$$

Note the misprint in [2] in the definition of $P(x)$. The coefficient of the term in x^6q^{11} should be $-q\sigma_2\sigma_5^2$, and not $-q\sigma_2\sigma_5$.

2. Three-term contiguous relation for ${}_8W_7$

We start with a balanced ${}_4\phi_3$ series

$$\phi := {}_4\phi_3\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; q\right) \tag{2.1}$$

where

$$efg = abcdq. \tag{2.2}$$

Wilson [15] has obtained a number of three-term contiguous relations satisfied by terminating balanced ${}_4F_3$'s. Following his procedure, we can obtain analogous results for terminating balanced ${}_4\phi_3$'s. We indicate below the derivation of a few of these which are relevant to our work.

If we subtract the series term-by-term, we obtain the following four relations:

$$\phi(a-,b+)-\phi=\frac{(bq-a)(1-c)(1-d)}{(1-e)(1-f)(1-g)}\,\phi_+(a-), \tag{2.3}$$

$$\phi(a-,e-)-\phi=\frac{(e-a)(1-b)(1-c)(1-d)}{(1-e/q)(1-e)(1-f)(1-g)}\,\phi_+(a-), \tag{2.4}$$

$$\phi(a+,e+)-\phi=\frac{q(a-e)(1-b)(1-c)(1-d)}{(1-e)(1-eq)(1-f)(1-g)}\,\phi_+(e+), \tag{2.5}$$

$$\phi(e+,f+)-\phi=\frac{q(f-eq)(1-a)(1-b)(1-c)(1-d)}{(1-e)(1-eq)(1-f/q)(1-f)(1-g)}\,\phi_+(e+). \tag{2.6}$$

In order to obtain another relation, we use Sear's transformation [4, (III.15)] for terminating balanced ${}_4\phi_3$ series; i.e., the transformation

$${}_4\phi_3\left(\begin{matrix}q^{-n},b,c,d\\ e,f,g\end{matrix};q\right)=\frac{(f/b)_n(g/b)_n}{(f)_n(g)_n}\,b^n\,{}_4\phi_3\left(\begin{matrix}q^{-n},b,e/c,e/d\\ e,bq^{1-n}/f,bq^{1-n}/g\end{matrix};q\right) \tag{2.7}$$

where $efg=bcdq^{1-n}$.

If we apply (2.7) to the series ϕ, $\phi_+(a-)$, $\phi_+(e+)$ with $a=q^{-n}$, $n>0$, we have, respectively, the series

$$\frac{(f/b)_n(g/b)_n}{(f)_n(g)_n}\,b^n\,{}_4\phi_3\left(\begin{matrix}a,b,e/c,e/d\\ e,ef/cd,eg/cd\end{matrix};q\right),$$

$$\frac{(f/b)_n(g/b)_n}{(fq)_n(gq)_n}\,(bq)^n\,{}_4\phi_3\left(\begin{matrix}a,bq,e/c,e/d\\ eq,ef/cd,eg/cd\end{matrix};q\right),$$

and

$$\frac{(f/b)_{n-1}(g/b)_{n-1}}{(fq)_{n-1}(gq)_{n-1}}\,(bq)^{n-1}\,{}_4\phi_3\left(\begin{matrix}aq,bq,eq/c,eq/d\\ eq^2,efq/cd,egq/cd\end{matrix};q\right).$$

It can be seen that the three ${}_4\phi_3$ series given above are connected by the relation (2.5). After writing this connecting equation, we reverse the transformations and thus arrive at the following contiguous relation:

$$(a-f)(a-g)\phi_+(a-)-\frac{aq}{e}\,\frac{(1-a)(b-e)(c-e)(d-e)}{(1-e)(1-eq)}\times\phi_+(e+)-a(1-f)(1-g)\phi=0. \tag{2.8}$$

Making appropriate substitutions from (2.3) through (2.6) into (2.8), we can easily derive the following:

$$\frac{(1-e)(b-f)(b-g)}{(b-aq)}\left[\phi(a+,b-)-\phi\right]+\frac{bq}{e}\,\frac{(1-f/q)(a-e)(c-e)(d-e)}{(1-a)(f-eq)} \times\left[\phi(e+,f-)-\phi\right]+b(1-c)(1-d)\phi=0, \quad (2.9)$$

$$(1-a)(b-e)\left[\phi(a+,e+)-\phi\right]=(1-b)(a-e)\left[\phi(b+,e+)-\phi\right], \quad (2.10)$$

$$a(1-a)(b-e)(c-e)(d-e)\left[\phi(a+,e+)-\phi\right]+e(1-e)(1-f/q)(a-e)(a-g) \times\left[\phi(a-,f-)-\phi\right]+ae(1-b)(1-c)(1-d)(a-e)\phi=0, \quad (2.11)$$

$$b(a-f)(a-g)\left[\phi(a-,e-)-\phi\right]-a(b-f)(b-g)\left[\phi(b-,e-)-\phi\right] +\frac{ab(a-b)(1-c)(1-d)}{(1-e/q)}\phi=0. \quad (2.12)$$

We now proceed to obtain a ${}_4\phi_3$ contiguous relation in which four of the seven parameters are made to vary. Replacing (b,g) by (bq,gq) in (2.9), interchanging e and g in (2.10), and then eliminating $\phi(a+,g+)$ from the resulting equations, we have

$$\left[(1-e)(1-g)(bq-f)+\frac{bq^2}{e}\,\frac{(1-f/q)(a-e)(c-e)(d-e)}{(f-eq)} - bq(1-a)(1-c)(1-d)\right]\phi(b+,g+) - \frac{bq^2}{e}\,\frac{(1-f/q)(a-e)(c-e)(d-e)}{(f-eq)}\phi(b+,e+,f-,g+) - (1-e)(1-g)(bq-f)\phi=0. \quad (2.13)$$

Next we replace (a,g) by $(a/q,g/q)$ in (2.9), interchange e and g in (2.12), and then eliminate $\phi(b-,g-)$ from the two relations. We get

$$\left[(ef-ab)(1-e)(b-g/q)-\frac{abq}{e}\,\frac{(1-f/q)(a/q-e)(b-e)(c-e)(d-e)}{(1-a/q)(f-eq)} + ab(b-e)(1-c)(1-d)\right]\phi(a-,g-) + \frac{abq}{e}\,\frac{(1-f/q)(a/q-e)(b-e)(c-e)(d-e)}{(1-a/q)(f-eq)}\phi(a-,e+,f-,g-) - (1-e)(b-g/q)\left[ef-ab+\frac{ab(1-c)(1-d)}{(1-g/q)}\right]\phi=0. \quad (2.14)$$

We now let $a\leftrightarrow b$, $f\leftrightarrow g$ in (2.13), $e\leftrightarrow g$ in (2.14), $e\leftrightarrow f$ in (2.11), and then eliminate $\phi(a+,f+)$ and $\phi(a-,e-)$ from the resulting three relations to obtain a three-term contiguous relation connecting ϕ, $\phi(a+,e+,f+,g-)$ and $\phi(a-,e-,f-,g+)$. We state this result below as Theorem 1.

Theorem 1. *If* $\phi := {}_4\phi_3\left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; q\right)$ *is a balanced terminating* ${}_4\phi_3$ *series, then*

$$\begin{aligned}
&\frac{a(1-a)(b-f)(c-f)(d-f)}{D_1}\left[\frac{aq^2}{e}\frac{(1-g/q)(b-e)(c-e)(d-e)}{(g-eq)}\right.\\
&\qquad\qquad \left.\times\,\phi(a+,e+,f+,g-) + (1-e)(1-f)(aq-g)\phi\right]\\
&\quad+\frac{f(1-f)(1-e/q)(a-f)(a-g)}{D_2}\\
&\quad\times\left[-\frac{abq}{g}\frac{(b-g)(1-f/q)(a/q-g)(c-g)(d-g)}{(1-a/q)(f-gq)}\phi(a-,e-,f-,g+)\right.\\
&\qquad\left.+(1-g)(b-e/q)\left\{(fg-ab)+\frac{ab(1-c)(1-d)}{(1-e/q)}\right\}\phi\right]\\
&\quad-\left[a(1-a)(b-f)(c-f)(d-f)+f(1-f)(1-e/q)(a-f)(a-g)\right.\\
&\qquad\left.-af(1-b)(1-c)(1-d)(a-f)\right]\phi = 0
\end{aligned}\tag{2.15}$$

where

$$\begin{aligned}
D_1 = \frac{(1-e)}{e(g-eq)}&\left[efg\left(q-aq-\frac{eq}{b}-\frac{eq}{c}-\frac{eq}{d}+\frac{eg}{bc}+\frac{eg}{cd}+\frac{eg}{bd}\right)\right.\\
&\left.+eg(eq-g-aeq)+ae^2fq^2\right]
\end{aligned}$$

and

$$\begin{aligned}
D_2 = \frac{(1-g)(f-a)}{g(1-a/q)(f-gq)}&\left[abg\left(bg+cg+dg-bc-bd-\frac{eg}{q}-\frac{fg}{q}+\frac{ef}{q^2}\right)\right.\\
&\left.+bg^2(e+f-gq)-\frac{efg^2}{q}\right].
\end{aligned}$$

Note that it requires some simple algebra to write expressions D_1 and D_2 in the form in which they are given.

It is now possible to obtain from (2.15), the required three-term contiguous relation for ${}_8W_7$ by applying the Watson's transformation [4, (III. 17)] of a terminating balanced ${}_4\phi_3$ to a terminating very-well-poised ${}_8\phi_7$:

$${}_4\phi_3\left(\begin{matrix} aq/bc, d, e, f \\ aq/b, aq/c, def/a \end{matrix}; q\right) = \frac{(aq/d, aq/e, aq/f, aq/def)_\infty}{(aq, aq/de, aq/df, aq/ef)_\infty}\,{}_8W_7(a; b, c, d, e, f) \tag{2.16}$$

with $f = q^{-n}$, say.

In (2.16), we make the parameter substitutions:

$$A = \frac{aq}{bc},\quad B = d,\quad C = e,\quad D = f,\quad E = \frac{aq}{b},\quad F = \frac{aq}{c},\quad G = \frac{def}{a} \tag{2.17}$$

where the balance condition requires

$$G = ABCDq/EF. \tag{2.18}$$

The transformations (2.16) therefore may be rewritten as

$$
{}_4\phi_3\left(\begin{matrix}A,B,C,D\\ E,F,G\end{matrix};q\right)=\frac{(CDq/G,BDq/G,BCq/G,q/G)_\infty}{(BCDq/G,Bq/G,Cq/G,Dq/G)_\infty}\times{}_8W_7\left(\frac{BCD}{G};\frac{BCDq}{GE},\frac{BCDq}{GF},B,C,D;\frac{EF}{BCD}\right).\quad(2.19)
$$

We replace small letters in relation (2.15) by capitals and apply (2.19) to the three series ϕ, $\phi(A+,E+,F+,G-)$, and $\phi(A-,E-,F-,G+)$.

Thus (2.15) will yield a relation connecting three ${}_8W_7$ series with parameters written in terms of A, B, C, D, E, F. With the help of (2.17), we substitute back the values of A, B, C, D, E, F in terms of a, b, c, d, e, f. After making simplifications, the result may be stated as:

Theorem 2. *If* $W := {}_8W_7\left(a;b,c,d,e,f;a^2q^2/bcdef\right)$ *is a terminating very-well-poised* ${}_8\phi_7$, *with say* $f=q^{-n}$, *then*

$$
\begin{aligned}
&\frac{a(1-aq)(1-qa/bc)(1-aq/bd)\cdots(1-aq/ef)}{(1-aq/b)(1-aq/c)\cdots(1-aq/f)}p(a/q)W(a+)\\
&+\Bigg[\frac{c}{q}(1-aq/bc)(1-aq/cd)(1-aq/ce)(1-aq/cf)(1-a^2q^2/bdef)(1-a^2q^2\sigma_5)p(a/q)\\
&\qquad+b(1-a/bc)(1-a/bd)(1-a/be)(1-a/bf)(1-a^2/cdef)(1-a^2q\sigma_5)p(a)\\
&+\left\{1-b-\frac{c}{q}+a\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right)-a^2q\sigma_5(d+e+f)-a^3q\sigma_5\left(1-\frac{q}{c}-\frac{1}{b}\right)\right\}p(a/q)p(a)\Bigg]W\\
&-\frac{(1-a/b)(1-a/c)\cdots(1-a/f)(1-a^2q\sigma_5)(1-a^2\sigma_5)}{(1-a)}p(a)W(a-)=0.\quad(2.20)
\end{aligned}
$$

We next proceed to remove the restriction of 'termination of series' in the statement of Theorem 2. We prove:

Theorem 3. *If* $W(a-) := {}_8W_7\left(a/q;b,c,d,e,f;a^2/bcdef\right)$ *is a convergent non-terminating very-well-poised* ${}_8\phi_7$ *series, then it satisfies the three-term contiguous relation* (2.20).

Proof. By Theorem 2, the relation (2.20) is valid for a terminating W. Suppose the termination is due to the parameter $f=q^{-n}$, n being a non-negative integer. We then replace q^{-n} by q^{-z} where z is a complex variable. We then multiply the left side of equation (2.20) by $(aq/f,a^2q^2\sigma_5)$, so that it becomes an analytic function of z which we denote by $F(z)$.

The function $F(z)$ is zero for $z=0,1,2,\ldots$, and the three ${}_8\phi_7$ series in $F(z)$ are absolutely convergent for $\Re(z)\geq 0$ since $|a^2/bcdef|<1$. Also, $F(z)$ is bounded in the half plane when $|z|\to\infty$.

Thus the conditions of Carlson's theorem [3, p. 39] are satisfied. Accordingly, $F(z)=0$ for all values of z. This completes the proof of the theorem. □

Remark 1. Note that, by the application of Carlson's theorem, we can remove the restriction of 'termination of series' in any three-term contiguous relation for a terminating ${}_8\phi_7$. However, Carlson's theorem cannot be applied to remove this restriction in a three-term relation for a terminating ${}_4\phi_3$. This is because of the fact that in this case, due to the balance condition, one cannot obtain a corresponding $F(z)$ which is both analytic and suitably bounded.

Remark 2. We now write the relation (1.5) given by Andrews and Bowman in the form

$$R(a/q)H(aq) + P(a/q)H(a) - Q(a/q)H(a/q) = 0, \tag{2.21}$$

with $Q(x)$, $P(x)$, $R(x)$ given by (1.6), (1.7), and (1.8), respectively. Using (1.4), if we change H s into W s in the relation (2.21), we observe that (2.21) and (2.20) are identical.

However, the equality of the middle term, i.e., the coefficient of W, is not so evident, and one needs to verify the following algebraic identity:

$$\begin{aligned}
&\frac{c}{q}\Big(1-\frac{aq}{bc}\Big)\Big(1-\frac{aq}{cd}\Big)\Big(1-\frac{aq}{ce}\Big)\Big(1-\frac{aq}{cf}\Big)\Big(1-\frac{a^2q^2}{bdef}\Big)(1-a^2q^2\sigma_5)p(a/q)\\
&\qquad + b\Big(1-\frac{a}{bc}\Big)\Big(1-\frac{a}{bd}\Big)\Big(1-\frac{a}{be}\Big)\Big(1-\frac{a}{bf}\Big)\Big(1-\frac{a^2}{cdef}\Big)(1-a^2q\sigma_5)p(a)\\
&+\Big[1-b-\frac{c}{q}+a\Big(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\Big)-a^2q\sigma_5(d+e+f)-a^3q\sigma_5\Big(-\frac{q}{c}-\frac{1}{b}+1\Big)\Big]p\Big(\frac{a}{q}\Big)p(a)\\
&= -a(1-a^2q\sigma_5)(\sigma_1-a\sigma_3+a^3\sigma_2\sigma_5-a^4\sigma_4\sigma_5)p(a)+(1+a\sigma_1-a^3q\sigma_5)p(a/q)p(a)\\
&\quad -\big[-a^6q^5\sigma_2\sigma_5^2+a^5q^4\sigma_1\sigma_4\sigma_5-a^5q^4\sigma_4\sigma_5+a^4q^3\sigma_3\sigma_5\\
&\qquad +a^4q^3\sigma_2\sigma_5-a^4q^3\sigma_4^2-a^3q^2\sigma_1\sigma_5+a^2q\sigma_4-a^2q\sigma_3+a\sigma_1\big]p(a/q).
\end{aligned} \tag{2.22}$$

We have verified the above identity with the help of the Maple Software program.

3. The q-difference equation and its solutions

Replacing a by aq^n in (2.21), we obtain the second-order difference equation

$$R(aq^{n-1})U_{n+1} + P(aq^{n-1})U_n - Q(aq^{n-1})U_{n-1} = 0 \tag{3.1}$$

where

$$U_n = H(aq^n) = \frac{(aq^{n+1}/b, aq^{n+1}/c, \ldots, aq^{n+1}/f)_\infty}{(aq^{n+1})_\infty}\, {}_8W_7\left(aq^n; b, c, d, e, f; \frac{a^2q^{2n+2}}{bcdef}\right). \tag{3.2}$$

Renormalizing (3.1), we obtain the equation

$$X_{n+1} + e_nX_n - f_nX_{n-1} = 0 \tag{3.3}$$

where

$$\begin{aligned}
e_n &= P(aq^{n-1}) \\
&= -aq^n(1-a^2q^{2n+1}\sigma_5)(\sigma_1 - aq^n\sigma_3 + a^3q^{3n}\sigma_2\sigma_5 - a^4q^{4n}\sigma_4\sigma_5)p(aq^n) \\
&\quad - (-1 - aq^n\sigma_1 + a^3q^{3n+1}\sigma_5)p(aq^{n-1})p(aq^n) \\
&\quad - \Big[-a^6q^{6n+5}\sigma_2\sigma_5^2 + a^5q^{5n+4}\sigma_1\sigma_4\sigma_5 - a^5q^{5n+4}\sigma_4\sigma_5 \\
&\qquad + a^4q^{4n+3}\sigma_3\sigma_5 + a^4q^{4n+3}\sigma_2\sigma_5 - a^4q^{4n+3}\sigma_4^2 - a^3q^{3n+2}\sigma_1\sigma_5 \\
&\qquad + a^2q^{2n+1}\sigma_4 - a^2q^{2n+1}\sigma_3 + aq^n\sigma_1\Big]p(aq^{n-1}), \qquad (3.4) \\
f_n &= R(aq^{n-2})Q(aq^{n-1}) \\
&= aq^{n-1}(1-aq^n/bc)(1-aq^n/bd)\cdots(1-aq^n/ef) \\
&\quad \times (1-a^2q^{2n}\sigma_5)(1-a^2q^{2n+1}\sigma_5)p(aq^{n-2})p(aq^n). \qquad (3.5)
\end{aligned}$$

From (3.2), we know that one solution of equation (3.3) is given by

$$\begin{aligned}
X_n^{(1)} = a^nq^{n(n-1)/2}\prod_{k=0}^{n} p(aq^{k-2})\frac{(aq^{n+1}/b, aq^{n+1}/c, \ldots, aq^{n+1}/f)_\infty}{(aq^{n+1})_\infty(aq^{n+1}/bc, aq^{n+1}/bd, \ldots, aq^{n+1}/ef)_\infty} \\
\times {}_8W_7\left(aq^n; b, c, d, e, f; \frac{a^2q^{2n+2}}{bcdef}\right), \qquad n \geq 0. \qquad (3.6)
\end{aligned}$$

A second solution to the equation (3.3) may be obtained by applying a 'reflection' transformation to (2.21). We make the parameter replacements

$$(a,b,c,d,e,f) \longrightarrow \left(\frac{q}{a}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{q}{f}\right) \qquad (3.7)$$

in the original equation, viz., the equation (2.21). Before making the parameter replacements in (2.21) we expand $P(a/q)$ in powers of 'a' to obtain the following:

$$P(a/q) = \sum_{k=0}^{13} C_k a^k \qquad (3.8)$$

where

$$\begin{aligned}
C_0 &= 1, \\
C_1 &= -\sigma_1, \\
C_2 &= (1+q)\sigma_3 - (1+q+q^2)\sigma_4, \\
C_3 &= (1+q+q^2+q^3)\sigma_1\sigma_5 - q\sigma_5, \\
C_4 &= (q+q^2+q^3)\sigma_4^2 - (q+q^2)\sigma_3\sigma_4 - (q+q^3)\sigma_3\sigma_5 - (1+q^3)\sigma_2\sigma_5,
\end{aligned} \qquad (3.9)$$

$$\begin{aligned}
C_5 &= (1+q+q^3+q^4)\sigma_4\sigma_5 + (q+q^3)\sigma_1\sigma_3\sigma_5 - (1+q^5)\sigma_5^2 \\
&\quad - (q+2q^2+2q^3+q^4)\sigma_1\sigma_4\sigma_5 + q^2\sigma_1\sigma_4^2, \\
C_6 &= (q^2+q^3)\sigma_2\sigma_4\sigma_5 + (q^2+q^3+q^4)\sigma_1^2\sigma_5^2 + (q+q^5)\sigma_2\sigma_5^2 \\
&\quad - (q^2+q^3)\sigma_1^2\sigma_4\sigma_5 - (q+q^4)\sigma_1\sigma_5^2 + 2q^3\sigma_3\sigma_4\sigma_5 - q^3\sigma_4^3, \\
C_7 &= -(q^2+q^3+q^4)\sigma_4^2\sigma_5 - (q+q^5)\sigma_3\sigma_5^2 - (q^3+q^4)\sigma_1\sigma_3\sigma_5^2 + (q^2+q^5)\sigma_4\sigma_5^2 \\
&\quad + (q^3+q^4)\sigma_1\sigma_4^2\sigma_5 - 2q^3\sigma_1\sigma_2\sigma_5^2 + q^3\sigma_1^3\sigma_5^2, \\
C_8 &= (q^2+2q^3+2q^4+q^5)\sigma_1\sigma_4\sigma_5^2 - (q^2+q^3+q^5+q^6)\sigma_1\sigma_5^3 + (q+q^6)\sigma_5^3 \\
&\quad - (q^3+q^5)\sigma_2\sigma_4\sigma_5^2 - q^4\sigma_1^2\sigma_4\sigma_5^2, \\
C_9 &= (q^4+q^5)\sigma_1\sigma_2\sigma_5^3 + (q^3+q^5)\sigma_2\sigma_5^3 + (q^3+q^6)\sigma_3\sigma_5^3 - (q^3+q^4+q^5)\sigma_1^2\sigma_5^3, \\
C_{10} &= -(q^3+q^4+q^5+q^6)\sigma_4\sigma_5^3 + q^5\sigma_5^4, \\
C_{11} &= (q^4+q^5+q^6)\sigma_1\sigma_5^4 - (q^5+q^6)\sigma_2\sigma_5^4, \\
C_{12} &= q^6\sigma_4\sigma_5^4, \\
C_{13} &= -q^6\sigma_5^5. \qquad (3.10)
\end{aligned}$$

Under the reflection transformation (3.7), we see that

$$a^k C_k \longrightarrow -a^{-13}q^{-6}\sigma_5^{-5}a^{13-k}C_{13-k}, \qquad (3.11)$$

and, hence,

$$P(a/q) \longrightarrow -a^{-13}\sigma_5^{-5}q^{-6}P(a/q). \qquad (3.12)$$

It is easy to obtain the reflected forms of Q and R. We find that

$$\begin{aligned}
Q(a/q) &\longrightarrow -q^{-5}\sigma_5^{-4}a^{-9}(1-a^2q^3\sigma_5)(1-a^2q^2\sigma_5)p(a/q), \\
R(a/q) &\longrightarrow -q^{-4}\sigma_5^{-6}a^{-16}p(a)(1-a/bc)(1-a/bd)\cdots(1-a/ef).
\end{aligned}$$

Thus, we obtain from (2.21), the equation

$$\begin{aligned}
&-a^{-16}\sigma_5^{-6}q^{-4}p(a)(1-a/bc)(1-a/bd)\cdots(1-a/ef)\tilde{H}(aq) - a^{-13}\sigma_5^{-5}q^{-6}P(a/q)\tilde{H}(a) \\
&\quad + a^{-9}\sigma_5^{-4}p(a/q)q^{-5}(1-a^2q^3\sigma_5)(1-a^2q^2\sigma_5)\tilde{H}(a/q) = 0 \qquad (3.13)
\end{aligned}$$

where $\tilde{H}(a)$ denotes the reflection of $H(a)$.

Again, replacing a by aq^n in (3.12) and writing $\tilde{H}(a/q) = V_{n+1}$, $\tilde{H}(a) = V_n$, and $\tilde{H}(aq) = V_{n-1}$, we have

$$\begin{aligned}
&a^4q^{4n+1}\sigma_5 p(aq^{n-1})(1-a^2q^{2n+2}\sigma_5)(1-a^2q^{2n+3}\sigma_5)V_{n+1} - P(aq^{n-1})V_n \\
&- a^{-4}q^{-4n+3}\sigma_5^{-1}aq^{n-1}p(aq^n)(1-aq^n/bc)(1-aq^n/bd)\cdots(1-aq^n/ef)V_{n-1} = 0 \\
&\qquad (3.14)
\end{aligned}$$

where

$$V_n = \frac{(bq^{-n+1}/a, cq^{-n+1}/a, \dots, fq^{-n+1}/a)_\infty}{(q^{-n+2}/a)_\infty} \times {}_8W_7\left(q^{-n+1}/a; q/b, q/c, q/d, q/e, q/f; \frac{bcdef}{a^2}q^{-2n-1}\right). \quad (3.15)$$

Renormalization of (3.13) yields the original equation (3.3). Hence, a second solution of (3.3) is given by

$$X_n^{(2)} = \text{const.}\ (-1)^n a^{4n} q^{2n^2-n} \sigma_5^n \prod_{k=0}^{n} p(aq^{k-2}) \times (a^2\sigma_5)_{2n+2} \frac{(bq^{-n+1}/a, cq^{-n+1}/a, \dots, fq^{-n+1}/a)_\infty}{(q^{-n+2}/a)_\infty} \times {}_8W_7\left(q^{-n+1}/a; q/b, q/c, \dots, q/f; \frac{bcdef}{a^2}q^{-2n-1}\right). \quad (3.16)$$

Omitting constant factors independent of n, $X_n^{(2)}$ also may be written as

$$X_n^{(2)} = (-1)^n \prod_{k=0}^{n} p(aq^{k-2}) \frac{(aq^{n-1})_\infty}{(aq^n/b, aq^n/c, \dots, aq^n/f)_\infty (a^2q^{2n+2}\sigma_5)_\infty} \times {}_8W_7\left(q^{-n+1}/a; q/b, q/c, \dots, q/f; \frac{bcdef}{a^2}q^{-2n-1}\right). \quad (3.17)$$

4. Asymptotics and the continued fraction

From (3.6), it is clear that as $n \to \infty$,

$$X_n^{(1)} \sim a^n q^{n(n-1)/2} \prod_{k=0}^{n} p(aq^{k-2}). \quad (4.1)$$

In order to find the asymptotics of the solution $X_n^{(2)}$, given by (3.16), we make use of the three-term transformation of a ${}_8W_7$ into two ${}_4\phi_3$ s. The standard transformation formula is given in [4, (III.36)]. However, it is more convenient to use the following form of the transformation derived in [10] as a limiting case of a generalized three-term transformation for complementary pairs of ${}_{10}\phi_9$ s:

$$\begin{aligned} &{}_8W_7\left(A; B, C, D, E, F; \frac{A^2q^2}{BCDEF}\right) \\ &= \frac{(Aq, B, Aq/DC, Aq/EC, Aq/FC, A^2q^2/BDEF)_\infty}{(Aq/C, Aq/D, Aq/E, Aq/F, B/C, A^2q^2/BCDEF)_\infty} \\ &\times {}_4\phi_3\left(\begin{matrix} Aq/DB, Aq/EB, Aq/FB, C \\ Aq/B, Cq/B, A^2q^2/BDEF \end{matrix}; q\right) + idem\,(B; C). \end{aligned} \quad (4.2)$$

Note that the above formula also can be derived by first applying the two-term transformation [4, (III.24)] to ${}_8W_7$ and then applying the three-term transformation [4,(III.36)] to the resulting ${}_8W_7$.

Applying (4.2) to the ${}_8W_7$ in (3.16), we obtain

$$
\begin{aligned}
&{}_8W_7\Big(q^{-n+1}/a; q/b, q/c, \ldots, q/f; \frac{bcdef}{a^2}\, q^{-2n-1}\Big) \\
&= \frac{\left(\frac{q^{-n+2}}{a}, \frac{q}{b}, \frac{cdq^{-n}}{a}, \frac{ceq^{-n}}{a}, \frac{cfq^{-n}}{a}, \frac{bdef}{a^2}\, q^{-2n}\right)_\infty}{\left(\frac{cq^{-n+1}}{a}, \frac{dq^{-n+1}}{a}, \frac{eq^{-n+1}}{a}, \frac{fq^{-n+1}}{a}, \frac{c}{b}, \frac{bcdef}{a^2}\, q^{-2n-1}\right)_\infty} \\
&\quad \times {}_4\phi_3\left(\begin{matrix} bdq^{-n}/a, beq^{-n}/a, bfq^{-n}/a, q/c \\ bq^{-n}/a, bq/c, bdefq^{-2n}/a^2 \end{matrix}; q\right) + idem\,(b; c).
\end{aligned} \tag{4.3}
$$

Applying

$$(aq^{-n})_\infty = (-1)^n a^n q^{-n(n+1)/2} (q/a)_n (a)_\infty$$

and letting $n \to \infty$, we find that the right side of (4.3) is asymptotic to

$$
\begin{aligned}
C &= \frac{\left(\frac{q^2}{a}, \frac{cd}{a}, \frac{ce}{a}, \frac{cf}{a}, \frac{a}{q}, \frac{aq}{cd}, \frac{aq}{ce}, \frac{aq}{cf}, \frac{q}{b}\right)_\infty \left(\frac{bdef}{a^2}, \frac{a^2q}{bdef}\right)_\infty}{\left(\frac{cq}{a}, \frac{dq}{a}, \frac{eq}{a}, \frac{fq}{a}, \frac{a}{c}, \frac{a}{d}, \frac{a}{e}, \frac{a}{f}, \frac{c}{b}\right)_\infty \left(\frac{bcdef}{a^2q}, \frac{a^2q^2}{bcdef}\right)_\infty} {}_1\phi_1\left(\begin{matrix} q/c \\ bq/c \end{matrix}; b\right) \\
&\quad + idem\,(b; c).
\end{aligned} \tag{4.4}
$$

Since ${}_1\phi_1\left(\begin{matrix} q/c \\ bq/c \end{matrix}; b\right) = \dfrac{(b)_\infty}{(bq/c)_\infty}$, we infer that

$${}_8W_7\left(q^{-n+1}/a; q/b, q/c, \ldots, q/f; \frac{bcdef}{a^2}\, q^{-2n-1}\right)$$

has constant asymptotics. Consequently from (3.16), we have, as $n \to \infty$,

$$X_n^{(2)} \approx (-1)^n C \prod_{k=0}^{n} \big(p(aq^{k-2})\big). \tag{4.5}$$

From (4.1) and (4.5), it follows that

$$\lim_{n\to\infty} \frac{X_n^{(1)}}{X_n^{(2)}} = 0, \tag{4.6}$$

and, therefore, $X_n^{(1)}$ is the minimal solution to the second-order difference equation (3.3).

We now apply Pincherle's theorem [12] to obtain the continued fraction associated with the equation (3.3). We use the formula

$$\frac{1}{CF} = \frac{X_1^{(1)}}{b_1 X_0^{(1)}} \tag{4.7}$$

where

$$CF = e_1 + \mathop{K}_{n=2}^{\infty} \frac{f_n}{e_n}. \tag{4.8}$$

Substituting from (3.4), (3.5), and (3.6), we have

$$\cfrac{1}{e_1 + \cfrac{f_2}{e_2 + \cfrac{f_3}{e_3 + \ddots}}}$$

$$= \frac{{}_8W_7(aq; b, c, d, e, f; a^2q^4/bcdef)}{{}_8W_7(a; b, c, d, e, f; a^2q^2/bcdef)} \times \frac{(1-aq)}{(1-\frac{aq}{b})(1-\frac{aq}{c})\cdots(1-\frac{aq}{f})(1-a^2q^2\sigma_5)(1-a^2q^3\sigma_5)p(aq)}. \tag{4.9}$$

If we replace a by x, this is Theorem 1 of Andrews and Bowman [2].

Remark 3. Andrews and Bowman [2, (3.16)] have obtained a limiting case of (4.9) by writing $f = q^{-N}$, and taking the limit as $N \to \infty$. This gives

$$\frac{{}_3\phi_2\left(\begin{matrix}a, b, c \\ d, e\end{matrix}; \frac{de}{abc}\right)}{{}_3\phi_2\left(\begin{matrix}aq, bq, cq \\ dq, eq^2\end{matrix}; \frac{de}{abc}\right)} = \frac{T'(a,b,c,d,e)}{S'(a,b,c,d,e)} + \cfrac{U'(a,b,c,d,e)/S'(a,b,c,d,e)}{\cfrac{T'(aq,bq,cq,dq,eq^2)}{S'(aq,bq,cq,dq,eq^2)} + \cdots} \tag{4.10}$$

where

$$\begin{aligned} S'(a,b,c,d,e) &= a^2b^2c^2(d)_2(e)_3(1-eq^2)(1-eq^3), \\ T'(a,b,c,d,e) &= abc(1-dq)(1-eq^2)(1-eq^3) \\ &\quad \times \Big[abc(e)_3 + de(1-eq^2)\{(ab+ac+bc+e-abc-e(a+b+c)\} \\ &\quad + d(1-e)\{eq(ab+ac+bc+e)-abc-e(a+b+c)\}\Big], \\ U'(a,b,c,d,e) &= d^2e(1-e)(1-aq)(1-bq)(1-cq)(a-eq)(b-eq)(c-eq). \end{aligned} \tag{4.11}$$

Note that we have corrected, in (4.11), a number of misprints in the expressions for S' and T' in [2, (3.14)]. The above result was obtained earlier in [5] and [6], where (4.10) is obtained with the help of the minimal solution to the three-term recurrence for the associated big q-Jacobi polynomials. We refer to [5, (3.3)], which is the same as (4.10). This is verified by writing

$$a = \frac{A}{q}, \quad b = \frac{B}{q}, \quad c = \frac{C}{q}, \quad d = -\frac{1}{q^x}, \quad e = \frac{D}{q^2}$$

in (4.10) and simplifying it to the other form.

5. Biorthogonal rational functions

Let us rewrite equation (3.1) with a new parameter 'y' inducted into the equation. Making parameter replacements

$$(a,b,c,d,e,f) \to \left(\frac{a}{y^2},\frac{b}{y},\frac{c}{y},\frac{d}{y},\frac{e}{y},\frac{f}{y}\right), \tag{5.1}$$

we find that

$$(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5) \to (y\sigma_1, y^2\sigma_2, y^3\sigma_3, y^4\sigma_4, y^5\sigma_5) \tag{5.2}$$

and

$$p(aq^n) \to p(aq^n). \tag{5.3}$$

Consequently, we can easily show that

$$\begin{aligned} R(aq^{n-1}) &\to y^{-2}R(aq^{n-1}),\\ Q(aq^{n-1}) &\to \left(1-\frac{\beta_n y}{q}\right)(1-\beta_n y)p(aq^n), \end{aligned}$$

and

$$P(aq^{n-1}) \to -\frac{1}{y}(A_n y - B_n) \tag{5.4}$$

where

$$\begin{aligned} A_n &= -p(aq^{n-1})p(aq^n) + \alpha_n\beta_n p(aq^n) + \delta_n p(aq^{n-1}),\\ B_n &= \mu_n p(aq^{n-1}) + \alpha_n p(aq^n) + \gamma_n p(aq^{n-1})p(aq^n), \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \alpha_n &= -aq^n(\sigma_1 - aq^n\sigma_3 + a^3q^{3n}\sigma_2\sigma_5 - a^4q^{4n}\sigma_4\sigma_5),\\ \beta_n &= a^2q^{2n+1}\sigma_5,\\ \gamma_n &= aq^n\sigma_1 - a^3q^{3n+1}\sigma_5,\\ \delta_n &= -a^6q^{6n+5}\sigma_2\sigma_5^2 + a^5q^{5n+4}\sigma_1\sigma_4\sigma_5 + a^4q^{4n+3}\sigma_3\sigma_5\\ &\quad - a^4q^{4n+3}\sigma_4^2 - a^3q^{3n+2}\sigma_1\sigma_5 + a^2q^{2n+1}\sigma_4,\\ \mu_n &= a^5q^{5n+4}\sigma_4\sigma_5 - a^4q^{4n+3}\sigma_2\sigma_5 + a^2q^{2n+1}\sigma_3 - aq^n\sigma_1. \end{aligned} \tag{5.6}$$

With the above substitutions made in equation (3.1), we have

$$\begin{aligned} \frac{aq^n}{y}p(aq^{n-1})\left(1-\frac{aq^{n+1}}{bc}\right)\left(1-\frac{aq^{n+1}}{bd}\right)\cdots\left(1-\frac{aq^{n+1}}{ef}\right)Y_{n+1} - (A_n y - B_n)Y_n\\ - y\left(1-\frac{\beta_n y}{q}\right)(1-\beta_n y)p(aq^n)Y_{n-1} = 0, \end{aligned} \tag{5.7}$$

with a solution given by

$$\begin{aligned} Y_n = \frac{(aq^{n+1}/by, aq^{n+1}/cy, \dots, aq^{n+1}/fy)_\infty}{(aq^{n+1}/y^2)_\infty}\\ \times {}_8W_7\left(aq^n/y^2; b/y, c/y, \dots, f/y; \frac{a^2q^{2n+2}y}{bcdef}\right). \end{aligned} \tag{5.8}$$

Renormalizing the equation (5.7), we obtain the equation in the form required for an R_{II}-fraction [11]. Namely

$$Z_n(y) - (y - c_n)Z_{n-1}(y) + \lambda_n(y - a_n)(y - b_n)Z_{n-2}(y) = 0 \tag{5.9}$$

where

$$a_n = q\beta_n^{-1} = a^{-2}q^{-2n}\sigma_5^{-1}, \quad b_n = \beta_n^{-1} = a^{-2}q^{-2n-1}\sigma_5^{-1}, \quad c_n = B_n/A_n,$$
$$\lambda_n = -\frac{a^5q^{5n}}{A_nA_{n-1}}(1 - aq^n/bc)(1 - aq^n/bd)\cdots(1 - aq^n/ef)\ \sigma_5^2p(aq^n)p(aq^{n-2}). \tag{5.10}$$

From (5.8), a solution of (5.9) is

$$Z_n^{(1)} = \left(\frac{a}{y}\right)^{n+1} q^{n(n+1)/2} \prod_{j=0}^{n}\left(\frac{p(aq^{j-1})}{A_j}\right)$$
$$\times \frac{(aq^{n+2}/by, aq^{n+2}/cy, \ldots, aq^{n+2}/fy)_\infty}{(aq^{n+2}/y^2)_\infty(aq^{n+2}/bc, aq^{n+2}/bd, \ldots, aq^{n+2}/ef)_\infty}$$
$$\times {}_8W_7\left(aq^{n+1}/y^2; b/y, c/y, \ldots, f/y; \frac{a^2q^{2n+4}y}{bcdef}\right). \tag{5.11}$$

We can obtain a second solution to the equation (5.9) from the formula we obtained for a second solution $X_n^{(2)}$ to equation (3.1). The second solution to (5.9) works out to be

$$Z_n^{(2)} = (-y)^n \prod_{j=0}^{n}\left(\frac{p(aq^{j-1})}{A_j}\right)$$
$$\times \frac{(aq^n/y^2)_\infty}{(aq^{n+1}/by, aq^{n+1}/cy, \ldots, aq^{n+1}/fy)_\infty(a^2q^{2n+4}\sigma_5 y)_\infty}$$
$$\times {}_8W_7\left(y^2q^{-n}/a; yq/b, yq/c, \ldots, yq/f; \frac{bcdef}{ya^2}q^{-2n-3}\right). \tag{5.12}$$

We now consider the R-fraction of the type R_{II} associated with the recurrence relation (5.9). From (5.9), we have

$$\frac{1}{\lambda_1(y-a_1)(y-b_1)}\frac{Z_0^{(1)}}{Z_{-1}^{(1)}} = \cfrac{1}{y - c_1 - \cfrac{\lambda_2(y-a_2)(y-b_2)}{y - c_2 - \cfrac{\lambda_3(y-a_3)(y-b_3)}{y - c_3 - \ddots}}}. \tag{5.13}$$

Substituting for $Z_0, Z_{-1}, \lambda_1, a_1, b_1$ from (5.9) and (5.11), we find that

$$\frac{1}{\lambda_1(y-a_1)(y-b_1)}\frac{Z_0^{(1)}}{Z_{-1}^{(1)}} = -\frac{A_1(y - a^{-2}q^{-2}\sigma_5^{-1})^{-1}(y - a^{-2}q^{-3}\sigma_5^{-1})^{-1}}{p(aq)a^4yq^5\sigma_5^2}$$
$$\times \frac{\left(\frac{aq}{y^2}\right)_\infty\left(\frac{aq^2}{by}, \frac{aq^2}{cy}, \ldots, \frac{aq^2}{fy}\right)_\infty {}_8W_7\left(\frac{aq}{y^2}; \frac{b}{y}, \frac{c}{y}, \ldots, \frac{f}{y}; \frac{a^2q^4y}{bcdef}\right)}{\left(\frac{aq^2}{y^2}\right)_\infty\left(\frac{aq}{by}, \frac{aq}{cy}, \ldots, \frac{aq}{fy}\right)_\infty {}_8W_7\left(\frac{a}{y^2}; \frac{b}{y}, \frac{c}{y}, \ldots, \frac{f}{y}; \frac{a^2q^2y}{bcdef}\right)}. \tag{5.14}$$

An application of the transformation [4, (III.36)] to the ${}_8W_7$ in the denominator above for $aq/bc = 1$ yields

$$\frac{(aq/y^2, aq/de, aq/df, aq/ef)_\infty}{(aq/dy, aq/ey, aq/fy, aqy/def)_\infty}. \tag{5.15}$$

Henceforth, we impose the restriction $aq = bc$. To the numerator ${}_8W_7$ in (5.14), we apply the ${}_8\phi_7$ to ${}_8\phi_7$ transformation [4, (III.23)]. We then obtain

$$\frac{1}{\lambda_1(y-a_1)(y-b_1)}\frac{Z_0^{(1)}}{Z_{-1}^{(1)}} = -\frac{1}{y}\frac{A_1}{p(aq)}\frac{\left(\frac{aq^2}{dy}, \frac{aq^2}{ey}, \frac{aq^2}{fy}\right)_\infty \left(q, \frac{a^2q^4}{bdef}, \frac{a^2q^4}{cdef}\right)_\infty}{\left(\frac{aq}{by}, \frac{aq}{cy}, \frac{a^2q^4}{defy}\right)_\infty \left(\frac{aq}{de}, \frac{aq}{df}, \frac{aq}{ef}\right)_\infty}$$
$$\times {}_8\phi_7\left(\begin{matrix}\frac{a^2q^3}{defy}, q\sqrt{\frac{a^2q^3}{defy}}, -q\sqrt{\frac{a^2q^3}{defy}}, \frac{aq^2}{ef}, \frac{aq^2}{df}, \frac{aq^2}{de}, \frac{b}{y}, \frac{c}{y} \\ \sqrt{\frac{a^2q^3}{defy}}, -\sqrt{\frac{a^2q^3}{defy}}, \frac{aq^2}{dy}, \frac{aq^2}{ey}, \frac{aq^2}{fy}, \frac{a^2q^4}{bdef}, \frac{a^2q^4}{cdef}\end{matrix}; q\right). \tag{5.16}$$

From (5.16), it is clear that the singularities are given by $(aq/by, aq/cy)_\infty = 0$, i.e.,

$$y = aq^{m+1}/b \quad \text{and} \quad y = aq^{m+1}/c \quad \text{for} \quad m = 0, 1, 2, \ldots.$$

The residue at $y = aq^{m+1}/b$ easily works out to be

$$-\frac{A_1}{p(aq)}\frac{(-1)^m q^{m(m+1)/2}}{(q)_m(q)_\infty}\frac{\left(\frac{bq^{-m+1}}{d}, \frac{bq^{-m+1}}{e}, \frac{bq^{-m+1}}{f}\right)_\infty \left(q, \frac{a^2q^4}{bdef}, \frac{a^2q^4}{cdef}\right)_\infty}{\left(\frac{bq^{-m}}{c}, \frac{abq^{-m+3}}{def}\right)_\infty \left(\frac{aq}{de}, \frac{aq}{df}, \frac{aq}{ef}\right)_\infty}$$
$$\times {}_8W_7\left(\frac{abq^{-m+2}}{def}; \frac{aq^2}{ef}, \frac{aq^2}{df}, \frac{aq^2}{de}, \frac{b^2}{a}, q^{-m-1}, q^{-m}; q\right). \tag{5.17}$$

The ${}_8W_7$ above is summed up by Jackson's q-analogue of Dougall's ${}_7F_6$-sum [4, II.22]. We thus obtain the residue at $y = aq^{m+1}/b$ as

$$\gamma_m^{(b)} = -\frac{A_1}{p(aq)}\frac{\left(\frac{a^2q^4}{bdef}, \frac{bq}{d}, \frac{bq}{e}, \frac{bq}{f}\right)_\infty}{\left(\frac{aq}{de}, \frac{aq}{df}, \frac{aq}{ef}, \frac{b^2}{aq}\right)_\infty}\frac{\left(\frac{aq^2}{bd}, \frac{aq^2}{be}, \frac{aq^2}{bf}\right)_m}{\left(q, \frac{cq}{b}, \frac{a^2q^4}{bdef}\right)_m} q^m. \tag{5.18}$$

The residue at $y = aq^{m+1}/c$ is similarly obtained. Using $aq = bc$, it is given by

$$\gamma_m^{(c)} = -\frac{A_1}{p(aq)}\frac{\left(\frac{a^2q^3}{def}, \frac{aq^2}{bd}, \frac{aq^2}{be}, \frac{aq^2}{bf}\right)_\infty}{\left(\frac{aq}{de}, \frac{aq}{df}, \frac{aq}{ef}, \frac{aq}{b^2}\right)_\infty}\frac{\left(\frac{bq}{d}, \frac{ab}{e}, \frac{bq}{f}\right)_m}{\left(q, \frac{b^2}{a}, \frac{abq^3}{def}\right)_m} q^m. \tag{5.19}$$

The sum of the residues at all singularities therefore is given by

$$\sum_{m=0}^{\infty}(\gamma_m^{(b)}+\gamma_m^{(c)}) = -\frac{A_1}{p(aq)}\frac{1}{\left(\frac{aq}{de},\frac{aq}{df},\frac{aq}{ef}\right)_\infty}\left[\frac{\left(\frac{a^2q^4}{bdef},\frac{bq}{d},\frac{bq}{e},\frac{bq}{f}\right)_\infty}{\left(\frac{b^2}{aq}\right)_\infty}\right.$$

$$\left.\times\, {}_3\phi_2\left(\begin{matrix}\frac{aq^2}{bd},\frac{aq^2}{be},\frac{aq^2}{bf}\\ \frac{aq^2}{b^2},\frac{a^2q^4}{bdef}\end{matrix};q\right) + \frac{\left(\frac{abq^3}{def},\frac{aq^2}{bd},\frac{aq^2}{be},\frac{aq^2}{bf}\right)_\infty}{\left(\frac{aq}{b^2}\right)_\infty}\,{}_3\phi_2\left(\begin{matrix}\frac{bq}{d},\frac{bq}{e},\frac{bq}{f}\\ \frac{b^2}{a},\frac{abq^3}{def}\end{matrix};q\right)\right]. \tag{5.20}$$

An explicit polynomial solution. A lot of simplification takes place in the case $aq = bc$. Applying the transformation [4, (III.36)] to the ${}_8W_7$ in (5.12), we can write $Z_n^{(2)}$ in the form

$$Z_n^{(2)} = (-y)^n\prod_{j=0}^{n}\left(\frac{p(aq^{j-1})}{A_j}\right)\frac{(aq^n/y^2)_\infty}{\left(\frac{aq^{n+1}}{by},\frac{aq^{n+1}}{cy},\ldots,\frac{aq^{n+1}}{fy}\right)_\infty(a^2q^{2n+4}\sigma_5 y)_\infty}$$

$$\times\frac{\left(\frac{qy^2}{a},\frac{de}{aq},\frac{df}{aq},\frac{ef}{aq}\right)_\infty}{\left(\frac{dy}{a},\frac{ey}{a},\frac{fy}{a},\frac{def}{ayq^2}\right)_\infty}\frac{\left(\frac{a}{y^2},\frac{aq^2}{de},\frac{aq^2}{df},\frac{aq^2}{ef}\right)_n}{\left(\frac{aq}{dy},\frac{aq}{ey},\frac{aq}{fy},\frac{ayq^3}{def}\right)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},\frac{yq}{d},\frac{yq}{e},\frac{yq}{f}\\ \frac{by}{a}q^{-n},\frac{cy}{a}q^{-n},\frac{ayq^{n+3}}{def}\end{matrix};q\right). \tag{5.21}$$

From (5.21), we write the expressions for $Z_0^{(2)}$ and $Z_1^{(2)}$, and consequently, we obtain the expression for $P_1 = Z_1^{(2)}/Z_0^{(2)}$. After simplification, we have

$$P_1 = \frac{p(a)}{A_1}\left(1-\frac{aq^2}{de}\right)\left(1-\frac{aq^2}{df}\right)\left(1-\frac{aq^2}{ef}\right)\left[b+\frac{aq}{b}+\frac{a^2q^5}{def}-aq^2\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right)\right.$$
$$\left.-y+aq^3y\left(\frac{1}{de}+\frac{1}{df}+\frac{1}{ef}\right)-\frac{a^2q^5y}{def}\left(\frac{b}{aq}+\frac{1}{b}\right)\right]. \tag{5.22}$$

In order to obtain a simplified expression for $p(a)/A_1$ in the case $aq = bc$, we prefer to write

$$\sigma_1 = \lambda + s_1,\quad \sigma_2 = \frac{1}{aq}+\lambda s_1 + s_2,\quad \sigma_3 = \frac{s_1}{aq}+\lambda s_2 + s_3,$$
$$\sigma_4 = \frac{s_2}{aq}+\lambda s_3,\quad \sigma_5 = \frac{s_3}{aq} \tag{5.23}$$

where

$$s_1 = 1/d+1/e+1/f,\quad s_2 = 1/de+1/df+1/ef,$$
$$s_3 = 1/def,\qquad \lambda = 1/b+b/aq. \tag{5.24}$$

Therefore, from (1.9), (5.5), (5.23), and (5.24), we get

$$\begin{aligned} p(a) &= 1 - aqs_2 + a^2q^2s_1s_3 - a^3q^3s_3^2 = (1-aq/de)(1-aq/df)(1-aq/ef),\\ p(aq) &= 1 - aq^3s_2 + a^2q^5s_1s_3 - a^3q^8s_3^2 - a^2q^4\lambda s_3(1-q), \\ \alpha_1 &= -aq\lambda p(a), \qquad \beta_1 = aq^2s_3, \end{aligned} \tag{5.25}$$

and

$$\delta_1 = a^2q^3(s_2/aq + \lambda s_3)p(aq) + a^2q^4s_3\Big[-\lambda(1 - aq^2s_2 + a^2q^5s_1s_3 - a^3q^7s_3^2) + (1-q)(-s_1 - a^2q^5s_2s_3 + aq^2s_3 + aq^3s_3)\Big].$$

Considerable algebra is required in determining the value of A_1 from (5.6) and then obtaining the value of $A_1/p(a)$ in the form below:

$$\frac{A_1}{p(a)} = -\left(1-\frac{aq^2}{de}\right)\left(1-\frac{aq^2}{df}\right)\left(1-\frac{aq^2}{ef}\right)(1 - aq^3s_2 + a^2q^5s_3\lambda). \tag{5.26}$$

Consequently, the monic polynomial P_1 can be written from (5.22) in the form

$$\begin{aligned} P_1(y) &= y - \frac{aq(\lambda - qs_2 + aq^4s_3)}{1 - aq^3s_2 + a^2q^5\lambda s_3} \\ &= y - \frac{aq\left(\frac{b}{aq} + \frac{1}{b} - q\left(\frac{1}{de} + \frac{1}{df} + \frac{1}{ef}\right) + \frac{aq^4}{def}\right)}{1 - aq^3\left(\frac{1}{de} + \frac{1}{df} + \frac{1}{ef}\right) + \frac{a^2q^5}{def}\left(\frac{b}{aq} + \frac{1}{b}\right)}. \end{aligned} \tag{5.27}$$

This provides a check of our calculations. In general, the monic polynomial solution of degree n to the recurrence (5.9) in the case $aq = bc$ is given explicitly by

$$P_n(y) = Z_n^{(2)}(y)\Big/Z_0^{(2)}(y) \tag{5.28}$$

with $Z_n^{(2)}$ given by (5.21). Thus,

$$\begin{aligned} P_n(y) = (-y)^n\left(\prod_{j=1}^{n}\frac{p(aq^{j-1})}{A_j}\right)&\frac{\left(\frac{aq}{by}, \frac{aq}{cy}, \frac{aq^2}{de}, \frac{aq^2}{df}, \frac{aq^2}{ef}\right)_n (a^2q^4\sigma_5 y)_{2n}}{(ayq^3/def)_n} \\ &\times {}_4\phi_3\left(\begin{matrix} q^{-n}, \frac{yq}{d}, \frac{yq}{e}, \frac{yq}{f} \\ \frac{by}{a}q^{-n}, \frac{cy}{a}q^{-n}, \frac{ay}{def}q^{n+3} \end{matrix}; q\right). \end{aligned} \tag{5.29}$$

The biorthogonality. We now record the biorthogonality associated with this R_{II}-fraction [11]. We first define the rational functions

$$S_n(y) = \frac{P_n(y)}{\prod_{k=1}^{n}(y-a_{k+1})(y-b_{k+1})} = \left(\frac{a}{def}\right)^{2n} q^{2n^2+5n}\frac{P_n(y)}{(ayq^3)_{2n}}. \tag{5.30}$$

The biorthogonality then is given by

$$\sum_{m=0}^{\infty} \gamma_m^{(b)} P_k(aq^{m+1}/b) S_n(aq^{m+1}/b) + \gamma_m^{(c)} P_k(aq^{m+1}/c) S_n(aq^{m+1}/c) = 0,$$
$$0 \le k < n, \quad bc = aq. \tag{5.31}$$

A $_3\phi_2$ summation formula. The large y asymptotics of (5.16) yields

$$\sum_{m=0}^{\infty}(\gamma_m^{(b)}+\gamma_m^{(c)}) = -\frac{A_1}{p(aq)}\frac{\left(q,\frac{a^2q^4}{bdef},\frac{a^2q^4}{cdef}\right)_\infty}{\left(\frac{aq}{de},\frac{aq}{df},\frac{aq}{ef}\right)_\infty}\,{}_3\phi_2\left(\begin{matrix}\frac{aq^2}{ef},\frac{aq^2}{df},\frac{aq^2}{de}\\ \frac{a^2q^4}{bdef},\frac{a^2q^4}{cdef}\end{matrix};q\right) \tag{5.32}$$

where $bc = aq$.

Comparing (5.32) with (5.20), we obtain a new $_3\phi_2$ summation formula. The $_3\phi_2$ s involved in (5.20) are of type I [4, p. 60] where the argument is 'q'. Also, they may be said to be 0-balanced because the product of the denominator parameters is equal to the product of the numerator parameters. The $_3\phi_2$ in (5.32) is also of type I, but is 1-balanced.

We thus arrive at a formula connecting a pair of 0-balanced, type I, $_3\phi_2$ s with a 1-balanced, type I, $_3\phi_2$. With a renaming of the parameters, we may write the identity as

$$\frac{(F,Aq/E,Bq/E,Cq/E)_\infty}{(q/E)_\infty}\,{}_3\phi_2\left(\begin{matrix}A,B,C\\ E,F\end{matrix};q\right)+\frac{(qF/E,A,B,C)_\infty}{(E/q)_\infty}$$
$$\times\,{}_3\phi_2\left(\begin{matrix}Aq/E,Bq/E,Cq/E\\ q^2/E,qF/E\end{matrix};q\right) = (q,F,qF/E)_\infty\,{}_3\phi_2\left(\begin{matrix}F/A,F/B,F/C\\ F,qF/E\end{matrix};q\right) \tag{5.33}$$

where

$$EF = ABC.$$

A limiting case of (5.33) for $F, C \to 0$ yields the non-terminating form of the q-Vandermonde sum (see [4,(II.23)].

Acknowledgment. This research was supported partially by NSERC (Canada).

References

1. G. E. Andrews, *On q-difference equations for certain well-poised basic hypergeometric series*, Quart. J. Math. Oxford Ser.(2) **19** (1968), 433–447.
2. G. E. Andrews and D. Bowman, *A full extension of the Rogers-Ramanujan continued fraction*, Proc. Amer. Math. Soc. **123** (1995), 3343–3350.
3. W. N. Bailey, *Generalized Hypergeometric Series*, reprinted by Hafner, New York, Cambridge University Press, Cambridge, 1972.
4. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.
5. D. P. Gupta, M. E. H. Ismail, and D. R. Masson, *Contiguous relations, basic hypergeometric functions and orthogonal polynomials II: associated big q-Jacobi polynomials*, J. Math. Anal, Appl. **171** (1992), 477–497.
6. ______, *Contiguous relations, basic hypergeometric functions and orthogonal polynomials III: associated continuous dual q-Hahn polynomials*, J. of Comp. and App. Math. **68** (1996), 115–149.
7. D. P. Gupta and D. R. Masson, *Solutions to the associated q-Askey-Wilson polynomial recurrence relation.* In: Approximation and Computation, (R. Zahar, ed.), ISNM, vol. 119, Birkhaüser, Boston, 1994, pp.273–284.
8. ______, *Watson's basic analogue of Ramanujan's Entry 40 and its generalization*, SIAM J. Math. Anal. **25** (1994), 429–440.
9. ______, *Contiguous relations, continued fractions and orthogonality: An* ${}_8\phi_7$ *model*, J. of Comp. and Appl. Math. **65** (1995), 157–164.
10. ______, *Contiguous relations, continued fractions and orthogonality*, Trans. Amer. Math. Soc. **350** (1998), 769–808.
11. M. E. H. Ismail and D. R. Masson, *Generalized orthogonality and continued fractions*, J. of Approx. Theory **83** (1995), 1–40.
12. W. B. Jones and W. J. Thron, *Continued Fractions: Analytic Theory and Applications*, Addison-Wesley, Reading, MA., 1980.
13. S. Ramanujan, *Notebook 2*, Chapter 12, Tata Institute of Fundamental Research, Bombay, 1957.
14. G. N. Watson, *Ramanujan's continued fraction*, Proc. Cambridge Philos. Soc. **31** (1935), 7–17.
15. J. A. Wilson, *Hypergeometric Series, Recurrence Relations and Some New Orthogonal Functions*, Ph.D. Thesis, University of Wisconsin, Madison, WI, 1978.

Department of Mathematics, University of Toronto, Toronto, M5S 3G3, Canada